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Abstract: In this paper, a control scheme is elaborated to perform the station keeping of a geostationary satellite equipped with electric propulsion. The use of electric thrusters imposes to take into account some additional mutually exclusive constraints that can be reformulated as logical constraints. The optimal station keeping problem is thus not solved with classical methods, either direct or indirect, but is transformed into a linear integer programming problem. The linearised relative velocity of the satellite is computed and some constraints on this velocity are added to the station keeping problem . Simulation results validate the efficiency of the optimal control thrusts obtained with these methods.

Keywords: Satellite control, GEO satellite station keeping, electric propulsion, integer programming.

1. INTRODUCTION

Satellites operating in Geostationary Earth Orbit (GEO) drifts outside its station keeping (SK) window (a rectangular box of a given geographical longitude and latitude range) due to orbital disturbing forces. Performing an accurate SK strategy is therefore necessary thanks to electric and/or chemical thrusters in order to fulfil that role.

Chemical propulsion systems have been and are widely used. For these propulsion systems with high thrust capabilities, SK control laws are usually designed assuming an impulsive idealisation of the thrust, as described for example in Soop (1994). Electric propulsion for station keeping has been used since the sixties (see, e.g. the work of Barrett (1967) and Hunziker (1970)) and some theoretical developments have been presented in the eighties by Anzel (1988) and Eckstein (1980). Nowadays, the SK of GEO satellites can be performed reliably by electric propulsion as well as the chemical one. The bigger specific impulsion of electric thrusters leads to a consequent savings in fuel consumption, what enables a reduction of the satellite mass and increased payload capacity and/or improved satellite longevity.

Considering these technological and operational features, optimal control strategies for electrical SK, taking various constraints into account (minimum elapsed time between two consecutive firings, on-off profile of the thrusters, thrust allocation) have to be carefully designed. Some references as for example Sidi (1997), Campan et al. (1995) or Soop (1994) have established some rule of thumb based on the physical analysis of the effect of the external forces in order to perform the SK. In general, the problem of station keeping can be expressed as an optimal control problem even if the above mentioned constraints, inherent to the use of electric thrusters, prevent us to solve it with classical methods.

Optimal SK control problems may be solved according to several approaches. When simplifying assumptions are used, analytical control laws may be obtained, as in Sukhanov and Prado (2012). Otherwise, it is in general necessary to resort to numerical methods, such as direct collocation based methods as described in Hull (1997) and Betts (1998). For this kind of approaches integrating the satellite dynamic, the state and the control variables are discretised to produce a non linear programming problem and get an optimal open loop control. To deal with on-off models of thrusts, rectangular profiles may be generated from a continuous one with the Pulse Width Modulation technique (see Vazquez et al. (2015) and the references therein). Losa et al. (2005) has formulated a method based on differential inclusion and Losa et al. (2006) and Gazzino et al. (2016) have implemented a decomposition technique.

In this paper, the idea is to use the particular form of the thrusters operational constraints that cannot be

taken into account easily with classical methods for solving the optimal control problem, in order to transform the linearised SK optimal control problem in a linear integer programming problem, using a discretisation of the state and control variables. Unlike the direct collocation method, the dynamic equation of the satellite evolution is integrated beforehand using its state transition matrix, allowing to remove the state vector from the optimisation variables set. The operational constraints are expressed as logical constraints such that the control variables are now be binary variables in this formulation. A simple way of transforming the logical constraints in binary variables constraints is investigated. A realistic numerical example illustrates the efficiency of the proposed approach and some comparison are made between a rule of thumb for SK strategies found in the literature and the systematic optimisation based approach presented in this paper.

