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# Input / Output Stability of a Damped String Equation coupled with Ordinary Differential System\*

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## Abstract

The input/output stability of an interconnected system composed of an ordinary differential equation and a damped string equation is studied. Issued from the literature on time-delay systems, an exact stability result is firstly derived using pole locations. Then, based on the Small-Gain theorem and on the Quadratic Separation framework, some robust stability criteria are provided. The latter follows from a projection of the infinite dimensional state on an orthogonal basis of Legendre polynomials. Numerical examples comparing these results with the ones in the literature are proposed and a comparison of its efficiency is made.

*Keywords:* String equation; Frequency approach; Robust stability; Coupled ODE/PDE; Quadratic Separation.

## 1 Introduction

The stability analysis of cyberphysical systems is a recent research area. The particular interconnection between a finite and an infinite dimension systems has a vast literature about it. This paper falls within the class which studies the interconnection between an ordinary differential equation (ODE) and a partial differential equation (PDE). We consider here only the string equation, which falls into the one dimension hyperbolic systems. Most of the work about the stability of this class of PDE has been conducted considering a Lyapunov functional. The main techniques are classified into two categories. Of the first kind, there is a backstepping methodology for boundary controlled PDEs. This approach has been used by many researchers and synthesized by [1, 2]. The other method relies on semi-group theory and has been widely studied in [3, 4, 5] and uses an energy argument to conclude.

These methods discuss the asymptotic/exponential stability of the coupled system. But the PDEs like transport and string are closely related to Time-Delay Systems (TDS). Another approach considered in the study of TDS is the input/output stability. The book by [6] explains its fundamentals. It relies on the derivation of a transfer function and the study of the characteristic equation to assess its input/output stability. For the wave equation, some works have been done by [7, 8, 9] for instance but they do not consider a coupled ODE/PDE system.

Based on the comparison between TDS and string equation, we have selected different tools to analyze this problem. Once the transfer functions are obtained, an exact stability criterion similar to the one developed in [10] can be stated. It is based on the poles location of a neutral system. Even if this approach gives an exact result, it is not robust and cannot be extended to synthesis purposes. Another complementary approach is then to use robust stability analysis tools as for example in [11] which considers

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an event-trigger control. In the same direction, the Small-Gain theorem ([12]) and Quadratic Separation ([13, 14]) provide efficient frameworks to estimate the stability area of this system.

The outline of the paper is as follow. Section 2 states the problem and gives some explanations about its important parameters. Section 3 deals with the input/output stability of the coupled system, its transfer function is derived, some properties are deduced. Based on this, an exact stability criterion is developed in Section 4. Section 5 is dedicated to two robust tools used in the stability analysis. More particularly, the Small-Gain theorem provides a very conservative but simple stability test while Quadratic Separation is developed in two stages: a first and then an extended analysis using projections of the infinite dimensional state. Finally, Section 6 presents examples and comparison with other techniques.

**Notations:** The following notations are used:  $\mathbb{R}^+ = [0, +\infty)$ ,  $\mathbb{C}^+ = \{z \in \mathbb{C} / \Re(z) \geq 0, s \neq 0\}$ ,  $L^2 = L^2([0, 1]; \mathbb{R})$ ,  $H^n = \{z \in L^2; \forall m \leq n, \frac{\partial^m z}{\partial x^m} \in L^2\}$  and  $i$  is the imaginary number.  $L^2$  is equipped with the norm  $\|z\|^2 = \int_0^1 |z(x)|^2 dx = \langle z, z \rangle$ . For  $u \in H^1$ , the notation  $u_x$  stands for  $\frac{\partial u}{\partial x}$ . For any square matrices  $A$  and  $B$ , let  $\text{diag}(A, B) = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ . A symmetric positive definite matrix  $P \in \mathbb{S}_+^n \subset \mathbb{R}^{n \times n}$  is written  $P \succ 0$ . For a matrix  $A \in \mathbb{R}^{m \times p}$ , then  $A(1 : N, :)$  ( $A(:, 1 : N)$ ) is the sub-matrix of  $A$  with the  $N$  first lines (columns respectively).  $A^\perp$  is the nullspace of  $A$  and  $*$  denotes the transconjugate.  $0_{m,p}$  and  $1_{m,p}$  are respectively the null and full of ones matrices with  $m$  lines and  $p$  rows and  $0_m = 0_{m,m}$ ,  $1_m = 1_{m,m}$ .

## 2 Problem Statement

### 2.1 Problem data

The input/output stability of the following interconnected system is studied.

$$\dot{X}(t) = AX(t) + B(u(1, t) + r(t)), \quad t \geq 0, \quad (1a)$$

$$u_{tt}(x, t) = c^2 u_{xx}(x, t), \quad x \in [0, 1], t \geq 0, \quad (1b)$$

$$u(0, t) = KX(t), \quad t \geq 0, \quad (1c)$$

$$u_x(1, t) = -c_0 u_t(1, t), \quad t \geq 0, \quad (1d)$$

$$u(x, 0) = u_0(x), \quad x \in [0, 1], \quad (1e)$$

$$u_t(x, 0) = v_0(x), \quad x \in [0, 1], \quad (1f)$$

$$X(0) = X_0, \quad (1g)$$

with the initial conditions  $X_0 \in \mathbb{R}^n$  and  $(u_0, v_0) \in H^2 \times H^1$  such that equations (1c) and (1d) are respected.  $(X_0, u_0, v_0)$  are compatibles with the boundary conditions.  $r$  is the input function.

This system is the interconnection of a linear time invariant system with a string equation. They are connected through a Dirichlet boundary condition, that means the output and the input are the state  $u$  at different positions and not its time derivative.  $u$  is the amplitude of the wave, it belongs to a functional space and consequently is of infinite dimension. We assume that the state  $u(x, t)$  belongs to  $\mathbb{R}$  but the analysis can be extended without difficulty to  $\mathbb{R}^m$ ,  $m > 1$ .

For the system to be well-posed, another boundary condition must be added. The choice there has been to use a boundary damping with equation (1d). Indeed, it has been shown in [15] for example that this condition ensures the stability of the wave equation itself if  $c_0 > 0$ , the case  $c_0 = 0$  suppresses the damping.

The wave equation operator associated with these boundary conditions is known to be diagonalizable (for more information on semi-group definitions and properties, the reader can refer to [16, 3] and references therein). Once diagonalized, it can be expressed as the composition of two transport equations, one going forward and another backward. The boundary condition (1d) implies a reflection of the forward wave with a coefficient  $\alpha = \frac{1-cc_0}{1+cc_0}$  as explained in [17]. Enforcing  $c_0 > 0$  implies  $|\alpha| < 1$  and the energy of the wave is then decreasing. These dynamics are the key part to understand the results coming from Quadratic Separation at the end of this paper.

The purpose of this paper is to study the input/output stability of this coupled system. The definition of input/output stability for infinite dimensional system is given in Definition 9.1.1 by [12] and can be formulated as follows:

**Definition 1** System (1) is said to be **input/output stable** if for all bounded input  $r$ , the output  $Y = KX$  is also bounded.

