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# Lyapunov stability analysis of a string equation coupled with an ordinary differential system

Mathieu Barreau, Alexandre Seuret, Frédéric Gouaisbaut and Lucie Baudouin

**Abstract**—In this paper we consider the stability of a linear time invariant system in feedback with a string equation. A new Lyapunov functional candidate is proposed based on the use of augmented states which enriches and encompasses the classical Lyapunov functional proposed in the literature. It results in a hierarchical tractable stability condition expressed in terms of linear matrix inequalities. This methodology follows from the application of the Bessel inequality together with Legendre polynomials. Two numerical examples illustrate the potential of our approach through two scenarios: a stable ODE perturbed by the PDE and an unstable ODE stabilized by the PDE.

**Index Terms**—String equation, Ordinary differential equation, Lyapunov functionals, LMI.

## I. INTRODUCTION

This paper presents a novel approach to assess stability of a heterogeneous system composed of the interconnection of a partial differential equation (PDE), more precisely a damped string equation with a linear ordinary differential equation (ODE). This class of interconnected systems has attracted many researchers. To cite only few related results in this area, one can read works on the coupling of ODEs with a transport equation [4], [5], [23], a heat equation [25], a string equation [13] or a beam equation [27].

While the area of stability and control of PDE systems has a rich literature at the boundary between applied mathematics [16], [7] and automatic control [18], the stability analysis and control of coupled PDE/ODE systems is a recent research area. The reader may refer to the backstepping method [13], [14] where the PDE equation is seen as a perturbation to an ODE system and where the main idea is to provide controllers that compensate the effects of the PDE equation. Conversely, the ODE system can be seen as a finite dimensional boundary controller for the PDE [21], [20], [10]. A last strategy describes a robust control approach, aiming at characterizing the robustness of the interconnection [11].

In the present paper, we consider a damped string equation, i.e. a stable one-dimensional wave equation which is perturbed at its boundary by a stable or unstable ODE. The proposed method to assess stability is inspired by the recent developments on the stability analysis of time-delay systems based on Bessel inequality and Legendre polynomials [24]. Since time-delay systems represent a particular class of a coupled transport PDE / ODE system (see for instance [2]), the main motivation of this work is to show how this methodology

can be adapted to a larger class of PDE / ODE systems as for the coupling between the heat equation [1] and an ODE. Compared to the literature on coupled PDE/ODE systems, the proposed methodology aims at designing a new Lyapunov functional, integrating some cross-terms merging the ODE's and the PDE's terms. This new class of Lyapunov functional encompasses the classical notion of energy usually proposed in the literature by offering more flexibility. Hence, it allows us to guarantee stability for a larger set of systems.

The paper is organized as follows. The next section formulates the problem and provides some general results on the existence of solutions and equilibrium. In Section 3, after a modeling phase using the Riemann coordinates, a generic form of Lyapunov functionals is introduced, and its associate analysis leading to a first stability theorem. Then, in Section 4, an extension using Bessel inequality is provided. Finally, Section 5 discussed the results on two examples. The last section draws some conclusion and perspectives.

**Notations:** In this paper,  $\Omega$  is the closed set  $[0, 1]$  and  $\mathbb{R}^+ = [0, +\infty)$ . Then,  $(x, t) \mapsto u(x, t)$  is a multi-variable function from  $\Omega \times \mathbb{R}^+$  to  $\mathbb{R}$ . The notation  $u_t$  stands for  $\frac{\partial u}{\partial t}$ . We also use the notations  $L^2 = L^2(\Omega; \mathbb{R})$  and for the Sobolov spaces:  $\mathcal{H}^n = \{z \in L^2; \forall m \leq n, \frac{\partial^m z}{\partial x^m} \in L^2\}$ . The norm in  $L^2$  is  $\|z\|^2 = \int_{\Omega} |z(x)|^2 dx = \langle z, z \rangle$ . For any square matrices  $A$  and  $B$ , the operations 'He' and 'diag' are defined as follow:  $\text{He}(A) = A + A^T$  and  $\text{diag}(A, B) = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ . A positive definite matrix  $P \in \mathbb{R}^{n \times n}$  belongs to the set  $\mathbb{S}_+^n$  or more simply  $P \succ 0$ .

## II. PROBLEM STATEMENT

We consider the coupled system described by

$$\dot{X}(t) = AX(t) + Bu(1, t), \quad t \geq 0 \quad (1a)$$

$$u_{tt}(x, t) = c^2 u_{xx}(x, t), \quad x \in \Omega, t \geq 0, \quad (1b)$$

$$u(0, t) = KX(t), \quad t \geq 0 \quad (1c)$$

$$u_x(1, t) = -c_0 u_t(1, t), \quad t \geq 0 \quad (1d)$$

$$u(x, 0) = u_0(x), \quad x \in \Omega \quad (1e)$$

$$u_t(x, 0) = v_0(x), \quad x \in \Omega \quad (1f)$$

$$X(0) = X^0, \quad (1g)$$

with the initial conditions  $X^0 \in \mathbb{R}^n$  and  $u_0 \in \mathcal{H}^1$ ,  $v_0 \in L^2$  such that equations (1c) and (1d) are respected. They are then called "compatibles" with the boundary conditions.

*Remark 1:* When there is no confusion possible, the parameter  $t$  may be omitted and so do the definition sets.

This system can be viewed as an interconnection in feedback between a linear time invariant system (1a) and an infinite dimensional system modeled by a wave equation (1b). The

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latter is a one dimension hyperbolic PDE, representing the evolution of a wave of speed  $c > 0$  and amplitude  $u$ . To keep the content clear, the dimension of  $u$  is assumed to be one but the calculus are done as if it was a vector of any dimension. The measure is the state  $u$  is for  $x = 1$  which is the end of the string and the control is a Dirichlet actuation (equation (1c)) because it affects directly the state  $u$ . To be well-posed, another boundary condition must be added. It is defined at  $x = 1$  by  $u_x(1) = -c_0 u_t(1)$ . This is a well-known damping condition for  $c_0 > 0$  (see for example the work by [15]). The case  $c_0 = 0$  suppresses the damping on the wave equation.

This system can be seen either as the control of the PDE by a finite dimensional dynamic control law generated by an ODE [6] or on the contrary the robustness of a linear closed loop system with a control signal conveyed by a damped wave equation. On the first scenario, both the ODE and the PDE are stable and the stability of the coupled system is studied. The second case is with an unstable but stabilizable ODE and the PDE is still stable. Notice that the Neumann actuation version of this problem, i.e. the control is acting on  $u_x(1)$  instead  $u(1)$ , has been studied by [13] using a backstepping transformation approach.

#### A. Existence and regularity of solutions

This subsection is dedicated to the existence and regularity of solutions  $(X, u)$  to system (1). Let consider the following space:  $H = \mathbb{R}^n \times \mathcal{H}^1 \times L^2$ .

We consider the classical norm on the Hilbert space  $H$ :

$$\|(X, u)\|_H^2 = |X|_n^2 + \|u\|^2 + c^2 \|u_x\|^2 + \|u_t\|^2.$$

This norm can be seen as the energy for the ODE system plus the one of the PDE. Indeed, the norm  $\|\cdot\|_H$  is constructed naturally in  $H$  as the norm for the PDE ( $L^2$ -norm of  $u_t$  and the  $\mathcal{H}^1$ -norm of  $u$ ) and the norm of the ODE.

