Interval Kalman filter enhanced by positive definite upper bounds

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Abstract: A method based on the interval Kalman filter for discrete uncertain linear systems is presented. The system under consideration is subject to bounded parameter uncertainties not only in the state and observation matrices, but also in the covariance matrices of Gaussian noises. The gain matrix provided by the filter is optimized to give a minimal upper bound on the state estimation error covariance for all admissible uncertainties. The state estimation is then determined by using interval analysis in order to enclose the set of all possible solutions with respect to the classical Kalman filtering structure.

Keywords: Uncertain linear systems, Kalman filter, Interval analysis, Gaussian noise, Covariance matrices.

1. INTRODUCTION

Extending the Kalman filter technique for uncertain linear systems is an active research topic that attracts increasing interest. For instance, a joint Zonotopic and Gaussian Kalman filter has been proposed in Combastel [2015] for discrete-time linear time-varying systems excited by Gaussian noises and bounded disturbances represented by zonotopes. The method minimizes a multi-objective optimization criterion which represents the compromise between the size of the zonotopic part and the covariance of the Gaussian distribution. Regarding to parameter uncertainties, several results have been derived on the design of optimal robust Kalman filters for discrete time-varying systems subject to norm-bounded uncertainties, for instance Zhe and Zheng [2006], Mohamed and Nahavandi [2012]. The optimality of these methods is based on finding an upper bound on the estimation error covariance for any acceptable modeling uncertainties and then to minimize the proposed upper bound. However, this approach may be conservative when there exist too many uncertainties in the system. Another possibility is to consider the model matrices (including the covariance matrices of Gaussian noises) as intervals containing all admissible values of parameters. A drawback of this approach is the singularity problem which may appear in interval matrix inversion. In Chen et al. [1997], the classical Kalman filter (Kalman [1960]) has been extended to this type of uncertain systems. The authors propose to bypass the singularity problem by using the upper bound of the interval matrix to be inverted. This method leads to a solution that is not guaranteed, i.e. the solution set may not include all the classical Kalman filter solutions consistent with the bounded uncertainties represented in the system. In Xiong et al. [2013], an improved interval Kalman filter (iIKF) has been proposed that solves the interval matrix inversion problem with the set inversion algorithm SIVIA (Set Inversion Via Interval Analysis) and constraint satisfaction problems (CSP) (see Jaulin et al. [2001]). Nevertheless, this algorithm demands high computational time if there exist large uncertainties affecting the considered system (see Tran et al. [2016]).

Motivated by the above observations, this paper proposes a new interval Kalman filtering algorithm with two main goals: minimizing an upper bound for the estimation error covariance and enclosing the set of possible solutions of the filtering problem for interval linear systems. This algorithm, called *Minimum Upper Bound of Variance Interval Kalman Filter* (UBIKF), achieves a reasonable computational time. It also achieves the goal to enclose all the estimates consistent with the parameter uncertainties with a much less conservative manner than Xiong et al. [2013].

This paper is organized as follows. The problem formulation is described in Section 2. Section 3 reviews the main notions of positive semidefinite matrix and interval analysis which are necessary for the development of the new algorithm. Then the new interval Kalman filter is derived in Section 4, followed by a numerical example in Section 5. Conclusions and future works are presented in Section 6.

2. PROBLEM FORMULATION

Considering the following class of uncertain linear discrete-time stochastic systems:

$$\begin{cases} \mathbf{x}_{k+1} = A_k \mathbf{x}_k + \mathbf{w}_k, \\ \mathbf{y}_k = C_k \mathbf{x}_k + \mathbf{v}_k, \end{cases} \tag{1}$$

where $\mathbf{x}_k \in \mathbb{R}^{n_x}$ is the state vector, $\mathbf{y}_k \in \mathbb{R}^{n_y}$ is the measurement vector and $\mathbf{w}_k$, $\mathbf{v}_k$ are white Gaussian noise sequences with zero mean and covariance matrices Q and R. The structure of the classical Kalman filter for system (1) is represented as follows (Kalman [1960]):

$$\hat{\mathbf{x}}_k^+ = \hat{\mathbf{x}}_k^- + K_k(\mathbf{y}_k - C_k \hat{\mathbf{x}}_k^-), \tag{2}$$

where $\hat{\mathbf{x}}_k^+$ is the posterior state estimate, $\hat{\mathbf{x}}_k^- = A_k \hat{\mathbf{x}}_{k-1}^+$ is the prediction of state a priori and K_k is the gain matrix

that minimizes the mean square error $E[(\hat{\mathbf{x}}_k^+ - \mathbf{x}_k)(\hat{\mathbf{x}}_k^+ - \mathbf{x}_k)^T]$. We consider the case where the matrices A_k, C_k and the covariance matrices Q, R are assumed bounded. Interval analysis is one of the tools to represent bounded uncertainties.

