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# Lyapunov stability analysis of a string equation coupled with an ordinary differential system

Mathieu Barreau, Alexandre Seuret, Frédéric Gouaisbaut and Lucie Baudouin

**Abstract**—This paper considers the stability problem of a linear time invariant system in feedback with a string equation. A new Lyapunov functional candidate is proposed based on the use of augmented states which enriches and encompasses the classical Lyapunov functional proposed in the literature. It results in tractable stability conditions expressed in terms of linear matrix inequalities. This methodology follows from the application of the Bessel inequality together with Legendre polynomials. Numerical examples illustrate the potential of our approach through three scenari: a stable ODE perturbed by the PDE, an unstable open-loop ODE stabilized by the PDE and an unstable closed-loop ODE stabilized by the PDE.

**Index Terms**—String equation, Ordinary differential equation, Lyapunov functionals, LMI.

## I. INTRODUCTION

This paper presents a novel approach to assess stability of a heterogeneous system composed of the interconnection of a partial differential equation (PDE), more precisely a damped string equation, with a linear ordinary differential equation (ODE). While the topic of stability and control of PDE systems has a rich literature at the boundary between applied mathematics [7], [16] and automatic control [18]; the stability analysis (and the control) of such a coupled system belongs to a recent research area. To cite a few related results, one can refer to [4], [5], [23] where an ODE is interconnected with a transport equation, to [25] for a heat equation, to [13] for the wave equation and to [27] for the beam equation.

Generally, the PDE is viewed as a perturbation to be compensated (using a backstepping method proposed by [14] for example where infinite dimensional controllers are provided to cope with the undesirable effect of the PDE). Another interesting point of view relies on the converse approach: the ODE system can be seen as a finite dimensional boundary controller for the PDE (see [10], [20], [21]). A last strategy describes a robust control approach, aiming at characterizing the robustness of the interconnection [11].

In the present paper, we consider a damped string equation, i.e. a stable one-dimensional wave equation which is perturbed at its boundary by a stable or unstable ODE. The proposed method to assess stability is inspired by the recent developments on the stability analysis of time-delay systems based on Bessel inequality and Legendre polynomials [24]. Since time-delay systems represent a particular class of systems coupling a transport PDE with a classical ODE system (see for instance

[2]), the main motivation of this work is to show how this methodology can be adapted to a larger class of PDE/ODE systems as demonstrated for instance with the heat equation in [1].

Compared to the literature on coupled PDE/ODE systems, the proposed methodology aims at designing a new Lyapunov functional, integrating some cross-terms merging the ODE's and the PDE's usual terms. This new class of Lyapunov functional encompasses the classical notion of energy usually proposed in the literature by offering more flexibility. Hence, it allows us to guarantee stability for a larger set of systems, for instance, instable open-loop ODE and, for the first time to the best of our knowledge, even an unstable closed-loop ODE.

The paper is organized as follows. The next section formulates the problem and provides some general results on the existence of solutions and equilibrium. In Section 3, after a modeling phase inspired by the Riemann coordinates, a generic form of Lyapunov functionals is introduced, and its associate analysis leads to a first stability theorem. Then, in Section 4, an extension using Bessel inequality is provided. Finally, Section 5 discusses the results on three examples. The last section draws some conclusion and perspectives.

**Notations:** In this paper,  $\Omega$  is the closed set  $[0, 1]$  and  $\mathbb{R}^+ = [0, +\infty)$ . Then,  $(x, t) \mapsto u(x, t)$  is a multi-variable function from  $\Omega \times \mathbb{R}^+$  to  $\mathbb{R}$ . The notation  $u_t$  stands for  $\frac{\partial u}{\partial t}$ . We also use the notations  $L^2 = L^2(\Omega; \mathbb{R})$  and for the Sobolov spaces:  $H^n = \{z \in L^2; \forall m \leq n, \frac{\partial^m z}{\partial x^m} \in L^2\}$ . The norm in  $L^2$  is  $\|z\|^2 = \int_{\Omega} |z(x)|^2 dx = \langle z, z \rangle$ . For any square matrices  $A$  and  $B$ , the operations 'He' and 'diag' are defined as follow:  $\text{He}(A) = A + A^T$  and  $\text{diag}(A, B) = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ . A symmetric positive definite matrix  $P$  of  $\mathbb{R}^{n \times n}$  belongs to the set  $\mathbb{S}_+^n$  or more simply  $P \succ 0$ .

## II. PROBLEM STATEMENT

We consider the coupled system described by

$$\dot{X}(t) = AX(t) + Bu(1, t), \quad t \geq 0 \quad (1a)$$

$$u_{tt}(x, t) = c^2 u_{xx}(x, t), \quad x \in \Omega, t \geq 0, \quad (1b)$$

$$u(0, t) = KX(t), \quad t \geq 0 \quad (1c)$$

$$u_x(1, t) = -c_0 u_t(1, t), \quad t \geq 0 \quad (1d)$$

$$u(x, 0) = u_0(x), \quad x \in \Omega \quad (1e)$$

$$u_t(x, 0) = v_0(x), \quad x \in \Omega \quad (1f)$$

$$X(0) = X^0, \quad (1g)$$

with the initial conditions  $X^0 \in \mathbb{R}^n$  and  $(u_0, v_0) \in H^1 \times L^2$  such that equations (1c) and (1d) are respected. They are then called "compatibles" with the boundary conditions.

*Remark 1:* When no confusion is possible, parameter  $t$  may be omitted and so do the definition sets.

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This system can be viewed as an interconnection in feedback between a linear time invariant system (1a) and an infinite dimensional system modeled by a wave equation (1b). The latter is a one dimension hyperbolic PDE, representing the evolution of a wave of speed  $c > 0$  and amplitude  $u$ . To keep the content clear, the dimension of  $x \mapsto u(x, \cdot)$  is assumed to be one but the calculus are done as if it was a vector of any dimension. The measure is the state  $u$  at  $x = 1$  which is the end of the string and the control is a Dirichlet actuation (equation (1c)) because it affects directly the state  $u$  and not its derivative. To be well-posed, another boundary condition must be added. It is defined at  $x = 1$  by  $u_x(1) = -c_0 u_t(1)$ . This is a well-known damping condition for  $c_0 > 0$  (see for example the work by [15]). The case  $c_0 = 0$  suppresses the damping on the wave equation.

This system can be seen either as the control of the PDE by a finite dimensional dynamic control law generated by an ODE [6] or on the contrary the robustness of a linear closed loop system with a control signal conveyed by a damped wave equation. On the first scenario, both the ODE and the PDE are stable and the stability of the coupled system is studied. The second case is with an unstable but stabilizable ODE and the PDE is still stable. Notice that the Neumann actuation version of this problem, i.e. the control  $KX$  is acting on  $u_x(0)$  instead of  $u(0)$ , has been studied by [13] using a backstepping transformation approach.

