Smooth Cubic Polynomial Trajectories for Human-Robot Interactions

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***Abstract*—With the growing importance of Human-Robot Interaction (HRI), the movement of the robots requires more and more attention to address the issues related to safety, efficiency and ergonomics. Trajectories are excellent candidates in the making of desirable motions designed for collaborative robots, because they allow to simply and precisely describe the motions. Despite the large number of works available for Online Trajectory Generation (OTG), there was, to our knowledge, no complete solution capable to simultaneously meet all the requirements of these new applications.**

In this paper we present the first complete trajectory generation algorithm that build trajectories from arbitrary initial and final conditions, subject to general asymmetric bounds on jerk, acceleration and velocity. A review of the state of the art exposes the limits of the previous OTG works and reveals the difficult problem of non-linearity related with short motions. We explain how these non-linearities introduce discontinuities and we propose a solution based on sequences of segment of third degree polynomial functions.

***Index Terms*—Online trajectory generation, trajectory planning, motion control, physical human robot interaction, non-linearity, third degree polynomial trajectories, security, ergonomics**

I. Introduction

TRAJECTORY generation then Online Trajectory Generation (OTG) has generated a lot of work and results from the beginning of robotic and until now. A good trajectory allows improving, not only, the performances of the robots in terms of speed, accuracy or reliability, but also, the safety and comfort of humans working with them. By giving a better description of the robot motion, this model improves the design of planners and controllers as well as the communication between them, thus leading to better architectures. As a trajectory describes both the path and the time evolution along it, the associated mathematical function is complex. Even in the case of a straight-line point-to-point motion, a double-S time trajectory is necessary to describe a smooth motion. Smoothness is associated to the number of derivative of the function, generally to the functions that are at least two times derivable.

With the new robotic application domains, beside the traditional constraints of collisions, accuracy, kinematic or dynamic, trajectories must comply with a growing number of requirements relative to safety, comfort, ergonomics, energy consumption, or flexibility. A large variety of mathematical functions have been proposed to address the multiple constraints that a motion has to satisfy. In this paper we thoroughly examine the sequences of third order polynomial functions, one of the simpler models to build smooth trajectories.

However, despite the many works in the field, trajectory generation still encounters serious difficulties especially for short and asymmetric moves. These harsh points have been encountered in several previous papers, we attempt to address them in this work. The time-optimal curve regarding the length of the motion might be non linear and present discontinuities, particularly for short motions or velocity shift.

We propose the first algorithm to generate smooth trajectories defined by a chain of cubic polynomial functions that joins two arbitrary conditions defined by position, velocity and acceleration in minimum time under asymmetric bounds on velocity, acceleration and jerk.

Although only the one-dimensional problem of time-optimal motion is considered in this paper, these results have direct consequences for control as well as for multi-axis synchronization. Solving this fundamental problem is necessary to solve more general problems in N-dimension and with several via-points.

The paper is organized as follows. In Section II, a state of the art is presented. The time-optimal cubic polynomials trajectories are introduced in Section III. The duration of the trajectories according to the length is presented in Section IV. The time-optimal trajectories are developed in Section V. Sections VI and VII gives some details in the solving of the related equations. Discussions are presented in Section VIII and Section IX concludes this paper.

II. Related works

A. Path planning vs Trajectory planning

Paths, which are pure geometric representation, have been the most widely used approach to describe movements. Among the numerous works that have covered this subject, [27, 38, 39] offer a good synthesis. However, trajectories are necessary to take into account the time. Trajectories are continuous and derivable functions of time, defining the evolution of the position of the robot. They can be defined in task space or joint space. A trajectory can also be seen as the combination of a path with a law of temporal evolution. A summary of trajectory generation in an industrial context can be found in [4], while [12, 28] offer the same for robotic manipulators.

Using trajectories instead of paths provides significant advantages [52]:

- Not only the position can be planned and controlled, but also the velocity, the acceleration, the jerk and eventually higher derivative, allowing to define the smoothness properties.

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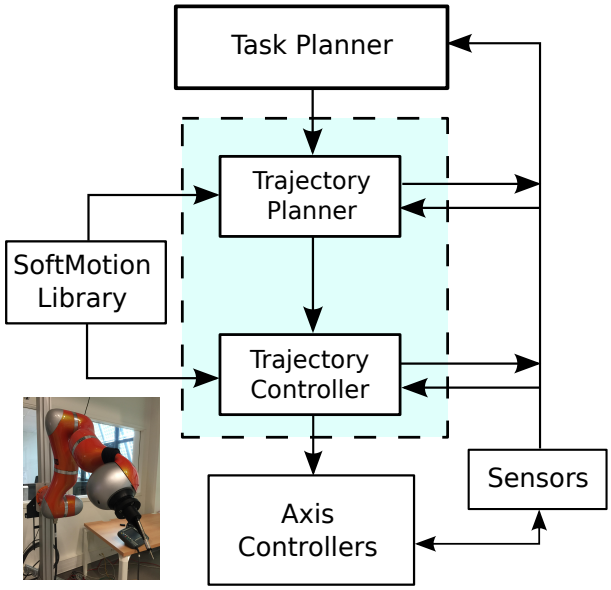


Fig. 1. A possible architecture of a robot that uses trajectories for the control and the communication between the different modules.

- The travel time can be optimized taking into account kinematic constraints.
- The movement is more precisely described allowing a better control.

Moreover, the use of trajectories as the main support of information between a slow task planner and the fast robot controllers leads to a more refined architecture that simplifies the communications (See Fig. 1). A higher granularity in the architecture is desirable as it simplifies the input for each node that can run at different frequencies according to the nature of the task. The overall communication is improved, and in case of unforeseen event, a part of the task can be replanned at a lower level, thus faster. A similar architecture is proposed in [41].

B. Trajectory planning in HRI context

With the emergence of new generations of robots that are always more sophisticated, one of the most challenging features of today's robotic, and even more in the future, is Physical Human-Robot Interaction (PHRI). These new robots will have to coexist and cooperate with humans in a variety of applications such as collaborative assembly, rehabilitation, domestic assistance, and many others. A survey on human–machine cooperation in assembly is proposed in [32]. Krüger et al. study the forms of cooperation between a Human and a Robot that can be used in assembly processes as well as the organizational and economic aspects. They describe the advantages of combining the respective strength of a Human and a Robot, and advocate in favor of these new kinds of cooperation, even compared to fully automated systems. These robots will have to be adaptable, flexible, and reusable.

To be able to work in proximity of humans, collaborative robots must meet a number of requirements that often differ from those required in a classic industrial environment, where robots are confined in cages with prohibited access for humans. While safety remains the main criterion, others emerge such as the level of stress and discomfort the human can feel in the vicinity of the robot. In this context, a robot should not cause excessive stress and discomfort to the human for extended periods of time [2, 9, 19, 35–37]. To fulfill these requirements, some works focus on the behavioral and societal aspects [44, 53]. They attempt to understand the implicit rules and codes that define Humans interactions. Such an approach can be used for the robot to anticipate Humans actions as a Human will be more efficient and more satisfied of the interaction if the robot can anticipate his actions [24]. It can also be used for a robot to communicate its intents. A user will feel more comfortable knowing the goal of the robot early in its movements [13, 14]. Most of these works are integrated at the highest layers of planning, but they make sense only if the movement has been designed beforehand at a lower level. Motions have to be built from a model satisfying strict criteria. This model have to be simple enough for real-time constraints that are linked to security. It should also be capable to generate human-friendly movements to satisfy ergonomics constraints for a large variety of applications. Smooth trajectories appear as excellent candidates since they possesses the advantages necessary to ensure safety, ergonomics, efficiency and adaptability required in the making of collaborative motions.

C. Smooth trajectories

Smooth trajectories were first introduced by Hogan in [26]. The first objective was to reduce wear on systems and improve path tracking [11, 33]. These qualities make smooth trajectories particularly interesting for machining. For instance the study [50] shows that tool-life can be improved up to 60%. One can resume the advantages of working with smooth trajectories as follow:

- Improve accuracy, thus moves can be executed more rapidly and accurately.
- Extend the life span of the manipulators as vibrations are reduced thereby preventing actuators to be damaged and reducing wear of the robot joints.
- Reduce stress and discomfort of human co-worker.

