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Refined exponential stability analysis of a coupled system ⋆

Mohammed Safi * **Lucie Baudouin** * **Alexandre Seuret** *

* *LAAS-CNRS, Université de Toulouse, CNRS, Toulouse, France.*

Abstract: The objective of this contribution is to improve recent stability results for a system coupling ordinary differential equations to a vectorial transport partial differential equation by proposing a new structure of Lyapunov functional. Following the same process of most of the investigations in literature, that are based on an a priori selection of Lyapunov functionals and use the usual integral inequalities (Jensen, Wirtinger, Bessel...), we will present an efficient method to estimate the exponential decay rate of this coupled system leading to a tractable test expressed in terms of linear matrix inequalities. These LMI conditions stem from the new design of a candidate Lyapunov functional, but also the inherent properties of the Legendre polynomials, that are used to build a projection of the infinite dimensional part of the state of the system. Based on these polynomials and using the appropriate Bessel-Legendre inequality, we will prove an exponential stability result and in the end, we will show the efficiency of our approach on academic example.

Keywords: Transport equation, Lyapunov stability, integral inequalities.

1. INTRODUCTION

When modeling a control problem phenomenon using a state space formulation, the trade off between capturing a certain level of complexity and obtaining a model on which the tools we have can be applied is inevitable. Considering systems modeled by ordinary differential equations (ODE) ensures huge literature and quantity of very well developed control tools whereas the choice of a partial differential equation (PDE) model brings in a very different set of approaches and references. Our work lies in the more narrow domain of the stability study of coupled ODE-PDE system and should be seen as a first step in the understanding of a possible way to get a simple common ground to consider at the same time both of the two states of such a heterogeneous system.

PDE systems stand out as having important applications in the modeling and control of physical networks: hydraulic (Coron et al. (2008)), gas pipeline networks (Gugat et al. (2011)) and road traffic (Coclite et al. (2005)) for instance. Systems coupling PDEs to ODEs have attracted some more attention in the last decade as in Krstic (2009), Hasan et al. (2016), Stinner et al. (2014) and Friedmann (2015). Stability analysis and stabilization of such systems have also appeared recently in e.g. Susto Gian and Krstic (2010), Prieur et al. (2008) and Tang et al. (2015).

To control these coupled systems, the backstepping approach can be considered. It was originally developed for parabolic and second-order hyperbolic PDEs, as well as for several challenging physical problems such as turbulent flows magnetohydrodynamics Vazquez and Krstic (2008). In many recent works, it is used to stabilize ODE-PDE systems. For example, a first order hyperbolic PDE coupled to an ODE has been stabilized by this approach in Krstic and Smyshlyaev (2008).

In this article, we are considering a vectorial transport equation, whose state is of infinite dimension, coupled with a classical system of ODEs. The stability of this specific kind of coupled systems has already been investigated recently in Castillo et al. (2015), Baudouin et al. (2016) and Safi et al. (2016) and one should mention that it can also be considered as specific formulation of a Time-Delay System (TDS). Actually, there is a very large literature in TDS and among many others, we can refer to Xu and Sallet (2002), Mondie et al. (2005) and Xu et al. (2006). In our paper, following the classical Lyapunov method for stability study (see e.g. Gu et al. (2003), Fridman (2014), Gyurkovics and Takacs (2016) and Seuret and Gouaisbaut (2015)), we will provide an efficient approach for assessing stability of this first ODE-PDE system.

Most of the contributions on stability in the TDS framework are based on the good selection of a Lyapunov-Krasovskii functional (LKF) leading to sufficient stability conditions (see Gu et al. (2003)), and a polynomial approximation to estimate the infinite dimensional state of the system, which is not a new idea (see Papachristodoulou and Peet (2006), Peet (2014) and Ahmadi et al. (2014)). But in this work, we aim at showing that a better design of the LKF may improve stability studies of such coupled systems and gives interesting results for the convergence rate of this system, getting closer to what an appropriate frequency analysis can give.