2. STATION KEEPING PROBLEM

2.1 Dynamic Modeling

Let us consider a satellite equipped with 4 electric thrusters mounted on the anti-nadir face. The position of the satellite on its orbit is described with the equinoctial orbit elements as defined in Battin (1999):

$$\boldsymbol{x}_{eoe} = \begin{bmatrix} a & e_x & e_y & i_x & i_y & \ell_{M\Theta} \end{bmatrix}^t \in \mathbb{R}^6, \tag{1}$$

where a is the semi-major axis, (e_x, e_y) the eccentricity vector components, (i_x, i_y) the inclination vector components, $\ell_{M\Theta} = \omega + \Omega + M - \Theta$ is the mean longitude where Ω is the right ascension of the ascending node, ω is the argument of perigee, M is the mean anomaly and $\Theta(t)$ is the right ascension of the Greenwich meridian. The dynamics of the satellite may be represented by the following non linear state-space model:

$$\frac{d\boldsymbol{x}_{eoe}}{dt} = \boldsymbol{f_L}(\boldsymbol{x}_{eoe}, t) + \boldsymbol{f_G}(\boldsymbol{x}_{eoe}, t)\boldsymbol{u}, \tag{2}$$

where $\boldsymbol{f_L} \in \mathbb{R}^6$ is the Lagrange contribution part of the external force model described by the CNES ORANGE model (cf. Campan and Brousse (1994)) and $\boldsymbol{f_G} \in \mathbb{R}^{6\times 3}$ is the Gauss contribution part.

In order to deal with the station keeping problem, the relative state of the satellite with respect to the station keeping state

$$\boldsymbol{x}_{sk} = \begin{bmatrix} a_{sk} & 0 & 0 & 0 & 0 & \ell_{M\Theta_{sk}} \end{bmatrix}^t, \tag{3}$$

is defined, where a_{sk} is the synchronous semi-major axis and $\ell_{M\Theta_{sk}}$ is the station mean longitude.

The relative dynamics equations are developed by linearisation of Equation (2) about the station keeping point (3). By denoting $\boldsymbol{x} = \boldsymbol{x}_{eoe} - \boldsymbol{x}_{sk}$ the relative state model reads:

$$\dot{\boldsymbol{x}} = A(t)\boldsymbol{x} + \boldsymbol{D}(t) + B(t)\boldsymbol{u}, \tag{4}$$

where the matrices:

$$A(t) = \frac{\partial}{\partial \boldsymbol{x}_{eoe}} \left(\overrightarrow{f_L}(\boldsymbol{x}_{eoe}(t), t) \right) \Big|_{\boldsymbol{x}_{eoe}=\boldsymbol{x}_{sk}} \in \mathbb{R}^{6\times 6}, \tag{5}$$

$$B(t) = f_G(\boldsymbol{x}_{sk}, t) \in \mathbb{R}^{6\times 3}, \tag{6}$$

$$\boldsymbol{D}(t) = \boldsymbol{f_L}(\boldsymbol{x}_{sk}, t) \in \mathbb{R}^6, \tag{7}$$

are obtained from the linearisation of functions $\boldsymbol{f_L}$ and $\boldsymbol{u} \mapsto f_G\boldsymbol{u}$.

As opposed to what is done in the literature, the linearisation was not performed with respect to an equilibrium point, but around a fictitious geostationary point. In other words, the equinoctial orbital elements $\boldsymbol{x}_{eoe}$ undergo all disturbing external forces whereas the geostationary equinoctial elements $\boldsymbol{x}_{sk}$ evolves under a keplerian motion. These two different dynamics for the state vector and the station keeping state vector explains the term $\boldsymbol{D}(t)$ in the relative state model (4).

Recalling that 4 thrusters are available to realize the control, the transcription of the station keeping problem expressed in terms of the control vector $\boldsymbol{u}$ would require to solve an allocation problem to find a right combination of thrusts. An alternative, consists in considering directly the 4 thrusts provided by the 4 engines in the satellite dynamic. The control $\boldsymbol{u}(t)$ can be written as a linear combination of the 4 thrusts such that $\boldsymbol{u} = \Gamma F_{max}\boldsymbol{F}$, where $\boldsymbol{F} \in \mathbb{R}^4$, $\Gamma \in \mathbb{R}^{3\times 4}$ and F_{max} is the maximum value of thrust of all the 4 thrusters. . In order to model the on-off nature of the control profile, the thrust vector $\boldsymbol{F} = F_{max}\begin{bmatrix} F_1 & F_2 & F_3 & F_4 \end{bmatrix}^t$ is composed of four normalized variables. The thrust direction matrix $\Gamma = \begin{bmatrix} \boldsymbol{\Gamma}_1 & \boldsymbol{\Gamma}_2 & \boldsymbol{\Gamma}_3 & \boldsymbol{\Gamma}_4 \end{bmatrix}^t \in \mathbb{R}^{3\times 4}$ is defined such that each thrust direction $\boldsymbol{\Gamma}_j \in \mathbb{R}^3$ are given by :