Because of their qualities, smooth trajectories can be used in many contexts and can be employed in the making of more efficient and flexible robots. The smoothness of a trajectory can be defined by the number of derivative of the position and the extreme values of these derivatives. It is generally accepted that a smooth trajectory has at least continuous speed and acceleration, hence a bounded jerk. Considering a constant jerk during a period of time and its triple integral with respect to time, the obtained trajectory is defined by a cubic polynomial function of time. As these cubic functions are simple and their properties well known, they are easy to manipulate, and thus are widely used.

However higher order polynomials, especially quintics, are often used in the literature to obtain smooth trajectories. The main reason being the need to compute trajectories with the possibility to specify position, velocity and acceleration at both ends, so that the robot is able to react quickly to unforeseen

events, an imperative in HRI context to ensure safety [29]. One quintic is enough to compute a trajectory that meet these criteria, hence justifying their use [11]. Unfortunately, quintics generate more computational burden than cubics and their behavior between the way-points is more unpredictable and less faithful to the expected trajectory. This is explained by the tendency to oscillate of the quintics and generally higher order trajectories [41]. A solution is to use more than one quintic, but doing so they loose all interest, as simpler solutions exist. More recently, sequences of cubic polynomial functions have been used to define a smooth trajectory joining arbitrary positions, velocities and accelerations [8, 57]. In the following paragraphs, we present the major results on smooth trajectories categorized in two main approaches.

1) Minimum-jerk model: In order to generate suitable movements for a Human-Robot interaction, human motion appears naturally as a source of inspiration. Among the numerous works that have covered this field, the model described by Hogan is very a popular. This mathematical model describes the organization of a class of voluntary arm movements. A major incertitude of this model is that it lays on a subjective criterion: since movements tends to be smoother and more graceful with skill and practice, they suppose the ideal motion should be the smoothest [17]. Since maximizing the smoothness implies minimizing the jerk, dynamic optimization is used with an objective function that minimizes the square of the jerk over the duration of the movement. This model was first verified on monkeys [6, 25] and later on humans [17]. This model was built to reproduce the bell-shaped tangential velocity profiles observed for human motions. However this was not experimented on a large variety of movements and was mainly verified for intermediate velocity profiles. Different works confirmed the fact that velocity and acceleration curves are asymmetric for a large variety of motions [3, 45, 47], especially for skilled motions. Thus a complete trajectory planner should be able to handle asymmetric kinematics. This approach was built around the hypothesis that the human behavior could be derived from a single organizing principle, wich provides a simple model, but also leads to a lack of flexibility and adaptability.

Since then, minimum-jerk model has been popular and used in order to obtain coordinated and natural human-like motion. In [33], Kyriakopoulos et al. used a minimax approach to minimize the maximum jerk. This work was later extended by Piazzi et al. in [48, 49] where interval analysis is used to globally minimizes the absolute maximum value of the jerk along a trajectory. This global minimization avoids a flaw present in minimum-jerk works, generally subjects to get stuck in local minima. In [1] fifth order polynomials for minimum-jerk control are used with symmetric or asymmetric jerk bounds.

In the different approaches listed above kinematic constraints are not considered and the time has to be set a priori. Some studies have considered minimizing a mixed criterion such as [20, 56]. In [20], Gasparetto et al. adopt a trade-off between a short execution time and smoothness of the motion by using a mixed criterion minimizing both the jerk and time. Kinematic constraints are taken as inputs, avoiding setting the time a priori. These works constitute a good attempt against the lack of flexibility of the model. However there is another model that is not affected by this drawback.

2) Constrained-jerk model: In this second approach, a suitable application dependent maximum jerk J_{max} is established through experiments or information provided by the manufacturer. This threshold is determined according to certain criteria related to the nature of the task. Once the maximum jerk is defined the problem left is that of a classic time's optimization. This is done by maximizing the jerk under the constraint $J < J_{max}$. It provides time-optimal trajectories under task dependent constraints defined by the user. This approach can also be extended to acceleration, velocity and other derivatives. The major disadvantage of this model is that kinematic constraints must be specified by the user. However, unlike the previous approach, it provides qualities that are essential to build the future generations of collaborative robots that we expect to be efficient, adaptable and reusable [32].

In [40], Liu proposes a real-time algorithm to generate smooth trajectories from current velocity under constraints on jerk, acceleration and velocity. Optimal in most cases, this paper points the difficulty of managing non-null initials and finals conditions. In the calculation steps for the maximum reachable speed, if the motion is too short to reach the maximum speed or acceleration, it becomes very difficult to compute an analytical solution online. A suboptimal strategy is then adopted by keeping the initial speed for a certain period. A similar approach is presented in [22] to generate third order time-optimal trajectories. The main contribution concerns the ability to handle arbitrary initial conditions, while end conditions must stay at rest. Nonetheless this work encounters numerical problems and produces infinite jerk for short displacements. In that case, second order trajectories are employed as a fallback solution. Similar results can be found in [31], except it is for quadratic trajectories and hence not smooth, since the jerk is not bounded. In [29], Kroger et al. introduced a general Online Trajectory Generation algorithm and an instance of it using third order polynomials. It brings a manipulator from an arbitrary initial state to an arbitrary final state except for the final acceleration which is always null. This work as been extended in [30] such that time-variant kinematic motion constraints are considered.

Third order polynomial trajectories offer a simple solution to generate jerk bounded trajectories as they are easy to manipulate, keep the jerk bounded and can be generated online. They also avoid some major drawbacks of higher degree polynomials such as the tendency to oscillate, and are better for approximation [41]. Moreover it has been demonstrated that a concatenation of at most seven cubic is enough to represent any time-optimal trajectory [7]. In [23] Herrera-Aguilar et al. propose seven cubic equations to obtain Soft Motions for robot service applications. It was extended later in [7] by Broquere et al. to compute online time-optimal trajectories given any initial and final conditions, under bounded jerk, acceleration and velocity, for any number of independently acting axis. A solution for time-imposed trajectories is presented in [8] which leads to a simpler axis-synchronization for multi-axis systems. Yet, this solution had drawbacks, since it was not

TABLE I
CLASSIFICATION OF MAIN TRAJECTORY PLANNING ALGORITHMS IN HRI CONTEXT

Reference	[40]	[41]	[34]	[31]	[46]	[7]	[22]	[29]	[15]	Ours
$J_{max} \in \mathbb{R}$	+	+	+		+	+	−	+	+	+
$V_I \in \mathbb{R}$	+	+		+		+	+	+	+	+
$A_I \in \mathbb{R}$				+		+	+	+	+	+
$V_F \in \mathbb{R}$		+		+		+		+	+	+
$A_F \in \mathbb{R}$				+		+			+	+
Asymmetric bounds									+	+
Online	+	+		+		+	+	+	+	+
Optimal	−			+		+	+	+		+

"−": the criterion is not satisfied for every scenario.

possible to both impose the time and keep the jerk bounded. An improvement was proposed by Zhao et al. in [57], which allows to have an imposed time and bounded jerk for sufficient large motions. For a point to point movement in an N-dimensional space, the time optimum straight-line motion is obtained by projecting the constraints on the line [52].

Polynomial and trigonometric models have also been combined in order to design smooth trajectories [41, 46]. In [41], Macfarlane et al. introduce an online method to compute smooth trajectories for industrial robots. Concatenations of fifth order polynomials are employed to join the waypoints approximating the desired trajectories. Oscillations due to the use of quintics are here corrected by sine-wave template for accelerations. The solution presented is not optimal, and the jerk continuity is not guaranteed either, despite the use of fifth order polynomials. Trigonometric and polynomial models are also combined to design s-curve motions of any order from rest to rest in [46].

In more recent works [15, 16], Ezair et al. display a new algorithm to generate smooth trajectories of any order under kinematic constraints and for multi-axis systems. A key point is its ability to deal with any general initial and final state. The algorithm builds trajectories from an input speed, which is updated iteratively by binary search until a near-optimal cruise speed is found (or peak when the motion is too short). A recursive approach is used to build higher order trajectories from lower ones. An interesting addition of this paper is the introduction of asymmetric constraints. Unfortunately this work presents some limits other than its non-optimality regarding a time criterion. Binary search in non-linear systems can lead to difficulties, such as being trapped in local minima. Moreover, it does not guarantee a solution can be found.

To our knowledge (See Table. I), there is no work proposing a complete answer to trajectory generation satisfying simultaneously all the following criteria to build safe, efficient, adaptable and human-friendly robots:

- Real-time capable.
- General initial and end conditions for both velocity and acceleration.
- General bounded jerk, acceleration and velocity.
- Asymmetric bounds.
- Time optimal.

