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# $\mathcal{H}_\infty$ Control Design for Synchronization of Identical Linear Multi-Agent Systems

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In this paper we study the state synchronization problem of multi-agent systems subject to external additive perturbations. We consider high-order linear time-invariant multi-agent systems whose communication topology is encoded by an undirected and connected graph. We propose an  $\mathcal{H}_\infty$  control design technique based on a decentralized output feedback controller. We give sufficient conditions to ensure state synchronization with bounded  $\mathcal{L}_2$  gain using a Lyapunov-based approach. These conditions are characterized in terms of matrix inequalities. Since these matrix inequality conditions are nonconvex and can not be solved straightforwardly, we propose a relaxation technique and an effective numerical procedure to design a suitable controller with guaranteed performance on the multi-agent distributed closed loop.

**Keywords:** consensus, state synchronization, multi-agent systems,  $\mathcal{H}_\infty$  design,  $\mathcal{L}_2$  gain, Lyapunov stability.

## 1. Introduction

In the past decade, the synchronization and consensus problems of multi-agent systems has received an increasing attention in the control community, due to their importance in a broad range of applications (see, e.g., (Dal Col, Tarbouriech, Zaccarian, & Kieffer, 2014), (Carli & Zampieri, 2014), (Leonard et al., 2007), and (D’Innocenzo, Di Benedetto, & Serra, 2013)).

Consensus refers to individuals coming to an agreement over a state variable, while synchronization refers to individuals reaching a temporal coincidence of some events (Wieland, 2010). To reach these goals, the individual systems exchange only relative information.

Consensus algorithms are primarily studied when the agents are modeled with integrator chains—that is, single- or double-integrator models (Ren & Beard, 2008a; Olfati-Saber & Murray, 2004). Recently, the consensus problem has been investigated considering agents modeled by general Linear Time-Invariant (LTI) systems (Fax & Murray, 2004; Wieland, Kim, Scheu, & Allgöwer, 2008; Seo, Shim, & J.Back, 2009). The underlying intuition behind these works is that the stability properties so far developed for a single dynamical system (see (Hespanha, 2009) for the linear case, and (Goebel, Sanfelice, & Teel, 2012) for the nonlinear and hybrid case) could be extended to networks of multi-agent systems by looking at the differential evolution of the agent states.

An interesting research direction of these works involves the extension to the case of linear multi-agent systems subject to disturbance signals. In (Wen, Hu, Yu, & Chen, 2014; Saboori & Khorasani, 2014), the authors solve the  $\mathcal{H}_\infty$  consensus problem for high-order agents with switching topology. Parallel derivation for multi-agents systems with fixed and undirected graph are investigated in (Li,

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Duan, & Chen, 2011). In (Dong, Xi, Shi, & Zhong, 2013), the authors investigate *practical consensus* in presence of time-varying external  $\mathcal{L}_2$  disturbances. In (Wang & Ding, 2016), the consensus problem is considered for multi-agent systems with input delays and external disturbances within a directed topology. Results in the same line of research for discrete-time systems are contained in (Massioni & Verhaegen, 2009). All of the above mentioned schemes address the case with full state measurements across the interconnection network. However, such an assumption is restrictive in practice since the agent states might not be available.

In this work we address the natural extension of the above mentioned setting, namely a decentralized dynamic output feedback synchronization problem in presence of external disturbances. So far, few works have addressed the case of distributed control protocols based on the agent outputs. Specifically, in (Q. Liu, Wang, He, & Zhou, 2015), the event-based consensus problem for multi-agent systems with external disturbances is tackled using a distributed cooperative estimator, based on the solution to a feasible backward recursive Riccati differential equation.

In (Zhu & Yang, 2016), the authors solve the robust  $\mathcal{H}_\infty$  consensus problem for high-order multi-agent systems using a dynamic output feedback controller. However, only sufficient analysis conditions are given to solve the problem, while the synthesis is based on a state feedback controller. In the same setting, (Y. Liu & Jia, 2010) proposed the synthesis of a distributed output feedback controller. However, this result is obtained constraining the Lyapunov matrix to have a block diagonal structure.

In this article, we investigate the synchronization problem of multi-agent systems subject to finite-energy disturbance signals. All agents are modeled by identical LTI high-order dynamical systems. The interconnections between the agents are modeled using a fixed, undirected and connected graph. The couplings among the agents are established via dynamic linear output feedback controllers. In this setup, this work makes the following contributions.

First, we present sufficient analysis conditions for  $\mathcal{L}_2$  synchronization of multi-agent systems, that guarantee a prescribed disturbance rejection level  $\gamma > 0$  in the interval  $[0, +\infty)$ . It is shown that the  $\mathcal{L}_2$  synchronization of the multi-agent system is achieved whenever a set of  $N - 1$  matrix inequality conditions parametrized by the Laplacian eigenvalues are satisfied. In other words, the fulfillment of  $N - 1$  matrix inequalities is a sufficient condition for the existence of a Lyapunov function for the  $\mathcal{L}_2$  synchronization stability, such that the closed-loop  $\mathcal{L}_2$  gain from the disturbance signal to the performance output of the multi-agent system is upper bounded.

Second, we propose an  $\mathcal{H}_\infty$  design procedure to select the matrices of the output feedback controller. Since the matrix inequality conditions obtained in the analysis part are not convex in the controllers parameters, we introduce a relaxation technique based on the completion-of-squares technique. The resulting relaxed conditions are bilinear in the unknown variables, and they depend only on the bounds on the spectrum of the Laplacian, and therefore they can be applied without the complete knowledge of the interconnected topology.

Third, we propose an Iterative Linear Matrix Inequality (ILMI) numerical procedure to solve the Bilinear Matrix Inequalities (BMI) relaxed conditions. Although this algorithm is not guaranteed to converge in general, it is systematic and numerically efficient in practice. With respect to the related work (Y. Liu & Jia, 2010), the proposed design leads to a less conservative solution, because we constrain the structure of some slack variable introduced in the completion-of-squares, instead of the Lyapunov function matrix.

The remainder of this paper is organized as follows. Section 2 presents the multi-agent system under consideration and states the problem that we intend to solve. Section 3 presents the dynamic output feedback protocol and the aggregate dynamics of the multi-agent system. Section 4 presents sufficient analysis conditions to solve the  $\mathcal{L}_2$  synchronization problem. Section 5 presents the  $\mathcal{H}_\infty$  relaxed conditions in terms of BMIs to perform the control design. Section 6 presents an iterative algorithm to solve the relaxed BMI conditions. Section 7 illustrates on an example the effectiveness of the algorithm described in Section 6. Finally, Section 8 concludes the paper with further directions of research.