$$\boldsymbol{\Gamma_j} = \frac{1}{m}\begin{bmatrix} -\sin\theta_j\cos\alpha_j & -\sin\theta_j\sin\alpha_j & -\cos\theta_j \end{bmatrix} \tag{8}$$

where angles θ_j and α_j are defined exactly as in Anzel (1988). Finally, the relative state model for the addressed SK problem is given by:

$$\dot{\boldsymbol{x}} = A(t)\boldsymbol{x} + \boldsymbol{D}(t) + B(t)\Gamma\boldsymbol{F}, \tag{9}$$

The geographical coordinates of the satellite:

$$\boldsymbol{y}_{eoe} = T(\boldsymbol{x}_{eoe}, t)\boldsymbol{x}_{eoe}, \tag{10}$$

are the variables of interest since the station keeping problem consists in constraining them in the vicinity of the station position $\boldsymbol{y}_{sk} = \begin{bmatrix} r_{sk} & 0 & \lambda_{sk} \end{bmatrix}^t$ where r_{sk} is the synchronous radius and λ_{sk} is the station keeping geographical longitude. The relative geographical position with respect to the station-keeping position is denoted by:

$$\boldsymbol{y} = \boldsymbol{y}_{eoe} - \boldsymbol{y}_{sk} = T(\boldsymbol{x}_{sk}, t)\boldsymbol{x} = C(t)\boldsymbol{x}, \tag{11}$$

by linearising Equation (10).

The relative geographical velocity of the satellite is computed by linearising the derivative of Equation (10). As done with the linearisation of the derivative of the state vector, the station-keeping position is a fictitious point supposed to evolve following a keplerian motion, whereas $\boldsymbol{y}_{eoe}$ is supposed to undergo all the external forces. The dynamics of the relative geographical position reads thus:

$$\dot{\boldsymbol{y}}(t) = E(t)\boldsymbol{x}(t) + C(t)\boldsymbol{D}(t) + F_{max}C(t)B(t)\Gamma\tilde{\boldsymbol{F}}(t), \quad (12)$$

$$\text{with: } E(t) = 2H(t) + C(t)A(t) + \dot{C}(t) \in \mathbb{R}^{3\times 6} \quad (13)$$

$$\text{and: } H(t) = \left.\frac{\partial[T(\boldsymbol{x}_{eoe}, t)]}{\partial \boldsymbol{x}_{eoe}}\right|_{\boldsymbol{x}=\boldsymbol{x}_{sk}} \boldsymbol{D}(t), \quad (14)$$

2.2 Station Keeping as Constrained Optimal Control Problem

The station keeping problem consists in maintaining the longitude and the latitude of the satellite in a box defined by its size δ on a fixed time horizon by acting on the orbital parameters via the 4 thrusters. An open loop control laws can be obtained by solving an optimal control problem. In this context, optimality means that a minimum fuel-solution is looked for to extend the operational life time of the satellite. Therefore, performing minimum-fuel station keeping amounts to minimizing the following performance index under the dynamic constraint (9):

$$\hat{J} = \int_0^T \sum_{\text{thruster } i=1}^{4} \left(|u_{R_i}(t)| + |u_{T_i}(t)| + |u_{N_i}(t)|\right) dt$$

$$\Leftrightarrow J = \sum_{\text{thruster } i=1}^{4} ||\boldsymbol{\Gamma}_i||_1 \int_0^T F_i(t)dt. \quad (15)$$