The algorithm we present here is the first, to our knowledge, to fulfill simultaneously all those criteria. We also attempt to understand the harsh points associated to non-linear systems that only few papers address [5, 10, 21].

III. TIME-OPTIMAL CUBIC POLYNOMIAL TRAJECTORIES

One key lesson to be drawn from this bibliography is that solutions are well known for large movements, but smaller movements exhibit more complex behavior. This problem has been approached in some works [22, 40, 41], but avoided by the majority. To better understand these small moves, we first develop a graphical approach based on the phase diagram and then plot the time-optimal solution as a function of the move length x_f. From these results we then introduce the first complete algorithm to compute the time-optimal motion.

A. Problem definition

The initial condition $c_i = (x_i, v_i, a_i)$ is defined by the position x_i, the velocity v_i and the acceleration a_i. The final condition is similarly defined by $c_f = (x_f, v_f, a_f)$. Without loss of generality, we choose $x_i = 0$, i.e. we define x_i as the origin. The trajectory must comply with the asymmetric bounds $(J_{min}, J_{max}, A_{min}, A_{max}, V_{min}, V_{max})$ for jerk, acceleration and velocity.

From these conditions, our objective is to find the time-optimal trajectory $\mathcal{T}(t)$, which is known to be a series of at most seven trajectory segments that each saturates one of the bounds [7, 40]. Each of these trajectory segments S_n is a polynomial cubic function of time t and it is characterized by a constant jerk $J_n \in \{J_{min}, J_{max}, 0\}$, an initial instant τ_n, a duration T_n and its initial condition $C_n = (x_n, v_n, a_n)$. For all t such that $\tau_n \leqslant t \leqslant (\tau_n + T_n)$:

$$\begin{aligned} S_n(t) &= \frac{1}{6}J_n(t-\tau_n)^3 + \frac{1}{2}a_n(t-\tau_n)^2 + v_n(t-\tau_n) + x_n \\ \dot{S_n}(t) &= \frac{1}{2}J_n(t-\tau_n)^2 + a_n(t-\tau_n) + v_n \\ \ddot{S_n}(t) &= J_n(t-\tau_n) + a_n \end{aligned} \tag{1}$$

B. The phase diagram

These trajectories are well described with a phase diagram (Fig. 2) where abscissa is velocity and ordinate is acceleration. In this diagram, constant jerk trajectories associated to third degree polynomial functions define horizontal parabolas, constant acceleration motions associated to second degree polynomial functions are horizontal lines and constant velocities motions are associated to points of the null acceleration axis. The equations of the parabolas are obtained by eliminating the time in the two last equation of (1):

$$\dot{S_n} = \frac{\ddot{S_n}^2}{2J_n} + v_n - \frac{{a_n}^2}{2J_n} \tag{2}$$

The area where acceleration and speed constraints are verified is plotted in green in the figures. The acceleration bounds

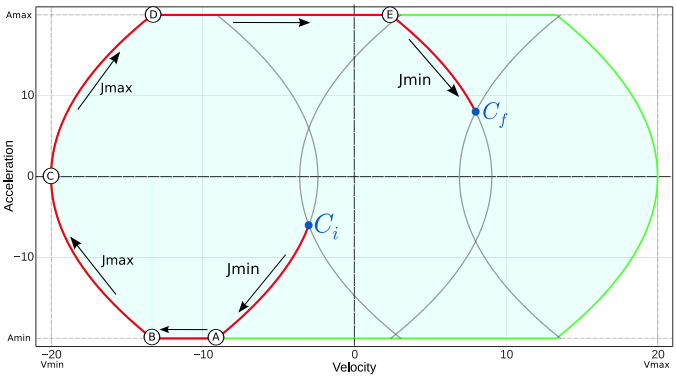


Fig. 2. Phase diagram of a classical seven segments trajectory for a large negative motion. The green area of admissible conditions is limited by the acceleration bounds (A_{min}, A_{max}) and the parabolas associated to velocity bounds (V_{min}, V_{max}). The jerks J_{min} and J_{max} define four condition parabolas passing throw the initial conditions C_i and the final condition C_f. The red curve joins C_f from C_i with the following 7 segments: $J_{min} \rightarrow A_{min} \rightarrow J_{max} \rightarrow V_{min} \rightarrow J_{max} \rightarrow A_{max} \rightarrow J_{min}$. The segment with saturated speed V_{min} holds on point C.

A_{min} and A_{max} define two horizontal boundary lines while the left and right boundaries are parabolas respectively defined by (V_{min}, J_{max}) and (V_{max}, J_{min}). By continuously varying the length x_f from a large negative value to a large positive one, the trajectory in the phase diagram reaches successively different limit shapes. These shapes are defined by a sequence of trajectory segments. For example, the large negative values are associated to the classical seven segments trajectory plotted in red in Fig. 2 and the limit case of this trajectory sequence is reached when the minimum velocity V_{min} segment lasts zero seconds. Just after this limit, five segments only define the trajectories. We associate the type (JAJVJAJ) to these sequence of trajectories and the type (JAJAJ) to the one without the V_{min}/V_{max} segment where J stands for jerk segment and A for acceleration segment.

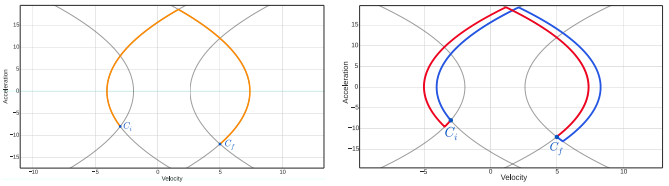
Introducing the notion of type allows decomposing the problem in three sub-problems, the last one being trivial:

1) Find the possible types of the solution.
2) For each possible type, compute the associated trajectory.
3) Select the faster trajectory.

In the following, we firstly develop a general method to compute the border trajectories between two types.

C. Local parabolas

From the initial condition C_i defined by v_i and a_i, the two jerk bounds allows only two possible optimal motions, which define two condition parabolas in the phase diagram (See Fig. 2). Similarly, only two motions i.e. two condition parabolas are possible to reach the final condition C_f defined by v_f and a_f. As these four condition parabolas are symmetric with respect to the horizontal axis of zero acceleration, they only define two relative configurations depending on whether only two parabolas intersect (Fig. 2) or the two interior parabolas also intersect (Fig. 4). A trajectory joining C_i to C_f and composed by two jerk bounds segments is called a direct trajectory. In the phase diagram, these direct trajectories are



(a) The direct trajectory $x_f = x_d$. (b) Trajectories with $x_f < x_d$ in red and $x_f > x_d$ in blue.

Fig. 3. Evolution of the phase of the trajectory with the length x_f around the direct trajectory.

associated to arcs of condition parabolas. When C_i and C_f are on the same parabola, the direct trajectory is reduced to one arc. If a part of these two arcs of parabolas is outside the admissible area, it is replaced by an horizontal segment associated to the acceleration A_{min} or A_{max}.

Each pair of initial and final conditions defines at least one direct trajectory, for example the orange trajectory of the Fig. 3a. When the internal parabolas also intersect defining shortcut trajectories, up to three direct trajectories can exists (See Fig. 4).

D. Varying the trajectory length

Now, we propose to explore the possibility of varying the trajectory length x_f in the neighborhood of a direct trajectory of length x_d by adding an optimum motion, i.e. a J_{min} or J_{max} parabolic arc A^p. For example, the red trajectory of the Fig. 3b is the result of adding a small arc, associated to a negative jerk segment, at the beginning of the motion that translate the negative parabola toward the left. The length of the red trajectory is $x_f < x_d$. Similarly, adding a small positive motion at the end of the trajectory increases the length of the motion, see for example the blue trajectory in Fig. 3b.

The modification of the length is less intuitive in the case of the Fig. 5a where direct trajectory doesn't reach the zero acceleration line. In this case, adding a short J_{min} jerk trajectory at the beginning of the motion, translates the parabola with J_{max} jerk to the right. The extension of this added segment brings the end of the first segment to progressively reach the zero acceleration axis until a cusp appears (Fig. 5b). Continuing to extend this added segment create a loop (Fig. 5c) that can be extended to the V_{min} limit. From the cusp, the second segment translates back to the left. Generally, during this extension the length x_f is not monotonically decreasing as we will see bellow.