Notations: $\mathbb{N}$ is the set of positive integers, $\mathbb{R}^n$ is the n-dimensional Euclidean space with vector norm $|\cdot|_n$. We denote $\mathbb{R}^{n\times m}$ the set of real matrices of dimension $n \times m$. $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix , $0_{n,m}$ the null matrix, and $\left[\begin{smallmatrix} A & B \\ \star & C \end{smallmatrix}\right]$ replaces the symmetric matrix $\left[\begin{smallmatrix} A & B \\ B^\top & C \end{smallmatrix}\right]$. We

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denote $\mathbb{S}^n \subset \mathbb{R}^{n\times n}$ (respectively $\mathbb{S}^n_+$) the set of symmetric (resp. symmetric positive definite) matrices and $\mathrm{diag}(A,B)$ is a bloc diagonal matrix equal to $\left[\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right]$. For any square matrix A, we define $\mathrm{He}(A) = A + A^\top$. Finally, $L^2(0,1;\mathbb{R}^n)$ represents the space of square integrable functions over the interval $]0,1[\subset \mathbb{R}$ with values in $\mathbb{R}^n$ and the partial derivative in time and space are denoted ∂_t and ∂_x, while the classical derivative are $\dot{X} = \frac{d}{dt}X$ and $\mathcal{L}' = \frac{d}{dx}\mathcal{L}$.

2. PROBLEM FORMULATION

2.1 System data

We consider the following system governed by the coupling of a transport PDE and a finite dimensional system of ODE:

$$\begin{cases} \dot{X}(t) = AX(t) + Bz(1,t) & t > 0, \\ \partial_t z(x,t) + \rho\partial_x z(x,t) = 0, & x \in (0,1), t > 0, \\ z(0,t) = CX(t) + Dz(1,t), & t > 0. \end{cases} \quad (1)$$

The state of this coupled system is composed not only of the finite dimensional state variable $X(t) \in \mathbb{R}^n$ of the ODE but also of the infinite dimensional state $z(x,t) \in \mathbb{R}^m$ of the vectorial transport-PDE. The pair $(X(t), z(x,t))$ forms the complete state of this coupled system and satisfies the initial condition $(X^0, z^0) \in \mathbb{R}^n \times L^2(0,1;\mathbb{R}^m)$. The matrices A, B, C and D are constant in $\mathbb{R}^{n\times n}$, $\mathbb{R}^{n\times m}$, $\mathbb{R}^{m\times n}$ and $\mathbb{R}^{m\times m}$. The parameter ρ of the transport-PDE represents the propagation velocity applied to the m components of the variable $z(x,t)$.

The total energy $E(t)$ of system (1) is given by :

$$E(X(t), z(t)) = |X(t)|_n^2 + \|z(t)\|^2_{L^2(0,1;\mathbb{R}^m)}.$$

2.2 Motivation

The purpose of studying (1) is twofold: as a system coupling infinite and finite dimensional states, the tools for stability assessment have to be shaped accordingly; and as the coupling of an ODE and a transport PDE, (1) mimics a TDS and could specifically take advantage of the last developments in this domain. More particularly, the study of system (1) is also motivated by its capacity to represent two types of time-delay systems:

- *Systems with single delay* ($D = 0_{m,m}$) that have been studied in many contributions on the subject (see Xu et al. (2006) and Seuret and Gouaisbaut (2015)).
- *Systems with commensurate (or, rationally dependent) delays* ($D \neq 0_{m,m}$), where a single delay and its multiples are involved(see e.g. Su (1995)).

Inspired by the stability study of these time-delay systems, we aim at leading a stability study for system (1) using the following type of Lyapunov functional

$$\begin{aligned} V(X(t), z(t)) = X^\top(t)PX(t) + 2X^\top(t)\int_0^1 \mathcal{Q}(x)z(x,t)dx \\ + \int_0^1\int_0^1 z^\top(x_1,t)\mathcal{T}(x_1,x_2)z(x_2,t)dx_1dx_2 \\ + \int_0^1 e^{\frac{-2\delta}{\rho}x} z^\top(x,t)(S + (1-x)R)z(x,t)dx, \quad (2) \end{aligned}$$

where the matrices $P \in \mathbb{S}^n_+, S, R \in \mathbb{S}^m_+$ and the functions $\mathcal{Q} \in L^2(0,1;\mathbb{R}^{n\times m})$ and $\mathcal{T} \in L^\infty((0,1)^2;\mathbb{S}^m)$ have to be specified.