## 2.2 Existence and Uniqueness of the solution

Before going further, the existence of a solution to system (1) with  $r = 0$  is proven. To do so, a step by step procedure is proposed. To ease the reading, the following sets are defined:

$$\begin{aligned}\mathbb{I}_n &= [nc^{-1}, (n+1)c^{-1}], \quad n \in \mathbb{N}, \\ \mathbb{H} &= \{(X, u, v) \in \mathbb{R}^n \times H^2 \times H^1\}, \\ \mathbb{D} &= \{(X, u, v) \in \mathbb{H} \text{ s.t. } u(0) = KX, u_x(1) = -c_0v(1)\},\end{aligned}$$

and using the same technique than [17] or [18] for  $(X_0, u_0, v_0) \in \mathbb{D}$ , a solution for  $t \in \mathbb{I}_0$  is:

$$u(1, t) = u_0(1 - ct) + \frac{\alpha}{2} (u_0(1 - tc) + u_0(1)) + \frac{1 + \alpha}{2} \int_{1-ct}^1 v_0(s) ds.$$

On  $\mathbb{I}_0$ , we notice that  $u(1, \cdot) \in H^2(\mathbb{I}_0)$  depends only on the initial conditions. Then,  $X$  on  $I_0$  is the solution of differential equation (1a), with initial condition  $X(0) = X_0$ . It is well known that the solution is :

$$\forall t \in \mathbb{I}_0, \quad X(t) = e^{At} X_0 + \int_0^t e^{A(t-s)} B u(1, s) ds.$$

Then,  $X \in C^1(\mathbb{I}_0)$  and consequently,  $u(0, \cdot)$  has the same regularity property. On the domain  $\Gamma_0 = \mathbb{I}_0 \times [0, 1]$ , with the boundary conditions defined previously and using the formulas of [17], equation (1b) has a unique solution  $(x, t) \in \Gamma_0 \mapsto u(x, t) \in H^2$  and for  $t \in \mathbb{I}_0$ ,  $u_t(\cdot, t) \in H^1$ . With all these considerations, it is possible to find  $(X(c^{-1}), u(\cdot, c^{-1}), u_t(\cdot, c^{-1})) \in \mathbb{D}$  and then repeat the same procedure for  $\mathbb{I}_1$  leading to the existence of a solution. We then get  $(X, u, u_t) \in L^2(0, +\infty, \mathbb{H})$ . Moreover,  $u, u_t, u_x, u_{tt}$  and  $u_{xx}$  are  $L^2$  on each compact set of  $\mathbb{R}^+$ . As the system is linear, energy consideration leads to the uniqueness property.

**Remark 1** *The set of solutions  $\mathbb{H}$  is smaller than the one in [19]. The main difference comes from considering the strong solution here while the weak solution will belong to the same space than in [19].*

## 3 Input / Output Analysis

One possible way to study the input/output stability of system (1) is to use Laplace transform and study the characteristic equation. This approach has been already used in [20, 21].

### 3.1 Laplace transform of the wave equation

The notion of Laplace transform extends easily from its traditional definition for finite dimension systems as noticed in [20]. The variables just need to be  $L^2$  on each compact set of  $\mathbb{R}^+$ , which has been ensured previously. As the transient response is not considered, we assume the initial conditions to be zero, i.e.  $u^0 = v^0 = 0$ . In the Laplace domain, equation (1b) can then be transformed into:

$$\forall x \in [0, 1], \quad s^2 U_{xx}(x, s) = c^2 U(x, s),$$

with  $s \in \mathbb{C}$  the Laplace variable and  $U(x, \cdot)$  the Laplace transform of  $u(x, \cdot)$ . The solutions are for  $x \in [0, 1]$ :

$$U(x, s) = C_1(s) \exp\left(\frac{s}{c}x\right) + C_2(s) \exp\left(-\frac{s}{c}x\right),$$

where  $C_1$  and  $C_2$  are space-independent transfer functions to be determined using the boundary conditions. Once calculated using equations (1c) and (1d), the transfer function for the string equation is:

$$\mathcal{W}(x, s) = \frac{U(x, s)}{U(0, s)} = \frac{e^{-\frac{s}{c}x} + \alpha e^{\frac{s}{c}(x-2)}}{1 + \alpha e^{-2\frac{s}{c}}}, \quad (2)$$

with  $\alpha = \frac{1-cc_0}{1+cc_0}$ ,  $x \in [0, 1]$  and  $s \in \mathbb{C}$ . The transfer function from  $U(0, s)$  to  $U(1, s)$  is then:

$$\mathcal{W}(s) = \frac{U(1, s)}{U(0, s)} = \frac{2e^{-s/c}}{1 + cc_0 + (1 - cc_0)e^{-2s/c}}.$$

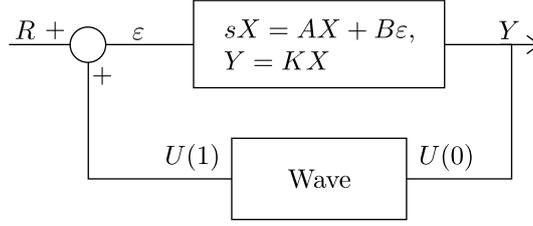


Figure 1: Bloc diagram for the system described by equation (1).

There are infinitely many poles of  $\mathcal{W}(x, s)$  for  $x \in [0, 1]$  but they are independent of  $x$  with a real part of  $\frac{c}{2} \log |\alpha|$  which is strictly negative if  $cc_0 \neq 1$  and  $c_0 > 0$ .

**Remark 2** *If  $c_0 = 0$ ,  $\mathcal{W}$  has infinitely many poles on the imaginary axis. Then Corollary 9.1.4 from [12] applies and there does not exist any finite-dimensional controller which exponentially stabilizes the wave equation. That is why only the case  $c_0 > 0$  is considered.*

### 3.2 Transfer function for the coupled system

Considering the bloc diagram in Figure 1, the transfer function of the finite dimensional system is:

$$\mathcal{H}(s) = \frac{Y(s)}{U(1, s) + R(s)} = K(sI - A)^{-1}B = \frac{N(s)}{D(s)},$$

where  $N$  and  $D$  are polynomials in  $s$  and  $R$  being the reference.

Using transfer function  $\mathcal{W}$  developed previously, we obtain the following closed-loop system:

$$\mathcal{F}(s) = \frac{Y(s)}{R(s)} = \frac{\mathcal{H}(s)}{1 - \mathcal{H}(s)\mathcal{W}(s)} = \frac{N(s)(1 + \alpha e^{-2s/c})}{(1 + cc_0)c_{eq}(s)}, \quad (3)$$

with

$$c_{eq}(s, \tau) = \left(1 + cc_0 + (1 - cc_0)e^{-2s/c}\right) D(s) - 2N(s)e^{-s/c}. \quad (4)$$

### 3.3 Input / Output stability

From an input/output approach, system (1) appears to be a neutral system. Indeed, the transfer function  $F$  derived earlier is the one of a neutral functional differential equation (NFDE) (see [22, 23] for a complete treatment of such systems). The formal definition is reminded there.

**Definition 2** *An NFDE is defined as follow:*

$$\frac{d}{dt}D(\dot{x}, F, \tau)(t) = Ax(t) + Bx(t - \tau), \quad (5)$$

with  $D(x, F, \tau)(t) = x(t) - Fx(t - \tau)$ ,  $A, B$  and  $F$  of appropriate dimension.

The following proposition gives a state-space representation of system (1) and then clearly shows the NFDE behavior of system (1).