#### A. Existence and regularity of solutions

This subsection is dedicated to the existence and regularity of solutions  $(X, u)$  to system (1). We consider the classical norm on the Hilbert space  $\mathcal{H} = \mathbb{R}^n \times H^1 \times L^2$ :

$$\|(X, u)\|_{\mathcal{H}}^2 = |X|_n^2 + \|u\|^2 + c^2 \|u_x\|^2 + \|u_t\|^2.$$

This norm can be seen as the sum of the energy of the ODE system and the one of the PDE.

*Remark 2:* A more natural norm for space  $\mathcal{H}$  would be  $|X|_n^2 + \|u\|^2 + \|u_x\|^2 + \|u_t\|^2$  which is equivalent to  $\|\cdot\|_{\mathcal{H}}^2$  for  $c > 0$ . The norm used here makes the calculus easier in the sequel.

Once the space is defined, we can translate System (1) using the following linear unbounded operator  $T : \mathcal{D}(T) \rightarrow \mathcal{H}$ :

$$T \begin{pmatrix} X \\ u \\ v \end{pmatrix} = \begin{pmatrix} AX + Bu(1) \\ v \\ c^2 u_{xx} \end{pmatrix} \text{ and}$$

$$\mathcal{D}(T) = \{(X, u, v) \in \mathcal{H}, u(0) = KX, \\ v(0) = AX + Bu(1), u_x(1) = -c_0 v(1)\}.$$

Then, from the semi-group theory, we propose the following result on the existence of solutions for (1).

*Proposition 1:* If there exists a norm on  $\mathcal{H}$  for which the linear operator  $T$  is dissipative, then there exists a unique solution  $(X, u, u_t)$  of system (1) with initial conditions  $(X^0, u_0, v_0) \in \mathcal{H}$  compatible with the boundary conditions. Moreover, the solution have the following regularity property:  $(X, u, u_t) \in C(0, +\infty, \mathcal{H})$ .

*Proof :* This proof follows the same lines than in [19]. Applying Lumer-Phillips theorem (p103 from [26]) and as the

norm is dissipative, it is enough to show that there exists a  $\lambda > 0$  for which the application  $\lambda I - T : \mathcal{H} \rightarrow \mathcal{H}$  is onto. This is quite technical and has already been done by Morgul in [19] for a slightly different system considering a Neumann actuation.  $\square$

*Remark 3:* An operator  $T$  is said to be dissipative with respect to a norm if the time-derivative along the trajectories generated by  $T$  is strictly negative.

The aim of this paper is then to find an equivalent norm to  $\|\cdot\|_{\mathcal{H}}$  which makes the operator  $T$  dissipative. Indeed, the dissipative property of an operator is relative to the norm under consideration. The norm constructed in this paper, that we will actually call Lyapunov functional, has variable parameters which are tuned using an optimization process to make the operator  $T$  dissipative for a given system.

#### B. Equilibrium point

Considering that all the derivatives along the time are set to zero, an equilibrium  $(X_e, u_e) \in \mathbb{R}^n \times H^1$  of System (1) verifies the following linear equations:

$$0 = AX_e + Bu_e(1), \quad (2a)$$

$$0 = c^2 \partial_{xx} u_e(x), \quad x \in (0, 1), \quad (2b)$$

$$u_e(0) = KX_e, \quad (2c)$$

$$\partial_x u_e(1) = 0. \quad (2d)$$

Using equation (2b), we get  $u_e$  as a first order polynomial in  $x$  but in accordance to equation (2d),  $u_e$  is proved to be a constant function. Then, using equation (2c), we get  $u_e = KX_e$ . Equation (2a) and the previous statement lead to:  $(A + BK)X_e = 0$ . We get then the following proposition:

*Proposition 2:* An equilibrium  $(X_e, u_e) \in \mathbb{R}^n \times H^1$  of system (1) verifies  $(A + BK)X_e = 0$  and  $u_e = KX_e$ . Moreover, if  $A + BK$  is not singular, system (1) admits a unique equilibrium  $(X_e, u_e) = (0_n, 0)$ .

### III. A FIRST STABILITY ANALYSIS BASED ON MODIFIED RIEMANN COORDINATES

This part is dedicated to the construction of a candidate Lyapunov functional. To do so, we need to introduce a new structure based on variables directly related to the states of the overall system (1).

#### A. Modified Riemann coordinates

The PDE considered in system (1) is of second order in time. As we want to use some tools already designed for first order systems, we propose to define some new states using modified Riemann coordinates, which satisfy a set of coupled first order PDEs and diagonalize the operator. Let us introduce these coordinates, defined as follow:

$$\chi(x) = \begin{bmatrix} u_t(x) + cu_x(x) \\ u_t(1-x) - cu_x(1-x) \end{bmatrix} = \begin{bmatrix} \chi^+(x) \\ \chi^-(1-x) \end{bmatrix}.$$

The introduction of such a variable is not new and the reader can refer to articles [22], [3] or [8] and references therein about Riemann invariant.  $\chi^+$  and  $\chi^-$  are eigenfunctions of equation (1b) associated respectively to the eigenvalues  $c$  and

–c. Therefore, using  $\chi^-(1-x)$ , the previous equation leads to a transport PDE for  $x \in \Omega$ :

$$\chi_t(x) = c\chi_x(x). \quad (3)$$

*Remark 4:* The norm of the modified state  $\chi$  can be directly related to the norm of the functions  $u_t$  and  $u_x$ . Indeed simple calculations and a change of variable give:

$$\|\chi\|^2 = 2(\|u_t\|^2 + c^2\|u_x\|^2). \quad (4)$$

The second step is to understand how the extra-variable  $\chi$  interacts with the ODE of the system (1). Hence, we notice:

$$\begin{aligned} \dot{X} &= AX + B(u(1) - u(0) + KX), \\ &= (A + BK)X + B \int_0^1 u_x(x)dx. \end{aligned}$$

It remains to express the last integral term using  $\chi$ . Let us first note that:

$$2c \int_0^1 u_x(x)dx = \int_0^1 \chi^+(x)dx - \int_0^1 \chi^-(x)dx.$$

This expression allows us to rewrite the ODE system as:

$$\dot{X} = (A + BK)X + \tilde{B}\tilde{\mathfrak{X}}_0,$$

where

$$\tilde{\mathfrak{X}}_0 = \int_0^1 \chi(x)dx, \quad \tilde{B} = \frac{1}{2c}B \begin{bmatrix} 1 & -1 \end{bmatrix}.$$

The extra-state  $\tilde{\mathfrak{X}}_0$  follows the dynamics:

$$\dot{\tilde{\mathfrak{X}}}_0 = c \int_0^1 \chi_x(x)dx = c[\chi(1) - \chi(0)].$$