**Notation.**  $\mathbf{1}_N$  and  $\mathbf{0}_N$  indicate the  $N$  dimensional column vectors with entries all equal to one and zero, respectively.  $I_N$  denotes the identity matrix of size  $N \times N$ .  $\mathcal{L}_2$  denotes the set of piecewise-continuous functions that are square integrable in the interval  $[0, +\infty)$ . The Euclidean norm is denoted by  $|\cdot|$ . For any matrix  $A$ ,  $A^\top$  indicates the transpose of  $A$  and  $\text{He}(A) = A + A^\top$ .  $\text{diag}(A, B)$  denotes the diagonal matrix which diagonal blocks are formed by the square matrices  $A$  and  $B$ . For two symmetric matrices  $A$  and  $B$  of the same dimensions, the notation  $A > B$  means that  $A - B$  is positive definite. In partitioned symmetric matrices,  $*$  denotes the symmetric block. Given a set  $\mathcal{A}$ ,  $|x|_{\mathcal{A}}$  denotes the standard point-to-set distance, i.e.,  $|x|_{\mathcal{A}} := \inf\{|x - a|, a \in \mathcal{A}\}$ , and  $\mathcal{A}^\perp$  the orthogonal complement of  $\mathcal{A}$ .

**Graph Theory.** Let  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$  be an undirected weighted graph. Any undirected graph  $\mathcal{G}$  is described by a node set  $\mathcal{V} = \{v_1, \dots, v_N\}$ , an edge set  $\mathcal{E} = \{e_1, \dots, e_q\} \subseteq \mathcal{V} \times \mathcal{V}$ , whose elements specify the incidence relation between distinct pairs of nodes. Let  $P = [p_{ij}]$  denote the adjacency matrix associated with  $\mathcal{G}$ . The adjacency elements associated with the edges of the graph are  $p_{ij} > 0$  if and only if  $(v_i, v_j) \in \mathcal{E}$ , otherwise  $p_{ij} = 0$ .  $\mathcal{N}$  indicates the index set of  $\mathcal{V}$ , and  $|\mathcal{N}|$  its cardinality. The diagonal matrix  $D = \text{diag}\{d_1, \dots, d_N\}$  is the degree matrix of  $\mathcal{G}$ , whose diagonal elements are  $d_i = \sum_{j=1}^N p_{ij}$ . The corresponding Laplacian of  $\mathcal{G}$  is defined as  $L := D - P$ . An undirected path is a sequence of ordered edges of the form  $(v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), \dots$ , with  $v_{i_j} \in \mathcal{V}$ . We denote with  $\mathcal{N}_i \subseteq \mathcal{N} \setminus \{i\}$  the set of nodes connected with node  $i$ , and with  $|\mathcal{N}_i|$  its cardinality, for  $i \in \mathcal{N}$ . A graph  $\mathcal{G}$  is called connected if and only if any two distinct nodes of  $\mathcal{G}$  can be connected via a path.

## 2. Problem Formulation

We consider multi-agent systems consisting of  $N$  identical LTI continuous-time plants of order  $n$ . Each agent in the network is identified by the subscript index  $i \in \mathcal{N} = \{1, \dots, N\}$ , where  $N > 1$  is the number of agents. The dynamics of each agent is described by the following linear state-space model

$$\begin{aligned} \dot{x}_{pi} &= A_p x_{pi} + B_{pu} \tilde{u}_i + B_{pw} w_i \\ y_i &= C_p x_{pi} + D_{pw} w_i \\ z_i &= C_{zp} x_{pi} + D_{zw} w_i, \end{aligned} \quad i \in \mathcal{N}, \quad (1)$$

where  $x_{pi} \in \mathbb{R}^n$  is the *agent state*,  $\tilde{u}_i \in \mathbb{R}^m$  is the *agent input*,  $y_i \in \mathbb{R}^p$  is the *agent output*, and  $z_i \in \mathbb{R}^\ell$  is the *agent performance output*.  $w_i \in \mathbb{R}^q$  is the *exogenous agent disturbance* (e.g., measurement noise, plant disturbances). The system matrices  $A_p$ ,  $B_{pu}$ ,  $B_{pw}$ ,  $C_p$ ,  $D_{pw}$ ,  $C_{zp}$  and  $D_{zp}$  are known matrices of appropriate dimensions. The communication topology in the multi-agent system (1) is described by an undirected graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ . Every node  $v_i \in \mathcal{V}$  is associated with one agent  $i \in \mathcal{N}$  in the group. Every edge  $(v_j, v_i) \in \mathcal{E}$  corresponds to a link between agent  $i$  and agent  $j$  in the network. In this paper, we make the following assumption on the graph  $\mathcal{G}$ .

**Assumption 1:** *The graph  $\mathcal{G}$  is undirected and connected.*

In this paper we consider the following definition of *state synchronization* of the multi-agent system (1).

**Definition 1:** *The multi-agent system (1) is said to achieve asymptotic state synchronization if, for any initial state  $x_{pi}(0) \in \mathbb{R}^n$ ,  $i \in \mathcal{N}$ , there exists a trajectory  $t \mapsto \bar{x}_p(t)$  such that*

$$\lim_{t \rightarrow +\infty} (x_{pi}(t) - \bar{x}_p(t)) = \mathbf{0}_n \quad (2)$$

*holds for every  $i \in \mathcal{N}$ , and  $\bar{x}_p$  is called synchronization trajectory.*

The goal of this paper is to design a distributed control law  $\tilde{u}_i$  that ensures synchronization of the multi-agent system (1), and attenuates the effect of the exogenous disturbance on the state

synchronization.

With the goal of studying disturbance attenuation requirements, we define the differential signals

$$\tilde{z}_i := \sum_{j \in \mathcal{N}_i} p_{ij}(z_i - z_j) = d_i \left( z_i - \frac{\sum_{j \in \mathcal{N}_i} p_{ij} z_j}{\sum_{j \in \mathcal{N}_i} p_{ij}} \right), \quad \tilde{w}_i := \sum_{j \in \mathcal{N}_i} p_{ij}(w_i - w_j) \quad i \in \mathcal{N}, \quad (3)$$

where  $\mathcal{N}_i$  denotes the set of neighbors of agent  $i$  in the graph  $\mathcal{G}$ , and  $|\mathcal{N}_i|$  its cardinality. The signals  $\tilde{z}_i$  and  $\tilde{w}_i$  are respectively the relative performance variable and relative disturbance of the agent  $i$  with respect to a suitably weighted average of its neighboring agents. The variable  $\tilde{w}_i$  in (3) is supposed to be limited in energy—that is, a function in  $\mathcal{L}_2$ . More precisely, we suppose that  $\tilde{w}_i$  are piecewise-continuous signals defined in  $[0, \infty)$  such that

$$\sum_{i=1}^N \|\tilde{w}_i\|_2^2 := \sum_{i=1}^N \int_0^\infty \tilde{w}_i(t)^\top \tilde{w}_i(t) dt < \infty. \quad (4)$$