This complex behavior appears each time one parabola of the direct trajectory doesn't cross the zero acceleration line to reach the condition C_i or C_f. In the particular case of the Fig. 5, it appears on both sides.

E. Internal parabolas also intersect

In the case of the Fig. 4 where the internal parabolas also intersect, three direct trajectories are possible: (C_i, D_3, C_f), (C_i, D_2, C_f) and (C_i, D_1, C_f). The last trajectory provides an evolution similar to the case of the Fig. 3 and can be extended to both V_{min} and V_{max}. The two first trajectories introduce

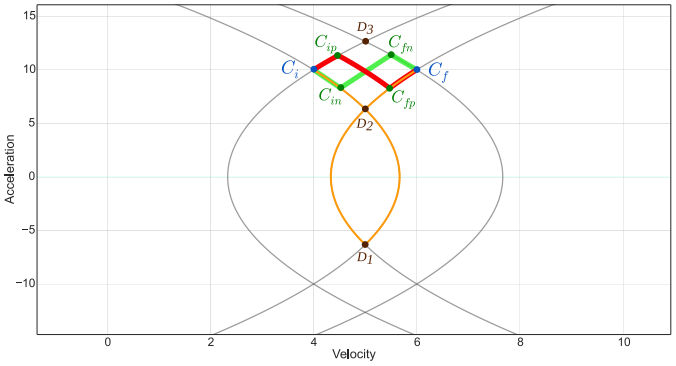


Fig. 4. Limits trajectories in the case where the two parabolas defined by C_i intersect the two parabolas defined by C_f. The main direct trajectory is in orange. The red and green trajectories are two shortcuts trajectories. ($J_{min}=-30$, $J_{max}=30$, $A_{min}=-20$, $A_{max}=20$, $V_{min}=-20$, $V_{max}=20$, $a_i=10$, $v_i=4$, $a_f=10$, $v_f=6$).

alternative trajectories for a range of values for the length x_f. Two samples, $(C_i, C_{ip}, C_{fp}, C_f)$ and $(C_i, C_{in}, C_{fn}, C_f)$, illustrate these shortcut trajectories in the Fig. 4.

The trajectory (C_i, D_1, C_f) is endlessly expandable, but $(C_i, C_{ip}, C_{fp}, C_f)$ and $(C_i, C_{in}, C_{fn}, C_f)$ have shorter curves in the phase diagram where they generally define faster trajectories for a small interval of values.

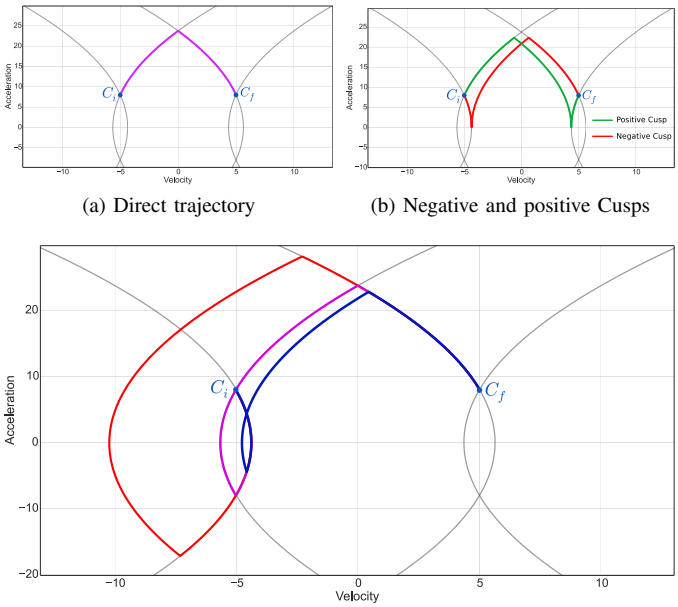
To summarize, in all cases the direct trajectory (Fig. 5a) can be respectively extended to long negative motions and to long positive motions. The behavior of these trajectories is more complex for short motions, i.e. around the direct trajectories, where shortcut trajectories can exist (Fig. 4). This point is developed further and constitutes a major contribution of our work.

IV. Duration of the Trajectories According to the Length

As expected, trajectories show a more complex behavior around the direct trajectory. We are going to plot the optimal time of these trajectories versus their length x_f to obtain the curve $\mathcal{C}_{opt}$. In a first stage, we propose to plot the duration of the trajectory according to its length as a parametrized curve $\mathcal{C}(x_f, t_f)$. We choose the time length of the added arc A^p (see III-D) as parameter, thereby defining a parameter t_n for left negative side and a parameter t_p for right positive.

A. A simple case

In order to simplify the presentation, we firstly plot the curves $\mathcal{C}(t_n)$ and $\mathcal{C}(t_f)$ (see Fig. 6) for the particular case of the Fig. 5 where internal parabolas doesn't intersect and without considering the bounds. Using the conditions of the Fig. 5, we compute the times of the trajectory segments and then apply the equations (1). The times t_n and t_p are null for the direct trajectory (Fig. 3a). A time $t_n > 0$ defines a parabolic arc A^p between the initial acceleration a_0 and the acceleration $a_n = a_0 + J_{min}t_n$. Respectively, $a_p = a_0 + J_{max}t_p$ for a positive parabolic arc. For example, in the case of the Fig. 5c, the first purple negative arc starts from C_i with acceleration a_i and ends at $-a_i$. Its duration is $t_n = -2a_i/J_{min}$.



(a) Direct trajectory (b) Negative and positive Cusps

(c) Negative loop expansion

Fig. 5. The direct trajectory displayed in (a) can be extended by adding a positive or a negative arc of parabola. In (b) two limit cases are displayed, which separate the trajectories with a cusp from the trajectories with a loop. Different trajectories with a loop are plotted in (c): a small loop in blue, the loop reaching the symmetric condition of C_i and a larger loop in red. ($J_{min}=-50$, $J_{max}=50$, $A_{min}=-30$, $A_{max}=30$, $V_{min}=-30$, $V_{max}=30$, $a_i=8$; $v_i=-5$; $a_f=8$; $v_f=5$).

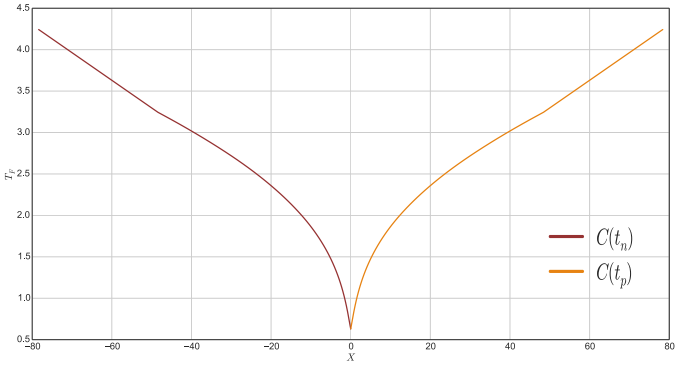


Fig. 6. The optimal (t_f) time solutions versus the trajectory length (x_f). A trajectory length of zero means the start and end positions are the same. For example the minimum here is associated to the direct trajectory of Fig. 5a, and the null length is explained by the presence of symmetries.

Using (1), we obtain the condition at the end of the first segment associated to the arc A^p. This condition defines the second parabolic arc, enabling the computation of the intersection of the two last parabolic arcs. From this intersection we determine the durations t_2 and t_3 of the two last segments and then the length x_f using (1). More details are given in appendix A. It is worth noting that these geometric constructions are simple but depend on the relative position of C_i and C_f, in particular, the arc A^p of parabola can be the first or the last segment.

This procedure defines the duration $t_f(t_n) = t_n + t_2 + t_3$ and the length $x_f(t_n)$ of the trajectory, namely a point of the parametric curve $\mathcal{C}(t_n)$. The repetition of this procedure allows

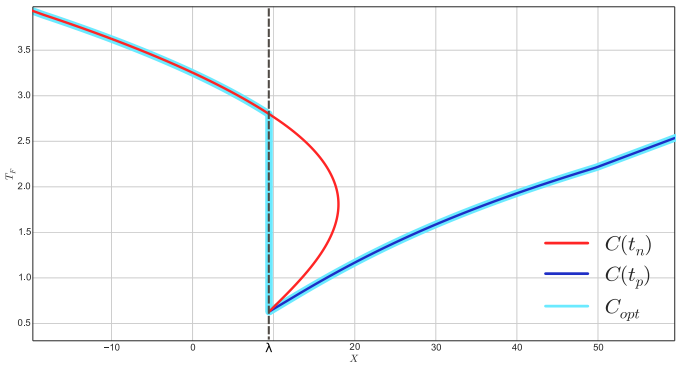


Fig. 7. Effect of velocity shift on the shape of the curve $\mathcal{C}_{opt}$.

to plot the curve $\mathcal{C}(t_n)$ defining the duration of the trajectory versus its length (Fig. 6). A similar procedure is used to plot $\mathcal{C}(t_p)$.