Interval analysis was developed by Moore [1966] and is useful to deal with bounded uncertainties. Most of the notions of interval analysis can be found in Jaulin et al. [2001] in which an interval $[x]$ is defined as a closed and connected subset of $\mathbb{R}$:

$$[x] = \{x \in \mathbb{R} \mid \underline{x} \leq x \leq \overline{x}\}, \tag{3}$$

where $\underline{x}$ and $\overline{x}$ are respectively the lower and upper bound. The center of $[x]$ is defined by $mid([x]) = (\overline{x} + \underline{x})/2$ and its radius is $rad([x]) = (\overline{x} - \underline{x})/2$. The set of all intervals in $\mathbb{R}$ is noted as $\mathbb{IR}$. An interval vector (or matrix) is a vector (or matrix) whose elements are considered as intervals. In this paper, an interval vector and an interval matrix are denoted as $[\mathbf{x}]$ and $[M]$. The set of $n-$dimensional interval vectors (or $m \times n$ interval matrices) is denoted as $\mathbb{IR}^n$ (or $\mathbb{IR}^{m \times n}$). Given $[M] \in \mathbb{IR}^{m \times n}$, the two functions $mid([M])$ and $rad([M])$ provide two $m \times n$ real matrices containing the centers and radius of elements of $[M]$.

The matrices A_k, C_k, Q and R of the system (1) are represented by interval matrices, denoted $[A_k]$, $[C_k]$, $[Q]$ and $[R]$, containing all possible values of each parameter. Since it is impossible to solve directly the Kalman filtering problem due to parameter uncertainties, our goal is to obtain an upper bound $\mathcal{P}_k^+$ such that:

$$E\left[(\hat{\mathbf{x}}_k^+ - \mathbf{x}_k)(\hat{\mathbf{x}}_k^+ - \mathbf{x}_k)^T\right] \preceq \mathcal{P}_k^+, \tag{4}$$

for the set of all models with parameters bounded by the above interval matrices. The envelope enclosing the set of state estimates corresponding to the gain K is then computed. This idea is similar to Xiong et al. [2013] in which the envelopes of the optimal gains and the state estimates given by the Kalman filtering procedure are determined by using interval analysis. In contrast, the proposed algorithm determines a gain matrix K which minimizes the trace of the upper bound on the error covariance instead of finding a set of gain in Xiong et al. [2013]. Since the gain matrix in our filter is punctual, it allows to reduce the conservatism and the computational time of interval operations.

3. PRELIMINARIES

This section introduces the notations used throughout the paper, including some definitions and properties of positive semidefinite matrices and some basic concepts of interval analysis.

3.1 Symmetric positive semidefinite matrices

Definition 1. A symmetric $n \times n$ real matrix M is positive semidefinite, denoted $M \succeq 0$, if for all non-zero n-dimensional column vector $\mathbf{z}$, the scalar $\mathbf{z}^T M \mathbf{z}$ is non-negative.

All the diagonal elements of a positive semi-definite matrix are real and non-negative. A special case of positive semidefinite matrices is given in the following remark.

Remark 1. For any real matrix M, the product MM^T is a positive semidefinite matrix.

Remark 1 comes from the fact that for any non-zero vector $\mathbf{z}$ of appropriate size, the expression $\mathbf{z}^T MM^T \mathbf{z} = \left\|M^T \mathbf{z}\right\|_2^2$ is non-negative, where $\|\mathcal{V}\|_2$ is the Euclidean norm of the real vector $\mathcal{V}$ defined as the square root of the sum of squares of its elements.

Definition 2. Given two symmetric real matrices M, N with the same size, M is said greater or equal to N, written as $M \succeq N$, if $M - N$ is positive semidefinite.

Proposition 1. Given two non null matrices M, N with the same size and an arbitrary real number $\beta > 0$, the following inequality holds:

$$\beta^{-1} MM^T + \beta NN^T \succeq MN^T + NM^T. \tag{5}$$

Proof. Developing $(M - \beta N)(M - \beta N)^T$ and using Remark 1, we obtain inequality (5).