**Proposition 1** *System (1) is a neutral system if  $\alpha \neq 0$ . If  $\alpha = 0$ , system (1) is a time-delay system.*

**Proof :** To prove this assertion, a state space representation is built. Transfer function (3) leads to the following coupled system:  $sX(s) = AX(s) + B(R(s) + \mathcal{W}(s)KX(s))$ . The inverse Laplace transform can be applied and gives:

$$\dot{X}(t) = AX(t) + \frac{2}{1 + cc_0}BKX(t - c^{-1}) + \alpha AX(t - 2c^{-1}) - \alpha \dot{X}(t - 2c^{-1}) + Br(t) + \alpha Br(t - c^{-1}), \quad (6)$$

with the difference operator is  $D(X, -\alpha, 2c^{-1})$ . □

When it comes to deal with NFDE, contrary to functional differential equation of retarded type, studying the pole location is not enough to guaranty the input/output stability. One important property is then the  $\tau$ -stabilization. This property ensures that the pole location becomes a sufficient condition for stability (see [24, 25] for more information).

**Definition 3** *An NFDE system is said to be  $\tau$ -stabilizable if the operator  $D$  is stable.*

This definition comes from [26, 27]. According to Theorem 12.5.1 from [26], a sufficient condition to assess the  $\tau$ -stabilization of NFDE (5) is that the spectral radius of  $F$  is strictly lower than 1. This condition becomes necessary if  $F$  is scalar (Corollary 12.5.1 from [26]). The  $\tau$ -stability of system (1) is assessed using the proof of Proposition 1 and we directly get the following proposition.

**Proposition 2** *System (1) is neutral and  $\tau$ -stabilizable if and only if  $c_0 > 0$  and  $cc_0 \neq 1$ .*

Finally, since the system is  $\tau$ -stabilizable under a given condition, the input/output stability can be discussed. The following corollary summarizes all this subsection.

**Corollary 1** *If  $(X_0, u_0, v_0) \in \mathbb{D}$  and all the poles of  $\mathcal{F}$  as defined in equation (3) are in the left side of the complex plane, then system (1) is input/output stable. The equilibrium points  $(X_e, u_e, v_e) \in \mathbb{H}$  are such that  $(A + BK)X_e = 0$ ,  $u_e = KX_e$  and  $v_e = 0$ .*

*Moreover, if the system  $\Sigma(A, B, K)$  is stabilizable and detectable such that  $A + BK$  is not singular, then system (1) is asymptotically stable.*

**Proof :** If system (1) is  $\tau$ -stabilizable, then it is input/output stable if all the poles of  $F$  are with a strictly negative real part (Theorem 9.9.1 from [25]). The link between asymptotic and input/output stability comes from the poles/zeros cancellation. If the system  $\Sigma(A, B, K)$  is stabilizable and detectable, then, the simplification poles/zeros in  $\mathcal{F}$  are with negative real parts. It means that the non-observable or non-controllable parts of system (1) are stable. Then, the whole state is indeed converging. The study of equilibrium points has been done in Proposition 2 of [19] and if  $A + BK$  is not singular, the whole state  $X$  is converging toward zero and the system is asymptotically stable.  $\square$

**Remark 3** *Compared to the article by [19], the condition  $A + BK$  not singular is not required for input/output stability.*

Once some general properties about the solution have been stated, the stability analysis can be pursued. We consider two main approaches, one using pole location and another with robust stability tools.

## 4 Stability using Poles Location

Once the characteristic equation is established, different methodologies can be applied to assert the input/output stability of system (1), see for example [28]. In this part, we focus on an exact treatment of this problem coming from the time-delay system analysis.

In this subsection, a pole location argument is used to assess the stability. We study the stability of the system described by transfer function  $\mathcal{F}$  using a Cluster Treatment of Characteristic Roots (CTCR) methodology originally proposed in [10]. Let  $\tau = c^{-1}$  and regrouping terms by their delay dependence, we get:

$$c_{eq}(s, \tau) = a_0(s) + a_1(s)e^{-\tau s} + a_2(s)e^{-2\tau s}, \quad (7)$$

with  $a_0(s) = \left(1 + \frac{co}{\tau}\right) D(s)$ ,  $a_1(s) = -2N(s)$  and  $a_2(s) = \left(1 - \frac{co}{\tau}\right) D(s)$ .

The idea behind CTCR is to find the number of unstable poles at  $\tau = 0$ , and then, by increasing slightly the delay, to detect the poles crossing of the imaginary axis and in which direction this crossing occurs. Then, for any delay  $\tau$ , the number of unstable poles can be counted and the intervals with no unstable poles are said to be input/output stable.

Moreover, the classical CTCR algorithm can be used directly on neutral systems, as shown in [27], as long as the system is  $\tau$ -stabilizable. Thanks to Proposition 2 related to invariance of root crossings over the imaginary axis, the CTCR paradigm can then be applied. This methodology is briefly described to point out the main difference introduced while studying system (1), but for more information, the reader may

refer to the articles cited afterwards. Following the process described in [10, 27], the Rekasius substitution is defined:

$$e^{-\tau s} = \frac{1 - Ts}{1 + Ts}, \quad \tau \in \mathbb{R}^+, \quad T \in \mathbb{R}, \quad s = i\omega, \quad (8)$$

which is an *exact* substitution of exponential terms when  $s = i\omega$ . This substitution is different than Padé approximation since in general  $T \neq \tau/2$ . Next, substituting (8) in the characteristic equation (4), we get

$c_{eq}^*(s, T) = \left(1 + \frac{2c_0}{\tau}Ts + T^2s^2\right)D(s) - N(s)(1 - T^2s^2)$ . We can then build  $\{b_k(T)\}_{k \in (0, n+2)}$ , such that:

$$c_{eq}^*(s, T) = \sum_{k=0}^{n+2} b_k(T, \tau) s^k. \quad (9)$$

Last but not least, the presence of the delay as a coefficient in (9) may not, in general, allow the application of Proposition 2 in [10]. To remove this constraint, we consider the following manipulation: as the delay  $\tau$  is always divided by  $c_0$  in  $b_k$ , if one defines a new positive variable  $c_1 = c_0\tau^{-1}$ , then  $b_k$  depends only on  $c_1$  and not on  $\tau$  anymore. Working at a given strictly positive  $c_1$  removes the dependence of  $b_k$  in  $\tau$ . Since they are now independent, the methodology applies. This manipulation shows that  $c_1$  is a variable of interest when it comes to studying boundary damped waves.

Using  $b_k$  and  $a_k$  as defined in equations (7) and (9), the CTCR methodology gives the boundaries of the input-output stability area for system (1). This method is summarized as follow for a given  $c_1 = c_0\tau^{-1}$ :

1. Using  $c_{eq}^*$ , find the roots  $s = i\omega$  depending on  $T \in \mathbb{R}$ . There are only a finite number of such solutions (Proposition 1 of [10]) ;
2. For each  $T \in \mathbb{R}$  obtained previously, we already have the points  $\omega$  on the imaginary axis where the poles may cross. For each of these points, there is a root tendency number indicating the direction of crossing (Proposition 2 of [10]).
3. Then, using the inverse transformation of Rekasius transformation, it is possible to find all the delays ( $\tau_\ell$ ) corresponding to each pair of  $(T, \omega)$  between 0 and a given delay  $\tau_{max}$  for which there is a crossing. Sorting them in ascending order, and starting with the number of unstable roots for  $\tau = 0$ , the number of unstable roots for a delay  $\tau < \tau_{max}$  can be calculated.
4. The stability areas of system (1) for a given  $c_1 = c_0\tau^{-1}$  are when there is no unstable pole.  $\tau$  is then known and  $c_0$  can be recovered:  $c_0 = c_1\tau$ .