We want to define a suitable  $\mathcal{L}_2$  gain index for the multi-agent system (1), as a measure of the disturbance attenuation level. We first notice that  $\tilde{z}_i$  in (3) can be rewritten as

$$\tilde{z}_i = C_{zp} \sum_{j \in \mathcal{N}_i} p_{ij}(x_i - x_j) + D_{zw} \sum_{j \in \mathcal{N}_i} p_{ij}(w_i - w_j) = C_{zp} \tilde{x}_i + D_{zi} \tilde{w}_i, \quad (5)$$

where we have defined the relative state vector  $\tilde{x}_i$  as the following mismatch between  $x_i$  and the weighted average of the neighboring states:

$$\tilde{x}_i := \sum_{j \in \mathcal{N}_i} p_{ij}(x_i - x_j) = d_i \left( x_i - \frac{\sum_{j \in \mathcal{N}_i} p_{ij} x_j}{\sum_{j \in \mathcal{N}_i} p_{ij}} \right), \quad i \in \mathcal{N}. \quad (6)$$

From (5), we can see that smaller values of the  $\mathcal{L}_2$  norm of  $\tilde{z}_i$  indicate a desirable behavior. Two factors cause small values of the  $\mathcal{L}_2$  norm of  $\tilde{z}_i$ . The first one is that the agent states of the multi-agent system (1) have similar values at the same time instants, that is, intuitively speaking, they are close to synchronization. The second one is that system (1) exhibits good disturbance rejection response with respect to the relative noise signals  $\tilde{w}_i$ . This last property is formalized in the following definition.

**Definition 2:** *The multi-agent system (1) has finite  $\mathcal{L}_2$  gain from  $\tilde{w}$  to  $\tilde{z}$ , with gain bound  $\gamma > 0$  if all solutions starting from  $x_{pi}(0) = \mathbf{0}_n$  satisfy*

$$\sum_{i=1}^N \|\tilde{z}_i\|_2^2 \leq \gamma^2 \sum_{i=1}^N \|\tilde{w}_i\|_2^2, \quad (7)$$

for all  $\tilde{w}_i \in \mathcal{L}_2$ , and  $i \in \mathcal{N}$ .

In other words, Definition 2 says that, for any relative disturbance signal  $\tilde{w}_i$  in  $\mathcal{L}_2$ , the response of the multi-agent system (1) starting from initial states  $x_i(0) = \mathbf{0}_n$ , is defined for all  $t \geq 0$ , and produces a performance variable  $\tilde{z}_i$  that is a function in  $\mathcal{L}_2$ , for all  $i \in \mathcal{N}$ . Moreover, the ratio between the  $\mathcal{L}_2$  norm of the relative performance signals  $\{\tilde{z}_i, i \in \mathcal{N}\}$  and the relative disturbance signals  $\{\tilde{w}_i, i \in \mathcal{N}\}$  is bounded by  $\gamma > 0$ . Note that the functions in  $\mathcal{L}_2$  represent signals having finite energy over the infinite time interval  $[0, +\infty)$ . Therefore the number  $\gamma$  in inequality (7) can be interpreted as an upper

bound on the ratio between the energy of the relative performance variables  $\{\tilde{z}_i, i \in \mathcal{N}\}$  over the energy of the relative disturbances  $\{w_i, i \in \mathcal{N}\}$ . Note that Definition 2 does not prevent signals  $w_i$  from possibly driving the outputs  $z_i$  to infinity, as long that these outputs diverge in a synchronized way. This is an important feature when characterizing evolutions of systems synchronizing around a nonconverging (but common) evolution induced by external disturbances acting on the formation. We will see in the next section that the fulfillment of inequality (7) is guaranteed by the fulfillment of suitable matrix inequalities involving the systems data, the controller data, the number  $\gamma$ , the Lyapunov matrix, and the topological parameters of the network.

The problem we intend to address in this paper is summarized as follows.

**Problem 1:** Consider the multi-agent system (1), with interconnection described by  $\mathcal{G}$ . The state synchronization problem in presence of external perturbation consists in finding a control law  $\tilde{u}_i$  such that

- (1) if  $w_i = \mathbf{0}_q$  for all  $i \in \mathcal{N}$ , there exists a trajectory  $\bar{x}_p$  such that (2) is satisfied—that is, the multi-agent system (1) reaches asymptotic state synchronization.
- (2) if  $w_i \neq \mathbf{0}_q$  for some  $i \in \mathcal{N}$ , and from initial conditions  $x_{pi}(0) = \mathbf{0}_n$ , for  $i \in \mathcal{N}$ , the multi-agent system (1) has finite  $\mathcal{L}_2$  gain, with prescribed gain bound  $\gamma > 0$ , that is, (7) is satisfied.

### 3. Distributed Dynamic Output Feedback

To solve Problem 1, we recognize that the problem at hand is an extension of the well known linear output feedback problem with  $\mathcal{H}_\infty$  performance, which inherits peculiar features of the classical  $\mathcal{H}_\infty$  scheme. In particular, when choosing the control architecture, we expect a convex characterization of the stability conditions if selecting a dynamic control structure, rather than a static one, which leads to nonconvex stability conditions (see, e.g., (Toker & Özbay, 1995)). Therefore, we choose the following dynamic output feedback control protocol:

$$\dot{x}_{ci} = A_k x_{ci} + B_k y_i \quad (8a)$$

$$u_i = C_k x_{ci} \quad (8b)$$

$$\tilde{u}_i = \sum_{j \in \mathcal{N}_i} p_{ij} (u_i - u_j), \quad (8c)$$

for  $i \in \mathcal{N}$ , where  $x_{ci} \in \mathfrak{R}^{n_k}$  is the *controller state*, and  $A_k, B_k$  and  $C_k$  are matrices to be designed. In the following, we suppose that the controller state and the plant state have the same dimensions  $n = n_k$ . Note that the coupling signal among the closed-loop multi-agent system (1) and (8) is the relative input  $\tilde{u}_i$ , which is based on the difference between the controller output  $u_i$  of agent  $i$  and the weighted average controller output of the neighboring agents  $\left(\sum_{j \in \mathcal{N}_i} p_{ij}\right)^{-1} \sum_{j \in \mathcal{N}_i} p_{ij} u_j$ , (see Figure 1). Since  $u_j$  only depends on  $x_{cj}$ , it is emphasized that control protocol (8) only requires single-hop information from the neighboring agents.

**Remark 1:** Note that the structure of the proposed controller has no direct feed-through term—that is,  $D_k = 0$ . This choice leads to useful simplifications in the ILMI relaxation proposed in Section 5.