Remark 2. A positive value of β can be computed to minimize the trace of the left-hand side of inequality (5). Differentiating the trace of the left-hand side of inequality (5) gives:

$$\frac{\partial tr\left(\beta^{-1} MM^T + \beta NN^T\right)}{\partial \beta} = -\frac{1}{\beta^2} tr\left(MM^T\right) + tr\left(NN^T\right), \tag{6}$$

$$\frac{\partial^2 tr\left(\beta^{-1} MM^T + \beta NN^T\right)}{\partial \beta^2} = \frac{2}{\beta^3} tr\left(MM^T\right) \succ 0, \tag{7}$$

where $tr(\mathcal{M})$ is the trace of the square matrix $\mathcal{M}$. The positiveness of the second derivative guarantees the existence of the global minimum. Setting the first derivative to zero, we obtain:

$$\beta = \sqrt{tr(MM^T)/tr(NN^T)}. \tag{8}$$

Proposition 2. Consider two $n \times n$ matrices M_1, M_2 and $M_1 \succeq M_2$, the following inequality holds for any $m \times n$ matrix N:

$$NM_1 N^T \succeq NM_2 N^T. \tag{9}$$

Proof. The definition of a positive semidefinite matrix is used to show that $N(M_1 - M_2)N^T \succeq 0$. For any non-zero m-dimensional vector $\mathbf{z}$, we have:

$$\mathbf{z}^T N(M_1 - M_2)N^T \mathbf{z} = \mathbf{x}^T (M_1 - M_2)\mathbf{x} \geq 0,$$

since $M_1 - M_2 \succeq 0$, where $\mathbf{x} = N^T \mathbf{z} \in \mathbb{R}^n$. Therefore, $N(M_1 - M_2)N^T \succeq 0$.

An operation used in this paper is the square root of a positive semidefinite matrix M.

Property 1. A symmetric positive semidefinite matrix M is orthogonally diagonalizable, i.e. there exist an orthogonal matrix V and a diagonal matrix D such that $M = VDV^T$. Thus, the square root of M is $M^{1/2} = VD^{1/2}V^T$.

Property 2. If $M \succeq N \succeq 0$ then $M^{1/2} \succeq N^{1/2} \succeq 0$.

3.2 Interval analysis

Definition 3. An interval matrix $[M] \in \mathbb{IR}^{n\times n}$ is said to be symmetric if the real matrices $mid([M])$ and $rad([M])$ are symmetric.

The following property is useful to describe a quantity in terms of its nominal value and a bounded uncertainty:

Property 3. (see Moore et al. [2009]). Given a real value x belonging to an interval $[x]$, there exists a real value $\alpha \in [-1,1]$ such that $x = mid([x]) + \alpha.rad([x])$.

Using Property 3 for matrices, the following proposition is obtained:

Proposition 3. Given an $m \times n$ real matrix M belonging to an interval matrix $[M]$, there exist mn real values $\alpha^{ij} \in [-1,1]$ with $i \in \{1,\ldots,m\}$, $j \in \{1,\ldots,n\}$ such that:

$$M = mid([M]) + \sum_{i=1}^{m}\sum_{j=1}^{n} \alpha^{ij} \left[rad([M^{ij}])\right], \text{ [1]} \qquad (10)$$

where $\left[rad([M^{ij}])\right]$ is an $m \times n$ matrix whose $ij-$th element contains the radius of the $ij-$th element interval value of $[M]$ and the other elements are zero.

In the case of symmetric matrices, the following representation should be considered:

Proposition 4. Given an $n \times n$ real symmetric matrix M belonging to a symmetric interval matrix $[M]$, there exist $n(n+1)/2$ real values $\alpha^{ij} \in [-1,1]$ such that:

$$\begin{aligned} M =& mid([M]) + diag\left(rad\left([M]\right)\right) diag\left(\alpha^{ii}\right) \\ &+ \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \alpha^{ij}\left[rad\left([M^{i,j}]\right)\right], \end{aligned} \qquad (11)$$

where $diag\left(rad\left([M]\right)\right)$ is a diagonal matrix containing the radius of diagonal elements of $[M]$, $\left[rad\left([M^{i,j}]\right)\right]$ is a symmetric matrix whose $ij-$ and $ji-$th elements are the radius of $[M^{ij}]$ and $[M^{ji}]$ and the other elements are zero. The matrix $diag\left(\alpha^{ii}\right)$ is diagonal and $\alpha^{ij} \in [-1,1]$, for all $1 \le i \le j \le n$.

Classical operations for intervals, interval vectors, interval matrices (e.g. $+,-,\times,\div,\ldots$) are extensions of the same operations for reals, real vectors, real matrices. For instance, the following results will be applied in the next sections of this paper:

$$\begin{aligned} [x] + [y] &= \left[\underline{x} + \underline{y}, \overline{x} + \overline{y}\right], \\ [x] - [y] &= \left[\underline{x} - \overline{y}, \overline{x} - \underline{y}\right], \\ [x] \times [y] &= \left[\min(\underline{xy}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{xy}), \max(\underline{xy}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{xy})\right]. \end{aligned}$$

For the interval matrix multiplication, the ij-th interval $[C^{ij}]$ of the product $[C] = [A][B]$ encloses all possible values of the ij-th element of real matrices $C = AB$, for any $A \in [A]$, $B \in [B]$. The interval matrix obtained $[C]$ may contain real matrices D that are not the result of the multiplication of any real matrices $A \in [A]$ and $B \in [B]$. This conservatism is generated by the multi-occurrence problem detailed in Moore et al. [2009].