The CTCR methodology presented above provides the general framework for an exact stability result. However, for degenerate cases, care must be paid[29]. These cases, for a slightly different  $c_1$  will however not arise.

We point out that the coefficients  $b_k$  depend linearly on  $c_1$  and then the roots of  $c_{eq}^*(\cdot, T)$  vary continuously subject to  $c_1$ . In other words, the delays  $\tau$  for which there is a crossing vary continuously relative to  $c_1$ . The border of each stability area on the map  $(c_1, \tau)$  is consequently continuous. This is one of the arguments for considering this mapping  $(c_1, \tau)$  of interest and not the natural  $(c_0, c)$ .

## 5 Robust Stability Analysis

The previous result is exact but it does not provide efficient tools when it comes to deal with robustness issues with respect to an uncertainty on  $A$  or  $c$  for example. It is also problem-specific and cannot be extended easily to other class of systems. That is why tools coming from the robust analysis are also considered. In [19], we built a Lyapunov functional to ensure the exponential stability. The main drawback was an important number of decision variables making the treatment long for large systems. This article aims at providing analysis tools with similar result but of lower complexity.

The studied system is an interconnection between two subsystems. The stability of the interconnection is stated under some conditions on each subsystem. Here, two tools coming from the robust analysis are considered: Small Gain theorem and Quadratic Separation.

## 5.1 Small-Gain Theorem

We consider here the bloc diagram of Figure 1 where the wave equation is seen as a disturbance. The transfer functions of the disturbance and the plant are defined in Section 3. The following stability criteria comes from a direct application of the Small-Gain Theorem.

**Theorem 1** *Let system (1) with  $A$  Hurwitz,  $\|\mathcal{H}\|_\infty < 1$  and  $c_0 > 0$ . The system is input-output stable for  $(c, c_0)$  if the following condition is satisfied:*

$$\|\mathcal{H}\|_\infty = \max_{w \in \mathbb{R}^+} |\mathcal{H}(iw)| < cc_0.$$

**Proof :** Beforehand, the infinity norm of the disturbance is computed. Some calculations lead to:  $\|\mathcal{W}\|_\infty = \frac{2}{(1+cc_0) \min_{\omega \geq 0} |1 + \alpha e^{-2i\omega/c}|}$ . In order to compute this norm, let note that the function  $\omega \mapsto 1 + \alpha e^{-2i\tau\omega}$  draws the circle centered at 1 and of radius  $|\alpha| < 1$ . Therefore, the minimum can be calculated as follow  $\min_{\omega \geq 0} |1 + \alpha e^{-2i\omega/c}| = \frac{2}{1+cc_0} \min\{1, cc_0\}$ . The infinity norm is then  $\|\mathcal{W}\|_\infty = \max((cc_0)^{-1}, 1)$ . Using the previous analysis and the small gain theorem for infinite dimensional systems as stated in Theorem 9.1.7 by [12], the interconnected system is asymptotically stable if  $\|\mathcal{H}\|_\infty \|\mathcal{W}\|_\infty < 1$  and each subsystem is stable. The second condition is ensured if  $\|\mathcal{H}\|_\infty < 1$  and  $c_0 > 0$ . Considering the case  $cc_0 \leq 1$  leads to  $\|\mathcal{W}\|_\infty = (cc_0)^{-1}$ , it follows that robust stability is ensured. If  $cc_0 > 1$ , then the stability is ensured if  $\|\mathcal{H}\|_\infty < 1$ , which is true by assumptions.  $\square$

This theorem is quite conservative, mostly because the Small-Gain Theorem provides only a sufficient condition. However the stability test is simple and some properties can be deduced.

**Proposition 3** *If  $\|\mathcal{H}\|_\infty < 1$ , then there exists a function  $c_0 \mapsto c_{min}(c_0)$  where  $c_{min}(c_0) \leq \|\mathcal{H}\|_\infty c_0^{-1}$ . That leads to three properties:*

1. *Since  $\|\mathcal{H}\|_\infty < 1$ ,  $A + BK$  is stable,*
2. *For a given  $c_0 > 0$ , system (1) is stable for all  $c \geq c_{min}(c_0)$ ,*
3.  $\lim_{c_0 \rightarrow +\infty} c_{min}(c_0) = 0$ .

To reduce the conservatism introduced in this subpart, another framework based on the same idea is proposed.

## 5.2 Quadratic Separation - Preliminary result

The Small-Gain Theorem ensures the exponential stability of an interconnected system composed of a disturbance a plant both stable. To decrease the conservatism and consider a broader class of interconnected systems, the Quadratic Separation (QS) framework can be used. It has been originally proposed in [30] and it studies the well-posedness of a closed-loop system made up of an unknown disturbance and a plant.

We describe the methodology of QS to provide a preliminary result on system (1) and then we extend this stability analysis to a more general case.

QS states the well-posedness of a generic system described in Figure 2.  $\nabla$  is called the uncertainty matrix and belongs to a set  $\mathbf{W}$ . The well-posedness is defined as follow:

**Definition 4** *The interconnected system described on Figure 2 is well-posed with respect to the norm  $\|\cdot\|$  if*

$$\exists \gamma > 0, \forall \nabla \in \mathbf{W}, \forall \bar{\omega}, \bar{z}, \quad \left\| \begin{bmatrix} \omega \\ z \end{bmatrix} \right\| \leq \gamma \left\| \begin{bmatrix} \bar{\omega} \\ \bar{z} \end{bmatrix} \right\|,$$

$\bar{\omega}$  and  $\bar{z}$  are the references.

In our case here, we consider the disturbance to be an operator acting on two signals  $w$  and  $z$  in Laplace domain. In other words, the following system is obtained:

$$\begin{cases} \Omega(s) = \nabla(s)Z(s), \\ \mathcal{E}Z(s) = \mathcal{A}\Omega(s), \end{cases} \quad (10)$$

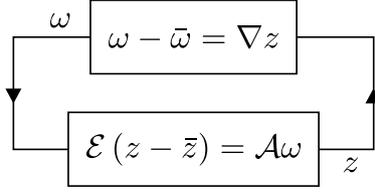


Figure 2: Feedback system for quadratic separation.  $\mathcal{A}$  and  $\mathcal{E}$  are matrices and  $\nabla \in \mathbb{W}$  is the uncertainty.  $\bar{\omega}$  and  $\bar{z}$  are the references.

with  $\mathcal{E}$  full column rank,  $Z$  and  $\Omega$  are the Laplace transformations of  $z$  and  $\omega$  respectively .

The well-posedness of system (10) is assessed in Theorem 1 and Corollary 2 from [13]. Following this formulation, the next theorem is stated.

**Theorem 2** *The system describes by Figure 2 and equation (10) is well-posed if and only if there exists a real matrix of appropriate dimension  $\Theta = \Theta^\top$  such that:*

$$\forall s \in \mathbb{C}^+, \begin{bmatrix} I \\ \nabla(s) \end{bmatrix}^* \Theta \begin{bmatrix} I \\ \nabla(s) \end{bmatrix} \preceq 0, \quad (11)$$

$$\begin{bmatrix} \mathcal{E} & -\mathcal{A} \end{bmatrix}^{\perp \top} \Theta \begin{bmatrix} \mathcal{E} & -\mathcal{A} \end{bmatrix}^{\perp} \succeq 0, \quad (12)$$

where  $\nabla^*$  is the trans-conjugate of  $\nabla$  and  $\mathcal{E}$  is full-rank.