2.2.1 Station Keeping Requirements (Orbital Constraints)

Station keeping constraints are imposed to ensure that the satellite stays in the geographical box. This box is defined in the plane (latitude,longitude) of width $2\delta \times 2\delta$ centered on the station keeping geographical position. By denoting $\boldsymbol{y} = \boldsymbol{y}_{eoe} - \boldsymbol{y}_{sk}$, the station-keeping constraints over a finite time interval can be written as:

$$|[0\ 1\ 0]\boldsymbol{y}(t)| \leqslant \delta \text{ and } |[0\ 0\ 1]\boldsymbol{y}(t)| \leqslant \delta\ \forall t \in [0, T]. \quad (16)$$

The initial condition is given by $\boldsymbol{x}(0) = \boldsymbol{x}_0$ whereas the final state $\boldsymbol{x}(t_f)$ is free. The constraints on the geographical position may not suffice to perform the station keeping over several time intervals. For instance, the final position at $t = T$ may be closed to the upper boundary of the SK window with the satellite velocity pointing outward. In such a case, it should not be possible for a limited-propulsion satellite to remain in the SK window for the following time interval. For this reason, a constraint on the final velocity is added:

$$|[0\ 1\ 0]\dot{\boldsymbol{y}}(T)| \leqslant \nu \text{ and } |[0\ 0\ 1]\dot{\boldsymbol{y}}(T)| \leqslant \nu, \quad (17)$$

where the parameter ν is small.

2.2.2 Operational Constraints on Actuation

Beside the station keeping geographical constraints and the usual bounds on the maximum thrust, electrical propulsion nature and additional technological operational constraints on the actuation have to be taken into account:

(a) actuators can only provide on-off thrusts;
(b) thrusters cannot be active simultaneously;
(c) each thrust must last at least T_l (s);
(d) two successive thrusts of a given thruster must be separated of an interval of latency equal to $T_s = mT_l$ (s) with $m \in \mathbb{N}^\star$;
(e) two thrusts of two different thrusters must be separated by an interval of latency equal to $T_d = T_l$ (s).

Finally, the optimal control problem reads:

$$\min_{\boldsymbol{F}(t)} J = \sum_{i=1}^{4} ||\boldsymbol{\Gamma}_i||_1 \int_0^T F_i(t)dt, \text{ such that:}$$

$$\begin{cases} \dot{\boldsymbol{x}} = A(t)\boldsymbol{x} + \boldsymbol{D}(t) + B(t)\Gamma\boldsymbol{F}, & \\ \boldsymbol{x}(0) = \boldsymbol{x}_0, & \text{initial condition,} \\ |\dot{y}_2(T)| \leqslant \nu, |\dot{y}_3(T)| \leqslant \nu, & \text{final conditions} \\ |y_2(t)| \leqslant \delta \text{ and } |y_3(t)| \leqslant \delta\ \forall t \in [0, T] & \text{SK. const.} \\ \text{conditions } (a) - (e) & \text{command const.} \end{cases} \quad (18)$$

3. INTEGER FORMULATION OF THE SK PROBLEM

As proposed earlier, the Optimal Control Problem (OCP) (18) will be addressed in the sequel by means of a direct methodology. Note that solving (18) with indirect methods by applications of the classical optimality conditions (Pontryagin principle) is difficult due to the complexity of the operational constraints involved and the presence of state constraints. However, in Gazzino et al. (2016), an equivalent problem has been tackled by means of a two-step methodology combining the application of the maximum principle on a simplified version of the SK problem and a transcription method initialized with the solution of the first step.

3.1 Control Parametrization and Operational Constraints

We first parametrize the control history profiles and then discretise the constraints in order to obtain an integer linear programming problem.

As mentioned before in condition (a), the electrical engines only produce on/off thrust so that $\boldsymbol{F}_i$ at time τ_i can be considered to be a 4-dimensional binary vector expressed by:

$$\boldsymbol{F}(t) = \begin{bmatrix} s_1 & s_2 & s_3 & s_4 \end{bmatrix}^t, \quad (19)$$

where $s_l \in \{0, 1\},\ l = 1, \ldots, 4$.