Proposition 5. Given a symmetric interval matrix $[M] \in \mathbb{IR}^{n\times n}$, there exists a symmetric positive semidefinite matrix $\mathcal{M}$ that bounds all symmetric positive semidefinite matrices $M \in [M]$, i.e. $\mathcal{M} - M \succeq 0$. We denote $\mathcal{M} \succeq [M]$. The expression of $\mathcal{M}^2$ is:

$$\begin{aligned} \mathcal{M}^2 =& \left(1 + \beta_{mid,ii}^{-1} + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \beta_{mid,ij}^{-1}\right) \left(mid\left([M]\right)\right)^2 \\ &+ \left(1 + \beta_{mid,ii} + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \beta_{ii,ij}^{-1}\right) \left(diag(rad([M]))\right)^2 \\ &+ \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \left(1 + \beta_{mid,ij} + \beta_{ii,ij} + \sum_{k=i}^{n-1}\sum_{l=j+1}^{n} \beta_{kl,ij}\right) \left[rad([M^{i,j}])\right]^2, \end{aligned}$$

where :

$$\beta_{mid,ii} = \sqrt{tr\left\{(mid([M]))^2\right\} / tr\{(diag(rad([M])))^2\}},$$

$$\beta_{mid,ij} = \sqrt{tr\{(mid([M]))^2\} / tr\{\left[rad([M^{i,j}])\right]^2\}},$$

$$\beta_{ii,ij} = \sqrt{tr\{(diag(rad([M])))^2\} / tr\{\left[rad([M^{i,j}])\right]^2\}},$$

$$\beta_{kl,ij} = \sqrt{tr\{\left[rad([M^{k,l}])\right]^2\} / tr\{\left[rad([M^{i,j}])\right]^2\}}.$$

The matrix $\mathcal{M}$ is computed using Property 1 for $\mathcal{M}^2$.

Proof. The representation of a symmetric matrix $M \in [M]$ in Proposition 4 is used to compute $M^2 = MM^T$:

$$\begin{aligned} M^2 =& \left(mid\left([M]\right)\right)^2 \\ &+ diag(rad([M])) \left(diag\left(\alpha^{ii}\right)\right)^2 diag\left(rad([M])\right) \\ &+ \left\{mid\left([M]\right)\right\} \left\{diag\left(\alpha^{ii}\right) diag(rad([M]))\right\} \\ &+ \left\{diag(rad([M]))diag\left(\alpha^{ii}\right)\right\} \left\{mid\left([M]\right)\right\} \\ &+ \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \left\{mid\left([M]\right)\right\} \left\{\alpha^{ij}\left[rad([M^{i,j}])\right]\right\} \\ &+ \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \left\{\alpha^{ij}\left[rad([M^{i,j}])\right]\right\} \left\{mid\left([M]\right)\right\} \\ &+ \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \left\{\alpha^{ij}\left[rad([M^{i,j}])\right]\right\} \left\{diag\left(\alpha^{ii}\right) diag(rad([M]))\right\} \\ &+ \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \left\{diag(rad([M]))diag\left(\alpha^{ii}\right)\right\} \left\{\alpha^{ij}\left[rad([M^{i,j}])\right]\right\} \\ &+ \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{k=1}^{n-1}\sum_{l=k+1}^{n} \left\{\alpha^{ij}\left[rad([M^{i,j}])\right]\right\} \left\{\alpha^{kl}\left[rad([M^{k,l}])\right]\right\}. \end{aligned}$$

The upper bound $\mathcal{M}^2$ of the matrix M^2 is obtained by using Proposition 1, Remark 2, Proposition 2 and noting that $0 \preceq \left(diag\left(\alpha^{ii}\right)\right)^2 \preceq I$ and $0 \le (\alpha^{ij})^2 \le 1$ for all $-1 \le \alpha^{ij} \le 1$. Property 2 gives $\mathcal{M} \succeq M$, for all symmetric positive semidefinite matrix M belonging to $[M]$.

[1] The expression is a developed form of the Hadamard product.

4. MAIN RESULTS

Like the classical Kalman filter, the filter proposed in this paper can be designed in two steps: prediction and correction.