The ODE dynamic can then be enriched by considering an extended system where  $\tilde{\mathfrak{X}}_0$  is viewed as a new dynamical state:

$$\dot{X}_0 = \begin{bmatrix} A+BK & \tilde{B} \\ 0_{2,n} & 0_2 \end{bmatrix} X_0 + \begin{bmatrix} 0_{n,2} \\ cI_2 \end{bmatrix} (\chi(1) - \chi(0)), \quad (5)$$

with

$$X_0 = \begin{bmatrix} X^\top & \tilde{\mathfrak{X}}_0^\top \end{bmatrix}^\top. \quad (6)$$

Hence, associated to the original system (1), we propose a set of equation (3)-(5)-(6). They are linked to system (1) but enriched by extra dynamics aiming at representing the interconnection between the extended finite dimensional system and the two transport equations. Nevertheless, these two systems are not equivalent, the second one just puts in the head a formulation for a candidate Lyapunov functional which is developed in the subsection below.

### B. Lyapunov functional and stability analysis

The main idea is to rely on the auxiliary variables satisfying equations (3) and (5) to define a Lyapunov functional for the original system (1). For the ODE (5), the associated Lyapunov function is a simple quadratic term on the state  $X_0^\top P_0 X_0$ , with  $P_0 \in \mathbb{S}_+^{n+2}$ . It introduces automatically a cross-term between the ODE and the original PDE. Hence, the auxiliary equations of the previous part show a coupling between a finite dimensional LTI system and an infinite dimension PDE seen as a transport equation.

For the infinite dimensional part, inspired from the literature on time-delay systems ([2], [8]), we provide a candidate

Lyapunov functional:

$$\mathcal{V}(u) = \int_0^1 \chi^\top(x) (S + xR) \chi(x) dx,$$

with  $S, R \in \mathbb{S}_+^2$ . The use of the modified Riemann coordinates enables us to consider full matrices  $S$  and  $R$ . If the classical  $\chi^+$  and  $\chi^-$  variables were used, it would have resulted in diagonal matrices with therefore more conservative stability conditions. As the transport described by the variable  $\chi$  is going backward,  $R$  is multiplied by  $x$ . Finally, we propose a Lyapunov functional for system (1) expressed with the extended state variable  $X_0$  defined in equation (6):

$$V_0(X_0, u) = X_0^\top P_0 X_0 + \mathcal{V}(u). \quad (7)$$

This Lyapunov functional falls actually into three terms:

- 1) The quadratic term  $X^\top P X$  introduced by the ODE;
- 2) The functional  $\mathcal{V}$  for the stability of the string equation;
- 3) The cross-term between  $\tilde{\mathfrak{X}}_0$  and  $X$  described by the extended state  $X_0$ .

The idea is that this last contribution is of great importance since it may enable the construction of non stable ODE leading to a stable coupled ODE/PDE. At this stage, a stability theorem can be derived using the Lyapunov functional  $V_0$ .

*Theorem 1:* Consider the system defined in (1) with a given speed  $c$ , a viscous damping  $c_0 > 0$  with initial conditions  $(X^0, u_0, v_0) \in \mathcal{H}$  compatible with the boundary conditions. Assume there exist  $P_0 \in \mathbb{S}_+^{n+2}$  and  $S, R \in \mathbb{S}_+^2$  such that the linear matrix inequality  $\Psi_0 \prec 0$  holds where

$$\Psi_0 = \text{He} \left( Z_0^\top P_0 F_0 \right) - c\tilde{R}_0 + c(H_0^\top (S + R)H_0 - G_0^\top S G_0) \quad (8)$$

$$\begin{aligned} F_0 &= \begin{bmatrix} I_{n+2} & 0_{n+2,2} \end{bmatrix}, & Z_0 &= \begin{bmatrix} \mathcal{N}_0^\top & c(H_0 - G_0)^\top \end{bmatrix}^\top, \\ \mathcal{N}_0 &= \begin{bmatrix} A+BK & \tilde{B} & 0_{n,2} \end{bmatrix}, & \tilde{R}_0 &= \text{diag}(0_n, R, 0_2), \\ G_0 &= \begin{bmatrix} 0_{2,n+2} & g \end{bmatrix} + \begin{bmatrix} -K \\ 0_{1,n} \end{bmatrix} \mathcal{N}_0, & g &= - \begin{bmatrix} 0 \\ 1+cc_0 & 0 \end{bmatrix}, \\ H_0 &= \begin{bmatrix} 0_{2,n+2} & h \end{bmatrix} + \begin{bmatrix} 0_{1,n} \\ K \end{bmatrix} \mathcal{N}_0, & h &= \begin{bmatrix} 1-cc_0 & 0 \\ 0 & -1 \end{bmatrix}. \end{aligned} \quad (9)$$

Then there exists a unique solution to System (1) and it is exponentially stable in the sense of norm  $\|\cdot\|_{\mathcal{H}}$  i.e. there exist  $\gamma \geq 1, \delta > 0$  such that the following estimate holds:

$$\forall t > 0, \|(X(t), u(t))\|_{\mathcal{H}}^2 \leq \gamma e^{-\delta t} \|(X^0, u_0)\|_{\mathcal{H}}^2. \quad (10)$$

*Remark 5:* It has been proven by [12] that the exponential stability of a similar coupled system is impossible if we consider an undamped string equation. Indeed, condition  $\Psi_0 \prec 0$  includes a necessary condition given by  $e_3^\top \Psi_0 e_3 \prec 0$ , with  $e_3 = [0_{n,n+2} \ I_2]^\top$ , which is  $h^\top (S + R)h - g^\top S g \prec 0$ . This inequality is guaranteed if and only if the matrix  $g^{-1}h$  has its eigenvalues in the unit cycle of the complex plan, which is guaranteed if and only if  $c_0 > 0$ .

### C. Proof of Theorem 1

The proof of stability is presented below and gives insights on the proof of existence.

1) *Preliminaries:* As a first step of this proof, let us introduce the following preliminary lemma that will be useful in the sequel.

*Lemma 1:* The following inequality holds:

$$\|u\|^2 \leq 2\|u_x\|^2 + 2|u(0)|^2, \quad \forall u \in \mathcal{H}^1(\Omega).$$

*Proof:* As  $u_x \in L^1(\Omega)$ , we have, for all  $x \in \Omega$ ,

$$u(x)^2 = \left( \int_0^x u_s(s) ds - u(0) \right)^2 \leq 2 \int_0^x u_s^2(s) ds + 2|u(0)|^2.$$

The last inequality is obtained using Young and Jensen inequalities.  $\square$

The proof of Theorem 1 consists in explaining how if the LMI condition presented in Theorem 1 is satisfied, there exist a norm  $V$  and three positive scalars  $\varepsilon_1, \varepsilon_2$  and  $\varepsilon_3$  such that inequalities

$$\begin{aligned} \varepsilon_1 \|(X, u)\|_{\mathcal{H}}^2 &\leq V(X, u) \leq \varepsilon_2 \|(X, u)\|_{\mathcal{H}}^2, \\ \dot{V}(X, u) &\leq -\varepsilon_3 \|(X, u)\|_{\mathcal{H}}^2, \end{aligned} \quad (11)$$

hold.