We want to give a compact representation of the interconnected system (1), (8). To this end, we define the extended state vectors

$$x_i := \begin{bmatrix} x_{pi} \\ x_{ci} \end{bmatrix} \in \mathfrak{R}^{2n}, \quad i \in \mathcal{N}. \quad (9)$$

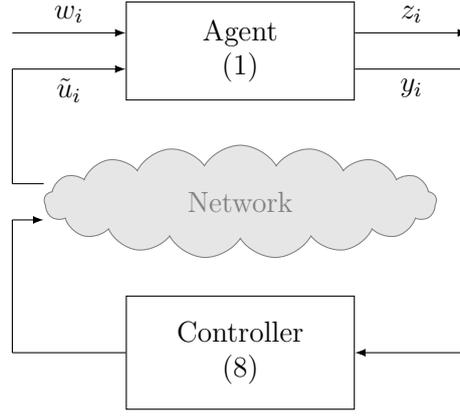


Figure 1. Block diagram of the closed-loop system of agents (1), with input (8) and coupling signal  $\tilde{u}_i$ .

The collective dynamics is obtained from (1) and (8), and corresponds to

$$\begin{aligned}
 \dot{x}_i &= \begin{bmatrix} A_p & 0 \\ B_k C_p & A_k \end{bmatrix} x_i + \begin{bmatrix} B_{pu} \\ 0 \end{bmatrix} \tilde{u}_i + \begin{bmatrix} B_{pw} \\ B_k D_{pw} \end{bmatrix} w_i \\
 u_i &= \begin{bmatrix} 0 & C_k \end{bmatrix} x_i \\
 z_i &= \begin{bmatrix} C_{zp} & 0 \end{bmatrix} x_i + D_{zw} w_i, \\
 \tilde{u}_i &= \sum_{j \in \mathcal{N}_i} p_{ij} (u_i - u_j),
 \end{aligned} \tag{10}$$

We want to simplify the collective dynamics (10) of the closed-loop system. To this end, it is convenient to use the Kronecker product (or tensor product) to describe the aggregate dynamics. We define the aggregate vectors

$$\begin{aligned}
 x &:= \begin{bmatrix} x_1^\top & \dots & x_N^\top \end{bmatrix}^\top \in \mathfrak{R}^{2Nn} \\
 u &:= \begin{bmatrix} u_1^\top & \dots & u_N^\top \end{bmatrix}^\top \in \mathfrak{R}^{Nm} \\
 w &:= \begin{bmatrix} w_1^\top & \dots & w_N^\top \end{bmatrix}^\top \in \mathfrak{R}^{Nq} \\
 z &:= \begin{bmatrix} z_1^\top & \dots & z_N^\top \end{bmatrix}^\top \in \mathfrak{R}^{N\ell},
 \end{aligned} \tag{11}$$

and similarly define the vectors  $\tilde{x}$ ,  $\tilde{u}$ ,  $\tilde{w}$  and  $\tilde{z}$ . These last vectors can be rewritten as a function of the ones defined in (11) according to (3), (6), and (8c) as

$$\tilde{x} = (L \otimes I_{2n})x, \quad \tilde{u} = (L \otimes I_m)u, \quad \tilde{w} = (L \otimes I_q)w, \quad \tilde{z} = (L \otimes I_\ell)z, \tag{12}$$

where  $L \in \mathfrak{R}^{N \times N}$  is the Laplacian matrix associated with  $\mathcal{G}$ . Combining (10), (11), and (12), we obtain the following collective closed-loop dynamics for the multi-agent system (1) and (8)

$$\begin{aligned}
 \dot{x} &= Ax + B_w w \\
 z &= C_z x + D_z w
 \end{aligned} \tag{13}$$

where the structure of  $A$ ,  $B_w$ ,  $C_z$ ,  $D_z$  is

$$\left( \begin{array}{c|c} A & B_w \\ \hline C_z & D_z \end{array} \right) := \left( \begin{array}{c|c} I_N \otimes \begin{bmatrix} A_p & 0 \\ B_k C_p & A_k \end{bmatrix} + L \otimes \begin{bmatrix} 0 & B_{pu} C_k \\ 0 & 0 \end{bmatrix} & I_N \otimes \begin{bmatrix} B_{pw} \\ B_k D_{pw} \end{bmatrix} \\ \hline I_N \otimes \begin{bmatrix} C_{zp} & 0 \end{bmatrix} & I_N \otimes D_{zw} \end{array} \right). \tag{14}$$

In Section 4, we will introduce a coordinates transformation for system (13), based on the spectral decomposition of the Laplacian matrix  $L$  of the network graph  $\mathcal{G}$ . This new set of coordinates system allows us to decouple the closed-loop dynamics into  $N$  subsystems of dimension  $2n$  parametrized by the eigenvalues of the Laplacian  $L$ .

With the goal of establishing synchronization of the multi-agent system (1) and (8), or, equivalently, (13), we define the *synchronization set* as

$$\mathcal{S} := \left\{ x \in \mathbb{R}^{2Nn} : \begin{bmatrix} x_{pi} \\ x_{ci} \end{bmatrix} - \begin{bmatrix} x_{pj} \\ x_{cj} \end{bmatrix} = 0, \forall i, j \in \mathcal{N} \right\}. \quad (15)$$

The synchronization set is the set of the extended space  $\mathbb{R}^{2Nn}$  in which the agent states  $x_{pi}$  in (1) and the controller states  $x_{ci}$  in (8) coincide. It is emphasized that item (1) of Problem 1 only requires synchronization of the plant state components, therefore it seems overly restrictive to require that also the controller states synchronize. Nevertheless, since we expect the controller states to be detectable from the control outputs  $u_i$ , requiring synchronization of the full plant-controller state is not more restrictive than plant-state synchronization only. In particular, it is evident that attractivity of the synchronization set  $\mathcal{S}$  for the unperturbed closed-loop dynamics (1) and (8) with  $w_i = \mathbf{0}_q$ , for all  $i \in \mathcal{N}$  implies item (1) of Problem 1, but it turns out that the converse is also true if the controller matrices design is performed following the strategy proposed here, which implies detectability of  $x_c$  from  $u$ .

#### 4. Sufficient Conditions for State Synchronization

In this section we provide sufficient conditions to solve Problem 1 in terms of matrix inequalities, involving the projection of the aggregate vectors (11) onto  $\mathcal{S}^\perp$ —that is, the orthogonal complement of  $\mathcal{S}$ . From a computational viewpoint our conditions are appealing because they do not involve the full dynamics over the network but exploit the symmetries in the network nodes and the Laplacian representation of the interconnection in order to perform a design arising from solutions to convex semidefinite programs having the dimension of the state space of each node (namely those of matrix  $A_p$ ). However, our design technique requires knowledge of the smallest and the largest eigenvalues of the Laplacian matrix ( $\lambda_2$  and  $\lambda_N$ ). Such knowledge might be hard to obtain in practice and several works have been published recently about distributed estimation of these parameters (see, e.g., (Kibangou & Commault, 2012; Franceschelli, Gasparri, Giua, & Seatzu, 2013) and references therein). Combining our synthesis method with those estimation schemes is beyond the scope of this work but is an interesting future direction.