B. Effects of the bounds

The acceleration bounds A_{min} and A_{max} limit the increase of the time parameters t_n and t_p and the arc A^p to some time t_j defining the intersecting point of the jerk parabolic trajectory with the boundary line. From this point, we follow the same process but by defining the intermediate parabola from the end of the acceleration segment of time t_s. To keep the same parameter t_n (resp. t_p) we define t_s by $t_s + t_j = t_n$ (resp. $t_s + t_j = t_p$).

Similarly, when the intermediate parabola reaches the velocity bound V_{min} or V_{max}, we define the duration of the constant velocity segment t_v by $t_v + t_a + t_j = t_n$ (resp. t_p) where t_a is the duration of the constant acceleration segment reaching the limit velocity parabola. In some cases, the parabola defined by t_j can reach the minimal or maximal velocity parabolas before the acceleration bounds: the geometric constructions are similar, but t_a is null.

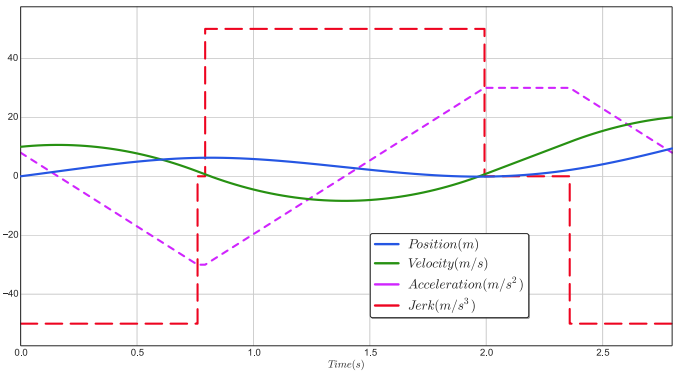
These bounds introduce changes in the nature of the portions of the curves $\mathcal{C}(t_n)$ and $\mathcal{C}(t_p)$ (Fig. 6), in particular the large values of t_n (resp. t_p) correspond to straight line segments associated to constant (minimum or maximum) velocity motions. The second parabola can also reach the second acceleration bound, resulting in a change in the trajectory sequence without affecting the parametrization of the curve.

V. The Time-Optimal Trajectories

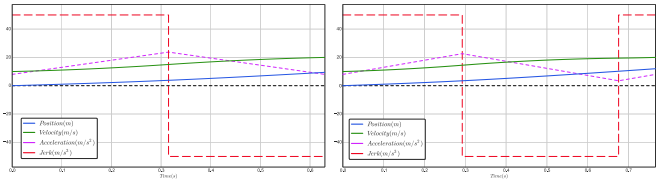
In the previous case (Fig. 6), the parametric curves $\mathcal{C}(t_n)$ and $\mathcal{C}(t_p)$ give directly the time-optimal function $\mathcal{C}_{opt}(x_f)$ that associates the optimal time to the length x_f. In general, the non-linearities in the definition of these functions generate far more complex curves. We will now consider two types of non-linearities: the influence of the velocity that distorts the curve and the presence of shortcuts that split the curve in two.

A. Influence of the velocity

Having now defined a tool to plot the time-optimal function versus the length of the motion, we can study the influence of the different parameters. Considering the case of the figure 5, we shift the initial and final velocities by 15 (from (-5, 5) to (10, 20)). This shift just translate the phase diagram to the right, but the parametric curves $\mathcal{C}(t_n)$ and $\mathcal{C}(t_p)$ plotted in the Fig. 7 have now more complex shapes and, for a range of values of x_f, there are multiple associated trajectories with different time t_f. Therefore the time-optimal function $\mathcal{C}_{opt}(x_f)$ is no more directly defined by the union of the two curves, but by the minimum time for each value of x_f. The resulting curve might exhibit discontinuities, which impact the definition of the trajectory.



(a) Solution in 5 segments for $x_f = 9.44920 < \lambda$



(b) Direct trajectory for $x_f = \lambda = 9.4492105$ (c) Solution in 3 segments $x_f \gg \lambda$

Fig. 8. Major effects of discontinuities on the time representation of the optimal trajectory and its derivative.

For example such discontinuity is present in the case of Fig. 7 at $x_f = \lambda$. For $x_f < \lambda$ the optimal solution begin with a jerk negative segment, but for $x_f >= \lambda$ the optimum trajectory begins with a positive jerk. The corresponding trajectories and their derivatives are plotted in the Fig. 8.

B. Duration in the presence of shortcut

The shortcut solutions associated with intersecting parabolas also introduce discontinuities in the optimal function $\mathcal{C}_{opt}(x_f)$. The curve of the Fig. 9, plotted from a similar case of the Fig. 4, shows a small lens shaped curve just below the cusp point of the parametric curves. Beside the main curve ($\mathcal{C}(t_n)$ and $\mathcal{C}(t_p)$) obtained by the previous procedure, the two parts ($\mathcal{C}(tl_n)$ and $\mathcal{C}(tl_p)$) of the lens shaped curve are similarly plotted from the two shortcut trajectories $(C_i, C_{ip}, C_{fp}, C_f)$ and $(C_i, C_{in}, C_{fn}, C_f)$, where the parameters tl_n and tl_p define the duration of the first of the two segments (C_i, C_{in}) and (C_i, C_{ip}).

In the neighborhood of the singular case where the interior parabolas become tangential, the lens shaped curve joins the main curve and disappears to extend the main curve on both sides (Fig. 10). It can be noted that for some values of x_f, the curve exhibit up to five solutions (Fig. 10).

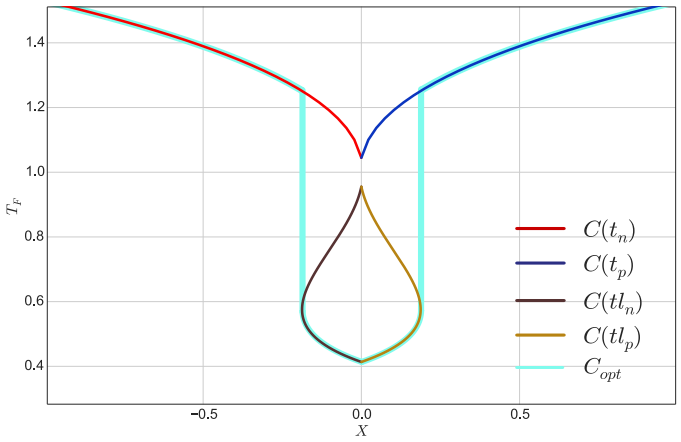


Fig. 9. Shortcut trajectories introduces discontinuities in the time-optimal curve $\mathcal{C}_{opt}$. ($J_{min}=-40$, $J_{max}=40$, $A_{min}=-30$, $A_{max}=30$, $V_{min}=-30$, $V_{max}=30$, $a_i=20$; $v_i=-4.99$; $a_f=20$; $v_f=4.99$).

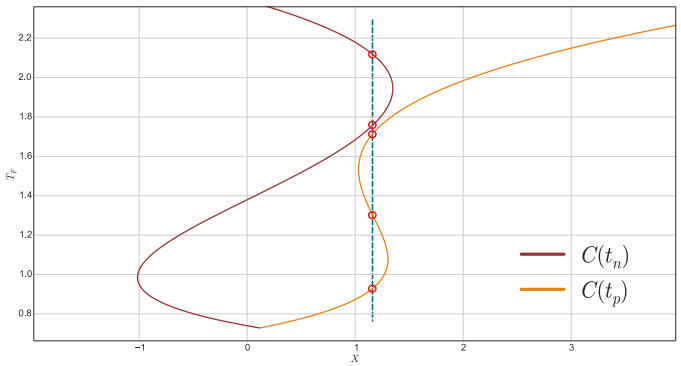


Fig. 10. Curve exhibiting up to 5 solutions when initial and final velocities are shifted. ($J_{min}=-40$, $J_{max}=50$, $A_{min}=-55$, $A_{max}=50$, $V_{min}=-40$, $V_{max}=70$, $a_i=-39.0$; $v_i=17.205$; $a_f=-39.0$; $v_f=-17.105$).