The previous theorem has been adapted to our system considering  $\nabla$  is an operator depending only on the Laplace variable  $s$  such that  $\mathbb{W} = \{\nabla(s)/s \in \mathbb{C}^+\}$ . In this case, the well-posedness of system (10) implies its input/output stability. Indeed, the well-posedness of system (10) with  $\nabla \in \mathbb{W}$  ensures that there is a unique solution and  $\omega$  and  $z$  are bounded by their references  $\bar{\omega}$  and  $\bar{z}$ . Then, there is no pole with a strictly positive real part and consequently, the system is input/output stable.

Now, we can transform the block diagram in Figure 1 into a suitable form to apply Theorem 2. Consider the following signals:

$$z(t) = \begin{bmatrix} \dot{X}^\top(t) & KX(t) & KX(t-\tau) & K\dot{X}(t) \end{bmatrix}^\top, \quad \omega(t) = \begin{bmatrix} X^\top(t) & KX(t-\tau) & u(1,t) & K(X(t) - X(t-\tau)) \end{bmatrix}^\top$$

To get the equivalence between (10) and transfer function (3),  $\nabla$ ,  $\mathcal{A}$  and  $\mathcal{E}$  are defined as follow:

$$\begin{aligned} \forall s \in \mathbb{C}^+, \nabla(s) &= \text{diag} \left( s^{-1}I_n, e^{-\tau s}, \delta(s), \delta_0(s) = \frac{1-e^{-\tau s}}{s} \right), \\ \delta(s) &= \frac{1}{1 + c_0\tau^{-1} + (1 - c_0\tau^{-1})e^{-2\tau s}}, \\ \mathcal{A} &= \begin{bmatrix} A & 0_{n,1} & B & 0_{n,1} \\ K & 0 & 0 & 0 \\ 0_{1,n} & 1 & 0 & 0 \\ 0_{1,n} & 0 & 0 & 0 \\ 0_{1,n} & 0 & 0 & 1 \end{bmatrix}, \quad \mathcal{E} = \begin{bmatrix} I_n & 0_{n,1} & 0_{n,1} & 0_{n,1} \\ 0_{1,n} & 1 & 0 & 0 \\ 0_{1,n} & 0 & 1 & 0 \\ -K & 0 & 0 & 1 \\ 0_{1,n} & 1 & -1 & 0 \end{bmatrix}. \end{aligned} \quad (13)$$

Note that  $\mathcal{E}$  is full column rank. The uncertainty matrix  $\nabla$  is defined for all  $s$  with a strictly positive real part and then, the well-posedness of system (10) implies the input/output stability of system (1).

**Remark 4** *The disturbance  $\delta$  is related to the neutral part of system (1). The last disturbance component  $\delta_0$  is related to Jensen inequality (see for instance [6]), a widely used inequality in analysis of time-delay systems.*

We need to propose a structure for the real-valued separator  $\Theta$  such that inequality (11) always holds for  $\nabla \in \mathbb{W}$ . For  $\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{12}^\top & \Theta_{22} \end{bmatrix}$ , the following structure is proposed:

$$\Theta_{11} = \text{diag}(0_n, -Q, R(1 - \alpha^2)\gamma^2, -\tau^2 S), \quad \Theta_{12} = \text{diag}(-P, 0, -R\gamma, 0), \quad \Theta_{22} = \text{diag}(0_n, Q, R, S), \quad (14)$$

with  $P \in \mathbb{S}_+^n$  and  $Q, R, S \in \mathbb{R}^+$ . This assumption is not new and is used in the examples presented in [30, 13]. With this specific structure, for all  $s \in \mathbb{C}^+$ , the following holds:

$$\begin{bmatrix} I \\ \nabla(s) \end{bmatrix}^* \Theta \begin{bmatrix} I \\ \nabla(s) \end{bmatrix} = \text{diag} \left\{ -2P\Re(s^{-1}), Q(|e^{-\tau s}|^2 - 1), \right. \\ \left. R \left( (1 - \alpha^2)\gamma^2 - 2\gamma\Re(\delta(s)) + |\delta(s)|^2 \right), \right. \\ \left. S \left( |\delta_0(s)|^2 - \tau^2 \right) \right\}.$$

The third diagonal block can be written differently:  $\delta_{-1}(s) = R(|\delta(s) - \gamma|^2 - \alpha^2\gamma^2)$ . Equation (13) implies that  $\delta$  is a circle, its center is  $\gamma = \frac{1}{1-\alpha^2}$  and its radius is  $\frac{|\alpha|}{1-\alpha^2} = |\alpha|\gamma$ , guarantying that  $\delta_{-1}(s) \leq 0$ . Noticing also that  $\forall s \in \mathbb{C}^+$ ,  $|e^{-\tau s}| \leq 1$ ,  $\left| \frac{1-e^{-\tau s}}{s} \right| \leq \tau$ , we get inequality (11). All these considerations lead to the following stability theorem:

**Theorem 3** *If there exist  $P \in \mathbb{S}_+^n$  and  $Q, R, S \in \mathbb{R}^+$  such that LMI (12) holds for  $\Theta$  defined in (14), then system (1) is input-output stable.*

## 5.3 Quadratic Separation - Extended stability analysis

### 5.3.1 Motivations and main theorem

The studies of time-delay systems during the last few years focused on reducing the conservatism of stability theorems (see [14] for example). Indeed, it has been showed (in [31, 32]), that Jensen's inequality is a conservative result. The idea developed here is to enrich  $\Omega$  and  $Z$  wisely to improve Theorem 3. Following this idea, we aim at capturing the infinite dimensional behavior of this system, described in equation (2), by adding new signals in the framework.

Indeed, quadratic separation clearly shows that the addition of more information will lead to a smaller kernel of  $[\mathcal{E} \ -\mathcal{A}]$  and then an improvement of the stability criterion at a price of a more complex  $\mathcal{A}$ ,  $\mathcal{E}$  and  $\nabla$ .

Following the methodology described in [31], the new signals are projections of the infinite dimensional state  $u$ . This state is projected on the orthogonal basis of shifted Legendre polynomials  $\{\mathcal{L}_k\}_{k \in [0, N]}$ . Their useful properties are reminded in the sequel but for more information, the reader can refer to [33]. Using this strategy leads to the following theorem.

**Theorem 4** *For a given  $N \in \mathbb{N}$ , if there exist  $P_N \in \mathbb{S}_+^{n+N}$  and  $Q, R, S \in \mathbb{R}^+$  such that the LMI:*

$$\begin{bmatrix} \mathcal{E}_N & -\mathcal{A}_N \end{bmatrix}^\perp \begin{bmatrix} \Theta_{N,1} & \Theta_{N,2} \\ \Theta_{N,2}^\top & \Theta_{N,3} \end{bmatrix} \begin{bmatrix} \mathcal{E}_N & -\mathcal{A}_N \end{bmatrix}^\perp \succeq 0, \quad (15)$$

holds for

$$\Theta_{N,1} = \text{diag}(0_{n+N}, -Q, R(1 - \alpha^2)\gamma^2, -\tau^2 S), \quad \Theta_{N,2} = \text{diag}(-P_N, 0, -R\gamma, 0_{1, N+1}), \\ \Theta_{N,3} = \text{diag}(0_{n+N}, Q, R, S I_{N+1}),$$