*Remark 2:* A more natural norm for space  $H$  would be  $|X|_n^2 + \|u\|^2 + \|u_x\|^2 + \|u_t\|^2$  which is equivalent to  $\|\cdot\|_H^2$  for  $c > 0$ . The norm used here makes the calculus easier in the sequel.

Once the space is defined, we can model system (1) using the following linear unbounded operator  $T : \mathcal{D}(T) \rightarrow H$ :

$$T \begin{pmatrix} X \\ u \\ v \end{pmatrix} = \begin{pmatrix} AX + Bu(1) \\ v \\ c^2 u_{xx} \end{pmatrix} \text{ and}$$

$$\mathcal{D}(T) = \{(X, u, v) \in H, u(0) = KX, v(0) = AX + Bu(1) \\ u_x(1) = -c_0 v(1)\}.$$

Then, from the semi-group theory, we propose the following result on the existence of solutions for (1).

*Proposition 1:* If there exists a norm on  $H$  for which the linear operator  $T$  is dissipative, then there exists a unique solution  $(X, u, u_t)$  of system (1) with initial conditions  $(X^0, u_0, v_0) \in H$  compatible with the boundary conditions. Moreover, the solution have the following regularity property:  $(X, u, u_t) \in C(0, +\infty, H)$ .

*Proof :* This proof follows the same lines than in [19]. Applying Lumer-Phillips theorem (p103 from [26]) and as

the norm is dissipative, it is enough to show that there exists a  $\lambda > 0$  for which the application  $\lambda I - T : H \rightarrow H$  is onto. This is quite technical and has already been done by Morgul in [19] for a slightly different system.  $\square$

*Remark 3:* A norm  $V$  is said to be dissipative with respects to an operator  $T$  if its time-derivative along the trajectories generated by  $T$  is strictly negative.

The aim of this paper is then to find an equivalent norm to  $\|\cdot\|_H$  which makes the operator  $T$  dissipative. Indeed, the dissipative property of an operator is relative to the norm considered. The norm constructed in this paper has variable parameters which are tuned by the computer to make the operator  $T$  dissipative for a large class of system's parameters.

#### B. Equilibrium point

Considering that all the derivatives along the time are set to zero, an equilibrium  $(X_e, u_e)$  of the interconnected system (1) verifies the following linear equations:

$$0 = AX_e + Bu_e(1), \quad (2a)$$

$$0 = c^2 \partial_{xx} u_e(x), \quad x \in (0, 1), \quad (2b)$$

$$u_e(0) = KX_e, \quad (2c)$$

$$\partial_x u_e(1) = 0. \quad (2d)$$

Using equation (2b), we get  $u_e$  as a first order polynomial in  $x$  but in accordance to equation (2d),  $u_e$  is proved to be a constant function. Then, using equation (2c), we get  $u_e = KX_e$ . Equation (2a) and the previous statement lead to:  $(A + BK)X_e = 0$ . We get then the following proposition:

*Proposition 2:* An equilibrium  $(X_e, u_e)$  of system (1) verifies  $(A + BK)X_e = 0$  and  $u_e = KX_e$ . Moreover, if  $A + BK$  is not singular, system (1) admits a unique equilibrium  $(X_e, u_e) = (0_n, 0)$ .

### III. A FIRST STABILITY ANALYSIS BASED ON RIEMANN COORDINATES

This part is dedicated to the construction of a candidate Lyapunov functional. To do so, we need to introduce a new structure based on variables directly related to the states of the overall system (1).

#### A. Riemann coordinates

The PDE considered in system (1) is of second order in time. As we want to use some tools already designed for first order systems, we propose to define some news states using modified Riemann coordinates, which satisfy a set of coupled first order PDEs and diagonalize the operator. Let us introduce these coordinates, defined as follow:

$$\chi(x) = \begin{bmatrix} u_t(x) + cu_x(x) \\ u_t(1-x) - cu_x(1-x) \end{bmatrix} = \begin{bmatrix} \chi^+(x) \\ \chi^-(1-x) \end{bmatrix}.$$

The introduction of such a variable is not new and the reader can refer to articles [22], [3] or [8] and references therein about Riemann invariant.  $\chi^+$  and  $\chi^-$  are eigenfunctions of equation (1b) associated respectively to the eigenvalues  $c$  and

–c. Therefore, using  $\chi^-(1-x)$ , the previous equation leads to a transport PDE:

$$\chi_t(x) = c\chi_x(x). \quad (3)$$

with appropriate boundary conditions.

*Remark 4:* The norm of the modified state  $\chi$  can be directly related to the norm of the functions  $u_t$  and  $u_x$ . Indeed simple calculations and a change of variable lead to:

$$\|\chi\|^2 = 2(\|u_t\|^2 + c^2\|u_x\|^2). \quad (4)$$

The second step is to understand how the extra-variable  $\chi$  interacts with the ODE of the system (1). Hence, we notice that

$$\begin{aligned} \dot{X} &= AX + B(u(1) - u(0) + KX), \\ &= (A + BK)X + B \int_0^1 u_x(x)dx. \end{aligned}$$

It remains to express the last integral term using the variable  $\chi$ . To do so, let us note that:

$$2c \int_0^1 u_x(x)dx = \int_0^1 \chi^+(x)dx - \int_0^1 \chi^-(x)dx.$$

This expression then allows us to rewrite the ODE system as:

$$\dot{X} = (A + BK)X + \tilde{B}\tilde{x}_0,$$

where

$$\tilde{x}_0 = \int_0^1 \chi(x)dx, \quad \tilde{B} = \frac{1}{2c}B \begin{bmatrix} 1 & -1 \end{bmatrix}.$$

Introducing  $\tilde{x}_0$  as an extra-state, its dynamic is generated by:

$$\dot{\tilde{x}}_0 = c \int_0^1 \chi_x(x)dx = c[\chi(1) - \chi(0)].$$

The ODE dynamic can then be enriched by considering an extended system where  $\tilde{x}_0$  is viewed as a new dynamical state:

$$\dot{X}_0 = \begin{bmatrix} A+BK & \tilde{B} \\ 0_{2,n} & 0_2 \end{bmatrix} X_0 + \begin{bmatrix} 0_{n,2} \\ cI_2 \end{bmatrix} (\chi(1) - \chi(0)), \quad (5)$$

with

$$X_0 = \begin{bmatrix} X^\top & \tilde{x}_0^\top \end{bmatrix}^\top. \quad (6)$$

Hence, associated to the original system (1), we propose a set of equation (3)-(5)-(6). They are linked to system (1) but enriched by extra dynamics such as an interconnection between the extended finite dimensional system and the two transport equations. Nevertheless, these two systems are not equivalent, the second one just puts in the head a formulation for a candidate Lyapunov functional which is developed in the subsection below.

### B. Lyapunov functional and stability analysis

The main idea is to rely on the auxiliary variables satisfying equations (3) and (5) to define a Lyapunov functional for the original system (1). The auxiliary equations of the previous part show a coupling between a finite dimensional LTI system and an infinite dimension PDE seen as a transport equation. For the LTI system, the associated Lyapunov function is a simple quadratic term on the state  $X_0^\top P_0 X_0$ , with  $P_0 \in \mathbb{S}_+^{n+2}$ . It introduces automatically a cross-term between the ODE and the original PDE representing the coupling between the two systems.