4.1 Prediction step

In the prediction step, the state estimate from the previous time step and the transition model are used to predict the state at the current time step. Thanks to natural interval extension, this step is performed as follows:

$$[\hat{\mathbf{x}}_k]^- = [A_k]\,[\hat{\mathbf{x}}_{k-1}]^+, \tag{12}$$

where the sign $(-)$ means the result of the prediction step and $(+)$ represents the estimations. The error covariance matrix can also be predicted using interval analysis:

$$[P_k]^- = [A_k]\,\mathcal{P}^+_{k-1}\,[A_k]^T + [Q]. \tag{13}$$

Equation (13) provides an interval matrix enclosing all possible covariance matrices given the interval matrices $[A_k]$, $[Q]$ and the estimated covariance matrix $\mathcal{P}^+_{k-1}$ of the previous step. An upper bound $\mathcal{P}^-_k$ of $[P_k]^-$ can be determined by applying Proposition 5, i.e. $\mathcal{P}^-_k \succeq [P_k]^-$.

4.2 Correction step

The state estimate at time step k is computed by the Equation (2), given $\hat{\mathbf{x}}^-_k \in [\hat{\mathbf{x}}_k]^-$ and $C_k \in [C_k]$. In order to reduce the effect of the dependency problem (Jaulin et al. [2001]), the Equation (2) is rearranged as follows:

$$[\hat{\mathbf{x}}_k]^+ = (I - K_k\,[C_k])\,[\hat{\mathbf{x}}_k]^- + K_k y_k. \tag{14}$$

The box $[\hat{\mathbf{x}}_k]^+$ encloses all possible values of $\hat{\mathbf{x}}^+_k$.

The gain matrix K_k is determined as follows. The expression of the error covariance matrix after the correction step, for any $C_k \in [C_k]$, $R \in [R]$, is:

$$P^+_k = (I - K_k C_k)\,P^-_k\,(I - K_k C_k)^T + K_k R K_k^T. \tag{15}$$

An upper bound of P^+_k can be obtained by using Proposition 3 for the matrix C_k, then developing Equation (15). The terms $(I - K_k mid([C_k]))\mathcal{P}^-_k(\alpha^{ij}\left[rad([M^{ij}])\right])^T$ can be rewritten as follows:

$$\left((I - K_k mid\,([C_k]))\,\left(\mathcal{P}^-_k\right)^{1/2}\right)\left(\alpha^{ij}\left[rad\left([M^{ij}]\right)\right]\left(\mathcal{P}^-_k\right)^{1/2}\right)^T.$$

Similarly, $(\alpha^{ij}\left[rad([M^{ij}])\right])\mathcal{P}^-_k(\alpha^{mn}\left[rad([M^{mn}])\right])^T$ are also divided into two parts:

$$\left(\left(\alpha^{ij}\left[rad([M^{ij}])\right]\right)\left(\mathcal{P}^-_k\right)^{1/2}\right)\left(\alpha^{mn}\left[rad([M^{mn}])\right]\left(\mathcal{P}^-_k\right)^{1/2}\right)^T,$$

for all $i, m \in \{1, \ldots, n_y\}$ and $j, n \in \{1, \ldots, n_x\}$. Proposition 1 is then applied to these pairs of matrices with $\beta = 1$ to obtain the following inequality:

$$\begin{aligned}
P^+_k \preceq\ & (n_x n_y + 1)\,(I - K_k mid([C_k]))\,\mathcal{P}^-_k\,(I - K_k mid([C_k]))^T \\
& + (n_x n_y + 1)\,K_k\left(\sum_{i=1}^{n_y}\sum_{j=1}^{n_x}\left[rad([C^{ij}_k])\right]\mathcal{P}^-_k\left[rad([C^{ij}_k])\right]^T\right)K_k^T \\
& + K_k \mathcal{R} K_k^T \\
=\ & \mathcal{P}^+_k.
\end{aligned} \tag{16}$$

where $\mathcal{R} \succeq [R]$ is determined with Proposition 5.

Having the expression of $\mathcal{P}^+_k$ as a function of K_k, we look for K_k, the value that minimizes the trace of $\mathcal{P}^+_k$. The first and second derivatives of $tr\left(\mathcal{P}^+_k\right)$ with respect to K_k are:

$$\begin{aligned}
\frac{dtr\left(\mathcal{P}^+_k\right)}{dK_k} =\ & -2\mathcal{P}^-_{k,ny} mid([C_k])^T \\
& + 2K_k mid([C_k])\mathcal{P}^-_{k,ny} mid([C_k])^T \\
& + 2K_k\left(\sum_{i=1}^{n_y}\sum_{j=1}^{n_x} rad([C^{ij}_k])\mathcal{P}^-_k rad([C^{ij}_k])^T\right) \\
& + 2K_k\mathcal{R},
\end{aligned}$$