	Lyapunov Functional	Quadratic Separation
	$\frac{n^2 + n}{2} + 2(N^2 + n) + 5N + Nn + 9$	$\frac{n^2 + n + N^2 + N}{2} + Nn + 3$
$N = 0$	27	13
$N = 2$	53	24
$N = 5$	122	48

Table 1: Number of variables for QS and LS for  $n = 4$  and an order  $N$ .

$$\mathcal{A}_N = \begin{bmatrix} A & 0_{n,N} & 0_{n,1} & B & 0_{n,N+1} \\ 0_{N,n} & 0_N & 0_{N,1} & 0_{N,1} & \tilde{I}_{N(1:N,:)} \\ K & 0_{1,N} & 0 & 0 & 0_{1,N+1} \\ 0_{1,n} & 0_{1,N} & 1 & 0 & 0_{1,N+1} \\ 0_{1,n} & 0_{1,N} & 0 & 0 & 0_{1,N+1} \\ 0_{N+1,n} & L_{N(:,1:N)} & 0_{N+1,1} & 0_{N+1,1} & \tilde{I}_N \end{bmatrix}, \mathcal{E}_N = \begin{bmatrix} I_n & 0_{n,N} & 0_{n,1} & 0_{n,1} & 0_{n,1} \\ 0_{N,n} & I_N & 0_{N,1} & 0_{N,1} & 0_{N,1} \\ 0_{1,n} & 0_{1,N} & 1 & 0 & 0 \\ 0_{1,n} & 0_{1,N} & 0 & 1 & 0 \\ -K & 0_{1,N} & 0 & 0 & 1 \\ 0_{N+1,n} & 0_{N+1,N} & 1_{N+1,1} & -1_{N+1} & 0_{N+1,1} \end{bmatrix},$$

with

$$\begin{aligned} \mathbb{1}_N &= [(-1)^0 \dots (-1)^k \dots (-1)^{N-1}]^\top, & \tilde{I}_N &= \text{diag} \left( \left\{ \frac{1}{\sqrt{2k+1}} \right\}_{k \in [0, N]} \right), \\ L_N &= [\ell_{ij}]_{i,j \in [0, N]}, & \ell_{ik} &= \begin{cases} 0, & \text{if } k \geq i, \\ (2k+1)(1 - (-1)^{k+i})c, & \text{otherwise,} \end{cases} \end{aligned} \quad (16)$$

then, system (1) is input/output stable.

**Remark 5** The case  $N = 0$  leads to Theorem 3. Theorem 4 also introduces a hierarchy of stability conditions. In other words, if system (1) is proven to be exponentially stable using LMI (15) for a given  $N = N_0$ , then for all  $N \geq N_0$ , LMI (15) also assesses the same stability.

Compared to classical stability analysis using the Lyapunov Stability (LS) obtained with the Lyapunov functional of [19], this method also uses an LMI solver. It usually results in LMIs with less decision variables than other techniques, which is critical for high dimension systems. Indeed, the difference between LS and quadratic separation in the number of variables is  $\frac{3}{2}(N^2 + N) + 2n + 6$ . This is due to the two dimensions state extension in LS. The double state extension leads to an increase of the number of variables and then to a slower computation. Table 1 shows that considering  $N = 2$  with LS has the same number of decision variables than considering quadratic separation at  $N = 5$ .

### 5.3.2 Proof of Theorem 4

Before going further, the tools used in the sequel are introduced. For the projections, the scalar product is the canonical inner product of the Hilbert space  $L^2([-\tau, 0], \mathbb{C})$  and is denoted:  $\langle f, g \rangle = \int_{-\tau}^0 f^*(\theta)g(\theta)d\theta$  for  $f, g \in L^2([-\tau, 0], \mathbb{C})$ .  $\|\cdot\|_2$  is the norm given by the previous inner-product. The shifted Legendre polynomials are defined as follows:

$$\mathcal{L}_k(u) = (-1)^k \sum_{l=0}^k (-1)^l \binom{k}{l} \binom{k+l}{l} \left( \frac{u+\tau}{\tau} \right)^l.$$

The Legendre polynomials basis has been chosen because of some interesting properties (see [31]). First, because the behavior of each Legendre polynomial at its boundaries 0 and  $-\tau$  is simple. Secondly, as it

is a polynomial basis, the following differentiation rule applies for  $N \in \mathbb{N}, x \in (-\tau, 0)$ :

$$\left[ \dot{\mathcal{L}}_0(x) \quad \cdots \quad \dot{\mathcal{L}}_N(x) \right]^\top = L_N \left[ \mathcal{L}_0(x) \quad \cdots \quad \mathcal{L}_N(x) \right]^\top.$$

Finally, the Legendre polynomials family is orthogonal with respect to the inner product  $\langle \cdot, \cdot \rangle$  and it is possible to use Bessel inequality. This inequality is the key part to build a separator and a version adapted to this problem is proposed below:

**Lemma 1** *Let  $\tau > 0$  and  $N \in \mathbb{N}$ , then the following inequality holds:*

$$\forall s \in \mathbb{C}^+, \quad \delta_N^*(s) \delta_N(s) \leq \tau^2,$$

$$\text{with } \delta_N(s) = \sqrt{\tau} \left[ \left\langle e^{\theta s}, \frac{\mathcal{L}_0(\theta)}{\|\mathcal{L}_0\|_2} \right\rangle \quad \cdots \quad \left\langle e^{\theta s}, \frac{\mathcal{L}_N(\theta)}{\|\mathcal{L}_N\|_2} \right\rangle \right]^\top.$$

**Proof :** Let  $s \in \mathbb{C}^+$ , Bessel inequality applied to function  $\theta \mapsto \exp(\theta s)$  gives:  $\sum_{k=0}^N \frac{1}{\|\mathcal{L}_k\|_2^2} \left| \langle e^{\theta s}, \mathcal{L}_k(\theta) \rangle \right|^2 \leq \|e^{\theta s}\|_2^2$ .

An identification with  $\delta_N$  leads to:  $\delta_N^*(s) \delta_N(s) \leq \tau \left| \int_{-\tau}^0 e^{\theta(s+s^*)} d\theta \right|$ . As  $s \in \mathbb{C}^+$ , the right hand side is bounded by  $\tau^2$  and that ends the proof.  $\square$

In order to ease the comparison with time delay-systems, we introduce a new variable  $\theta$  defined as follows:  $\theta = -\tau x = -\frac{x}{c}$ . If the neutral part is not taken into consideration, a new infinite dimensional state can be defined:  $\tilde{U}(\theta, s) = U(-c\theta, s) \delta^{-1}(s)$ . Using equation (2), its expression is then:

$$\tilde{U}(\theta, s) = \left( 1 + \frac{c_0}{\tau} \right) \left( e^{\theta s} + \alpha e^{-(\theta+2\tau)s} \right) KX(s), \quad (17)$$

with  $t > 0$  and  $\theta \in (-\tau, 0)$ . There are several starting points to enrich  $\Omega$ , and the one we choose in this paper is to consider only the projection of the first part of the infinite dimensional state, i.e.  $\theta \mapsto e^{s\theta} KX(s)$ .