First, we suppose that the controller matrices  $A_k, B_k, C_k$  in (8) are given, and we look at a suitable Lyapunov function to perform the  $\mathcal{L}_2$  synchronization stability analysis. The problem of designing a suitable controller in the form (8) to solve Problem 1 is then addressed in Section 6.

To perform this analysis, we first introduce a coordinates transformation of the closed-loop dynamics (13) and we translate the  $\mathcal{L}_2$  gain condition (7) in the new coordinates.

##### 4.1 Decoupling Change of Coordinates

Let us introduce a change of coordinates of the closed-loop system (13) (see, for example, (Fax & Murray, 2004)). This coordinates transformation is induced by a specific unitary matrix  $U$ , related to the Laplacian matrix  $L$ . According to Assumption 1, the information topology of the considered multi-agent system is encoded by an undirected and connected graph  $\mathcal{G}$ . As shown in (Ren & Beard, 2008b), zero is a simple eigenvalue of  $L$  if and only if  $\mathcal{G}$  is connected. Under these assumptions we can state the following lemma, which is a well-established result in the consensus and synchronization literature (see, e.g., (Lin, Jia, Du, & Yu, 2008) for more details).

**Lemma 1:** Let  $L = L^\top \in \mathfrak{R}^{N \times N}$  be the symmetric Laplacian matrix of an undirected and connected graph  $\mathcal{G}$ . Let  $0 = \lambda_1 < \lambda_2 \leq \dots \leq \lambda_N$  denote the ordered eigenvalues of  $L$ . The set of eigenvectors  $\nu_1, \nu_2, \dots, \nu_N$  corresponding to  $\lambda_1, \lambda_2, \dots, \lambda_N$  forms an orthonormal basis of  $\mathfrak{R}^N$ , and  $\nu_1 := \frac{\mathbf{1}_N}{\sqrt{N}}$ .

Define the unitary matrix  $U := [\nu_1 \ U_2]$ , where  $U_2 := [\nu_2 \ \dots \ \nu_N] \in \mathfrak{R}^{N \times (N-1)}$ , such that  $U^\top U = U U^\top = I_N$ . Then, based on Lemma 1, we can decompose the Laplacian matrix  $L$  as follows

$$\Delta := \begin{bmatrix} 0 & * \\ \mathbf{0}_{N-1} & \Delta_1 \end{bmatrix} = U^\top L U, \quad (16)$$

where  $\Delta_1 := \text{diag}(\lambda_2, \dots, \lambda_N) \in \mathfrak{R}^{(N-1) \times (N-1)}$  is positive definite.

We are ready to introduce the following coordinates transformation

$$\hat{x} := (U^\top \otimes I_{2n})x, \quad \hat{w} := (U^\top \otimes I_q)w, \quad \hat{z} := (U^\top \otimes I_\ell)\tilde{z}. \quad (17)$$

Consider now the following partition of vectors (17)

$$\begin{aligned} \hat{x} &= \begin{bmatrix} (\nu_1^\top \otimes I_{2n})x \\ (U_2^\top \otimes I_{2n})x \end{bmatrix} := \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} \in \mathfrak{R}^{2n} \times \mathfrak{R}^{2(N-1)n}, \\ \hat{w} &= \begin{bmatrix} (\nu_1^\top \otimes I_q)w \\ (U_2^\top \otimes I_q)w \end{bmatrix} := \begin{bmatrix} \hat{w}_1 \\ \hat{w}_2 \end{bmatrix} \in \mathfrak{R}^q \times \mathfrak{R}^{(N-1)q}, \\ \hat{z} &= \begin{bmatrix} (\nu_1^\top \otimes I_\ell)\tilde{z} \\ (U_2^\top \otimes I_\ell)\tilde{z} \end{bmatrix} := \begin{bmatrix} \hat{z}_1 \\ \hat{z}_2 \end{bmatrix} \in \mathfrak{R}^\ell \times \mathfrak{R}^{(N-1)\ell}, \end{aligned} \quad (18)$$

where  $\hat{x}, \hat{w}, \hat{z}$  have the same partition as  $U$  in (16). Applying the transformation (17) to (13), we obtain the following dynamics of the closed-loop system

$$\begin{aligned} \dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}_w\hat{w} \\ \dot{\hat{z}} &= \hat{C}_z\hat{x} + \hat{D}_z\hat{w}, \end{aligned} \quad (19)$$

where the structure of  $\hat{A}, \hat{B}_w, \hat{C}_z, \hat{D}_z$  is as follows

$$\left( \begin{array}{c|c} \hat{A} & \hat{B}_w \\ \hline \hat{C}_z & \hat{D}_z \end{array} \right) := \left( \begin{array}{c|c} I_N \otimes \begin{bmatrix} A_p & 0 \\ B_k C_p & A_k \end{bmatrix} + \Delta \otimes \begin{bmatrix} 0 & B_{pu} C_k \\ 0 & 0 \end{bmatrix} & I_N \otimes \begin{bmatrix} B_w \\ B_k D_{pw} \end{bmatrix} \\ \hline \Delta \otimes [C_{zp} \ 0] & \Delta \otimes D_{zw} \end{array} \right), \quad (20)$$

where we have used the orthonormality of  $U$  and relation (16) for the derivations, together with the associative properties of the Kronecker product. In the new coordinates system, matrices (20) have a block diagonal structure. Based on this observation, using the partitioned vectors in (18), we can decouple the closed-loop dynamics (19) into the following subsystems

$$\begin{aligned} \dot{\hat{x}}_1 &= \begin{bmatrix} A_p & 0 \\ B_k C_p & A_k \end{bmatrix} \hat{x}_1 + \begin{bmatrix} B_w \\ B_k D_{pw} \end{bmatrix} \hat{w}_1 \\ \hat{z}_1 &= \mathbf{0}_\ell, \end{aligned} \quad (21)$$

and

$$\begin{aligned} \dot{\hat{x}}_2 &= \left( I_{N-1} \otimes \begin{bmatrix} A_p & 0 \\ B_k C_p & A_k \end{bmatrix} + \Delta_1 \otimes \begin{bmatrix} 0 & B_{pu} C_k \\ 0 & 0 \end{bmatrix} \right) \hat{x}_2 + \left( I_{N-1} \otimes \begin{bmatrix} B_w \\ B_k D_{pw} \end{bmatrix} \right) \hat{w}_2 \\ \dot{\hat{z}}_2 &= (\Delta_1 \otimes [C_{zp} \ 0]) \hat{x}_2 + (\Delta_1 \otimes D_{zw}) \hat{w}_2. \end{aligned} \quad (22)$$

Note that, from the structure of matrix  $U$  in (16), the first component of  $\hat{x}$  in (17) is

$$\hat{x}_1 = (\nu_1^\top \otimes I_{2n})x = \frac{1}{\sqrt{N}} \sum_{i=1}^N \begin{bmatrix} x_{pi} \\ x_{ci} \end{bmatrix}. \quad (23)$$

From (Dal Col, Tarbouriech, Zaccarian, & Kieffer, 2015, Lemma 3), we see that vector  $\frac{\hat{x}_1}{\sqrt{N}}$  is the orthogonal projection of the aggregate state vector  $x \in \mathfrak{R}^{2Nn}$  in (12) onto the synchronization set  $\mathcal{S}$  in (15), and (21) is the projection of the closed-loop dynamics (13) onto  $\mathcal{S}^\perp$ .