A Lyapunov functional is usually constructed from the complete state of the system, and in this one, the first quadratic term is dedicated to the vector state $X(t)$ and the last term to the infinite dimensional state $z(x,t)$. The two terms that remain in between are formed through the functions $\mathcal{Q}(x)$ and $\mathcal{T}(x_1,x_2)$, and have the specificity of building a link between those finite and infinite dimensional states, both constitutive of the ODE-PDE coupling system. The challenge in the precise design of this candidate Lyapunov functional (2) relies on the choice of the functions $\mathcal{Q}$ and $\mathcal{T}$ that depend on the integration parameters x, x_1 and x_2. In our approach, we will use polynomial functions of a given degree (a basis of Legendre polynomials) to construct a truncated decomposition of those functions as follows

$$\begin{aligned} \mathcal{Q}(x) &= \textstyle\sum_{k=0}^N Q(k)\mathcal{L}_k(x), \\ \mathcal{T}(x_1,x_2) &= \textstyle\sum_{i=0}^N\sum_{j=0}^N T(i,j)\mathcal{L}_i(x_1)\mathcal{L}_j(x_2), \end{aligned} \quad (3)$$

where $N \in \mathbb{N}$, and where $\mathcal{L}_k$, for $k \in \mathbb{N}$, denote the shifted Legendre polynomials of degree k considered over the interval $[0,1]$. These polynomials and their properties will be detailed in the next section. Using the decomposition (3) of the functions $\mathcal{Q}$ and $\mathcal{T}$, the Lyapunov functional becomes

$$\begin{aligned} V_N(X(t), z(t)) = \begin{bmatrix} X(t) \\ Z_N(t) \end{bmatrix}^\top \begin{bmatrix} P & Q_N \\ Q_N^\top & T_N \end{bmatrix} \begin{bmatrix} X(t) \\ Z_N(t) \end{bmatrix} \\ + \int_0^1 e^{\frac{-2\delta}{\rho}x} z^\top(x,t)(S + (1-x)R)z(x,t)dx, \quad (4) \end{aligned}$$

where

$$\begin{aligned} Q_N &= [Q(0) \ \dots \ Q(N)] \text{ in } \mathbb{R}^{n,m(N+1)}, \\ T_N &= [T(i,j)]_{i,j=0..N} \quad \text{in } \mathbb{R}^{m(N+1),m(N+1)} \end{aligned}$$

and

$$Z_N(t) = \begin{bmatrix} \int_0^1 z(x,t)\mathcal{L}_0(x)\,dx \\ \vdots \\ \int_0^1 z(x,t)\mathcal{L}_N(x)\,dx \end{bmatrix} \in \mathbb{R}^{m(N+1)} \quad (5)$$

is the projection of the m components of the infinite dimensional state $z(x,t)$ over the $N+1$ first Legendre polynomials.

In this article, we aim specifically at showing that taking the last integral term in (4) with an exponential weight $e^{\frac{-2\delta}{\rho}x}$, allows to derive a better estimate of the decay rate of the system compared to Baudouin et al. (2016), which uses a different method. In addition to the contribution of Baudouin et al. (2016), an additional term (i.e. $Dz(1,t)$) has been included in the boundary condition of system (1) so that a larger class of systems can be covered, as, for instance, systems with commensurate delays.

3. PRELIMINARIES

3.1 Legendre polynomials

The shifted Legendre polynomials (see for instance Courant and Hilbert (1953)) we will use are denoted $\{\mathcal{L}_k\}_{k\in\mathbb{N}}$ and

act over $[0,1]$. The family $\{\mathcal{L}_k\}_{k\in\mathbb{N}}$ forms an orthogonal basis of $L^2(0,1;\mathbb{R})$ and we have precisely

$$\int_0^1 \mathcal{L}_j(x)\mathcal{L}_k(x)\,dx = \frac{1}{2k+1}\delta_{jk},$$

where δ_{jk} represents Kronecker's coefficient, equal to 1 if $j=k$ and 0 otherwise. The boundary values are given by:

$$\mathcal{L}_k(0) = (-1)^k, \qquad \mathcal{L}_k(1) = 1. \tag{6}$$

Moreover, the derivative of those polynomials is given by

$$\frac{d}{dx}\mathcal{L}_k(x) = \begin{cases} 0, & k=0, \\ \sum_{j=0}^{k-1} \ell_{kj}\mathcal{L}_j(x), & k\geq 1. \end{cases} \tag{7}$$

with

$$\ell_{kj} = \begin{cases} (2j+1)(1-(-1)^{k+j}), & \text{if} \quad j\leq k-1, \\ 0, & \text{if} \quad j\geq k. \end{cases} \tag{8}$$

3.2 Bessel-Legendre inequality

The following lemma gives a Bessel-type inequality that compares an $L^2(0,1)$ scalar product with the corresponding finite dimensional approximation product.