**Remark 6** *We can justify the choice of projecting only part of the state  $\tilde{U}(\cdot, t)$ . First, in equation (17), the state  $\tilde{U}$  at a given  $\theta$  and  $t$  is made up of two contributions. The first one is a result of a wave going forward  $\tilde{U}_1(\theta, s) = e^{s\theta} KX(s)$  and another one going backward  $\tilde{U}_2(\theta, s) = e^{-s(\theta+2\tau)} KX(s)$ . We decided here to project only  $\tilde{U}_1$ . Another option is to consider the two components  $\tilde{U}_1$  and  $\tilde{U}_2$  independently and then to enrich the state by two projections at each order. This leads to an increasing number of variables and the same performances. Indeed, with a change of variable, one can notice that  $\tilde{U}_2$  shares the same projections than  $\tilde{U}_1$  and can be then omitted.*

Let  $N \geq 0$  be the number of projections used. The purpose is now to build  $\Omega_N$  and  $Z_N$  such that:

$$\begin{aligned} \Omega_N(s) &= \nabla_N(s) Z_N(s), \\ \mathcal{E}_N Z_N(s) &= \mathcal{A}_N \Omega_N(s), \end{aligned} \quad (18)$$

with  $\mathcal{E}_N$  full column-rank where  $\Omega_N$  and  $Z_N$  are extended versions of  $\Omega$  and  $Z$ . The projections for  $k \in [0, N]$  are introduced as follow:

$$\begin{aligned} \chi_k(s) &= \langle e^{\theta s}, \mathcal{L}_k(\theta) \rangle KX(s), & \mathcal{X}_N(s) &= \left[ \chi_0(s) \quad \cdots \quad \chi_{N-1}(s) \right]^\top, \\ \nu_k(s) &= s \sqrt{2k+1} \chi_k(s), & \mathcal{V}_N(s) &= \left[ \nu_0(s) \quad \cdots \quad \nu_N(s) \right]^\top. \end{aligned} \quad (19)$$

Following the differentiation rule reminded below, the dynamic of the extra-states  $\mathcal{X}_N$  can be easily calculated:

$$\dot{\mathcal{X}}_N(t) = L_N \mathcal{X}_N(t). \quad (20)$$

The new state in Laplace domain is composed of  $X(s)$  and the projections of  $\theta \mapsto KX(s) e^{\theta s}$  on the orthogonal basis of Legendre polynomials using the projections defined in equation (19).

Noting that  $\mathcal{L}_k(-\tau) = (-1)^k$  and  $\mathcal{L}_k(0) = 1$ , the derivation rule in equation (20) and an integration by part give the following result:

$$s\chi_k(s) = \left(1 - (-1)^k e^{-\tau s} - \int_{-\tau}^0 e^{\theta s} \mathcal{L}'_k(\theta) d\theta\right) KX(s) = (1 - (-1)^k e^{-\tau s}) KX(s) - \sum_{i=0}^{k-1} \ell_{ik} \chi_i(s),$$

for  $k \in [0, N-1]$  and  $\ell_{ik}$  as defined in equation (16). Noting that  $\|\mathcal{L}_k\|^2 = \tau(2k+1)^{-1}$ ,  $\mathcal{V}_N$  can be expressed as follows:  $\mathcal{V}_N(s) = \delta_N(s) s K X(s)$ . The new states for the quadratic separation are:

$$\begin{aligned} z_N(t) &= \left[ \dot{X}^\top(t) \quad \dot{\mathcal{X}}_N^\top(t) \quad KX(t) \quad KX(t-\tau) \quad K\dot{X}(t) \right]^\top, \\ \omega_N(t) &= \left[ X^\top(t) \quad \mathcal{X}_N^\top(t) \quad KX(t-\tau) \quad u(1,t) \quad \mathcal{V}_N^\top(t) \right]^\top. \end{aligned}$$

With the previous signals, equalities (18) hold for  $\mathcal{A}_N, \mathcal{E}_N$  defined in the theorem and

$$\nabla_N(s) = \text{diag} \left( s^{-1} I_{n+N}, e^{-\tau s}, \delta(s), \delta_N(s) \right).$$

Now, we need to find a separator  $\Theta_N$ . The state extension is based on projections on a orthogonal basis such that the Bessel inequality holds. The result presented in Lemma 1 guaranties that the  $\Theta_N$  proposed in the theorem is a solution to LMI (11).

**Remark 7** *The main problem of this method comes from the inclusion of  $\delta$  into a circle. This is a bad bound for systems with a high reflexion coefficient  $\alpha$ . So for  $c$  and  $c_0$  small or  $c$  and  $c_0$  high, weaker results are expected.*

## 6 Examples

In this part, we aim at presenting the difference between the exact stability area and the one obtained using other theorems. The same system has also been studied in [19] using a Lyapunov-based stability approach. It also uses a hierarchy of LMI conditions, so the notation “LS,  $N = i$ ” refers to the stability obtained using the other methodology for an order  $i$ . A comparison of efficiency between all the methods is of course proposed.

The estimation of the stability area of system (1) is provided on some chosen examples with different behaviors. We consider the interconnection of a stable wave equation ( $c_0 > 0$ ) with different systems. First, the interconnection with a stable finite dimension LTI system is proposed. Then, an unstable ODE is interconnected. We then aim at proving that there exist unstable systems stabilized thanks to the wave equation. To finish, we analyze a system known to possess stability pockets for  $cc_0 = 1$ . The LMI solver used in the examples is “sdpt3” with Yalmip (developed by [34]).

### 6.1 $A$ and $A + BK$ Hurwitz with $\|H\|_\infty < 1$

To apply the Small-Gain Theorem, system (1) with the following matrices is proposed:

$$A = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad K = \begin{bmatrix} 0 & -\frac{20}{21} \end{bmatrix}. \quad (21)$$

It is easy to verify that  $A$  and  $A + BK$  are indeed Hurwitz. Moreover, the infinity norm of the open-loop system is less than 1, then Theorem 1 applies and  $\lim_{c_0 \rightarrow \infty} c_{min}(c_0) = 0$ .

The quadratic separation (QS) and the method with Lyapunov-based stability (LS) at order  $N = 0$  developed in [19] are added to the chart of Figure 3. As expected, the small gain theorem provides the worst estimation of the stability area but with stronger properties. The QS and LS approaches provide similar results. The QS does not use an extended state so the results for high  $c_0$  are further from the CTCR curve than the one obtained with LS.

Finally, the result obtained with CTCR shows a non-continuous behavior for small  $cc_0$  and this observation discourages us to use the  $(c_0, c)$  chart to estimate the stability area of system (1). This is not a numerical issue, another mapping can be used to correct this problem. Indeed, an appropriate choice of system

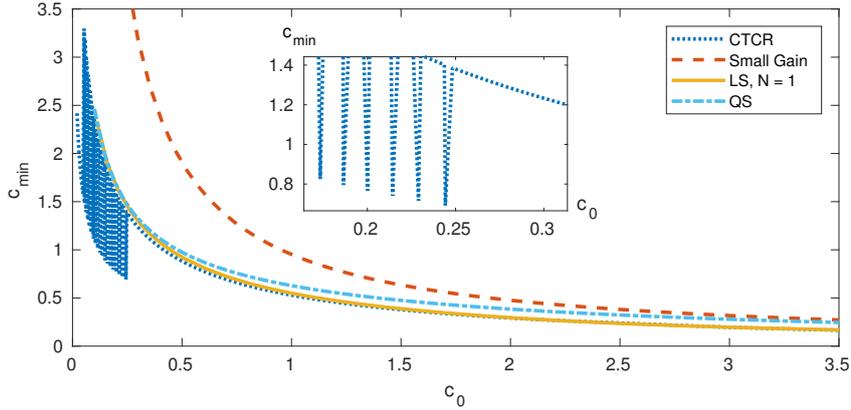


Figure 3: Minimum wave speed  $c_{min}$  as a function of  $c_0$  for system (1) to be stable. The values for  $A$ ,  $B$  and  $K$  are given by equation (21).

coordinates is  $(cc_0, c^{-1})$ . In order to make a comparison with time-delay systems, the delay  $c^{-1}$  needs to be considered as an axis. It is natural to add  $cc_0$  as the other variable of interest as it pilots the behavior of the system. Then, on the line  $cc_0 = 1$ , we get a time-delay system (see the state-space representation (6)), and it is possible to compare the results with the literature. From this point, the mapping  $(cc_0, c^{-1})$  is used.