For the infinite dimensional part, inspired from the literature on time-delay systems, we will provide a candidate Lyapunov functional. In [2], the proposed functional is the following one:

$$\mathcal{V}(u) = \int_0^1 \chi^\top(x) (S + xR) \chi(x)dx,$$

with  $S, R \in \mathbb{S}_+^2$ . The use of the modified Riemann coordinates enables us to consider full matrices  $S$  and  $R$ . If the classical  $\chi^+$  and  $\chi^-$  variables were used, it would have resulted in diagonal matrices with therefore more conservative stability conditions.

As the transport described by the variable  $\chi$  is going backward,  $R$  is multiplied by  $x$ . Finally, we propose a Lyapunov functional for system (1) expressed with the extended state variable  $X_0$  defined in equation (6):

$$V_0(X_0, u) = X_0^\top P_0 X_0 + \mathcal{V}(u). \quad (7)$$

The Lyapunov functional falls actually into three terms:

- 1) The quadratic one introduced by the ODE which is  $X^\top P X$ ;
- 2) The functional  $\mathcal{V}$  for the stability of the string equation;
- 3) The cross-term between  $\tilde{x}_0$  and  $X$  described by the extended state  $X_0$ .

The idea is that this last contribution is of great importance since it may enable the construction of non stable ODE leading to a stable coupled ODE/PDE. At this stage, a stability theorem can be derived using the Lyapunov functional  $V_0$ .

*Theorem 1:* Consider the system defined in (1) with a given speed  $c$ , a viscous damping  $c_0 > 0$  with initial conditions  $(X^0, u_0, v_0) \in H$  compatible with the boundary conditions. If there exist  $P_0 \in \mathbb{S}_+^{n+2}$  and  $S, R \in \mathbb{S}_+^2$  such that denoting

$$\Psi_0 = \text{He} \left( Z_0^\top P_0 F_0 \right) - c\tilde{R}_0 + c(H_0^\top (S + R)H_0 - G_0^\top S G_0) \quad (8)$$

The linear matrix inequality  $\Psi_0 \prec 0$  holds with

$$\begin{aligned} F_0 &= \begin{bmatrix} I_{n+2} & 0_{n+2,2} \end{bmatrix}, & Z_0 &= \begin{bmatrix} \mathcal{N}_0^\top & c(H_0 - G_0)^\top \end{bmatrix}^\top, \\ \mathcal{N}_0 &= \begin{bmatrix} A+BK & \tilde{B} & 0_{n,2} \end{bmatrix}, & \tilde{R}_0 &= \text{diag}(0_n, R, 0_2), \\ G_0 &= \begin{bmatrix} 0_{2,n+2} & g \end{bmatrix} + \begin{bmatrix} -K \\ 0_{1,n} \end{bmatrix} \mathcal{N}_0, & g &= - \begin{bmatrix} 0 & 1 \\ 1+cc_0 & 0 \end{bmatrix}, \\ H_0 &= \begin{bmatrix} 0_{2,n+2} & h \end{bmatrix} + \begin{bmatrix} 0_{1,n} \\ K \end{bmatrix} \mathcal{N}_0, & h &= \begin{bmatrix} 1-cc_0 & 0 \\ 0 & -1 \end{bmatrix}. \end{aligned} \quad (9)$$

then the following statements hold:

- (i) There exists a unique solution to the system described by equation (1).
- (ii) The coupled infinite dimensional system (1) is exponentially stable in the sense of norm  $\|\cdot\|_H^2$  i.e. there exist  $\gamma \geq 1, \delta > 0$  such that the following estimate holds:

$$\forall t > 0, \|(X(t), u(t))\|_H^2 \leq \gamma e^{-\delta t} \|(X^0, u_0)\|_H^2. \quad (10)$$

*Remark 5:* It has been proven by [12] that the exponential stability of a similar coupled system is impossible if we consider an undamped string equation. Indeed, condition  $\Psi_0 \prec 0$  includes another necessary condition given by  $\Psi_0(3, 3) = e_3^\top \Psi_0 e_3 \prec 0$ , with  $e_3 = [0_{n,n+2} \ I_2]^\top$ , which is  $h^\top (S + R)h - g^\top S g \prec 0$ . This inequality is guaranteed if and only if the matrix  $g^{-1}h$  has its eigenvalues in the unit cycle

of the complex plan, which is guaranteed by the condition  $c_0 > 0$ .

### C. Proof of Theorem 1

The proof of item (ii) is presented below and gives insights on the proof of item (i).

1) *Preliminaries:* As a first step of this proof, let us introduce the following preliminary lemma that will be useful in the sequel.

*Lemma 1:* The following inequality holds:

$$\|u\|^2 \leq 2\|u_x\|^2 + 2|u(0)|^2, \quad \forall u \in \mathcal{H}^1(\Omega).$$

*Proof:* As  $u_x \in L^1(\Omega)$ , we have, for all  $x \in \Omega$ ,

$$u(x)^2 = \left( \int_0^x u_s(s) ds - u(0) \right)^2 \leq 2 \int_0^x u_s^2(s) ds + 2|u(0)|^2.$$

The last inequality is obtained using Young and Jensen inequalities.  $\square$

The proof of Theorem 1 consists in proving that if the LMI condition presented in Theorem 1 is satisfied, then there exist a norm  $V$  and  $\varepsilon_1, \varepsilon_2, \varepsilon_3$  three positive scalars such that inequalities

$$\begin{aligned} \varepsilon_1 \|(X, u)\|_H^2 &\leq V \leq \varepsilon_2 \|(X, u)\|_H^2, \\ \dot{V} &\leq -\varepsilon_3 \|(X, u)\|_H^2, \end{aligned} \quad (11)$$

The next paragraphs aim at proving these in order to prove the convergence of the states to the equilibrium.

2) *Existence of  $\varepsilon_1$ :* The LMIs conditions,  $P_0 \succ 0, S \succ 0$  and  $R \succ 0$  mean that there exists  $\varepsilon_1 > 0$ , such that for all  $x \in [0, 1]$ :

$$\begin{aligned} P_0 &\succeq \varepsilon_1 (I_{n+2} + 2K^\top K), \\ S + xR &\succeq S \succeq \varepsilon_1 \frac{2+c^2}{2c^2} I_2. \end{aligned}$$

These inequalities lead to:

$$\begin{aligned} V_0(X, u) &\geq \varepsilon_1 \left( |X|_n^2 + |KX|^2 + \frac{2+c^2}{2c^2} \|\chi\|^2 \right) \\ &\quad + \int_0^1 \chi^\top(x) \left( S + xR - \varepsilon_1 \frac{2+c^2}{2c^2} I_2 \right) \chi(x) dx \\ &\geq \varepsilon_1 \left( |X|_n^2 + |KX|^2 + \frac{2+c^2}{2c^2} \|\chi\|^2 \right). \end{aligned}$$

Noting the boundary condition (1c) and norm equality (4), the previous inequality becomes

$$\begin{aligned} V_0(X, u) &\geq \varepsilon_1 \left( |X|_n^2 + \|u\|^2 + \|u_t\|^2 + c^2 \|u_x\|^2 \right) \\ &\quad + \frac{2\varepsilon_1}{c^2} \|u_t\|^2 + \varepsilon_1 \left( 2\|u_x\|^2 + 2|u(0)|^2 - \|u\|^2 \right). \end{aligned}$$

Then, we apply Lemma 1 to ensure that the last term of the previous inequality is positive so that it yields  $V_0 \geq \varepsilon_1 \|(X, u)\|_H^2$ , which ends the proof for the existence of  $\varepsilon_1$ .