The  $N - 1$  subsystems in (22) correspond to the projection of the closed-loop dynamics (13) onto  $\mathcal{S}^\perp$ , that is, the subspace generated by  $\nu_2 \otimes I_{2n}, \dots, \nu_N \otimes I_{2n}$ , where  $\nu_2, \dots, \nu_N$  are the eigenvectors of  $L$  corresponding to  $\lambda_2, \dots, \lambda_N$ . Note that the  $N - 1$  decoupled systems (22) are obtained combining (1) and (8), while replacing (8c) with the input  $\tilde{u}_i = \lambda_i u_i$ , for  $i = 2, \dots, N$ .

We want to write the  $\mathcal{L}_2$  bound (7) in terms of the transformed variables (17). The  $\mathcal{L}_2$  gain condition (7) can be written in the aggregate vectors  $\tilde{z}$  and  $\tilde{w}$  in (12) as

$$\|\tilde{z}\|_2^2 = \sum_{i=1}^N \|\tilde{z}_i\|_2^2 \leq \gamma^2 \sum_{i=1}^N \|\tilde{w}_i\|_2^2 = \gamma^2 \|\tilde{w}\|_2^2, \quad (24)$$

Consider the variable  $\hat{z}$  in (17). From (12) and from the properties of the Laplacian  $L$ , we have that the first component  $\hat{z}_1$  of  $\hat{z}$  satisfies

$$\hat{z}_1 = (\nu_1^\top \otimes I_\ell)\tilde{z} = (\nu_1^\top L \otimes I_\ell)z = \mathbf{0}_\ell. \quad (25)$$

Based on these considerations, we conclude that the  $\mathcal{L}_2$  norm of  $\hat{z}$  satisfies

$$\|\hat{z}\|_2^2 = \|\hat{z}_2\|_2^2 \quad (26)$$

On the other hand, from the orthonormality of  $U$  in (16), definition (12), and using (25), we have

$$\|\tilde{z}\|_2^2 = \|(U^\top \otimes I_\ell)\tilde{z}\|_2^2 = \|\hat{z}\|_2^2 = \|\hat{z}_1\|_2^2 + \|\hat{z}_2\|_2^2 = \|\hat{z}_2\|_2^2, \quad (27)$$

Consider now the disturbance variable  $\hat{w}$  in (17) and  $\tilde{w}$  in (12). Using (16), we have

$$(U^\top \otimes I_q)\tilde{w} = (U^\top LU \otimes I_q)\hat{w} = (\Delta \otimes I_q)\hat{w}. \quad (28)$$

We obtain the following bound on the  $\mathcal{L}_2$  norm of the variables of subsystem (22)

$$\begin{aligned} \|\hat{w}_2\|_2^2 &= \|(\Delta_1^{-1} \Delta_1 \otimes I_q)\hat{w}_2\|_2^2 \leq \lambda_2^{-2} \|(\Delta_1 \otimes I_q)\hat{w}_2\|_2^2 \\ &= \lambda_2^{-2} \|(\Delta \otimes I_q)\hat{w}\|_2^2 = \lambda_2^{-2} \|\tilde{w}\|_2^2, \end{aligned} \quad (29)$$

where we used relation (28), and the fact that the entries of  $\Delta^{-1}$  are smaller than  $\lambda_2^{-1}$ —that is, the inverse of the smallest eigenvalue of the Laplacian matrix  $L$ . We conclude that the  $\mathcal{L}_2$  gain property

$$\|\hat{z}_2\|_2^2 \leq \hat{\gamma}^2 \|\hat{w}_2\|_2^2, \quad (30)$$

where  $\hat{\gamma}^2 = \lambda_2^2 \gamma^2$ , ensures that the desired  $\mathcal{L}_2$  gain property (7) is satisfied. In fact, combin-

ing (26), (27), (29) and (24), we obtain

$$\|\tilde{z}\|_2^2 = \|\hat{z}_2\|_2^2 \leq \hat{\gamma}^2 \|\hat{w}_2\|_2^2 \leq \gamma^2 \lambda_2^2 \|\hat{w}_2\|_2^2 \leq \gamma^2 \|\tilde{w}\|_2^2. \quad (31)$$

Note that inequality (31) only involves the variables of subsystem (22)—that is, the projections of the aggregate variables (11) onto  $\mathcal{S}^\perp$ .

## 4.2 State Synchronization Analysis

In this section, we characterize Problem 1 in terms of matrix inequality conditions. These conditions are obtained by using the decoupling change of coordinates (17), the  $\mathcal{L}_2$  gain condition (30), and the Lyapunov synchronization stability results contained in (Dal Col et al., 2014, Theorem 2) (See also (Dal Col et al., 2015, Theorem 1) and (Dal Col, 2016, Theorem 2.1) for more details). We point out that the presented conditions for state synchronization analysis are only sufficient, due to the conservatism introduced in the  $\mathcal{L}_2$  bounds (26) and (29).

The result provided in the following theorem is an extension to the multi-agent framework of the well-known results on  $\mathcal{L}_2$  gain stability, and  $\mathcal{H}_\infty$  control for isolated LTI systems (see, for example, (Scherer, Gahinet, & Chilali, 1997)).

**Theorem 1:** *Given a desired bound  $\gamma > 0$ , if there exist matrices  $A_k \in \mathbb{R}^{n \times n}$ ,  $B_k \in \mathbb{R}^{n \times p}$ ,  $C_k \in \mathbb{R}^{m \times n}$ ,  $N - 1$  positive definite matrices  $P_i = P_i^\top \in \mathbb{R}^{2n \times 2n}$ ,  $i = 2, \dots, N$  such that the following matrix inequalities*

$$\begin{bmatrix} \text{He} \left( P_i \begin{bmatrix} A_p & \lambda_i B_{pu} C_k \\ B_k C_p & A_k \end{bmatrix} \right) & P_i \begin{bmatrix} B_{pw} \\ B_k D_{pw} \end{bmatrix} & \begin{bmatrix} \lambda_i C_{zp}^\top \\ 0 \end{bmatrix} \\ * & -I_q & \lambda_i D_{zw}^\top \\ * & * & -\hat{\gamma}^2 I_\ell \end{bmatrix} < 0, \quad (32)$$

for  $i = 2, \dots, N$  are satisfied, where  $\hat{\gamma}^2 = \lambda_2^2 \gamma^2$ , then the controller (8) with  $x_{ci}(0) = \mathbf{0}_n$ , for  $i \in \mathcal{N}$ , solves Problem 1.