The time interval $[0,\ T]$ is divided in N equal intervals $[\tau_i, \tau_{i+1}]$ with:

$$0 = \tau_0 < \tau_1 < \cdots < \tau_{N-1} < \tau_N = T. \quad (20)$$

The thrust vector $\boldsymbol{F}(t)$ will be considered constant over each intervals $[\tau_i,\ \tau_{i+1}]$ so that $\boldsymbol{F}(t)$ is completely parametrized by the parameter vectors $\{\boldsymbol{F}_1, \ldots, \boldsymbol{F}_N\}$:

$$\boldsymbol{F}(t) = \boldsymbol{F}_i, \quad \text{if } t \in [\tau_{i-1},\ \tau_i],\ i = 1, \ldots, N. \quad (21)$$

In order to cope with condition (c), the length of each interval is chosen such that $\tau_{i+1} - \tau_i = T_l$.

Moreover, it is forbidden to thrust simultaneously according to conditions (b). Thus, the control profile of the

satellite can be considered at each time to be a binary word that has to be chosen among the five possibilities:

$$\forall i,\ \boldsymbol{F}_i = \begin{bmatrix}(s_1)_i\\(s_2)_i\\(s_3)_i\\(s_4)_i\end{bmatrix} \in \left\{ \begin{bmatrix}0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \right\}. \quad (22)$$

With such a parametrisation of the control, it is possible to write the operational constraints (b)-(e) as logical constraints. Let us consider a discretisation $\{\tau_i\}$ of the time interval $[0, T]$ in sub-intervals of length T_l. Defining $F_i = S(\tau_i)$, the constraints (c) is automatically satisfied. The constraint (d) can be transformed to:

$$\forall i,\ \Big[(\boldsymbol{F}_i \neq 0)\ \&\ (\boldsymbol{F}_{i+1} = 0)\Big] \Rightarrow \Big[(\boldsymbol{F}_{i+2} \neq \boldsymbol{F}_i) \ \&\ (\boldsymbol{F}_{i+3} \neq F_i)\ \&\ \ldots\ \&\ (\boldsymbol{F}_{i+m} \neq F_i)\Big]. \quad (23)$$

The constraint (e) can be expressed as:

$$\forall i,\ \boldsymbol{F}_i \neq 0 \Rightarrow \boldsymbol{F}_{i+1} = \{0, \boldsymbol{F}_i\}, \quad (24)$$

The previous transformation of the operational constraint enables to write the station keeping problem as a linear integer programming problem.

3.2 State Transition and States Constraints Transcription

Using the transition matrix of the differential equation of the satellite state vector evolution (9) as defined in Antsaklis (2003), the state vector trajectory, the geographical position and velocity can be evaluated at each discretisation point: $\forall j = 0, \ldots, N$,

$$\boldsymbol{x}_j = \Phi_{j,0}\boldsymbol{x}_0 + \sum_{i=1}^{j} \Phi_{j,i} B_i^{\Phi} \boldsymbol{F}_i + \boldsymbol{D}^{\Phi}(\tau_j), \quad (25a)$$

$$\boldsymbol{y}_j = C_j \Phi_{j,\tau_0}\boldsymbol{x}_0 + \sum_{i=1}^{j} C_j \Phi_{j,i} B_i^{\Phi} \boldsymbol{F}_i + C_j \boldsymbol{D}_j^{\Phi}, \quad (25b)$$

$$\dot{\boldsymbol{y}}_j = \sum_{i=1}^{j} E_j \Phi_{ji} B_i^{\Phi} \boldsymbol{F}_i + C_j \tilde{B}_j \boldsymbol{F}_j + E_j \Phi_{j,0}\boldsymbol{x}_0 + E_j \boldsymbol{D}_j^{\Phi} + C_j \boldsymbol{D}_j, \quad (25c)$$

with $\boldsymbol{x}_j = \boldsymbol{x}(\tau_j)$, $\boldsymbol{y}_j = \boldsymbol{y}(\tau_j)$, $\dot{\boldsymbol{y}}_j = \dot{\boldsymbol{y}}(\tau_j)$, $C_j = C(\tau_j)$, $\boldsymbol{D}_j = \boldsymbol{D}(\tau_j)$, $E_j = E(\tau_j)$, $\Phi_{j,0} = \Phi(\tau_j, 0)$, $\Phi_{j,i} = \Phi(\tau_j, \tau_i)$, $B_i^{\Phi} = \left(\int_{\tau_{i-1}}^{\tau_i} \Phi(\tau_i, s) B(s) ds\right) F_{max}\Gamma$, $\tilde{B} = F_{max} B \Gamma$ and $\boldsymbol{D}_i^{\Phi} = \int_0^t \Phi(\tau_i, s)\boldsymbol{D}(s) ds$.