We have seen the influence of the initial and final velocity and acceleration on the optimal curve $\mathcal{C}_{opt}(x_f)$. The jerks J_{min} and J_{max} deform also the curve but do not introduce new particular case. The values of the bounds influence the shape of the area of the admissible conditions, which change the nature of the function defining the curve $\mathcal{C}_{opt}(x_f)$ but do not introduces new types of discontinuities.

Considering only the jerk bounds, it is possible to compute analytically the parametric curve $\mathcal{C}(x_f(t_n), t_f(t_n))$ (See Appendix A). The expression obtained is large and complicated and generates an even more complicated derivative. Unfortunately, we could not manage to analytically compute the zero of the derivative of $\mathcal{C}_{opt}(x_f)$ with respect to x_f associated to its points of discontinuity.

In the algorithm presented below, we chose to compute all the solutions (up to 5) and then select the optimal one. An alternative approach would be to numerically compute the zero of the derivative of $\mathcal{C}(t_n)$ and $\mathcal{C}(t_p)$ relatively to x_f and then compute directly the optimum.

C. The time-optimal algorithm

From the previous elements, several strategies are possible to compute the time optimum trajectory $T_{opt}(x_f)$. However, as our main motivation is real-time control, we propose now a fast algorithm:

1) Compute the local parabolas.
2) Compute the singular limit trajectories.
3) From x_f, determine what are the possible sequences of trajectories.
4) For each possible sequence compute the time optimum trajectory.
5) Select the faster trajectory.

The first three stages of this algorithm were previously detailed and the last one is trivial, therefore the following sections will detail how to compute a sequence of trajectory segments from the length x_f. We begin by showing how to transform the associated systems of equations in a quartic polynomial equation and then we will detail how to solve these quartic equations.

The seven segment trajectory labelled as (JAJVJAJ) doesn't generally include all the seven segments. When the V segment exists, it is the only one for which the duration is varying accordingly with x_f. If the V segment is not reached, the sequence comprises at most five elements (JAJAJ). This problem can always be reduced further to a sequence of three segments: Each time an acceleration segment is reached, the first or the last jerk segment duration is fixed and defined by $t_j = (A_b - a_i)/J$ where $A_b \in \{A_{min}, A_{max}\}$ and $J \in \{J_{min}, J_{max}\}$. Therefore after simplification, the four problems left to solve are: (JJJ) when no acceleration bounds are reached, (AJJ) or (JJA) when only one acceleration bound is reached and (AJA) when both the acceleration bounds are reached.

The next paragraph details the (JJJ) case, the three others being similar and introduced in appendix B and the (V) case is trivial because only one segment is varying.

VI. Solving the (JJJ) problem

After reducing the problem, the three jerk trajectories problem (JJJ) is defined by seven parameters: the initial condition (a_i, v_i), the final condition (a_f, v_f, x_f) and two jerks J_a and J_b. The unknowns are the durations of the three segments (t_1, t_2, t_3). Two conditions (a_1, v_1, x_1) and (a_2, v_2, x_2) are associated to the transitions between the segments and defined by:

$$\begin{aligned} a_1 &= J_a * t_1 && +a_i \\ v_1 &= J_a * {t_1}^2/2 && +a_i * t_1 && +v_i \\ x_1 &= J_a * {t_1}^3/6 && +a_i * {t_1}^2/2 && +v_i * t_1 \end{aligned} \tag{3}$$

$$\begin{aligned} a_2 &= J_b * t_2 && +a_1 \\ v_2 &= J_b * {t_2}^2/2 && +a_1 * t_2 && +v_1 \\ x_2 &= J_b * {t_2}^3/6 && +a_1 * {t_2}^2/2 && +v_1 * t_2 && +x_1 \end{aligned} \tag{4}$$

Then the system of polynomial equations to solve can be written as:

$$\begin{aligned} a_f &= J_a * t_3 && +a_2 \\ v_f &= J_a * {t_3}^2/2 && +a_2 * t_3 && +v_2 \\ x_f &= J_a * {t_3}^3/6 && +a_2 * {t_3}^2/2 && +v_2 * t_3 && +x_2 \end{aligned} \tag{5}$$

Using algebra systems like Maple [42] and Maxima [43], the solution of this system of equation can be expressed from the roots of a quartic equation. Firstly we define a set of intermediate variables (k_1 to k_4) and the coefficients of the polynomial:

$$
\begin{aligned}
k_1 &= 2 * J_b - J_a \\
k_2 &= {J_b}^2 + J_a * (J_a - 2 * J_b) \\
k_3 &= J_a - J_b \\
k_4 &= 2 * J_a * v_f \\
c_4 &= -J_b * ({J_b}^3 + J_a * (J_a * (5 * J_b - 2 * J_a) - 4 * {J_b}^2)) \\
c_3 &= 0 \\
c_2 &= -6 * (2 * J_a * k_2 * v_i - {J_b}^2 * {a_i}^2 \\
&\quad + J_a * (k_1 * {a_i}^2 + 2 * k_2 * v_f) + {a_f}^2 * (J_a * k_1 - {J_b}^2)) \\
c_1 &= -8 * (a_i * (3 * J_a * J_b * v_i - (3 * {J_a}^2 * v_i + J_b * {a_i}^2)) \\
&\quad + J_a * {a_i}^3 - 3 * J_a * k_3 * x_f) \\
&\quad + a_f * (3 * J_a * k_3 * v_f - {a_f}^2 * k_3)) \\
c_0 &= 3 * (4 * J_a * v_i * (J_a * v_i - ({a_i}^2 + k_4 - {a_f}^2)) + {a_i}^4 \\
&\quad + 2 * ((k_4 - {a_f}^2) * {a_i}^2 + 2 * {J_a}^2 * v_f * v_f) \\
&\quad + {a_f}^2 * ({a_f}^2 - 4 * J_a * v_f))
\end{aligned}
$$

The quartic polynomial equation is then defined by: $c_4 * x^4 + c_2 * x^2 + c_1 * x + c_0 = 0$. If r_i is one of its roots, the solution can be written as:

$$
\begin{aligned}
t_1 &= -(k_6 * (k_5 - {a_i}^2 + 2 * (J_a * (r_i * a_i - v_f) \\
&\quad - Jb * r_i * a_i) + k_3 * J_b * {r_i}^2 + {a_f}^2)) * 0.5 \\
t_2 &= r_i \\
t_3 &= (k_6 * (k_5 - ({a_i}^2 + k_4) - J_b * k_3 * {r_i}^2 \\
&\quad + a_f * (2 * k_3 * r_i + a_f))) * 0.5
\end{aligned}
$$

with:

$$
\begin{aligned}
k_5 &= 2 * J_a * v_i \\
k_6 &= 1/(J_a * k_3 * r_i)
\end{aligned}
$$

As the times t_1, t_2 and t_3 must be positive, only the positive solutions define a valid trajectory. This system can have up to four solutions, but we never find a particular case with more than three admissible triplets. It must be noted that the trajectories can begin with one of the two jerk bounds, defining two different problems, which defines up to three solutions each. So, in some cases like the one of the Fig. 10, five different trajectories composed of a sequence of potentially optimum segments can be computed.

The last step is to compute the time-optimal solution, which is the one that minimize $t_1 + t_2 + t_3$.

The approach to compute the polynomial equation in the case of the sequences (AJJ), (JJA) or (AJA) is really similar and presented in appendix B.

VII. Solving the quartic polynomial equation

It is well known that solving a quartic polynomial equation is difficult. The analytical solutions have been known since the 16th century, but this approach is time consuming and can fail for some particular equations. Numerical algorithms like Newton-Raphson based algorithms are efficient but require initial information about the root, precisely the information we do not have in our case. Recent works have proposed to associate the two approaches: the analytical results are used as inputs for a numerical solver [54]. This has generated a new class of faster and more accurate algorithms [18, 55].

We used a solver derived from the one of Schwarze [51] to compute a first approximation of the solution. To improve the accuracy, we directly applied a three dimensional Newton method to the durations of the three segments. The analytical expression of the derivative of the polynomial functions (1) is given in appendix C.

TABLE II
Comparison of computation times for different trajectory lengths. Results for 5×10^5 runs.