*Proof.* To prove Theorem 1, we invoke (Dal Col et al., 2014, Theorem 2), once we observe that the Laplacian  $L$  satisfies the hypotheses of this Theorem. First, we consider the case  $w_i = \mathbf{0}_q$  for all  $i \in \mathcal{N}$ . From (32) we have, in particular, that the first  $2n \times 2n$  block of the  $N - 1$  inequalities (32) is negative definite, that is

$$\text{He} \left( P_i \begin{bmatrix} A_p & \lambda_i B_{pu} C_k \\ B_k C_p & A_k \end{bmatrix} \right) < 0, \quad i = 2, \dots, N. \quad (33)$$

Since  $P_i$  are positive definite matrices, (33) is equivalent to matrices

$$\begin{bmatrix} A_p & \lambda_i B_{pu} C_k \\ B_k C_p & A_k \end{bmatrix}$$

being Hurwitz, for  $i = 2, \dots, N$ . From the equivalence between items (i) and (iv) of (Dal Col et al., 2014, Theorem 2), we conclude that there exists a Lyapunov function  $V(x)$  for dynamics (22), such that

$$\begin{aligned} \alpha_1 |x|_{\mathcal{S}}^2 &\leq V(x) \leq \alpha_2 |x|_{\mathcal{S}}^2 \\ \dot{V}(x) &\leq -\beta |x|_{\mathcal{S}}^2, \end{aligned} \quad (34)$$

for some positive scalars  $\alpha_1, \alpha_2, \beta$ ,  $\mathcal{S}$  is defined in (15), and  $|x|_{\mathcal{S}}$  is the standard point-to-set distance. Such a Lyapunov function has the following expression

$$V(x) = x^\top (U \otimes I_{2n}) \bar{P} (U^\top \otimes I_{2n}) x = \hat{x}_2^\top \text{diag}(P_i) \hat{x}_2, \quad (35)$$

where  $\bar{P} := \text{diag}(0, P_i)$ , for  $i = 2, \dots, N$ , and  $U \in \mathfrak{R}^{N \times N}$  is the unitary matrix in (16). The uniform global exponential stability of the consensus set  $\mathcal{S}$  in (15) with respect to the unperturbed dynamics (1) and (8) (or, equivalently (13)) with  $w_i = \mathbf{0}_q$  for all  $i \in \mathcal{N}$ , follows from the equivalence between items (i) and (ii) of (Dal Col et al., 2014, Theorem 2). This implies, in particular, the convergence of the state of the agents (1) to a common trajectory—that is, (2) is satisfied.

Consider now the case where there exists an index  $i$ , such that  $w_i \neq \mathbf{0}_q$ . Denote with  $\hat{x}_2^{(i)} \in \mathfrak{R}^{2n}$ ,  $\hat{z}_2^{(i)} \in \mathfrak{R}^\ell$ ,  $\hat{w}_2^{(i)} \in \mathfrak{R}^q$ , for  $i = 2, \dots, N$ , the vector components of  $\hat{x}_2$ ,  $\hat{z}_2$ ,  $\hat{w}_2$  in (18). From (22), we obtain that  $\hat{x}_2^{(i)}$  and  $\hat{w}_2^{(i)}$  are related to  $\hat{z}_2^{(i)}$  according to

$$\hat{z}_2^{(i)} = \lambda_i [C_{zp} \quad 0] \hat{x}_2^{(i)} + \lambda_i D_{zw} \hat{w}_2^{(i)}, \quad i = 2, \dots, N \quad (36)$$

By applying the Schur complement to (32) and pre- and post-multiplying by  $[\hat{x}_2^{(i)\top} \quad \hat{w}_2^{(i)\top}]^\top$  and its transpose, we obtain

$$\begin{aligned} 2\hat{x}_2^{(i)\top} P_i \begin{bmatrix} A_p & \lambda_i B_{pu} C_k \\ B_k C_p & A_k \end{bmatrix} \hat{x}_2^{(i)} \\ - \hat{w}_2^{(i)\top} \hat{w}_2^{(i)} + 2\hat{x}_2^{(i)\top} P_i \begin{bmatrix} B_{pw} \\ B_k D_{pw} \end{bmatrix} \hat{w}_2^{(i)} + \frac{1}{\hat{\gamma}^2} \hat{z}_2^{(i)\top} \hat{z}_2^{(i)} < 0, \quad \forall i = 2, \dots, N, \end{aligned} \quad (37)$$

where we used relation (36) for the derivations. Stacking the  $N - 1$  inequalities in (37), from (22) we obtain

$$\frac{d}{dt} \left( \hat{x}_2^\top \text{diag}(P_i) \hat{x}_2 \right) + \frac{1}{\hat{\gamma}^2} \hat{z}_2^\top \hat{z}_2 - \hat{w}_2^\top \hat{w}_2 < 0, \quad (38)$$

By integration of (38) over the interval  $[0, T]$ , with  $T > 0$ , and from (35), we obtain

$$V(x(T)) - V(x(0)) + \frac{1}{\hat{\gamma}^2} \int_0^T \hat{z}_2(\tau)^\top \hat{z}_2(\tau) d\tau - \int_0^T \hat{w}_2(\tau)^\top \hat{w}_2(\tau) d\tau < 0. \quad (39)$$

Since from the first equation in (34),  $V(x(T)) \geq 0$  for all  $T > 0$ , and with the hypothesis  $x_i(0) = \begin{bmatrix} x_{pi}(0) \\ x_{ci}(0) \end{bmatrix} = \mathbf{0}_{2n}$  for all  $i = 1, \dots, N$ , we get  $V(x(0)) = 0$ . Taking the limit of (39) as  $T \rightarrow \infty$  we obtain

$$\|\hat{z}_2\|_2^2 \leq \hat{\gamma}^2 \|\hat{w}_2\|_2^2. \quad (40)$$

Hence, by using (24), (30) and (31), it follows that the  $\mathcal{L}_2$  gain property (7) is satisfied, and then item (2) of Problem 1 is solved. This completes the proof.  $\square$

We are now interested in characterizing the synchronization trajectory  $\bar{x}_p$  in (2). This is possible only when the dynamics in (1) and (8) is not perturbed. To this end we state the following result, that is obtained particularizing Theorem 1 when  $w_i = \mathbf{0}_q$ ,  $i \in \mathcal{N}$ .