The notation  $c_{min}$  is used in the chart, meaning that for all  $c > c_{min}$ , the system is also stable. This property was proven only for the Small-Gain theorem.

## 6.2 $A$ and $A + BK$ are not Hurwitz

The tool developed using QS is powerful enough to deal with more complex systems. In this example, the case of  $A$  and  $A + BK$  both unstable is studied:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad K = \begin{bmatrix} 1 & 0 \end{bmatrix}. \quad (22)$$

For  $\tau = 0$ , there are indeed two unstable poles. It means that a simple feedback cannot stabilize the system. This system for  $cc_0 = 1$  has been studied in [14], and CTCR recovers the same exact stability area. QS at  $N = 0$  does not provide any results, but, as we can see in Figure 4, higher orders of QS detect a stability area which is getting closer as the order increases to the one provided by CTCR around  $c_1 = 1$ . Despite the state augmentation, it is not possible to recover the whole stability area and simulations show that for  $cc_0 < 0.5$ , QS at any order does not assess stability. This can be a consequence of considering a “bad” bound of  $\delta$ .

It seems then that QS allows to detect stability pockets. To compare LS and QS, another example with more stability pockets is studied.

## 6.3 An example with stability pockets

For this last example, the matrices are:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -11 & 10 & 0 & 0 \\ 5 & -15 & 0 & -0.25 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad K = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^\top. \quad (23)$$

This system taken from [6] is known to possess some stability pockets if  $c_1 = 1$ . We can compare the efficiency of LS in Figure 5a. The hierarchy property can be seen and the stability pockets are indeed

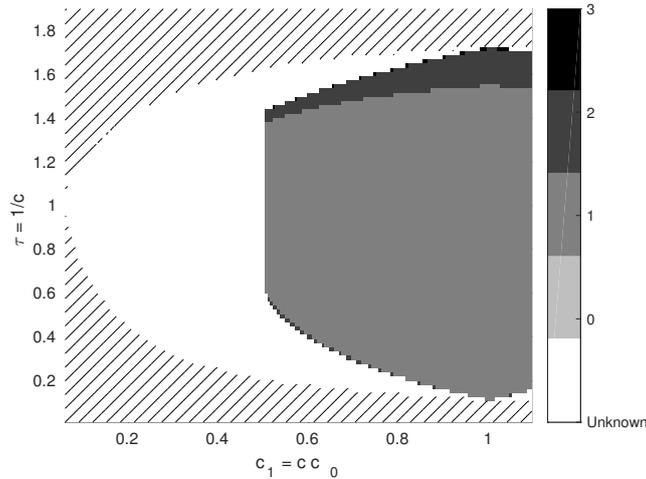


Figure 4: Stability areas for system (1) with matrices defined in (22) and  $c_1 = cc_0$  and  $\tau = c^{-1}$  obtained using CTCR and Theorem 4. The hatched area is the exact unstable area, the white area corresponds to the unstable area for Theorem 4 at order  $N = 3$  but stable with CTCR. The gray scale represents the stability area depending on the order  $N$  considered.

recovered as the order increases.

Figure 5b shows the stability result with QS at an order  $N$ . QS and LS at the same order provides similar results but with lower decision variables for QS.

Moreover, for small  $c_1$ , the reflection coefficient is not close to 0 and then, as noticed in Remark 7, QS should have a worse estimation. That is why we get poor result on the left side compared to LS. But if  $c_1$  is close to 1, results similar to LS are obtained using QS with a subsequently lower number of decision variables.

That means at the same number of decision variables, QS can detect more stability pockets around  $c_1 = 1$  but LS detects a wider stability area.

## 7 Conclusion

In this paper, we proposed an exact method to find the input/output stability area for a system made up of an ODE and a string equation. We also proposed two robust stability results: a simple one obtained with the Small-Gain theorem and another one using Quadratic Separation. The latter is based on a hierarchy of LMI conditions and is more conservative than some others approaches; but it proposes a subsequently reduced number of decision variables. A perspective would be to enhance the stability results for Quadratic Separation while keeping its low computing burden.

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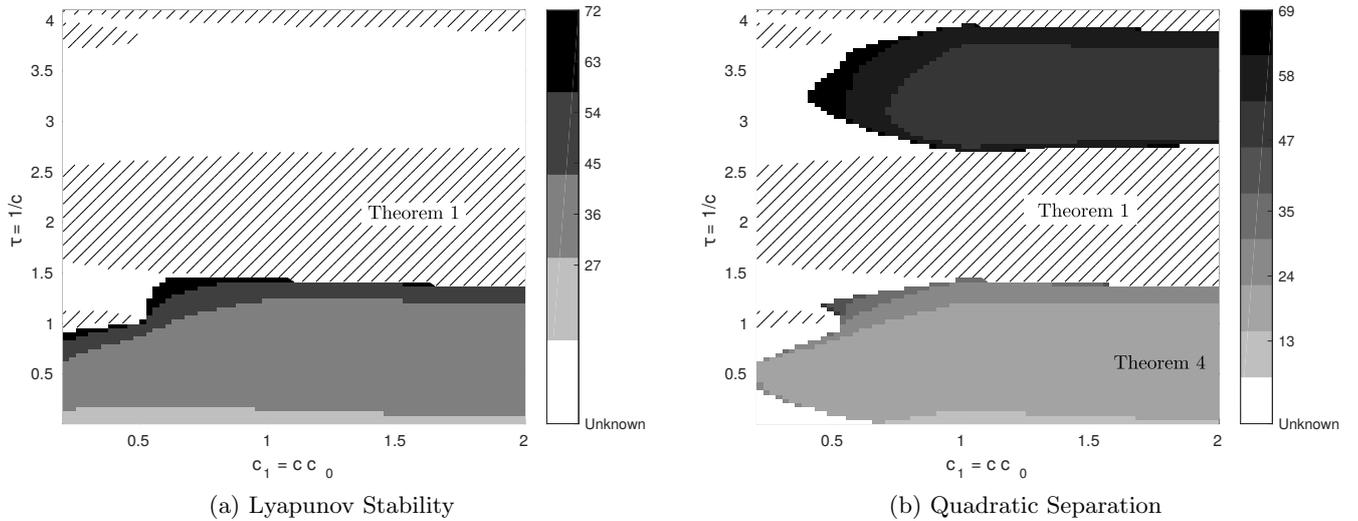


Figure 5: Stability areas for system (1) with matrices defined in (23) and  $c_1 = cc_0$  and  $\tau = c^{-1}$  obtained using CTCR, LS and QS. The hatched area is the exact unstable area, the white area corresponds to the unknown area according to Theorem 4 but stable with CTCR. The gray scale represents the stability area depending on the number of variables considered. To enable comparison,  $N$  varies between 0 and 3 for LS and between 0 and 7 for QS.

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