The next paragraphs aim at proving (11) in order to prove the convergence of the states to the equilibrium.

2) *Existence of  $\varepsilon_1$ :* The LMI conditions,  $P_0 \succ 0, S \succ 0$  and  $R \succ 0$  mean that there exists  $\varepsilon_1 > 0$ , such that for all  $x \in [0, 1]$ :

$$\begin{aligned} P_0 &\succeq \varepsilon_1 (I_{n+2} + 2K^\top K), \\ S + xR &\succeq S \succeq \varepsilon_1 \frac{2+c^2}{2c^2} I_2. \end{aligned}$$

These inequalities lead to:

$$\begin{aligned} V_0(X, u) &\geq \varepsilon_1 \left( |X|_n^2 + |KX|^2 + \frac{2+c^2}{2c^2} \|\chi\|^2 \right) \\ &\quad + \int_0^1 \chi^\top(x) \left( S + xR - \varepsilon_1 \frac{2+c^2}{2c^2} I_2 \right) \chi(x) dx \\ &\geq \varepsilon_1 \left( |X|_n^2 + |KX|^2 + \frac{2+c^2}{2c^2} \|\chi\|^2 \right). \end{aligned}$$

Noting the boundary condition (1c) and norm equality (4), the previous inequality becomes

$$\begin{aligned} V_0(X, u) &\geq \varepsilon_1 \left( |X|_n^2 + \|u\|^2 + \|u_t\|^2 + c^2 \|u_x\|^2 \right) \\ &\quad + \frac{2\varepsilon_1}{c^2} \|u_t\|^2 + \varepsilon_1 \left( 2\|u_x\|^2 + 2|u(0)|^2 - \|u\|^2 \right). \end{aligned}$$

Then, we apply Lemma 1 to ensure that the last term of the previous inequality is positive so that it yields  $V_0(X, u) \geq \varepsilon_1 \|(X, u)\|_{\mathcal{H}}^2$ , which ends the proof for the existence of  $\varepsilon_1$ .

3) *Existence of  $\varepsilon_2$ :* Since  $P_0 \in \mathbb{S}_+^{n+2}$  and  $S, R \in \mathbb{S}_+^2$ , there exists  $\varepsilon_2 > 0$  such that for  $x \in (0, 1)$ :

$$\begin{aligned} P_0 &\preceq \text{diag}(\varepsilon_2 I_n, \frac{\varepsilon_2}{4} I_2), \\ S + xR &\preceq S + R \preceq \frac{\varepsilon_2}{4} I_2. \end{aligned}$$

From equation (7), we get:

$$\begin{aligned} V_0(X, u) &\leq \varepsilon_2 \left( |X|_n^2 + \frac{1}{4} \mathfrak{X}_0^\top \mathfrak{X}_0 + \frac{1}{4} \|\chi\|^2 \right) \\ &\quad + \int_0^1 \chi^\top(x) \left( S + xR - \frac{\varepsilon_2}{4} I_2 \right) \chi(x) dx \quad (12) \\ &\leq \varepsilon_2 \left( |X|_n^2 + \frac{1}{2} \|\chi\|^2 \right) \end{aligned}$$

where we have used Jensen's inequality which ensures that  $\mathfrak{X}_0^\top \mathfrak{X}_0 \leq \|\chi\|^2$ . The proof of the existence of  $\varepsilon_2$  ends by

using norm equality (4) so that we get:

$$V_0(X, u) \leq \varepsilon_2 \left( |X|_n^2 + \|u_t\|^2 + c^2 \|u_x\|^2 \right) \leq \varepsilon_2 \|(X, u)\|_{\mathcal{H}}^2.$$

4) *Existence of  $\varepsilon_3$ :* Differentiating  $V_0$  in (7) along the trajectories of system (1) leads to

$$\dot{V}_0(X, u) = \text{He} \left( \begin{bmatrix} \dot{X} \\ \dot{\mathfrak{X}}_0 \end{bmatrix}^\top P_0 \begin{bmatrix} X \\ \mathfrak{X}_0 \end{bmatrix} \right) + \dot{V}(u).$$

Our goal is to expressed an upper bound of  $\dot{V}_0$  thanks to the extended vector  $\xi_0$  defined as follow:

$$\xi_0 = \begin{bmatrix} X^\top & \mathfrak{X}_0^\top & u_t(1) & cu_x(0) \end{bmatrix}^\top. \quad (13)$$

Let us first concentrate on  $\dot{V}$ . Equation (3) yields:

$$\dot{V}(u) = 2c \int_0^1 \chi_x^\top(x, t) (S + xR) \chi(x, t) dx. \quad (14)$$

Integrating by parts the last expression leads to:

$$\begin{aligned} \dot{V}(u) &= c \left( \chi^\top(1) (S + R) \chi(1) - \chi^\top(0) S \chi(0) \right. \\ &\quad \left. - \int_0^1 \chi^\top(x) R \chi(x) dx \right). \end{aligned} \quad (15)$$

Then we note that  $\dot{X} = N_0 \xi_0, \dot{\mathfrak{X}}_0 = c(H_0 - G_0) \xi_0, \chi(1) = H_0 \xi_0, \chi(0) = G_0 \xi_0$ , with  $\xi_0$  defined in (13) and the matrices above in equation (9). We get  $X_0 = F_0 \xi_0$  and  $\dot{X}_0 = Z_0 \xi_0$  and the resulting expression for  $\dot{V}_0$  is obtained:

$$\begin{aligned} \dot{V}_0(X, u) &= \xi_0^\top \left( \text{He} (Z_0^\top P_0 F_0) + cG_1^\top (S + R) G_1 - cG_0^\top S G_0 \right) \xi_0 \\ &\quad - c \int_0^1 \chi^\top(x) R \chi(x) dx. \end{aligned} \quad (16)$$

Then, using the definition of matrix  $\Psi_0$  given in (8), the previous expression can be rewritten as follows:

$$\dot{V}_0(X, u) = \xi_0^\top \Psi_0 \xi_0 + c \mathfrak{X}_0^\top R \mathfrak{X}_0 - c \int_0^1 \chi^\top(x) R \chi(x) dx. \quad (17)$$