Lemma 1. Let $z\in L^2(0,1;\mathbb{R}^m)$ and $R\in\mathbb{S}_+^m$. The following integral inequality holds for all $N\in\mathbb{N}$:

$$\int_0^1 z^\top(x)Rz(x)\,dx \geq Z_N^\top R^N Z_N, \tag{9}$$

with

$$R^N = \mathrm{diag}(R, 3R, \ldots, (2N+1)R), \tag{10}$$

Proof : The proof is easily conducted, as we show in Baudouin et al. (2016), by considering the difference between the state z and its projections over the $N+1$ first Legendre polynomials. Indeed, denoting $y_N(x,t) = z(x,t) - \sum_{k=0}^N (2k+1)\mathcal{L}_k(x)\int_0^1 z(\xi,t)\mathcal{L}_k(\xi)\,d\xi$, the orthogonality of the Legendre polynomials and the Bessel inequality allows to obtain (9) from the positive definiteness and the expansion of $\int_0^1 y_N^\top(x,t)Ry_N(x,t)dx$, as in e.g. Seuret and Gouaisbaut (2015). □

4. STABILITY RESULTS

4.1 Exponential stability

To assess exponential stability of system (1), we will show that the Lyapunov functional (4) satisfies the following inequalities:

$$\varepsilon_1 E(t) \leq V_N(X(t),z(t)) \leq \varepsilon_2 E(t), \tag{11}$$

$$\dot{V}_N(X(t),z(t)) + 2\delta V_N(X(t),z(t)) \leq -\varepsilon_3 E(t) \tag{12}$$

for some positive scalars ε_1, ε_2 and ε_3. From now on, we will use the shorthand notation $V_N(t)$. Since V_N partly depends on a projection Z_N of the state z, as written in (4), then in order to compute the time derivative of $V_N(t)$ in (12) we will need the time derivative of $Z_N(t)$. The following property thus provides a simple expression.

Property 1. Consider $z\in C(\mathbb{R}_+;L^2(0,1;\mathbb{R}^m))$ satisfying the transport equation in system (1). The time derivative of the projection vector Z_N is given by :

$$\dot{Z}_N(t) = \rho L_N Z_N(t) + \rho(\mathbb{1}_N^* D - \mathbb{1}_N)z(1,t) + \rho\mathbb{1}_N^* CX(t), \tag{13}$$

where we used the notations

$$\begin{aligned} \mathbb{1}_N &= [I_m \ \ldots \ I_m]^\top \in \mathbb{R}^{m(N+1),m}, \\ \mathbb{1}_N^* &= \left[I_m \ -I_m \ \ldots \ (-1)^N I_m\right]^\top \in \mathbb{R}^{m(N+1),m}, \\ L_N &= [\ell_{kj}I_m]_{j,k=0..N} \in \mathbb{R}^{m(N+1),m(N+1)}, \end{aligned} \tag{14}$$

the ℓ_{kj} being defined in (8).

Proof : First, let us compute the time derivative of the projection of the infinite dimensional state $z(x,t)$ over the kth Legendre polynomial $\mathcal{L}_k$. Using the transport equation in (1), integration by parts and properties (6) and (7) of the Legendre polynomials, we obtain the following expression:

$$\begin{aligned} &\frac{d}{dt}\int_0^1 z(x,t)\mathcal{L}_k(x)\,dx = -\int_0^1 \rho\partial_x z(x,t)\mathcal{L}_k(x)\,dx \\ &= -\left[\rho z(x,t)\mathcal{L}_k(x)\right]_0^1 + \int_0^1 \rho z(x,t)\frac{d}{dx}\mathcal{L}_k(x)\,dx \\ &= -\rho z(1,t) + (-1)^k\rho z(0,t) \\ &\qquad + \textstyle\sum_{j=0}^{max[0,k-1]} \ell_{kj}\rho\int_0^1 z(x,t)\mathcal{L}_j(x)\,dx. \end{aligned}$$