$$\begin{aligned}
\frac{d^2 tr\left(\mathcal{P}^+_k\right)}{dK_k^2} =\ & 2mid([C_k])\mathcal{P}^-_{k,ny} mid([C_k])^T \\
& + 2\sum_{i=1}^{n_y}\sum_{j=1}^{n_x}\left[rad([C^{ij}_k])\right]\mathcal{P}^-_k\left[rad([C^{ij}_k])\right]^T \\
& + 2\mathcal{R},
\end{aligned}$$

where $\mathcal{P}^-_{k,n_y} = (n_x n_y + 1)\mathcal{P}^-_k$. The second derivative is always definite positive that guarantees the existence of a minimum value for $tr\left(\mathcal{P}^+_k\right)$ and K_k is obtained from the first derivative:

$$K_k = \mathcal{P}^-_{k,n_y} mid([C_k])^T S_k^{-1}, \tag{17}$$

where:

$$\begin{aligned}
S_k =\ & mid([C_k])\mathcal{P}^-_{k,n_y} mid([C_k])^T + \\
& + \sum_{i=1}^{n_y}\sum_{j=1}^{n_x}\left[rad([C^{ij}_k])\right]\mathcal{P}^-_{k,n_y}\left[rad([C^{ij}_k])\right]^T + \mathcal{R}.
\end{aligned} \tag{18}$$

The expression of the covariance matrix bound $\mathcal{P}^+_k$ is obtained by Equation (16) using K_k given in Equation (17):

$$\mathcal{P}^+_k = (I - K_k mid([C_k]))\,\mathcal{P}^-_{k,n_y}. \tag{19}$$

In summary, the UBIKF allows to find a gain matrix K_k, a covariance upper bound $\mathcal{P}^+_k$ and an interval state estimate $[\hat{\mathbf{x}}_k]^+$ at each iteration such that for any real matrices A_k, C_k, Q, R belonging to the interval matrices $[A_k]$, $[C_k]$, $[Q]$, $[R]$, we have:

$$\begin{aligned}
\hat{\mathbf{x}}^+_k &\in [\hat{\mathbf{x}}^+_k] \\
P^+_k &\preceq \mathcal{P}^+_k,
\end{aligned} \tag{20}$$

where $\hat{\mathbf{x}}^+_k$ and P^+_k are the state estimate and the covariance matrix given by the Kalman filtering procedure for the linear system (A_k, C_k, Q, R) and the gain matrix K_k. The proposed filter has the similar objective as Xiong et al. [2013], i.e. to bound the set of all state estimates. Nevertheless, it should provide less conservative results since the gain matrix is punctual and so the conservatism of interval analysis is reduced. This is illustrated in the case study of Section 5.

5. NUMERICAL EXAMPLE

The filter proposed in this paper is applied to a state estimation problem of a typical proton exchange membrane fuel cell system (PEMFC). The main components of the

system are shown in Figure 1. The fuel cell converts the chemical energy into electricity by consuming oxygen and hydrogen provided by air and hydrogen supply system. The linear model used in this paper can be found in Rotondo et al. [2016]. It is derived from the nonlinear model presented in Pukrushpan et al. [2004] under the following assumptions: the stack temperature T_{st} is constant; the temperature and humidity of the inlet reactant flows are perfectly controlled; the anode and cathode volumes of multiple fuel cells are lumped as a single stack anode and cathode volumes; all the reactant behave as ideal gases.