Since  $R \succ 0$  and  $\Psi_0 \prec 0$ , there exists  $\varepsilon_3 > 0$  such that:

$$R \succeq \frac{\varepsilon_3}{2c} \frac{2+c^2}{c^2} I_2, \quad (18a)$$

$$\Psi_0 \preceq -\varepsilon_3 \text{diag} \left( I_n + 2K^\top K, \frac{2+c^2}{2c^2} I_2, 0_2 \right). \quad (18b)$$

Using (18b) and the boundary condition  $u(0) = KX$ , equation (17) becomes:

$$\begin{aligned} \dot{V}_0(X, u) &\leq -\varepsilon_3 \left( |X|_n^2 + 2|u(0)|^2 + \frac{2+c^2}{2c^2} \|\chi\|^2 \right) \\ &\quad + c \mathfrak{X}_0^\top \left( R - \frac{\varepsilon_3}{2c} \frac{2+c^2}{c^2} I_2 \right) \mathfrak{X}_0 \\ &\quad - c \int_0^1 \chi^\top(x) \left( R - \frac{\varepsilon_3}{2c} \frac{2+c^2}{c^2} I_2 \right) \chi(x) dx \end{aligned}$$

So that we get by application of Jensen's inequality:

$$\dot{V}_0(X, u) \leq -\varepsilon_3 \left( |X|_n^2 + 2|u(0)|^2 + \frac{2+c^2}{2c^2} \|\chi\|^2 \right). \quad (19)$$

The most important part of the proof lies in the following

trick. Since (4) holds, we get:

$$\begin{aligned}\dot{V}_0(X, u) &\leq -\varepsilon_3 \left( |X|_n^2 + \|u\|^2 + \|u_t\|^2 + c^2 \|u_x\|^2 \right) \\ &\quad - \varepsilon_3 \frac{2}{c^2} \|u_t\|^2 - \varepsilon_3 \left( 2|u(0)|^2 + 2\|u_x\|^2 - \|u\|^2 \right) \\ &= -\varepsilon_3 \| (X, u) \|_{\mathcal{H}}^2 - \varepsilon_3 \frac{2}{c^2} \|u_t\|^2 \\ &\quad - \varepsilon_3 \left( 2|u(0)|^2 + 2\|u_x\|^2 - \|u\|^2 \right).\end{aligned}$$

Moreover, Lemma 1 ensures that the last term of the previous expression is negative so that we have  $\dot{V}_0(X, u) \leq -\varepsilon_3 \| (X, u) \|_{\mathcal{H}}^2$ , which concludes this proof of existence.

5) *Conclusion:* Finally, there exist  $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$  such that equation (11) holds for a functional  $V_0$ . Hence  $V_0(\cdot)$  is an equivalent norm of  $\| \cdot \|_H$  which is strictly decreasing. It means, according to Propositions 1 and 2 that there exists a unique solution to system (1) converging in  $H$  to the solution equilibrium  $(0, 0)$ . These conditions also bring :  $\forall t > 0, \dot{V}_0(t) + \frac{\varepsilon_3}{\varepsilon_2} V_0(t) \leq 0$  and

$$\begin{aligned}\| (X(t), u(t)) \|_{\mathcal{H}}^2 &\leq \frac{1}{\varepsilon_1} V_0(t) \leq \frac{1}{\varepsilon_1} V_0(0) e^{-\frac{\varepsilon_3}{\varepsilon_2} t} \\ &\leq \frac{\varepsilon_2}{\varepsilon_1} \| (X(0), u(0)) \|_{\mathcal{H}}^2 e^{-\frac{\varepsilon_3}{\varepsilon_2} t},\end{aligned}$$

which shows the exponential convergence of all trajectories of system (1). In other words, the norm  $\| \cdot \|_{\mathcal{H}}$  is exponentially decreasing along the trajectories of system (1).

*Remark 6:* There is no need to specify  $A+BK$  non singular in the previous theorem. Indeed, if the conditions of Theorem 1 are satisfied, then for  $e_1 = [I_n \ 0_{n,4}]^\top$ , the inequality  $\Psi_0(1, 1) = e_1^\top \Psi_0 e_1 < 0$  holds. After some simplifications, we get:

$$\Psi_0(1, 1) = \text{He} \left( (A+BK)^\top Q \right) < 0, \quad (20)$$

for some matrix  $Q$  depending on  $R, S$  and  $P_0$ . Inequality (20) implies that  $A+BK$  is not singular. As  $Q$  is apparently not symmetric, it is not possible to conclude directly on the stability of  $A+BK$  but an example will show that it does not need to be stable.

*Remark 7:* It is also worth noting that LMI (8) is affine with respect to matrices  $A, B$ , which allows us, in a straightforward manner, to extend this theorem to uncertain ODE systems subject, for instance, to polytopic-type uncertainties.

#### IV. EXTENDED STABILITY ANALYSIS

##### A. Main motivation

In the previous analysis, we have proposed an auxiliary system presented in (5) helping us to define a new Lyapunov functional for system (1). The notable aspect is that the term  $\mathfrak{X}_0 = \int_0^1 \chi(x) dx$  appears naturally in the dynamics of system (1). In light of the previous work on integral inequalities by [24], this term can also be interpreted as the projection of the modified state  $\chi$  over the set of constant functions in the sense of the canonical inner product in  $L_2$ . One may therefore enrich system (5) by additional projections of  $\chi$  over the next Legendre polynomials, as one can read in [24] in the context of time-delay systems.

The family of Legendre polynomials, denoted  $\{\mathcal{L}_k\}_{k \in \mathbb{N}}$ , is an orthogonal family respect to the  $L^2$  inner product. They are shifted from the traditional ones as described in [9], and

their definition is not required. Only some of their properties are considered.

##### B. Preliminaries

The previous discussion leads us to define additional vectors for any function  $\chi$  in  $L^2$ :

$$\forall k \in \mathbb{N}, \quad \mathfrak{X}_k = \int_0^1 \chi(x) \mathcal{L}_k(x) dx,$$

and the augmented vector  $X_N$ , at a given order  $N \in \mathbb{N}$ , as follow:

$$X_N = \left[ X^\top \quad \mathfrak{X}_0^\top \quad \dots \quad \mathfrak{X}_N^\top \right]^\top. \quad (21)$$

Following the same methodology as for Theorem 1, this specific structure leads us to introduce a new Lyapunov functional, inspired from (7), with  $P_N \in \mathbb{S}_+^{n+2N+2}$ :

$$V_N(X, u) = X_N^\top P_N X_N + \mathcal{V}(u). \quad (22)$$

In order to follow the same procedure, several technical extensions are required. Indeed, the stability conditions issued from the functional  $V_0$ , are coming from Jensen's inequality and an explicit expression of the time derivative of  $\mathfrak{X}_0$ . Therefore, it is necessary to provide an extended version of the Jensen inequality and of this differentiation rule. These technical steps are summarized in the two following lemmas.