Consequently, using the notations recently introduced and omitting the time variable t, we have

$$\frac{d}{dt}Z_N(t) = \rho L_N Z_N(t) - \rho\mathbb{1}_N z(1,t) + \rho\mathbb{1}_N^* z(0,t).$$

The proof is concluded by injecting the boundary condition $z(0,t) = CX(t) + Dz(1,t)$ in the previous expression. □

Remark 1. As mentioned before, the vector Z_N corresponds to the projections of the state z of the PDE dynamics over a set of polynomials of limited degree in $L^2(0,1;\mathbb{R}^m)$. We can note in (13) that the components of Z_N are computed by several integration of a combination of $z(1,t)$ and $z(0,t)$, since the matrices L_N are strictly lower triangular for any integer N. Therefore, the augmented variable Z_N cannot be exponentially stable. However, in this work we are interested in stability of the global coupled system and not only the augmented system given in Property 1.

Based on the previous discussions, the following theorem is stated.

Theorem 2. Consider system (1) with a given transport speed $\rho>0$. Recall that the matrices L_N, $\mathbb{1}_N$ and $\mathbb{1}_N^*$ are defined in (14), the matrix R^N is given by (10) and define the following $\mathbb{R}^{m(N+1),m(N+1)}$ matrices

$$\begin{aligned} S^N &= \mathrm{diag}(S, 3S, \ldots, (2N+1)S), \\ \mathcal{I}^N &= \mathrm{diag}(I_m, 3I_m, \ldots, (2N+1)I_m). \end{aligned} \tag{15}$$

If there exists an integer $N>0$ such that there exists $\delta>0$, $P\in\mathbb{S}_+^n$, $Q_N\in\mathbb{R}^{n,(N+1)m}$, $T_N\in\mathbb{S}^{(N+1)m}$, S and $R\in\mathbb{S}_+^m$, satisfying the following LMIs

$$\Phi_N = \begin{bmatrix} P & Q_N \\ Q_N^\top & T_N + e^{-\frac{2\delta}{\rho}}S^N \end{bmatrix} \succ 0, \tag{16}$$

$$\Psi_N(\rho,\delta) = \begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} \\ * & \Psi_{22} & \Psi_{23} \\ * & * & \Psi_{33} \end{bmatrix} \prec 0, \tag{17}$$

where

$$\begin{aligned}
\Psi_{11} &= He(PA + \rho Q_N \mathbb{1}_N^* C) + 2\delta P + \rho C^T (S+R)C, \\
\Psi_{12} &= PB + \rho Q_N(\mathbb{1}_N^* D - \mathbb{1}_N) + \rho C^\top (S+R)D, \\
\Psi_{13} &= A^\top Q_N + \rho C^\top \mathbb{1}_N^{*\top} T_N + \rho Q_N L_N + 2\delta Q_N, \\
\Psi_{22} &= -\rho e^{-\frac{2\delta}{\rho}} S + \rho D^\top (S+R)D, \\
\Psi_{23} &= B^\top Q_N + \rho(\mathbb{1}_N^* D - \mathbb{1}_N)^\top T_N, \\
\Psi_{33} &= \rho \mathrm{He}(T_N L_N) + 2\delta T_N - \rho e^{-\frac{2\delta}{\rho}} R^N,
\end{aligned}$$

then system (1) is exponentially stable. Indeed, there exists a constant $K > 0$ and a guaranteed decay rate $\delta^* > \delta$ such that the energy of the system verifies, for all $t > 0$,

$$E(t) \le K e^{-2\delta^* t} \left(|X^0|_n^2 + \|z^0\|^2_{L^2(0,1;\mathbb{R}^m)} \right). \tag{18}$$

Proof : To prove this stability result, we have to show that the Lyapunov functional V_N given in (4) verifies the inequalities (11) and (12) for some positive scalars ε_1, ε_2 and ε_3. The proof falls then into four steps.