3) *Existence of  $\varepsilon_2$ :* Since  $P_0 \in \mathbb{S}_+^{n+2}$  and  $S, R \in \mathbb{S}_+^2$ , there exists  $\varepsilon_2 > 0$  such that for  $x \in (0, 1)$ :

$$\begin{aligned} P_0 &\preceq \text{diag}(\varepsilon_2 I_n, \frac{\varepsilon_2}{4} I_2), \\ S + xR &\preceq S + R \preceq \frac{\varepsilon_2}{4} I_2. \end{aligned}$$

From equation (7), we get:

$$\begin{aligned} V_0(X, u) &\leq \varepsilon_2 \left( |X|_n^2 + \frac{1}{4} \mathfrak{X}_0^\top \mathfrak{X}_0 + \frac{1}{4} \|\chi\|^2 \right) \\ &\quad + \int_0^1 \chi^\top(x) \left( S + xR - \frac{\varepsilon_2}{4} I_2 \right) \chi(x) dx \\ &\leq \varepsilon_2 \left( |X|_n^2 + \frac{1}{2} \|\chi\|^2 \right) \end{aligned} \quad (12)$$

where we have used Jensen's inequality which ensures that  $\mathfrak{X}_0^\top \mathfrak{X}_0 \leq \|\chi\|^2$ . The proof of the existence of  $\varepsilon_2$  ends by using norm equality (4) so that we get:

$$V_0(X, u) \leq \varepsilon_2 \left( |X|_n^2 + \|u_t\|^2 + c^2 \|u_x\|^2 \right) \leq \varepsilon_2 \|(X, u)\|_H^2.$$

4) *Existence of  $\varepsilon_3$ :* Differentiating  $V_0$  in (7) along the trajectories of system (1) leads to

$$\dot{V}_0(X, u) = \text{He} \left( \begin{bmatrix} \dot{X} \\ \dot{\mathfrak{X}}_0 \end{bmatrix}^\top P_0 \begin{bmatrix} X \\ \mathfrak{X}_0 \end{bmatrix} \right) + \dot{V}(u).$$

Our goal is to expressed an upper bound of  $\dot{V}_0$  thanks to the extended vector  $\xi_0$  defined as follow:

$$\xi_0 = \begin{bmatrix} X^\top & \mathfrak{X}_0^\top & u_t(1) & cu_x(0) \end{bmatrix}^\top. \quad (13)$$

Let us first concentrate on  $\dot{V}$ . Equation (3) yields:

$$\dot{V}(u) = 2c \int_0^1 \chi_x^\top(x, t) (S + xR) \chi(x, t) dx. \quad (14)$$

Integrating by parts the last expression leads to:

$$\begin{aligned} \dot{V}(u) &= c \left( \chi^\top(1) (S + R) \chi(1) - \chi^\top(0) S \chi(0) \right. \\ &\quad \left. - \int_0^1 \chi^\top(x) R \chi(x) dx \right). \end{aligned} \quad (15)$$

Then we note that  $\dot{X} = \mathcal{N}_0 \xi_0, \dot{\mathfrak{X}}_0 = c(H_0 - G_0) \xi_0, \chi(1) = H_0 \xi_0, \chi(0) = G_0 \xi_0$ , with  $\xi_0$  defined in (13) and the matrices above in equation (9). We get  $X_0 = F_0 \xi_0$  and  $\dot{X}_0 = Z_0 \xi_0$  and the resulting expression for  $\dot{V}_0$  is obtained:

$$\begin{aligned} \dot{V}_0(X, u) &= \xi_0^\top \left( \text{He} (Z_0^\top P_0 F_0) + cG_1^\top (S + R) G_1 - cG_0^\top S G_0 \right) \xi_0 \\ &\quad - c \int_0^1 \chi^\top(x) R \chi(x) dx. \end{aligned} \quad (16)$$

Then, using the definition of matrix  $\Psi_0$  given in (8), the previous expression can be rewritten as follows:

$$\dot{V}_0(X, u) = \xi_0^\top \Psi_0 \xi_0 + c \mathfrak{X}_0^\top R \mathfrak{X}_0 - c \int_0^1 \chi^\top(x) R \chi(x) dx. \quad (17)$$

Since  $R \succ 0$  and  $\Psi_0 \prec 0$ , there exists  $\varepsilon_3 > 0$  such that:

$$R \succeq \frac{\varepsilon_3}{2c} \frac{2+c^2}{c^2} I_2, \quad (18a)$$

$$\Psi_0 \preceq -\varepsilon_3 \text{diag} \left( I_n + 2K^\top K, \frac{2+c^2}{2c^2} I_2, 0_2 \right). \quad (18b)$$

Using (18b) and the boundary condition  $u(0) = KX$ , equation (17) becomes:

$$\begin{aligned} \dot{V}_0(X, u) &\leq -\varepsilon_3 \left( |X|_n^2 + 2|u(0)|^2 + \frac{2+c^2}{2c^2} \|\chi\|^2 \right) \\ &\quad + c \mathfrak{X}_0^\top \left( R - \frac{\varepsilon_3}{2c} \frac{2+c^2}{c^2} I_2 \right) \mathfrak{X}_0 \\ &\quad - c \int_0^1 \chi^\top(x) \left( R - \frac{\varepsilon_3}{2c} \frac{2+c^2}{c^2} I_2 \right) \chi(x) dx \end{aligned}$$

So that we get by application of Jensen's inequality:

$$\dot{V}_0(X, u) \leq -\varepsilon_3 \left( |X|_n^2 + 2|u(0)|^2 + \frac{2+c^2}{2c^2} \|\chi\|^2 \right). \quad (19)$$

The most important part of the proof lies in the following trick. Since (4) holds, we get:

$$\begin{aligned} \dot{V}_0(X, u) &\leq -\varepsilon_3 (|X|_n^2 + \|u\|^2 + \|u_t\|^2 + c^2 \|u_x\|^2) \\ &\quad -\varepsilon_3 \frac{2}{c^2} \|u_t\|^2 - \varepsilon_3 (2|u(0)|^2 + 2\|u_x\|^2 - \|u\|^2) \\ &= -\varepsilon_3 \|(X, u)\|_H - \varepsilon_3 \frac{2}{c^2} \|u_t\|^2 \\ &\quad -\varepsilon_3 (2|u(0)|^2 + 2\|u_x\|^2 - \|u\|^2). \end{aligned}$$

Moreover, Lemma 1 ensures that the last term of the previous expression is negative so that we have  $\dot{V}_0(X, u) \leq -\varepsilon_3 \|(X, u)\|_H^2$ , which concludes this proof of existence.