*Lemma 2:* For any function  $\chi \in L^2$  and symmetric positive matrix  $R \in \mathbb{S}_+^2$ , the following Bessel-like integral inequality holds for all  $N \in \mathbb{N}$ :

$$\int_0^1 \chi^\top(x) R \chi(x) dx \geq \sum_{k=0}^N (2k+1) \mathfrak{X}_k^\top R \mathfrak{X}_k. \quad (23)$$

This inequality includes Jensen's inequality as the particular case  $N = 0$ , which was one of the key element in the proof of Theorem 1. This comment allows us to think that the previous lemma is the appropriate extension of the Jensen's inequality to address the stability analysis using the new Lyapunov functional (22) with the augmented state  $X_N$ .

Even if the proof can be found in [2], we would like to point out that it is based on the following equality, which results from the orthogonality of the Legendre polynomials:

$$\begin{aligned}\int_0^1 \chi^\top(x) R \chi(x) dx - \sum_{k=0}^N (2k+1) \mathfrak{X}_k^\top R \mathfrak{X}_k \\ = \int_0^1 \chi_N^\top(x) R \chi_N(x) dx \geq 0,\end{aligned}$$

where  $\chi_N(x) = \chi(x) - \sum_{k=0}^N (2k+1) \mathfrak{X}_k \mathcal{L}_k(x)$  can be interpreted as the error approximation between function  $\chi$  and its orthogonal projection over the family  $\{\mathcal{L}_k\}_{k \leq N}$ .

The next lemma is concerned by the differentiation of  $\mathfrak{X}_k$ .

*Lemma 3:* For any function  $\chi \in L^2$ , the following expression holds for any  $N$  in  $\mathbb{N}$ :

$$\begin{bmatrix} \dot{\mathfrak{X}}_0 \\ \vdots \\ \dot{\mathfrak{X}}_N \end{bmatrix} = c \mathbb{1}_N \chi(1) - c \bar{\mathbb{1}}_N \chi(0) - c L_N \begin{bmatrix} \mathfrak{X}_0 \\ \vdots \\ \mathfrak{X}_N \end{bmatrix},$$

where

$$L_N = \begin{bmatrix} \ell_{0,0} I_2 & \cdots & 0_2 \\ \vdots & \ddots & \vdots \\ \ell_{N,0} I_2 & \cdots & \ell_{N,N} I_2 \end{bmatrix}, \mathbb{1}_N = \begin{bmatrix} I_2 \\ \vdots \\ I_2 \end{bmatrix}, \bar{\mathbb{1}}_N = \begin{bmatrix} I_2 \\ \vdots \\ (-1)^N I_2 \end{bmatrix}, \quad (24)$$

and

$$\ell_{k,j} = \begin{cases} (2j+1)(1 - (-1)^{j+k}), & j \leq k-1, \\ 0, & \text{otherwise.} \end{cases} \quad (25)$$

*Proof:* The proof of this lemma is presented in the appendix of this paper because of its technical nature. It relies on the properties of the Legendre polynomials that are detailed there as well.  $\square$

### C. Main result

Taking advantage of the previous lemmas, the following extension to Theorem 1, using the extra-state  $X_N$ , is stated:

*Theorem 2:* Consider the system defined in (1) for a given speed  $c > 0$ , a viscous damping  $c_0 > 0$  with initial conditions  $(X^0, u_0, v_0) \in \mathcal{H}$  compatible with the boundary conditions. Assume that for a given integer  $N \in \mathbb{N}$ , there exist  $P_N \in \mathbb{S}_+^{n+2N+2}$  and  $S, R \in \mathbb{S}_+^2$  such that inequality

$$\Psi_N = \text{He} \left( Z_N^\top P_N F_N \right) - c \tilde{R}_N + c \left( H_N^\top (S + R) H_N - G_N^\top S G_N \right) \prec 0$$

holds, where

$$\begin{aligned} F_N &= \begin{bmatrix} I_{n+2N+2} & 0_{n+2N+2,2} \end{bmatrix}, Z_N = \begin{bmatrix} \mathcal{N}_N^\top & c \mathcal{Z}_N^\top \end{bmatrix}^\top, \\ \mathcal{N}_N &= \begin{bmatrix} A + BK & \tilde{B} & 0_{n,2(N+1)} \end{bmatrix}, \\ \mathcal{Z}_N &= \mathbb{1}_N H_N + \bar{\mathbb{1}}_N G_N - \begin{bmatrix} 0_{2N+2,n} & L_N & 0_{2N+2,2} \end{bmatrix}, \\ G_N &= \begin{bmatrix} 0_{2,n+2N+2} & g \end{bmatrix} + \begin{bmatrix} -K \\ 0_{1,n} \end{bmatrix} \mathcal{N}_N, \\ H_N &= \begin{bmatrix} 0_{2,n+2N+2} & h \end{bmatrix} + \begin{bmatrix} 0_{1,n} \\ K \end{bmatrix} \mathcal{N}_N, \\ \tilde{R} &= \text{diag} (0_n, R, 3R, \dots, (2N+1)R, 0_2). \end{aligned} \quad (26)$$

and where matrices  $L_N$ ,  $\mathbb{1}_N$  and  $\bar{\mathbb{1}}_N$  are given in (24). Then, the coupled infinite dimensional system (1) is exponentially stable in the sense of norm  $\|\cdot\|_{\mathcal{H}}^2$  and there exist  $\gamma > 1$  and  $\delta > 0$  such that the energy estimate (10) holds.

*Remark 8:* Remarks 5 and 6 also apply for this theorem. More particularly, the condition of the previous theorem requires, to be feasible, that matrix  $A + BK$  is non singular.

*Remark 9:* The article [24] shows that this methodology introduces a hierarchy. In other words, the set

$$\mathcal{C}_N = \left\{ c > 0 \text{ s.t. } \exists P_N \in \mathbb{S}_+^{n+2(N+1)}, S, R \in \mathbb{S}_+^2, \Psi_N \prec 0 \right\}$$

which represents the values of parameter  $c$  such that the LMIs of Theorem 2 are feasible for a given system (1) satisfies  $\mathcal{C}_N \subseteq \mathcal{C}_{N+1}$ . That means, if there exists a solution to Theorem 2 at an order  $N_0$ , then there also exists a solution for any order  $N \geq N_0$ . The proof is very similar to the one given in [24]. We can proceed by induction with  $P_{N+1} = \begin{bmatrix} P_N & 0 \\ 0 & \varepsilon I_2 \end{bmatrix}$  and a sufficiently small  $\varepsilon > 0$ . Then,  $\Psi_N \prec 0 \Rightarrow \Psi_{N+1} \prec 0$ . The calculus are tedious and technical and we do not intend to give them in this article. They relies in particular on the derivation of the Legendre polynomials which can be expressed in terms of strictly lower order Legendre polynomials (because  $L_N$  is strictly lower triangular).