This leads to the following discretisation of the station keeping constraints (16) and of the velocity constraints (17):

$$\begin{cases} \sum_{i=1}^{j} \beta C_j \Phi_{ji} B_i^{\Phi} \boldsymbol{F}_i \leqslant \delta - C_j \Phi_{j,0}\boldsymbol{x}(0) - C_j \boldsymbol{D}_j^{\Phi}, \\ -\sum_{i=1}^{j} \beta C_j \Phi_{ji} B_i^{\Phi} \boldsymbol{F}_i \leqslant \delta + C_j \Phi_{j,0}\boldsymbol{x}(0) + C_j \boldsymbol{D}_j^{\Phi}, \\ \sum_{i=1}^{N} \beta E_N \Phi_{Ni} B_i^{\Phi} \boldsymbol{F}_i \leqslant \nu - E_N \Phi_{N,0}\boldsymbol{x}_0 - E_N \boldsymbol{D}_j^{\Phi} - C_N \boldsymbol{D}_N, \\ -\sum_{i=1}^{N} \beta E_N \Phi_{Ni} B_i^{\Phi} \boldsymbol{F}_i \leqslant \nu + E_N \Phi_{N,0}\boldsymbol{x}_0 + E_N \boldsymbol{D}_j^{\Phi} + C_N \boldsymbol{D}_N, \end{cases} \quad (26)$$

with β being either [0 1 0] or [0 0 1]. The discretisation of the station keeping requirements are rewritten in matrix format as:

$$\begin{cases} \mathcal{B}_s \bar{F} \leqslant d - \mathcal{B}_x x(0) - \mathcal{B}_c, \\ -\mathcal{B}_s \bar{F} \leqslant d + \mathcal{B}_x x(0) + \mathcal{B}_c, \\ \mathcal{G}_s \bar{F} \leqslant \upsilon - \mathcal{G}_x x(0) - \mathcal{G}_c, \\ -\mathcal{G}_s \bar{F} \leqslant \upsilon + \mathcal{G}_x x(0) + \mathcal{G}_c, \end{cases} \quad (27)$$

where $\bar{\boldsymbol{F}} = [\boldsymbol{F}_1^t, \ldots, \boldsymbol{F}_N^t]^t$. With the transformation of the thrusters operational constraints of Section 2.2.2 and the previous discretisation of the dynamics equation, it is possible to use an integer programming formulation to solve the minimum fuel SK problem.

3.3 Actuators Constraints in Integer Formalism

Considering that the parameter vectors F_i are composed of binary variables s_i (cf. (19)), the constraint (b) is expressed as:

$$\forall i,\ (s_1)_i + (s_2)_i + (s_3)_i + (s_4)_i \leqslant 1. \quad (28)$$

Constraint (c) is satisfied by definition of the control parametrization. In order to express constraints (d) and (e), an auxiliary binary variable $\gamma \in \{0, 1\}^N$ is used and is defined as:

$$\forall i,\ (s_1)_i + (s_2)_i + (s_3)_i + (s_4)_i = \alpha_i. \quad (29)$$

The variable γ_i expresses the fact that one of the four thrusters is firing on interval $[\tau_i, \tau_{i+1}]$. If $\gamma_i = 0$, all thrusters are off. Thanks to γ, the constraint (d) reads:

$$\forall k = 1, \ldots, m,\ \forall i = 1, \ldots, N-k-1,\ \forall l = 1, \ldots, 4,\ (s_l)_{i+k} + (s_l)_i + \gamma_i - \gamma_{i+1} \leqslant 0, \quad (30)$$

and the constraint (e) reads:

$$\forall i = 1, \ldots, N-1,\ \forall l = 1, \ldots, 4,\ (s_l)_{i+1} - (s_l)_i + \gamma_i \leqslant 1. \quad (31)$$