5) *Conclusion:* Finally, there exist  $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$  such that equation (11) holds for a functional  $V_0$ . Hence  $V_0(\cdot)$  is an equivalent norm of  $\|\cdot\|_H$  which is strictly decreasing. It means, according to Propositions 1 and 2 that there exists a unique solution to system (1) converging in  $H$  to the solution equilibrium  $(0, 0)$ . These conditions also bring :  $\forall t > 0, \dot{V}_0(t) + \frac{\varepsilon_3}{\varepsilon_2} V_0(t) \leq 0$  and

$$\begin{aligned} \|(X(t), u(t))\|_H^2 &\leq \frac{1}{\varepsilon_1} V_0(t) \leq \frac{1}{\varepsilon_1} V_0(0) e^{-\frac{\varepsilon_3}{\varepsilon_2} t} \\ &\leq \frac{\varepsilon_2}{\varepsilon_1} \|(X(0), u(0))\|_H^2 e^{-\frac{\varepsilon_3}{\varepsilon_2} t}, \end{aligned}$$

which shows the exponential convergence of all trajectories of system (1). In other words, the norm  $\|\cdot\|_H$  is exponentially decreasing along the trajectories of system (1).

*Remark 6:* There is no need to specify  $A+BK$  non singular in the previous theorem. Indeed, if the conditions of Theorem 1 are satisfied, then for  $e_1 = [I_n \ 0_{n,4}]^\top$ , the following inequality holds:

$$\Psi_0(1, 1) = e_1^\top \Psi_0 e_1 \prec 0.$$

After some simplifications, we get:

$$\Psi_0(1, 1) = \text{He}((A+BK)^\top Q) \prec 0, \quad (20)$$

for some matrix  $Q$  depending on  $R, S$  and  $P_0$ . Inequality (20) implies that  $A+BK$  is not singular. As  $Q$  is apparently not symmetric, it is not possible to conclude directly on the stability of  $A+BK$ .

*Remark 7:* It is also worth noting that LMI (8) is affine with respect to matrices  $A, B$ , which allows us, in a straightforward manner, to extend this theorem to uncertain ODE systems subject, for instance, to polytopic-type uncertainties.

#### IV. EXTENDED STABILITY ANALYSIS

##### A. Main motivation

In the previous analysis, we have proposed an auxiliary system presented in (5) which helps us to define a new Lyapunov functional for system (1). The notable aspect is that the term  $\mathfrak{X}_0 = \int_0^1 \chi(x) dx$  appears naturally in the dynamics of this system. In light of the previous work on integral inequalities by [24], this term can also be interpreted as the projection of the modified state of system (1) over the set of constant functions in the sense of the inner product in  $L_2$

defined in the introduction. One may therefore enrich system (5) by some projections over other functions. As done with time-delay systems, we propose then to do the projection of the state function  $\chi$  on the Legendre polynomials.

The family of Legendre polynomials, denoted  $\{\mathcal{L}_k\}_{k \in \mathbb{N}}$ , is an orthogonal family of polynomials with respect to the classical inner product of  $L^2$ . They are shifted from the traditional ones as described in [9]. Their definition is not required, only some of their properties are considered. The reader may look at [24] where a definition is provided.

##### B. Preliminaries

The previous discussion leads us to define additional vectors for any function  $\chi$  in  $L^2$ :

$$\forall k \in \mathbb{N}, \quad \mathfrak{X}_k = \int_0^1 \chi(x) \mathcal{L}_k(x) dx,$$

and the augmented vector  $X_N$ , defined for a given order  $N \in \mathbb{N}$ , as follow:

$$X_N = \begin{bmatrix} X^\top & \mathfrak{X}_0^\top & \dots & \mathfrak{X}_N^\top \end{bmatrix}^\top. \quad (21)$$

Following the same methodology as for Theorem 1, this specific structure leads us to introduce a new Lyapunov functional with  $P_N \in \mathbb{S}_+^{n+2N+2}$ :

$$V_N(X, u) = X_N^\top P_N X_N + \mathcal{V}(u). \quad (22)$$

In order to follow the same procedure as in Theorem 1, several technical extensions are required. Indeed, the stability conditions issued from the functional  $V_0$ , are coming from Jensen's inequality and an explicit expression of the time derivative of  $\mathfrak{X}_0$ . Therefore, it is necessary to provide an extended version of the Jensen inequality and of this differentiation rule. These technical steps are summarized in the two following lemmas.

*Lemma 2:* For any function  $\chi \in L^2$  and for a symmetric positive matrix  $R \in \mathbb{S}_+^2$  the following integral inequality holds for all  $N \in \mathbb{N}$ :

$$\int_0^1 \chi^\top(x) R \chi(x) dx \geq \sum_{k=0}^N (2k+1) \mathfrak{X}_k^\top R \mathfrak{X}_k. \quad (23)$$

This inequality includes Jensen's inequality as the particular case  $N = 0$ , which was one of the key element to the proof of Theorem 1. This comment allows us to think that the previous lemma is the appropriate extension of the Jensen's inequality to address the stability analysis using the new Lyapunov functional (22) with the augmented state  $X_N$ .

Even if the proof can be found in [2], we would like to point out that it is based on the following equality, which results from the orthogonality of the Legendre polynomials:

$$\begin{aligned} \int_0^1 \chi^\top(x) R \chi(x) dx &- \sum_{k=0}^N (2k+1) \mathfrak{X}_k^\top R \mathfrak{X}_k \\ &= \int_0^1 \chi_N^\top(x) R \chi_N(x) dx \geq 0, \end{aligned}$$

where  $\chi_N(x) = \chi(x) - \sum_{k=0}^N (2k+1) \mathfrak{X}_k \mathcal{L}_k(x)$  can be interpreted as the error approximation between the function  $\chi$  and its orthogonal projection over the family  $\{\mathcal{L}_k\}_{k \leq N}$ .

The next lemma is concerned by the differentiation of  $\mathfrak{X}_k$ .

*Lemma 3:* For any function  $\chi \in L^2$ , the following expression holds for any  $N$  in  $\mathbb{N}$ :

$$\begin{bmatrix} \dot{\mathfrak{x}}_0 \\ \vdots \\ \dot{\mathfrak{x}}_N \end{bmatrix} = c\mathbb{1}_N\chi(1) - c\bar{\mathbb{1}}_N\chi(0) - cL_N \begin{bmatrix} \mathfrak{x}_0 \\ \vdots \\ \mathfrak{x}_N \end{bmatrix},$$

where

$$L_N = \begin{bmatrix} \ell_{0,0}I_2 & \cdots & 0_2 \\ \vdots & \ddots & \vdots \\ \ell_{N,0}I_2 & \cdots & \ell_{N,N}I_2 \end{bmatrix}, \mathbb{1}_N = \begin{bmatrix} I_2 \\ \vdots \\ I_2 \end{bmatrix}, \bar{\mathbb{1}}_N = \begin{bmatrix} I_2 \\ \vdots \\ (-1)^N I_2 \end{bmatrix}, \quad (24)$$

and

$$\ell_{k,j} = \begin{cases} (2j+1)(1 - (-1)^{j+k}), & j \leq k-1, \\ 0, & \text{otherwise.} \end{cases} \quad (25)$$

*Proof:* The proof of this lemma is presented in the appendix of this paper because of its technical nature.  $\square$