### D. Proof of Theorem 2

The proof is following the same reasoning than for Theorem 1 and consists in proving the existence of positive scalars  $\varepsilon_1, \varepsilon_2$  and  $\varepsilon_3$  such that the functional  $V_N$  verifies inequalities (11).

1) *Existence of  $\varepsilon_1$ :* It strictly follows the same line as in Theorem 1 and is therefore omitted.

2) *Existence of  $\varepsilon_2$ :* For  $\varepsilon_2$ , as  $P_N, S, R \succ 0$ , there exists  $\varepsilon_2 > 0$  such that:

$$\begin{aligned} P_N &\preceq \text{diag} \left( \varepsilon_2 I_n, \frac{\varepsilon_2}{4} \text{diag} \{ (2k+1) I_n \}_{k \in (0, N)} \right), \\ (S + xR) &\preceq S + R \preceq \frac{\varepsilon_2}{4} I_2, \quad \forall x \in (0, 1). \end{aligned}$$

From equation (22), we get:

$$\begin{aligned} V_N(X, u) &\leq \varepsilon_2 |X|_n^2 + \frac{\varepsilon_2}{4} \left( \sum_{k=0}^N (2k+1) \mathfrak{X}_k^\top \mathfrak{X}_k + \|\chi\|^2 \right) \\ &\leq \varepsilon_2 \left( |X|_n^2 + \frac{1}{2} \|\chi\|^2 \right). \end{aligned}$$

While the first inequality is guaranteed by the constraint  $(S + xR) \preceq \frac{\varepsilon_2}{4} I_2$ , for all  $x \in (0, 1)$ , the last inequality results from the application of Bessel inequality (23). Therefore, following the same procedure as in the proof of Theorem 1 after equation (12), there indeed exists  $\varepsilon_2 > 0$  such that  $V_N(X, u) \leq \varepsilon_2 \|(X, u)\|_{\mathcal{H}}^2$ .

3) *Existence of  $\varepsilon_3$ :* Differentiating  $V_N$  defined in (22) along the trajectories of system (1) leads to:

$$\dot{V}_N(X, u) = \text{He} \left( \begin{bmatrix} \dot{X} \\ \dot{\mathfrak{X}}_0 \\ \vdots \\ \dot{\mathfrak{X}}_N \end{bmatrix}^\top P_N \begin{bmatrix} X \\ \mathfrak{X}_0 \\ \vdots \\ \mathfrak{X}_N \end{bmatrix} \right) + \dot{V}(u).$$

The aim here is to find an upper bound of  $\dot{V}_N$  using the following extended state:  $\xi_N = \begin{bmatrix} X_N^\top & u_t(1) & cu_x(0) \end{bmatrix}^\top$ . Using equation (15) and Lemma 3, we note that  $X_N = F_N \xi_N$ ,  $\dot{X}_N = Z_N \xi_N$ ,  $\chi(1) = H_N \xi_N$ ,  $\chi(0) = G_N \xi_N$  where the matrices  $F_N, Z_N, H_N, G_N$  are given in (26). Then we can write:

$$\begin{aligned} \dot{V}_N(X, u) &= \xi_N^\top \Psi_N \xi_N + c \sum_{k=0}^N \mathfrak{X}_k^\top (2k+1) R \mathfrak{X}_k \\ &\quad - c \int_0^1 \chi^\top(x) R \chi(x) dx. \end{aligned} \quad (27)$$

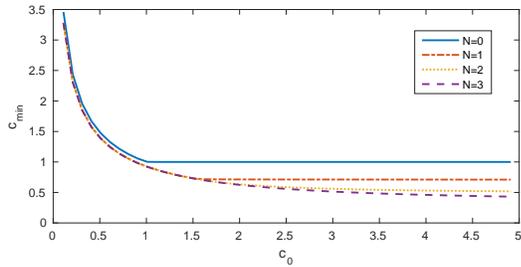
Since  $R \succ 0$  and  $\Psi_N \prec 0$ , there exists  $\varepsilon_3 > 0$  such that:

$$\begin{aligned} R &\succeq \frac{\varepsilon_3}{2c} \frac{2+c^2}{c^2} I_2, \\ \Psi_N &\preceq -\varepsilon_3 \text{diag} \left( I_n + K^\top K, \right. \\ &\quad \left. \frac{2+c^2}{2c^2} \text{diag} \{ I_2, 3I_2, \dots, (2N+1)I_2 \}, 0_2 \right). \end{aligned} \quad (28)$$

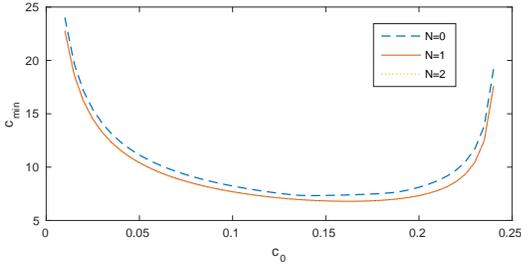
Using (28) in order to apply Bessel's inequality, equation (27) becomes:

$$\dot{V}_N(X, u) \leq -\varepsilon_3 \left( |X|_n^2 + 2|u(0)|^2 + \frac{2+c^2}{2c^2} \|\chi\|^2 \right),$$

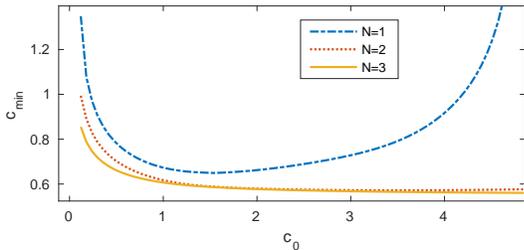
which is similar to equation (19) in the proof of Theorem 1. Therefore, following the same procedure, we obtain  $\dot{V}_N(X, u) \leq -\varepsilon_3 \|(X, u)\|_{\mathcal{H}}^2$ .



(a) System (29)



(b) System (30)



(c) System (31)

Fig. 1: Minimum wave speed  $c_{min}$  as a function of  $c_0$  for System (1) to be stable. The values for  $A$ ,  $B$  and  $K$  are given by equations (29), (30) and (31).

4) *Conclusion:* There exist  $\varepsilon_1, \varepsilon_2$  and  $\varepsilon_3$  positive reals such that inequalities (11) are satisfied and the exponential stability of system (1) is therefore guaranteed.

## V. EXAMPLES

Three examples of stability for problem (1) are provided here. The solver used for the LMIs is `sdp3` with the `YALMIP` toolbox (by [17]).