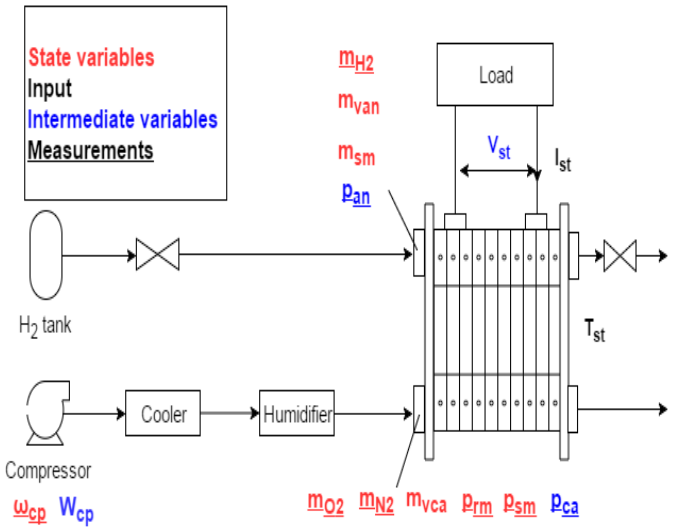

Fig. 1. Fuel cell system

The stack current I_{st} is considered as the system input. A state space representation of order 9 is presented in Rotondo et al. [2016] with the following state variables: mass of oxygen (m_{O_2}), mass of hydrogen (m_{H_2}), mass of nitrogen (m_{N_2}), air mass in the supply manifold (m_{sm}), air pressure in the supply manifold (p_{sm}), air pressure in the return manifold (p_{rm}), compressor speed (ω_{cp}), mass of water in the anode ($m_{w,an}$) and in the cathode ($m_{w,ca}$). An Euler discretization has been performed with the sample time $T_s = 0.04s$ to obtain the following discrete model:

$$\begin{cases} \mathbf{x}_{k+1} = A_k \mathbf{x}_k + B_k I_{st,k} + D_k + \mathbf{w}_k, \\ \mathbf{y}_k = C_k \mathbf{x}_k + \mathbf{v}_k, \end{cases} \tag{21}$$

where $\mathbf{w}_k$ and $\mathbf{v}_k$ are zero-mean Gaussian process noise and measurement error vectors. The measured outputs of the system are compressor speed (ω_{cp}), mass of oxygen (m_{O_2}), mass of hydrogen (m_{H_2}), mass of nitrogen (m_{N_2}), air pressure in the supply manifold (p_{sm}), air pressure in the return manifold (p_{rm}), pressure in the anode (p_{an}), pressure in the cathode (p_{ca}). The covariance matrices Q and R of process noises and measurement errors are given by:

$$Q = diag(10^{-12}; 10^{-12}; 10^{-12}; 10^{-12}; 10^{2}; 10^{2}; 10; 10^{-12}; 10^{-12}),$$
$$R = diag(10; 10^{-10}; 10^{-10}; 10^{-10}; 10^{2}; 10^{2}; 10^{2}).$$

A test with time-varying parameters has been designed. The elements a_{1i} (first line), a_{2i} (second line), and a_{7i} (seventh line) of the matrix A_k oscillate with sinusoidal law having the magnitude of 5% of their nominal values. These variation laws are assumed unknown and must be considered as uncertainties by the filters. The interval matrix $[A_k]$ containing all admissible parameters is chosen as follows:

$$mid([A_k]) = A_k^0 \quad , \quad rad([A_k]) = 0.05\left|A_k^0\right|, \tag{22}$$