### C. Main result

Considering the previous lemmas, the following extension to Theorem 1, using the extra-state  $X_N$ , is stated:

*Theorem 2:* Consider the system defined in (1) with a given speed  $c$ , a viscous damping  $c_0 > 0$  with initial conditions  $(X^0, u_0, v_0) \in H$  compatible with the boundary conditions. Assume that for a given integer  $N \in \mathbb{N}$ , there exist  $P_N \in \mathbb{S}_+^{n+2N+2}$  and  $S, R \in \mathbb{S}_+^2$  such that inequality

$$\begin{aligned} \Psi_N = \text{He} \left( Z_N^\top P_N F_N \right) - c\tilde{R}_N \\ + c \left( H_N^\top (S + R) H_N - G_N^\top S G_N \right) \prec 0 \end{aligned}$$

holds where

$$\begin{aligned} F_N &= \begin{bmatrix} I_{n+2N+2} & 0_{n+2N+2,2} \end{bmatrix}, Z_N = \begin{bmatrix} \mathcal{N}_N^\top & c\mathcal{Z}_N^\top \end{bmatrix}^\top, \\ \mathcal{N}_N &= \begin{bmatrix} A + BK & \tilde{B} & 0_{n,2(N+1)} \end{bmatrix}, \\ \mathcal{Z}_N &= \mathbb{1}_N H_N + \bar{\mathbb{1}}_N G_N - \begin{bmatrix} 0_{2N+2,n} & L_N & 0_{2N+2,2} \end{bmatrix}, \\ G_N &= \begin{bmatrix} 0_{2,n+2N+2} & g \end{bmatrix} + \begin{bmatrix} -K \\ 0_{1,n} \end{bmatrix} \mathcal{N}_N, \\ H_N &= \begin{bmatrix} 0_{2,n+2N+2} & h \end{bmatrix} + \begin{bmatrix} 0_{1,n} \\ K \end{bmatrix} \mathcal{N}_N, \\ \tilde{R} &= \text{diag}(0_n, R, 3R, \dots, (2N+1)R, 0_2). \end{aligned} \quad (26)$$

and where matrices  $L_N$ ,  $\mathbb{1}_N$  and  $\bar{\mathbb{1}}_N$  are given in (24), then, the coupled infinite dimensional system (1) is exponentially stable in the sense of norm  $\|\cdot\|_H^2$  and there exist  $\gamma > 1$  and  $\delta > 0$  such that equation (10) holds.

*Remark 8:* Remarks 5 and 6 also apply for this theorem. More particularly, the condition of the previous theorem requires that matrix  $A + BK$  is non singular to be feasible.

*Remark 9:* The article from [24] shows that this methodology introduces a hierarchy. In other words, the set

$$\mathcal{C}_N = \left\{ c > 0 \text{ s.t. } \exists P_N \in \mathbb{S}_+^{n+2(N+1)}, S, R \in \mathbb{S}_+^2, \Psi_N \prec 0 \right\}$$

which represents the values of parameter  $c$  such that the LMIs are feasible for a given system (1) satisfies  $\mathcal{C}_N \subseteq \mathcal{C}_{N+1}$ .

That means, if there is solution to Theorem 2 at an order  $N_0$ , then there also exists a solution for any order  $N \geq N_0$ . The proof is very similar to the one by [24]. We can proceed by

induction with  $P_{N+1} = \begin{bmatrix} P_N & 0 \\ 0 & \varepsilon I_2 \end{bmatrix}$  and a sufficiently small  $\varepsilon > 0$ . Then,  $\Psi_N \prec 0 \Rightarrow \Psi_{N+1} \prec 0$ . The calculus are quite long and technical so they are not written fully in the paper. This is due to the derivation of the Legendre polynomials which can be expressed in strictly lower order Legendre polynomials (because  $L_N$  is strictly lower triangular).

### D. Proof of Theorem 2

The proof is following the same reasoning than for the previous theorem and consists in proving the existence of positive scalars  $\varepsilon_1, \varepsilon_2$  and  $\varepsilon_3$  such that the functional  $V_N$  verifies inequalities (11).

1) *Existence of  $\varepsilon_1$ :* It strictly follows the same line as in Theorem 1 and is therefore omitted.

2) *Existence of  $\varepsilon_2$ :* For  $\varepsilon_2$ , as  $P_N, S, R \succ 0$ , there exists  $\varepsilon_2 > 0$  such that:

$$\begin{aligned} P_N &\preceq \text{diag} \left( \varepsilon_2 I_n, \frac{\varepsilon_2}{4} \text{diag} \{ (2k+1)I_n \}_{k \in (0,N)} \right), \\ (S + xR) &\preceq S + R \preceq \frac{\varepsilon_2}{4} I_2, \quad \forall x \in (0,1). \end{aligned}$$

From equation (22), we get:

$$\begin{aligned} V_N(X, u) &\leq \varepsilon_2 |X|_n^2 + \frac{\varepsilon_2}{4} \left( \sum_{k=0}^N (2k+1) \mathfrak{X}_k^\top \mathfrak{X}_k + \|\chi\|^2 \right) \\ &\leq \varepsilon_2 \left( |X|_n^2 + \frac{1}{2} \|\chi\|^2 \right). \end{aligned}$$

While the first inequality is guaranteed by the constraint  $(S + xR) \preceq \frac{\varepsilon_2}{4} I_2$ , for all  $x \in (0,1)$ , the last inequality results from the application of Bessel inequality (23). Therefore, following the same procedure than in the proof of Theorem 1 since equation (12), there indeed exists  $\varepsilon_2$  such that  $V_N(X, u) \leq \varepsilon_2 \|(X, u)\|_H^2$ .

3) *Existence of  $\varepsilon_3$ :* Differentiating  $V_N$  in (22) along the trajectories of system (1) leads to:

$$\dot{V}_N(X, u) = \text{He} \left( \begin{bmatrix} \dot{X} \\ \dot{\mathfrak{x}}_0 \\ \vdots \\ \dot{\mathfrak{x}}_N \end{bmatrix}^\top P_N \begin{bmatrix} X \\ \mathfrak{x}_0 \\ \vdots \\ \mathfrak{x}_N \end{bmatrix} \right) + \dot{V}(u).$$

The aim of this part is to find an upper bound of  $\dot{V}_N$  using the following extended state:  $\xi_N = \begin{bmatrix} X_N^\top & u_t(1) & cu_x(0) \end{bmatrix}^\top$ . Using equation (15) and Lemma 3, we note that  $X_N = F_N \xi_N$ ,  $\dot{X}_N = Z_N \xi_N$ ,  $\chi(1) = H_N \xi_N$ ,  $\chi(0) = G_N \xi_N$  where the matrices  $F_N, Z_N, H_N, G_N$  are given in (26). Then the expression of  $\dot{V}_N$  holds

$$\begin{aligned} \dot{V}_N(X, u) &= \xi_N^\top \Psi_N \xi_N + c \sum_{k=0}^N \mathfrak{X}_k^\top (2k+1) R \mathfrak{X}_k \\ &\quad - c \int_0^1 \chi^\top(x) R \chi(x) dx. \end{aligned} \quad (27)$$