Exponential stability: As soon as we will obtain that the Lyapunov functional V_N satisfies (11) and (12), we can prove the exponential stability of system (1), since we get easily

$$\dot{V}_N(t) + (2\delta + \frac{\varepsilon_3}{\varepsilon_2}) V_N(t) \le 0.$$

Indeed, integrating on the interval $[0,t]$ and using $2\delta^* = 2\delta + \frac{\varepsilon_3}{\varepsilon_2}$, inequality $V_N(t) \le V_N(0)e^{-2\delta^* t}$ holds for all $t > 0$ and using(11) once again, we get

$$\varepsilon_1 E(t) \le V_N(t) \le V_N(0) e^{-2\delta^* t} \le \varepsilon_2 E(0) e^{-2\delta^* t},$$

which yields (18).

Existence of ε_1: On the one hand, since $S \succ 0$ and $\Phi_N \succ 0$, there exists a sufficiently small $\varepsilon_1 > 0$ such that

$$S \succ \varepsilon_1 e^{\frac{2\delta}{\rho}} I_m, \quad \Phi_N = \begin{bmatrix} P & Q_N \\ * & T_N + e^{-\frac{2\delta}{\rho}} S^N \end{bmatrix} \succ \varepsilon_1 \begin{bmatrix} I_n & 0 \\ * & \mathcal{I}^N \end{bmatrix}.$$

On the other hand, V_N defined by (4) satisfies, $\forall t \ge 0$,

$$\begin{aligned}
V_N(t) \ge & \begin{bmatrix} X(t) \\ Z_N(t) \end{bmatrix}^\top \Phi_N \begin{bmatrix} X(t) \\ Z_N(t) \end{bmatrix} - e^{-\frac{2\delta}{\rho}} Z_N^\top(t) S^N Z_N(t) \\
& + e^{-\frac{2\delta}{\rho}} \int_0^1 z^\top(x,t) S z(x,t) dx.
\end{aligned}$$

Replacing Φ_N by its lower bound depending on ε_1 and introducing ε_1 in the last integral term, we have

$$\begin{aligned}
V_N(t) \ge & \ \varepsilon_1 |X(t)|_n^2 + \ \varepsilon_1 \int_0^1 z^\top(x,t) z(x,t) dx \\
& - Z_N^\top(t)(e^{-\frac{2\delta}{\rho}} S^N - \varepsilon_1 \mathcal{I}^N) Z_N(t) \\
& + \int_0^1 z^\top(x,t)(e^{-\frac{2\delta}{\rho}} S - \varepsilon_1 I_m) z(x,t) dx.
\end{aligned}$$

By noting that $S - \varepsilon_1 e^{\frac{2\delta}{\rho}} I_m \succ 0$, Lemma 1 ensures that the sum of the two last terms is positive and we thus obtain that there exists $\varepsilon_1 > 0$ such that $V_N(t) \ge \varepsilon_1 E(t)$.

Existence of ε_2: There exists a sufficiently large scalar $\beta > 0$ that allows

$$\begin{bmatrix} P & Q_N \\ Q_N^\top & T_N \end{bmatrix} \preceq \beta \begin{bmatrix} I_n & 0 \\ * & \mathcal{I}^N \end{bmatrix},$$

such that, under the assumptions $S \succ 0$ and $R \succ 0$, we get

$$\begin{aligned}
V_N(t) \le & \ \beta \ |X(t)|_n^2 + \beta Z_N^\top(t) \mathcal{I}^N Z_N(t) \\
& + \int_0^1 e^{\frac{-2\delta}{\rho} x} z^\top(x,t)(S + (1-x)R) z(x,t) dx \\
\le & \ \beta |X(t)|_n^2 + \beta Z_N^\top(t) \mathcal{I}^N Z_N(t) \\
& + \int_0^1 z^\top(x,t)(S+R) z(x,t) dx.
\end{aligned}$$

Applying Lemma 1 to the second term of the right-hand side gives

$$\begin{aligned}
V_N(t) \le & \ \beta |X(t)|_n^2 + \int_0^1 z^\top(x,t)(\beta I_m + S + R) z(x,t) dx \\
\le & \ \beta |X(t)|_n^2 + \varepsilon_2 \|z\|^2_{L^2(0,1;\mathbb{R}^m)} \le \varepsilon_2 E(t),
\end{aligned}$$

where $\varepsilon_2 = \beta + \lambda_{\max}(S) + \lambda_{\max}(R)$. Therefore, the proof of (11) is complete.