### A. Problem (1) with $A$ and $A + BK$ Hurwitz

In this first part, the considered system is defined as follow:

$$A = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad K = \begin{bmatrix} 0 & -2 \end{bmatrix}. \quad (29)$$

Matrices  $A$  and  $A + BK$  are stable. The ODE and the PDE are then stable if they are not coupled. As shown in Figure 1a, there exists a minimum wave speed called here  $c_{min}$  which is a function of the damping  $c_0$  for the system to be stable.

The phenomenon induced by the coupling can be understood as the robustness of the ODE to a disturbance generated by a wave equation. Intuitively, if the wave speed is large enough, the perturbation tends to 0 fast enough for the ODE to keep its stability behavior. Another important thing to notice is the hierarchy property i.e. the decrease of  $c_{min}$  as  $N$  increases.

### B. Problem (1) with $A + BK$ Hurwitz but an unstable $A$ .

This time, the system is described by the following matrices:

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad K = \begin{bmatrix} -10 & 2 \end{bmatrix}. \quad (30)$$

As  $A$  is not Hurwitz, we are studying the stabilization of the ODE through a communication modeled by the wave equation. For the same reason as before, the wave must be fast enough for the control not to be too much delayed but also with a moderated dumping to transfer the state variable  $X$  through the PDE equation. Intuitively, we are lead to introduce a trade-off between  $c_{min}$  and  $c_0$ , introducing then a  $c_{0max}$  as it is possible to see in Figure 1b.

Some numerical simulations have been done on this last example. Figure 1b shows that for system (30) with  $c_0 = 0.15$ , the minimum wave speed is  $c_{min} = 6.83$ . The numerical stability can also be seen in Figure 2 and indeed, the system is at the boundary of the stable area in Figure 2b and unstable for smaller values.

### C. Problem (1) with $A$ and $A + BK$ not Hurwitz.

Consider an open loop unstable system defined by:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad K = \begin{bmatrix} 1 & 0 \end{bmatrix}. \quad (31)$$

Gain  $K$  has been chosen such that the closed loop is also unstable. Surprisingly, the proposed algorithm give some couples  $(c, c_0)$  for which the whole system (1) is stable. The results are depicted in Figure 1c. Notice that for  $N = 0$ , the LMIs do not give any stability results. For  $N \geq 1$ , there is a stability area for which the slope of the right asymptotic branch is decreasing at each order. Hence, it appears that the introduction of the string equation in the feedback loop helps the stabilization of the closed loop.

## VI. CONCLUSION

A hierarchy of stability criteria has been provided for the stability of systems described by the interconnection between a finite dimensional linear system and an infinite dimensional system modeled by a string equation. The proposed methodology relies on an extensive use of Bessel's inequality which allows us to design a new and accurate Lyapunov functional. This new class encompasses the classical notion of energy proposed in that case. In particular, the stability of the closed-loop or open-loop system is not a requirement anymore. Future works include the study of robustness of this approach and the controller design.

## APPENDIX

### A. Proof of Lemma 3

For a given integer  $k$  in  $\mathbb{N}$ , the differentiating of  $\mathfrak{X}_k$  along the trajectories of (3) yields

$$\dot{\mathfrak{X}}_k = \int_0^1 \chi_t(x) \mathcal{L}_k(x) dx = c \int_0^1 \chi_x(x) \mathcal{L}_k(x) dx.$$

Then, integrating by parts, we get

$$\dot{\mathfrak{X}}_k = c \left( [\chi(x) \mathcal{L}_k(x)]_0^1 - \int_0^1 \chi(x) \mathcal{L}'_k(x) dx \right). \quad (32)$$

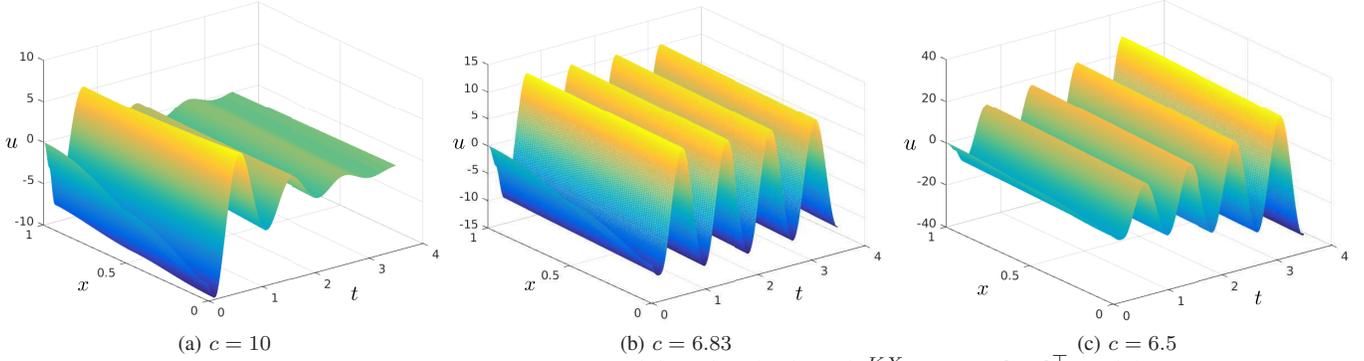


Fig. 2: Chart of  $u$  for system (30) with the parameters:  $u_0(x) = (\cos(\pi x) + 1) \frac{KX_0}{2}$ ,  $X_0 = [1 \ 1]^\top$ ,  $v_0(x) = 0$  and  $c_0 = 0.15$  for 3 values of  $c$ . These results are obtained using Euler forward as a numerical scheme.

In order to derive the expression of  $\dot{\mathfrak{X}}_k$ , we will use the following properties of the Legendre polynomials. On the one hand, the values of Legendre polynomials at the boundaries of  $[0 \ 1]$  are given by  $\mathcal{L}_k(0) = (-1)^k$  and  $\mathcal{L}_k(1) = 1$ . On the other hand, the Legendre polynomials verifies the following differentiation rule:

$$\frac{d}{dx} \mathcal{L}_k(x) = \begin{cases} \sum_{j=0}^{k-1} (2j+1)(-1)^{j+k} \mathcal{L}_j(x), & \text{if } k \geq 1, \\ 0, & \text{if } k = 0. \end{cases}$$

Hence, injecting these expressions into (32) leads to:

$$\dot{\mathfrak{X}}_k = c(\chi(1, t) - (-1)^k \chi(0)) - c \sum_{j=0}^N \ell_N^{kj} \mathfrak{X}_j$$

where the coefficient  $\ell_N^{kj}$  are defined in equation (25). The end of the proof consists in gathering the previous expression from  $k = 1$  to  $k = N$ , leading to the definition of matrices  $L_N$ ,  $\mathbb{1}_N$  and  $\bar{\mathbb{1}}_N$  given in (24).

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