Since  $R \succ 0$  and  $\Psi_N \prec 0$ , there exists  $\varepsilon_3 > 0$  such that:

$$\begin{aligned} R &\succeq \frac{\varepsilon_3}{2c} \frac{2+c^2}{c^2} I_2, \\ \Psi_N &\preceq -\varepsilon_3 \text{diag} \left( I_n + K^\top K, \right. \\ &\quad \left. \frac{2+c^2}{2c^2} \text{diag} \{ I_2, 3I_2, \dots, (2N+1)I_2 \}, 0_2 \right). \end{aligned} \quad (28)$$

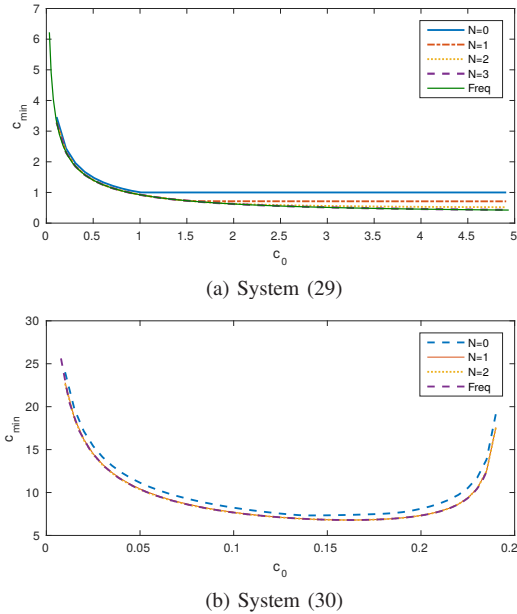


Fig. 1: Minimum wave speed  $c_{min}$  as a function of  $c_0$  for System (1) to be stable. The values for  $A$ ,  $B$  and  $K$  are given by equations (29) or (30).

Using (28) in order to apply Bessel's inequality, equation (27) becomes then:

$$\dot{V}_N(X, u) \leq -\varepsilon_3 \left( |X|_n^2 + 2|u(0)|^2 + \frac{2+c^2}{2c^2} \|\chi\|^2 \right),$$

which is similar to equation (19) in the proof of Theorem 1. Therefore, following the same procedure, there exists  $\varepsilon_3 > 0$  such that  $\dot{V}_N(X, u) \leq -\varepsilon_3 \|(X, u)\|_H^2$ .

4) *Conclusion:* Thus, there exist  $\varepsilon_1, \varepsilon_2$  and  $\varepsilon_3$  positive such that inequalities (11) are satisfied and the exponential stability of system (1) is guaranteed.

## V. EXAMPLES

Two examples of stability for problem (1) will be provided here. The solver used for the LMIs is `sdp3` with the `YALMIP` toolbox (by [17]).

### A. Problem (1) with $A$ and $A + BK$ Hurwitz

In this first part, the considered system is defined as follow:

$$A = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad K = \begin{bmatrix} 0 & -2 \end{bmatrix}. \quad (29)$$

$A$  is then stable and  $A + BK$  has two poles in  $-3$  and  $-2$ . The ODE and the PDE are then stable when they are not coupled. As shown in Figure 1a, there exists a minimum speed for the wave called here  $c_{min}$  which is a function of the damping  $c_0$  for the system to be stable.

The phenomenon induced by the coupling can be understood if we consider the robustness of the ODE to a disturbance generated by a wave equation. The wave equation is then seen as a disturbance for the ODE. If the wave speed is large enough, the perturbation tends to 0 fast enough for the ODE to keep its stability behavior. Another important thing to notice is the hierarchy property i.e. the decrease of  $c_{min}$

as  $N$  increases. The curve denoted ‘‘Freq’’ is obtained using a frequentional analysis and displays the exact stability area. This method will be explained in another paper.

### B. Problem (1) with $A + BK$ Hurwitz but an unstable $A$ .

This time, the system is described by the following set of matrices:

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad K = \begin{bmatrix} -10 & 2 \end{bmatrix}. \quad (30)$$

If  $K$  was a null matrix, the ODE system would not be stable. We are then studying the stabilization of the ODE through a communication modeled by the wave equation. For the same reason than before, the wave must be fast enough for the control not to be too much delayed but also with a moderated dumping to transfer the state variable  $X$  through the PDE equation. This is then a trade-off between  $c_{min}$  and  $c_0$ , introducing a  $c_{0max}$  as it is possible to see in Figure 1b.

Some numerical simulations have been done on this last example. The energy is then not strictly decreasing even if the system is stable (the choice of the couple  $(c, c_0)$  is in the stability area of system (30)). This is not the case for the Lyapunov functional in which assesses the stability of the coupled system. Figure 1b shows that for system (30) with  $c_0 = 0.15$ , the minimum wave speed is  $c_{min} = 6.83$ . The numerical stability can also be seen in Figure 2 and indeed, the system is at the boundary of the stable area in Figure 2b and unstable for smaller values. The frequentional criterion has been designed to assess stability for system (30) and the results are close to the one provided by the general framework with LMI conditions. That means the stability area provided with  $N = 1$  is a good estimation of the maximum stability set.

## VI. CONCLUSION

A hierarchy of stability criteria has been provided for the stability of systems described by the interconnection between a finite dimensional linear system and an infinite dimensional system modeled by a string equation. The proposed methodology relies on an extensive use of Bessel inequality which allow to design a new Lyapunov functional. This new class encompasses the classical notion of energy proposed in that case. In particular, the stability of the ODE is not a requirement anymore.

## APPENDIX

### A. Proof of Lemma 3

For a given integer  $k$  in  $\mathbb{N}$ , the differentiating of  $\mathfrak{X}_k$  along the trajectories of (3) yields

$$\dot{\mathfrak{X}}_k = \int_0^1 \chi_t(x) \mathcal{L}_k(x) dx = c \int_0^1 \chi_x(x) \mathcal{L}_k(x) dx.$$

Then by integrating by parts, we get

$$\dot{\mathfrak{X}}_k = c \left( [\chi(x) \mathcal{L}_k(x)]_0^1 - \int_0^1 \chi(x) \mathcal{L}'_k(x) dx \right). \quad (31)$$

In order to derive the expression of  $\dot{\mathfrak{X}}_k$ , we will use the following properties of the Legendre polynomials. On the one hand, the values of Legendre polynomials at the boundaries of  $[0, 1]$  are given by  $\mathcal{L}_k(0) = (-1)^k$  and  $\mathcal{L}_k(1) = 1$ . On the second hand, the Legendre polynomials verifies the following