Existence of ε_3: Defining a kind of finite dimensional augmented state vector, of size $n + (N+2)m$ given by

$$\xi_N(t) = \left[X^\top(t) \ z^\top(1,t) \ Z_N^\top(t) \right]^\top,$$

and using Property 1 and the definition of $\Psi_N(\rho,\delta)$, several calculations, based on (1), lead to the following expression of the time derivative of V_N, we obtain

$$\begin{aligned}
& \dot{V}_N(t) + 2\delta V_N(t) \le \xi_N^\top(t) \Psi_N(\rho,\delta) \xi_N(t) \\
& + \rho e^{-\frac{2\delta}{\rho}} Z_N^\top(t) R^N Z_N(t) - \rho e^{-\frac{2\delta}{\rho}} \int_0^1 z^\top(x,t) R z(x,t) dx.
\end{aligned} \tag{19}$$

These calculations are omitted because of space limitations. Following the same procedure as for the existence of ε_1, the LMI (17) ensures that there exists a sufficiently small $\varepsilon_3 > 0$ such that

$$R \succ \frac{1}{\rho} \varepsilon_3 e^{\frac{2\delta}{\rho}} I_m, \quad \Psi_N(\rho,\delta) \prec -\varepsilon_3 \begin{bmatrix} I_n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathcal{I}^N \end{bmatrix}.$$

Hence, using these two LMIs in (19) yields

$$\begin{aligned}
\dot{V}_N(t) + 2\delta V_N(t) \le & -\varepsilon_3 \left(|X(t)|_n^2 + \int_0^1 |z(x,t)|^2 dx \right) \\
& + Z_N^\top(t)(\ \rho e^{-\frac{2\delta}{\rho}} R^N - \varepsilon_3 \mathcal{I}^N) Z_N(t) \\
& - \int_0^1 z^\top(x,t)(\rho e^{-\frac{2\delta}{\rho}} R - \varepsilon_3 I_m) z(x,t) dx.
\end{aligned}$$

Since $R - \frac{1}{\rho}\varepsilon_3 e^{\frac{2\delta}{\rho}} I_m \succ 0$, Lemma 1 ensures that the sum of the two last terms of the previous equation is negative. Thus the Lyapunov functional V_N satifies $\dot{V}_N(t) + 2\delta V_N(t) \le -\varepsilon_3 E(t)$, which concludes on the exponential stability of system (1). □

5. NUMERICAL EXAMPLE

To test our approach, we consider the following academic time-delay system

$$\dot{X}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} X(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} X(t-h), \tag{20}$$

which is given under the form of system (1) by $\rho = \frac{1}{h}$ and

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

This example was studied in Baudouin et al. (2016) using a different Lyapunov functional. Figure 1 gives the maximum decay rate δ for each value of the transport speed ρ and for several values of N using the LF (4). We can note that accelerating the system by a refind value of the transport speed ρ may improve its decay rate. The value of δ reaches a maximum value $\delta_{max} > 2.5$ at the order $N = 1$, which corresponds to the particular case Wirtinger-based inequality (see Seuret and Gouaisbaut (2013)), rather than $\delta_{max} \simeq 2.5$ at the order $N = 4$ in Baudouin et al. (2016).

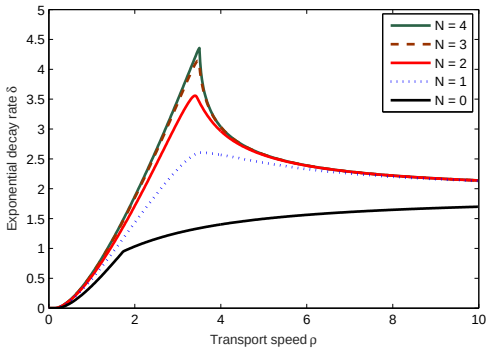

Fig. 1. Evolution of the decay rate δ obtained with Theorem 2 with respect to ρ for several value of N.

To explain the large difference between the two studies, we can note in Figure 2 that the analysis in Baudouin et al. (2016) was limited by the constraint $(\rho - 2\delta)R - 2\delta S \succ 0$ which is presented by the dashed line $\delta = \frac{\rho}{2}$. In fact, we have $-2\delta S \prec 0$ and to satisfy this constraint and obtain a positive term in $(\rho - 2\delta)R$, the decay rate δ has to be strictly lower than the half of the transport speed ρ. The new analysis with Lyapunov functional (4) allow us to avoid this constraint and reduce the conservatism of our study and which is illustrated with the blue curve in Figure 2.