The fuel-consumption is expressed with the control profile S:

$$J_s = \sum_{i=1}^{N} [1\ 1\ 1\ 1]\boldsymbol{F}_i, \quad (32)$$

and the SK linear integer programming problem reads:

$$\min_{(s_l)_i, \gamma_i} J_s = \sum_{i=1}^{N} [1\ 1\ 1\ 1]\boldsymbol{F}_i$$
$$\text{s.t.} \begin{cases} \mathcal{B}_s \bar{\boldsymbol{F}} \leqslant d - \mathcal{B}_x \boldsymbol{x}(0) - \mathcal{B}_c, \\ -\mathcal{B}_s \bar{\boldsymbol{F}} \leqslant d + \mathcal{B}_x \boldsymbol{x}(0) + \mathcal{B}_c, \\ \mathcal{G}_s \bar{\boldsymbol{F}} \leqslant \upsilon - \mathcal{G}_x \boldsymbol{x}(0) - \mathcal{G}_c \\ -\mathcal{G}_s \bar{\boldsymbol{F}} \leqslant \upsilon + \mathcal{G}_x \boldsymbol{x}(0) + \mathcal{G}_c, \\ (s_1)_i + (s_2)_i + (s_3)_i + (s_4)_i \leqslant 1, \\ (s_1)_i + (s_2)_i + (s_3)_i + (s_4)_i = \gamma_i, \\ (s_l)_{i+k} + (s_l)_i + \gamma_i - \gamma_{i+1} \leqslant 0, \\ (s_l)_{i+1} - (s_l)_i + \gamma_i \leqslant 1. \end{cases} \quad (33)$$

4. NUMERICAL RESULTS

In this section, simulation results obtained with the proposed methodology are presented. Let consider a satellite

of mass 4850 kg equipped with 4 electric thrusters oriented in the directions North-East, North-West, South-East and South-West. This satellite has to be controlled in order to remain close to its geostationary position at a fixed longitude $\bar{\lambda}$ and a fixed latitude $\bar{\varphi} = 0$. The SK problem has been solved with using Gurobi (see Gu et al. (2010)) with a Matlab interface done with Yalmip (see Löfberg (2004)). The problem horizon was of few days. The initial relative geographical position is $\boldsymbol{y}(0) = [0\ 0.03^\circ\ -0.04^\circ]^t$ and the relative initial geographical velocity is $\dot{\boldsymbol{y}}(0) = [0\ 0\ 0]^t$. The initial orbit elements $\boldsymbol{x}_0$ are recover by inversion of Equations (9) and (12) written at $t = 0$ and $\boldsymbol{F}(0) = [0\ 0\ 0\ 0]^t$

In order to discuss the benefit of taken the relative velocity into account, the integer formulation (33) of the SK problem has been solved on one hand with the final velocity constraint and on the second hand without the this final velocity constraints. On Figure 1 and 3, the geographical position for the problem with the final velocity constraint and without the final velocity constraint are drawn. It can be easily noticed that taking this final constraint into account or not can have a high impact on the final trajectory, as well as on the control profile (see Figure 4). The consumption for the case with the final velocity constraint is three times as high as the consumption for the case without

However, as shown on Figure 2, adding the final velocity constraint changes the velocity profile, and the difficulties already mentioned in Section 2.2.1 concerning the high final velocities can occur for a starting point too close to the SK boundary.

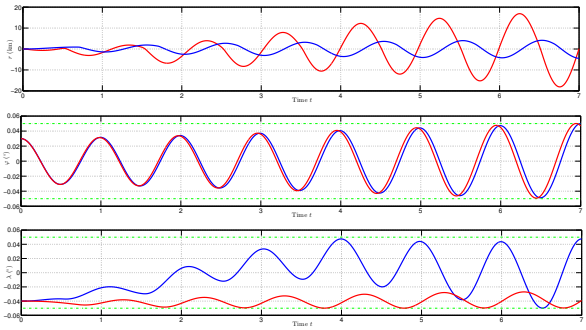

Fig. 1. Relative geographical position.
– : with final velocity constraint.
– : without final velocity constraint.