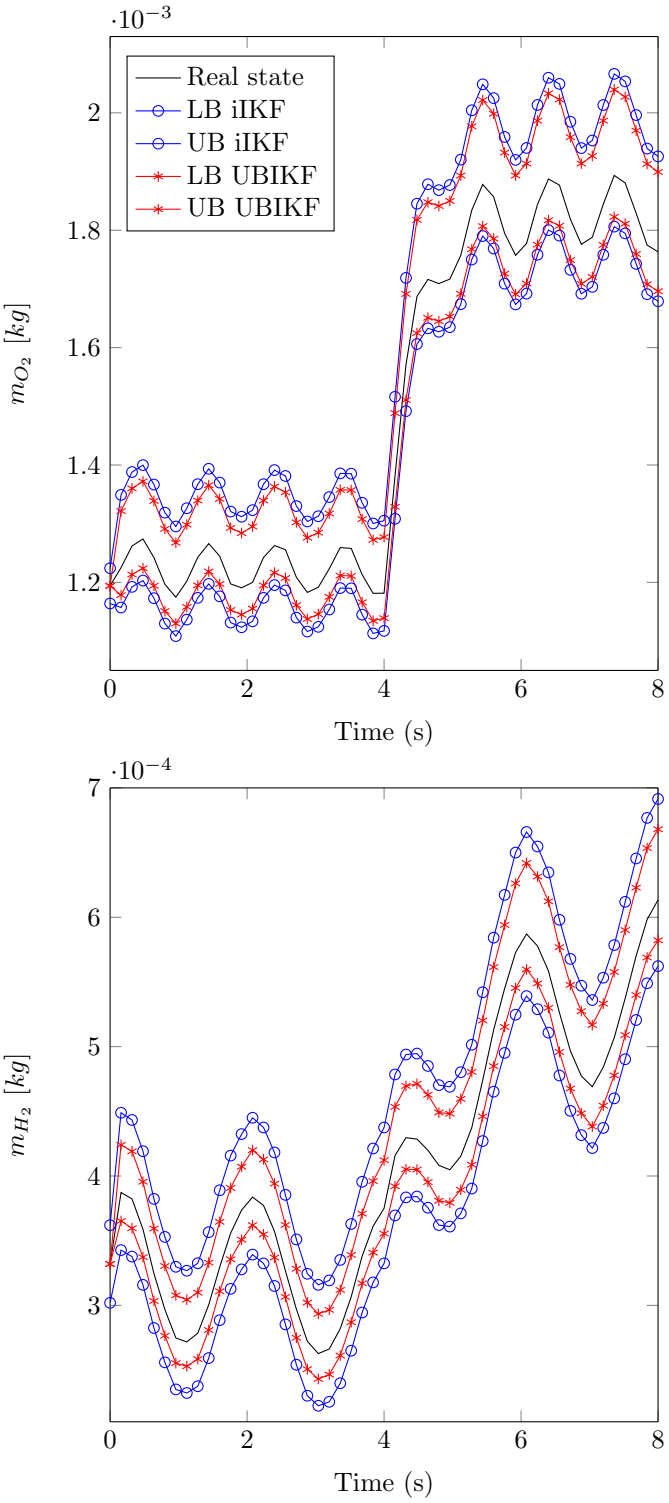

Fig. 2. Real state and 99.7% confidence intervals (upper bound (UB) and lower bound (LB)) given by iIKF and the UBIKF for mass of oxygen m_{O_2} and mass of hydrogen m_{H_2}.

where A_k^0 is the nominal value of A_k given in Rotondo et al. [2016].

Moreover, the compressor speed sensor, mass of hydrogen sensor, mass of oxygen sensor are assumed to provide 2% of uncertainty. Therefore, the interval matrix $[C_k]$ is defined as:

$$mid([C_k]) = C_k^0 \quad , \quad rad([C_k]) = 0.02\left|C_k^0\right|, \tag{23}$$

where C_k^0 is the nominal value of C_k.

The covariance matrix of process noise is considered to be certain, i.e. $[Q] = Q$, while the measurement error covariance matrix is assumed to be uncertain with 50% uncertainty.

$$mid([R]) = R \quad , \quad rad([R]) = 0.5|R|. \tag{24}$$

Besides, this numerical example is also tested with the improved interval Kalman filter (iIKF) presented in Xiong et al. [2013]. The results of this algorithm are compared with those of the UBIKF in Figure 2 and in Table 1, including the root mean square error (RMSE) for each state, the percentage of time steps when the confidence interval contains the real state O, and the computational time (for 7500 time steps). Both of these two filtering algorithms have the same goal to provide the envelopes of the set of all possible state estimates of the filtering problem in a linear system with bounded uncertainties. The 99.7% confidence intervals [2] given by these two filters (Figure 2) enclose the real states at every time step ($O = 100\%$). The confidence intervals of the proposed filter are tighter than those given by iIKF. Moreover, the gain in computational time of the new filter compared to iIKF is about 3.8 (Table 1). These advantages come from the fact that the UBIKF deals with the real matrices while the iIKF applies the constraint satisfaction problem and the set inversion techniques for interval matrices. Therefore, the UBIKF allows to obtain less conservative result with reasonable computational time.

Table 1: Results of the iIKF and the proposed UBIKF

	m_{O_2}		m_{H_2}		ω_{cp}		Time
	RMSE	O(%)	RMSE	O(%)	RMSE	O(%)	
iIKF	$5.4.10^{-5}$	100	$1.5.10^{-5}$	100	156	100	511 s
UBIKF	$4.8.10^{-5}$	100	$1.4.10^{-5}$	100	177	100	133 s

The initial interval Kalman filter in Chen et al. [1997] has also been applied to this example. The results given by this algorithm diverge due to the large uncertainties in the model.

6. CONCLUSION

This paper presents an efficient method to find a positive definite upper bound of an interval matrix, that is applied to propose an interval filter for uncertain linear systems. This approach allows to bound the set of all possible state estimations given by the Kalman filtering structure for any admissible parameter uncertainties. It has been applied to a model-based state estimation of a PEMFC and has been compared with the improved interval Kalman filter Xiong et al. [2013]. The new filter provides convergent state confidence intervals with reasonable width and small computational time.

Further work will target complexity analysis of the proposed filter and the other interval Kalman filters. It will give a theoretical result about the computational time of the considered filtering algorithms.

[2] The 99.7% confidence intervals of a multivariate Gaussian distribution $\mathcal{N}(\boldsymbol{\mu}, \Sigma)$ are $\left[\boldsymbol{\mu} - 3\sqrt{diag(\Sigma)}, \boldsymbol{\mu} + 3\sqrt{diag(\Sigma)}\right]$. For UBIKF, the 99.7% confidence intervals are computed using the box $[\hat{\mathbf{x}}_k]^+$ and the upper bound of the covariance matrix.

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