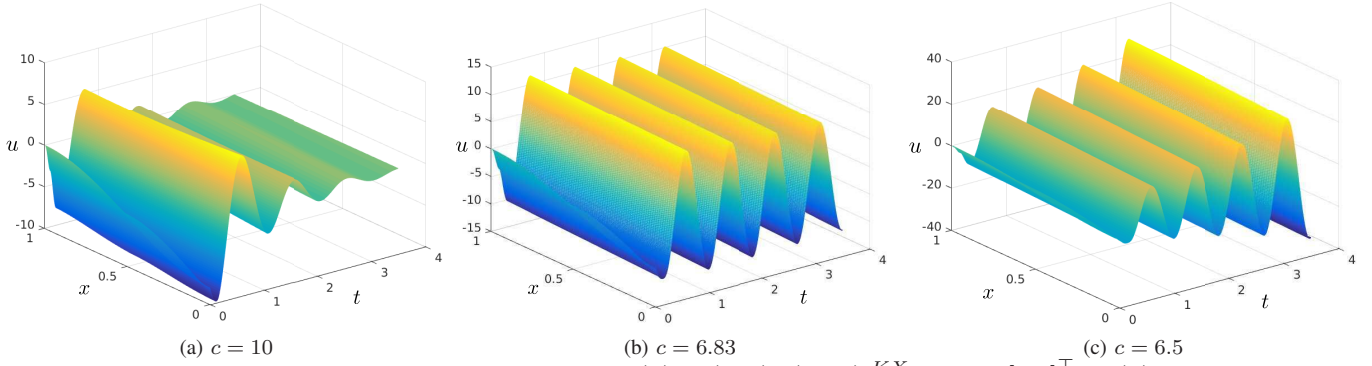


Fig. 2: Chart of  $u$  for system (30) with the parameters:  $u_0(x) = (\cos(\pi x) + 1) \frac{KX_0}{2}$ ,  $X_0 = [1 \ 1]^\top$ ,  $v_0(x) = 0$  and  $c_0 = 0.15$  for 3 values of  $c$ . These results are obtained using Euler forward as a numerical scheme.

differentiation rules. For  $k \geq 1$ , the derivation of a Legendre polynomial  $\mathcal{L}_k$  can be expressed in the basis  $\{\mathcal{L}_j\}_{j \in (0, k-1)}$ :

$$\frac{d}{dx} \mathcal{L}_k(x) = \begin{cases} \sum_{j=0}^{k-1} (2j+1)(-1)^{j+k} \mathcal{L}_j(x), & \text{if } k \geq 1, \\ 0, & \text{if } k = 0. \end{cases}$$

Hence, injecting these expression into (31) leads to:

$$\dot{\mathfrak{X}}_k = c(\chi(1, t) - (-1)^k \chi(0)) - c \sum_{j=0}^N \ell_N^{kj} \mathfrak{X}_j$$

where the coefficient  $\ell_N^{kj}$  are defined in equation (25). The end of the proof consists in merging the previous expression from  $k = 1$  to  $k = N$ , leading to the definition of matrices  $L_N$ ,  $\mathbb{1}_N$  and  $\bar{\mathbb{1}}_N$  given in (24). We just want to point out that  $L_N$  is a strictly lower triangular matrix.

## REFERENCES

- [1] L. Baudouin, A. Seuret, and F. Gouaisbaut. Lyapunov stability analysis of a linear system coupled to a heat equation. To be published for IFAC World Congress, 2017.
- [2] L. Baudouin, A. Seuret, and M. Safi. Stability analysis of a system coupled to a transport equation using integral inequalities. 2016. 2nd IFAC Workshop on Control of Systems Governed by Partial Differential Equations (CPDE) in Italy.
- [3] D. Bresch-Pietri and M. Krstic. Output-feedback adaptive control of a wave PDE with boundary anti-damping. *Automatica*, 50(5):1407–1415, 2014.
- [4] F. Castillo, E. Witrant, C. Prieur, and L. Dugard. Dynamic Boundary Stabilization of Linear and Quasi-Linear Hyperbolic Systems. In *Decision and Control (CDC), 2012 IEEE 51st Annual Conference on*, pages 2952–2957. IEEE, 2012.
- [5] F. Castillo, E. Witrant, C. Prieur, and L. Dugard. Dynamic boundary stabilization of first order hyperbolic systems. In *Recent Results on Time-Delay Systems*, pages 169–190. Springer International Publishing, 2016.
- [6] G. Chen and J. Zhou. The wave propagation method for the analysis of boundary stabilization in vibrating structures. *SIAM Journal on Applied Mathematics*, 50(5):1254–1283, 1990.
- [7] J. M. Coron. *Control and nonlinearity*. Number 136. American Mathematical Soc., 2007.
- [8] J. M. Coron, B. d’Andrea Novel, and G. Bastin. A strict lyapunov function for boundary control of hyperbolic systems of conservation laws. *IEEE Transactions on Automatic Control*, 52(1):2–11, Jan 2007.
- [9] R. Courant and D. Hilbert. *Methods of mathematical physics*. John Wiley & Sons, Inc., 1989.
- [10] B. d’Andréa Novel, F. Boustany, F. Conrad, and B. P. Rao. Feedback stabilization of a hybrid pde-ode system: Application to an overhead crane. *Mathematics of Control, Signals and Systems*, 7(1):1–22, 1994.
- [11] N. Espitia, A. Girard, N. Marchand, and C. Prieur. Event-based control of linear hyperbolic systems of conservation laws. *Automatica*, 70:275–287, 2016.
- [12] A. Helmicki, C. A. Jacobson, and C. N. Nett. Ill-posed distributed parameter systems: A control viewpoint. *IEEE transactions on automatic control*, 36(9):1053–1057, 1991.
- [13] M. Krstic. *Delay compensation for nonlinear, adaptive, and PDE systems*. Springer, 2009.
- [14] M. Krstic. Dead-time compensation for wave/string pdes. *Journal of Dynamic Systems, Measurement, and Control*, 133(3), 2011.
- [15] J. Lagnese. Decay of solutions of wave equations in a bounded region with boundary dissipation. *Journal of Differential equations*, 50(2):163–182, 1983.
- [16] J.-L. Lions. Exact controllability, stabilization and perturbations for distributed systems. *SIAM review*, 30(1):1–68, 1988.
- [17] J. Löfberg. Yalmip: A toolbox for modeling and optimization in matlab. pages 284–289, 2005.
- [18] Z.-H. Luo, B.-Z. Guo, and Ö. Morgül. *Stability and stabilization of infinite dimensional systems with applications*. Springer Science & Business Media, 2012.
- [19] Ö. Morgül. A dynamic control law for the wave equation. 30(11):1785–1792, 1994.
- [20] Ö. Morgül. On the stabilization and stability robustness against small delays of some damped wave equations. *IEEE Transactions on Automatic Control*, 40(9):1626–1630, 1995.
- [21] Ö. Morgül. An exponential stability result for the wave equation. *Automatica*, 38(4):731–735, 2002.
- [22] C. Prieur, S. Tarbouriech, and J. M. G. da Silva. Wave equation with cone-bounded control laws. *IEEE Transactions on Automatic Control*, 61(11):3452–3463, 2016.
- [23] M. Safi, L. Baudouin, and A. Seuret. Stability analysis of a linear system coupled to a transport equation using integral inequalities. To be published for IFAC World Congress, 2017.
- [24] A. Seuret and F. Gouaisbaut. Hierarchy of LMI conditions for the stability analysis of time delay systems. *Systems & Control Letters*, 81:1–7, 2015.
- [25] S. Tang and C. Xie. State and output feedback boundary control for a coupled PDE–ODE system. *Systems & Control Letters*, 60(8):540–545, 2011.
- [26] M. Tucsnak and G. Weiss. *Observation and control for operator semigroups*. Springer Science & Business Media, 2009.
- [27] H.-N. Wu and J.-W. Wang. Static output feedback control via PDE boundary and ODE measurements in linear cascaded ODE-beam systems. *Automatica*, 50(11):2787–2798, 2014.