It is possible to find in the literature some rules of thumb for geostationary SK strategies, in particular in Campan et al. (1995), Soop (1994) and Sidi (1997). In these references, the effect of the perturbing forces on the orbit is studied, and some general SK laws are derived. The North-South effect of the Sun and the Moon attractions are the most pregnant forces and these forces must be corrected each half orbit, once in the North direction and half an orbit later in the South direction. The East-West drift is meanwhile corrected by setting different thrust durations for each thrusters. This SK strategy was usefully used in

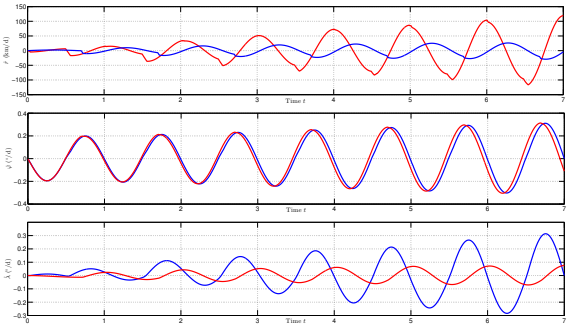

Fig. 2. Relative geographical velocity.
– : with final velocity constraint.
– : without final velocity constraint.

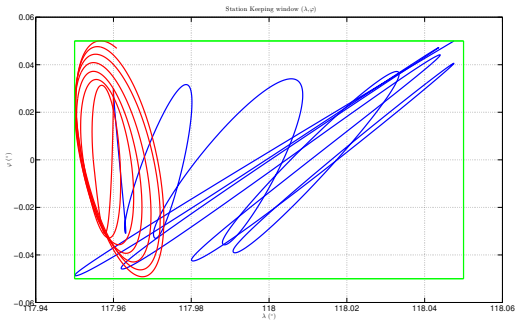

Fig. 3. Station Keeping window.
– : with final velocity constraint.
– : without final velocity constraint.

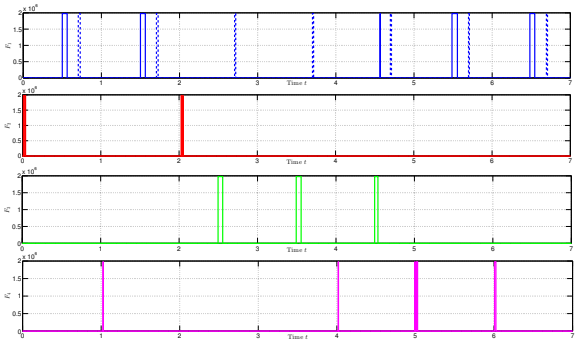

Fig. 4. Control profiles.
– : with final velocity constraint.
- - : without final velocity constraint.

an industrial context by Anzel (1995). The control profile of Figure 4 shows that the two South thrusters have a thrust at the beginning of each day, and the two North thrusters have a thrust half a day later. On the control profile, the thrusts have always different durations in order to compensate for the East-West drift. The physical rule of thumb can thus be recovered from a systematic optimisation process.

5. CONCLUSION

In this paper, the optimal control formulation of the minimum fuel station keeping problem has been transformed into a linear integer programming problem to cope with the on-off characteristics of the electrical thrusters. Moreover we show that such a formulation permits to nicely describe the operational constraints on these actuators. The methodology has been illustrated with examples that highlight the interest of constraining the velocity at the final moment of the week in order to prevent the satellite to fly out the station keeping window. The optimal SK strategy based on an optimisation method that could be derived in this paper matches the general rules for station keeping that have been derived by analysing the physical effect of the disturbing forces on the satellite orbit. The transition matrices used to build up the linear integer programming problem have been obtained under zero-holder assumption. However, in order to improve the accuracy of our results, further investigations have to be made to provide either closed-form expression or numerically certified approximations for the linearised dynamic. Another venue of improvement can be the use of advanced integer transcription or branch-and-bound algorithm in order to perform more effectively the numerical solution of the linear integer programming problem.

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