Case	General	Near the direct trajectory	With cruising velocity phase
Times (μs)	1.04	2.45	0.854

VIII. Discussion

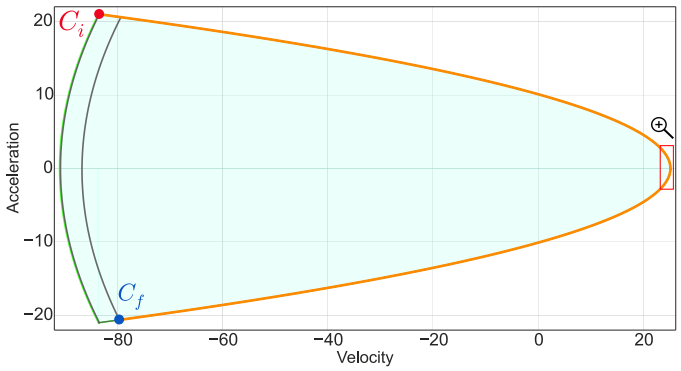
The characteristics of our method are summarized in the classification table I, which compare the possibilities of the online trajectory generators. The implementation[1] of this algorithm on a system equipped with a Intel Core i7 processor running at 2.2 GHz gives a mean time of 1.02 μs with a standard deviation of 0.81 μs observed for 10^8 random tests, allowing to use it in real time and for planning (See table II). These performances are comparable or better than the previous algorithms that do not always give the optimal solution.

The longest times are relative to particular case where the Newton-Raphson method have difficulty to improve the accuracy of the solution. The figure 11 gives an example of such a configuration where the analytical solution is not precise enough and the numerical one struggle to converge. Fortunately these cases are hardly relevant and the system always return a solution for such configurations, eventually a sub-optimal solution.

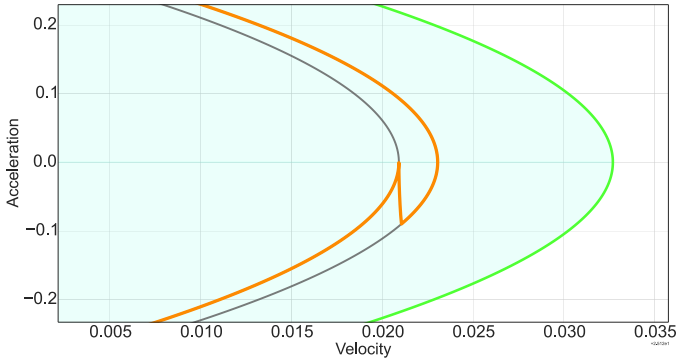
Solving the optimal trajectory problem in the vicinity of the direct trajectory opens the way to an intensive use of these simple trajectories for control and planning. Concerning the trajectory control, where the objective is to compute in real time a trajectory to bring back smoothly the mobile to the target trajectory from the current state, we can notice that the connection trajectories are short and consequently close to the direct trajectories. In this case, the proposed trajectory generator provides a good solution.

Similarly for multi-axes trajectory generation, one classical solution is to compute the time optimal trajectory for each axes, select the slowest one and synchronise the other axis with the selected one. The proposed trajectory generator can improve the calculation of the time optimal trajectories.

[1]The documentation, the softMotion library and examples are available at https://git.openrobots.org/projects/softmotion/wiki.



(a) In this case parabolas are almost identical.



(b) The zoomed area where the positive cusp that produces a very short jerk segment.

Fig. 11. Illustration of a problematic case where newton doesn't converge. ($J_{min}=-2.02754$, $J_{max}=29.7968$, $a_i=20.9815$; $v_i=-83.4179$; $a_f=-20.6076$; $v_f=-79.5853$).

Sampling-based motion planners are really efficient to find a polygonal path, even in the case of cluttered and high-dimensional space, but planning efficient and smooth trajectories is more difficult. Here also the proposed trajectory generator could improve the smoothing of an initial trajectory built from the polygonal path.

The generator can be used in joint or operational spaces. In the first case kinematic bounds can be directly deduced from joint characteristics, whereas Cartesian space is suitable to incorporate the constraints related to safety and ergonomics. The non-symmetrical bounds can be employed to enhance the ergonomics properties of our model as they allow to design more natural human-like motions. They can also be very useful in the making of vertical motions under gravity, or motions in the presence of a human, for which an approaching move is more scaring than a withdrawal move.

As this work explains the discontinuities of the time-optimal curve $\mathcal{C}_{opt}$ and solves the optimum time problem, it will contribute to the development of trajectory based robotic architectures. In these architectures trajectories will be used as the main support of communication and facilitate the link between planning and control, leading towards an improvement of robots motions.

By explaining the complex behavior of the jerk bounded trajectories, this work defines also a step in the solving of the snap bounded optimal trajectory problems, where the snap is the derivative of the jerk.

IX. Conclusion

With the emergence of HRI, the problem of the generation of safe, efficient and human-friendly movements has to be addressed. From the review of the state of the art it appears that no complete solution for the making of collaborative motions exists yet. To the best of our knowledge, the algorithm presented herein that uses trajectory of class $\mathcal{C}^2$ defined by a chain of cubic polynomial functions is the first one that:

- joins two arbitrary conditions defined by position, velocity and acceleration,
- in minimum time under general and asymmetric bounds on velocity, acceleration and jerk.

By explaining graphically the behavior of the optimal trajectories, this work allows to explain and solve the difficulties highlighted by the previous works. The proposed trajectory generator completes the existing tools for planning and controlling multi-axes cubic polynomial trajectories, which open the way for more flexible and friendly robots. This applies particularly to the HRI domain where the underlying constrained jerk model approach makes easier the consideration of the different types of constraints related to safety and ergonomics. More specifically, close cooperation between humans and industrial robots needs more flexibility, adaptability and reusability that can be improved by the models and tools developed in this work.

Even if the model of the chain of cubic polynomial trajectories is efficient, the questions relative to higher degrees polynomial still remains. These models are necessary to solve specific problems. The underactuated vehicles, for example, need one more derivative to control the motion obtained by integration. It is the case, for example, to obtain a jerk bounded horizontal move with a quadrotor. The same problem appears also for double and, more generally, multiple pendulum. Given the difficulties encountered to solve the cubic trajectories, the higher degree appears as really challenging.

Appendix A
Analytic expression of the parametric curve $\mathcal{C}(x_f(t_n), t_f(t_n))$

We consider here only a particular case defined by:

- Three non null jerk segments associated to parabolas in the phase diagram.
- The first and last segment are associated to $J_{min} < 0$..
- The third segment begin with a positive acceleration a_2.

The first trajectory segment is entirely defined by the initial conditions a_i and v_i and the duration t_1 using the equations (3). The parabola associated to the second segment cross the abscissa axis at v_1' and the one associated to the third segment at v_f' with:

$$v_1' = v_1 - \frac{{a_1}^2}{2J_{max}} \qquad v_f' = v_f - \frac{{a_f}^2}{2J_{min}}$$

And the acceleration at the intersection point of the two parabolas is a_2 defined by:

$$a_2 = \sqrt{\frac{2(v_1' - v_f') \times J_{max} \times J_{min}}{J_{max} - J_{min}}}$$

We choose the positive solution as defined in the hypothesis and compute the durations t_2 and t_3 of the two last segments.

$$t_2 = \frac{(a_2 - a_1)}{J_{max}} \qquad t_3 = \frac{(a_f - a_2)}{J_{min}}$$

From the equations (4) and (5) and using an algebraic calculator, the values of x_f and $t_f = t_1+t_2+t_3$ can be easily computed. Then the zeros of the derivative of the x_f function with respect to t_1 would define the points of discontinuity of the optimal curve C_{opt}. Unfortunately, we cannot manage to obtain this points in an usable form.