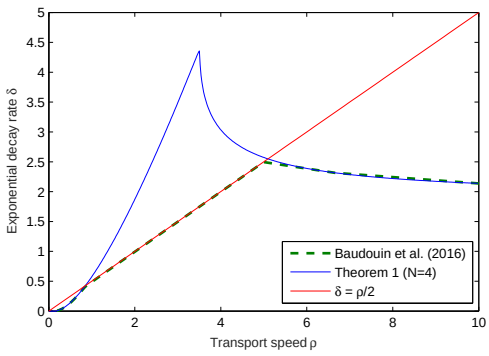

Fig. 2. Comparison between the decay rate δ obtained in Theorem 2 and in (Baudouin et al., 2016) with respect to ρ and for N=4.

The most important result to illustrate our approach appears in Figure 3 which shows that we obtain the same values as frequency analysis in Breda et al. (2015), for the exponential decay rate δ of the energy at the order $N = 4$. We remark also that we can improve the convergence rate by limiting the transport speed in system (1) since we have a maximum decay rate $\delta_{max} = 4.3562$ for the transport speed $\rho = 3.5010$.

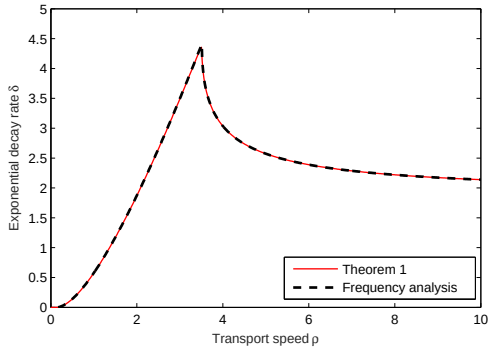

Fig. 3. Comparison of the decay rate δ obtained by Theorem 2 with $N = 4$ and obtained by Breda et al. (2015) using a frequency approach for several values of ρ.

Now, to test a more general case, we take the same dynamic matrix A with $D \neq 0$ by considering the following system

$$\begin{aligned} \dot{X}(t) &= \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} X(t) \\ &+ \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} (0.05X(t - \tfrac{h}{2}) + 0.95X(t - h)), \end{aligned}$$

which takes the shape of(1) for $\rho = \frac{2}{h}$ and

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 & -1 & 0 \\ -1 & -1 & -1 & -1 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.05 & 0 \\ 0 & 0.05 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 19 & 0 & 0 & 0 \\ 0 & 19 & 0 & 0 \end{bmatrix}.$$

Using Theorem 2, the maximum transport speed for stability is estimated as follows

	N=0	N=1	N=2	N=3
Variables	41	75	125	191
$\delta = 0$	$\rho_{min} = 0.253$	0.233	0.232	0.232
$\delta = 0.01$	$\rho_{min} = 0.391$	0.367	0.366	0.365

Table 1. Minimal allowable transport speed

When considering this example as a time delay system, the following table shows that the results are very close to those found in Gu et al. (2003), and to the analytical limit, even for small values of N.

Gu et al. (2003)	$N_{d1} = N_{d2} = 1$	$N_{d1} = N_{d2} = 2$
h_{max}	8.585	8.596
Theorem 2	N=4	N=5
h_{max}	8.594	8.596

Table 2. Comparison in term of time delay with Gu et al. (2003)

The variables N_{d1} and N_{d2} in Gu et al. (2003) present the discretization degrees, respectively for the delay terms $h_1 = \frac{h}{2}$ and $h_2 = h$.

6. CONCLUSION

In this document, we show as a first result that the structure of the proposed Lyapunov functional may improve the estimation of the convergence rate of system (1) and gives better assessment stability results. By changing the design of the LF, we give a novel approach for the stability analysis of coupled ODE - transport PDE systems issued from recent developments on time-delay systems.

Indeed, we provide an efficient result of stability of a coupled ODE-transport PDE system in terms of tractable LMIs depending on the transport speed ρ and the order N of a polynomial approximation of the infinite dimensional part of the state.

Moreover, the estimation of the decay rate that we generate with this approach is very close, in a definite order N, to the frequency results which are more precise, and we show that the maximum of this estimation is greater than the one found in (Baudouin et al., 2016).

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