APPENDIX B
SOLVING (AJA), (JJA) AND (AJJ)

Solving the (AJA) problem

In this case the jerk J_a of the first and last segments is null in the equations (3) and (5) that respectively become (6) and (7):

$$\begin{aligned} a_1 &= a_i \\ v_1 &= a_i \times t_1 + v_i \\ x_1 &= a_i \times {t_1}^2/2 + v_i \times t_1 \end{aligned} \tag{6}$$

$$\begin{aligned} a_f &= a_2 \\ v_f &= a_2 \times t_3 + v_2 \\ x_f &= a_2 \times {t_3}^2/2 + v_2 \times t_3 + x_2 \end{aligned} \tag{7}$$

Using an algebra system, we obtain the results:

$$\begin{aligned} k_1 &= a_i - a_f \\ k_2 &= {k_1}^{-1} \\ k_3 &= a_i \times ((-J_a \times v_i) + {a_f}^2/2 + a_i \times (-a_f + a_i/2)) \\ k_4 &= a_f \times J_a \times v_i \\ k_5 &= 12 \times {J_a}^2 \times {v_i}^2 \\ k_6 &= a_f - a_i \\ k_7 &= (3 \times (12 \times a_i \times {J_a}^2 \times (2 \times a_f \times k_6 \times x_f + k_1 \times {v_f}^2) \\ &\quad + a_f \times (a_i \times (a_i \times (4 \times {a_f}^3 + a_i \times (a_i \times (4 \times a_f - a_i) \\ &\quad - 6 \times {a_f}^2)) - (k_5 + a_f^4)) + 12 \times a_f \times {J_a}^2 \times {v_i}^2)))^{0.5} \\ k_8 &= k_6 \times {J_a}^{-1} \\ k_9 &= k_1 \times a_f \\ k_{10} &= 2 \times J_a \times v_f \\ k_{11} &= 1/\sqrt{3} \\ k_{12} &= (k_6 \times (12 \times a_i \times {J_a}^2 \times (2 \times a_f \times x_f - {v_f}^2) \\ &\quad + a_f \times (k_5 + a_i \times (a_i \times (3 \times {a_f}^2 + a_i \times (a_i - 3 \times a_f)) \\ &\quad - {a_f}^3))))^{0.5} \end{aligned}$$

And finally the system has two solutions (t_1, t_2, t_3) and (t'_1, t_2, t'_3) with:

$$\begin{aligned} t_1 &= {a_i}^{-1} \times k_2 \times {J_a}^{-1} \times ((-k_7/6) + k_4 + k_3) \\ t_2 &= k_8 \\ t_3 &= ({a_f}^{-1} \times {J_a}^{-1} \times (k_{11} \times k_2 \times k_{12} + k_{10} + k_9))/2 \\ t'_1 &= {a_i}^{-1} \times k_2 \times {J_a}^{-1} \times (k_7/6 + k_4 + k_3) \\ t'_3 &= {a_f}^{-1} \times {J_a}^{-1} \times ((k_{10} + k_9)/2 - (k_{11} \times k_2 \times k_{12})/2) \end{aligned}$$

Solving the (JJA) problem

The system of polynomial equations to solve is defined by (6), (4) and (5). Using an algebra system, we obtain the following result where r_i is one solution of the quartic equation defined by the coefficients c_i with $0 \leq i \leq 4$:

$$\begin{aligned} k_1 &= 1 - 2 \times a_f \\ k_2 &= 2 \times a_f \\ k_3 &= {a_f}^2 \times J_b + k_1 \times J_a \\ c_4 &= 3 \times {a_f}^4 \times {J_b}^4 \\ &\quad + J_a \times (2 \times k_1 \times {a_f}^2 \times J_b + (4 \times (a_f - 1) \times a_f + 1) \times J_a)) \\ c_3 &= 4 \times a_f \times J_b \times ({a_f}^3 \times {J_b}^2 \\ &\quad + J_a \times ((3 \times a_f - 1) \times J_a - 3 \times {a_f}^2 \times J_b)) \\ c_2 &= 6 \times J_b \times (2 \times J_a \times k_3 \times v_i + ((k_2 - 1) \times J_a - {a_f}^2 \times J_b) \times {a_i}^2) \\ c_1 &= 12 \times a_f \times (2 \times J_a \times (a_f \times J_b - J_a) \times v_i + (J_a - a_f \times J_b) \times {a_i}^2) \\ c_0 &= 3 \times (4 \times {J_a}^2 \times {v_i}^2 + a_i \times (4 \times J_a \times (k_2 - a_i) \times v_i + {a_i}^3)) \\ &\quad + 4 \times (3 \times {J_a}^2 \times (2 \times a_f \times x_f - {v_f}^2) - 2 \times a_f \times {a_i}^3) \\ t1 &= -{J_a}^{-1} \times (a_i + a_f \times J_b \times r_i) \\ t2 &= r_i \\ t3 &= -\frac{Ja^{-1} \times (2 \times Ja \times v_i - ({a_i}^2 + 2 \times Ja \times v_f) + J_b \times k_3 \times {r_i}^2)}{2 \times a_f} \end{aligned}$$

Solving the (AJJ) problem

An (AJJ) system can be solved similarly as a (JJA) one or using a symmetry with respect to the acceleration axis to build an equivalent (JJA) system.

APPENDIX C
DERIVATIVE OF THE CUBIC POLYNOMIAL FUNCTIONS

The derivatives of the functions x_f, v_f and a_f relatively to the three times t_1, t_2 and t_3 can be grouped in a matrix. The newton method uses the inverse of this matrix to compute the times increments that, at the first oder, compensate the errors of the functions.

$$\begin{pmatrix} \frac{\partial x_f}{\partial t_1} & \frac{\partial x_f}{\partial t_2} & \frac{\partial x_f}{\partial t_3} \\ \frac{\partial v_f}{\partial t_1} & \frac{\partial v_f}{\partial t_2} & \frac{\partial v_f}{\partial t_3} \\ \frac{\partial a_f}{\partial t_1} & \frac{\partial a_f}{\partial t_2} & \frac{\partial a_f}{\partial t_3} \end{pmatrix} = \begin{pmatrix} lx_1 & lv_1 & la_1 \\ lx_2 & lv_2 & la_2 \\ lx_3 & lv_3 & la_3 \end{pmatrix}^{-1}$$

In the (JJJ) case defined by the equations (3), (4) (5), using an algebraic calculator we obtain:

$$\begin{aligned} k_1 &= J_a \times t_1 \\ k_2 &= J_b \times t_2 \\ k_3 &= J_a \times t_3 \\ k_4 &= a_i + k_3 + k_2 + k_1 \\ k_5 &= a_i + J_b \times (t_3 + t_2) + k_1 \\ k_6 &= J_a \times k_5 - J_b \times k_4 \\ k_7 &= {t_1}^2 \\ k_8 &= {t_2}^2 \end{aligned}$$

$$
\begin{aligned}
k_9 &= {t_3}^2 \\
k_{10} &= (J_a \times (k_9 + k_8 + k_7))/2 \\
k_{11} &= t_1 \times a_i \\
k_{12} &= t_2 \times (a_i + k_1) \\
k_{13} &= t_3 \times (a_i + J_a \times (t_2 + t_1)) \\
k_{14} &= v_i + k_{13} + k_{12} + k_{11} + k_{10} \\
k_{15} &= a_i + J_a \times (t_3 + t_2 + t_1) \\
k_{16} &= J_b \times k_{15} - J_a \times k_5 \\
k_{17} &= J_a \times k_7 \\
k_{18} &= J_a \times k_9 + J_b \times k_8 + k_{17} \\
k_{19} &= t_3 \times (a_i + k_2 + k_1) \\
k_{20} &= v_i + k_{19} + k_{12} + k_{11} + k_{18}/2 \\
k_{21} &= (-k_{15}) + a_i + k_3 + k_2 + k_1 \\
k_{22} &= v_i + k_{19} + k_{12} + k_{11} + (J_b \times (k_9 + k_8) + k_{17})/2 \\
k_{23} &= 1/(J_a \times k_{21} \times k_{22} + k_{16} \times k_{20} + k_6 \times k_{14}) \\
lx_1 &= k_6 \times k_{23} \\
lv_1 &= (J_b \times k_{20} - J_a \times k_{22}) \times k_{23} \\
la_1 &= (k_4 \times k_{22} - k_5 \times k_{20}) \times k_{23}] \\
lx_2 &= J_a \times k_{21} \times k_{23} \\
lv_2 &= J_a \times (-(k_{19} + k_{12} + k_{11} - (-k_{18}/2)) \\
&\quad + k_{13} + k_{12} + k_{11} + k_{10}) \times k_{23} \\
la_2 &= (k_{15} \times k_{20} - k_4 \times k_{14}) \times k_{23}] \\
lx_3 &= k_{16} \times k_{23} \\
lv_3 &= (J_a \times k_{22} - J_b \times k_{14}) \times k_{23} \\
la_3 &= (k_5 \times k_{14} - k_{15} \times k_{22}) \times k_{23}]))
\end{aligned}
$$

The computation is done similarly in the case of (AJA), (AJJ) and (JJA) sequences.

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