**Corollary 1:** *If  $w_i = \mathbf{0}_q$  for all  $i \in \mathcal{N}$ , and there exist matrices  $A_k \in \mathfrak{R}^{n \times n}$ ,  $B_k \in \mathfrak{R}^{n \times p}$ ,  $C_k \in \mathfrak{R}^{m \times n}$ ,  $N - 1$  positive definite matrices  $P_i = P_i^\top \in \mathfrak{R}^{2n \times 2n}$  such that*

$$\text{He} \left( P_i \begin{bmatrix} A_p & \lambda_i B_{pu} C_k \\ B_k C_p & A_k \end{bmatrix} \right) < 0, \quad i = 2, \dots, N, \quad (41)$$

*then, for any initial condition  $x_i(0) = [x_{pi}(0)^\top \quad x_{ci}(0)^\top]^\top \in \mathfrak{R}^{2n}$ , the trajectories of the closed-loop system (1), (8) asymptotically synchronize to the solution to the following initial values problem:*

$$\dot{\bar{x}} = \begin{bmatrix} A_p & 0 \\ B_k C_p & A_k \end{bmatrix} \bar{x}, \quad \bar{x}(0) = \frac{1}{N} \sum_{i=1}^N \begin{bmatrix} x_{pi}(0) \\ x_{ci}(0) \end{bmatrix}, \quad (42)$$

where  $\bar{x} = \begin{bmatrix} \bar{x}_p \\ \bar{x}_c \end{bmatrix}$ .

*Proof.* From item (iii) of (Dal Col et al., 2014, Theorem 2), combined with Theorem 1, the agent states exponentially synchronize to the same trajectory, described by the state  $\bar{x} = \frac{1}{N} \sum_{i=1}^N x_i = \frac{1}{N} (\mathbf{1}_N^\top \otimes I_{2n}) x$ . Based on (13), (14), the time evolution of  $\bar{x}$  is

$$\begin{aligned} \dot{\bar{x}} &= \frac{1}{N} (\mathbf{1}_N^\top \otimes I_{2n}) \left[ \left( I_N \otimes \begin{bmatrix} A_p & 0 \\ B_k C_p & A_k \end{bmatrix} \right) + \left( L \otimes \begin{bmatrix} 0 & B_{pu} C_k \\ 0 & 0 \end{bmatrix} \right) \right] x \\ &= \frac{1}{N} \left( \mathbf{1}_N^\top I_N \otimes \begin{bmatrix} A_p & 0 \\ B_k C_p & A_k \end{bmatrix} \right) x = \frac{1}{N} \left( \mathbf{1}_N^\top \otimes \begin{bmatrix} A_p & 0 \\ B_k C_p & A_k \end{bmatrix} \right) x \\ &= \begin{bmatrix} A_p & 0 \\ B_k C_p & A_k \end{bmatrix} \frac{1}{N} (\mathbf{1}_N^\top \otimes I_{2n}) x = \begin{bmatrix} A_p & 0 \\ B_k C_p & A_k \end{bmatrix} \bar{x}, \end{aligned} \quad (43)$$

where we used the relation  $\mathbf{1}_N^\top L = 0$ , and  $\bar{x}(0) = \frac{1}{N} (\mathbf{1}_N^\top \otimes I_{2n}) x(0) = \frac{1}{N} \sum_{i=1}^N x_i(0)$ .  $\square$

Note that, taking the first component of the vector  $\bar{x}$  of (42), we obtain

$$\dot{\bar{x}}_p = A_p \bar{x}_p, \quad \bar{x}_p(0) = \frac{1}{N} \sum_{i=1}^N x_{pi}(0). \quad (44)$$

Moreover, from Corollary 1, we deduce that also the controller states  $x_{ci}$  in (8) converge to a common trajectory  $\bar{x}_c$ , that is the solution to

$$\dot{\bar{x}}_c = B_k C_p \bar{x}_p + A_k \bar{x}_c, \quad \bar{x}_c(0) = \frac{1}{N} \sum_{i=1}^N x_{ci}(0). \quad (45)$$

If the controller matrices  $A_k$ ,  $B_k$ ,  $C_k$  are given, conditions (32) and (41) are convex in the Lyapunov matrices  $P_i$ ,  $i = 2, \dots, N$ . However, if we consider the controller matrices as variables of the problem, (32) and (41) become nonlinear matrix inequalities, and then these conditions cannot straightforwardly be used for the controller design. Nevertheless, using some relaxation techniques, (32) and (41) can be converted to BMI feasibility problems in the controller matrices and the Lyapunov matrices. A possible method to solve feasibility problems involving BMIs is using an iterative LMI procedure. Such a procedure is described in the next section.

## 5. $\mathcal{H}_\infty$ State Synchronization Design

In this Section, we use some relaxation techniques in order to convert (32) and (41) into BMI feasibility problems. In fact, conditions (32) and (41) are hardly tractable from a numerical standpoint, because the controller parameters  $A_k, B_k, C_k$  are coupled with the Lyapunov matrices  $P_i$  for  $i = 2, \dots, N$  and this leads to a possible NP-hard problem without desirable design guarantees. More precisely, the nonlinear term  $P_i \begin{bmatrix} \lambda_i B_{pu} & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} 0 & C_k \\ B_k & A_k \end{bmatrix} \begin{bmatrix} C_p & 0 \\ 0 & I_n \end{bmatrix}$  in the first diagonal block of (32) has been a long-standing obstacle to the derivation of suitable conditions for the dynamic output-feedback design (see (Chilali & Gahinet, 1996)). Thus, in general, the direct design of the controller matrices solving (32) and (41) is unlikely doable. Nevertheless, we are able to provide a dynamic  $H_\infty$  design technique, that is effective in practice in satisfying the requirements of Problem 1.

Based on a suitable congruence transformation of the controller variables (see, e.g., (Masubuchi, Ohara, & Suda, 1998) and (Fichera, Prieur, Tarbouriech, & Zaccarian, 2012)), and relaxation techniques, we give sufficient conditions for (32) and (41). These new conditions are bilinear in the unknown variables. Although those conditions are not convex, they are more tractable from a numerical point of view, and they provide a first step towards a design algorithm for the proposed controller, which is presented in Section 6. The relaxed synchronization conditions are contained in the following theorem.

**Theorem 2:** *Given a desired performance bound  $\gamma > 0$ , if there exist symmetric positive definite matrices  $Y, W \in \mathfrak{R}^{n \times n}$ , matrices  $\hat{A} \in \mathfrak{R}^{n \times n}$ ,  $\hat{B} \in \mathfrak{R}^{n \times p}$ ,  $\hat{C} \in \mathfrak{R}^{m \times n}$ , matrices  $M \in \mathfrak{R}^{2n \times 2n}$  and  $H \in \mathfrak{R}^{2n \times 2n}$ , and a scalar  $\lambda_c$  such that*

$$\begin{bmatrix} Y & I \\ I & W \end{bmatrix} > 0, \quad (46)$$

$$\begin{bmatrix} \Gamma & H & \hat{W}M & \begin{bmatrix} B_{pw} \\ WB_{pw} - \hat{B}D_{pw} \end{bmatrix} & \lambda_i \begin{bmatrix} Y \\ I_n \end{bmatrix} C_{zp}^\top \\ * & (\lambda_i - \lambda_c)\Sigma - M & 0 & 0 & 0 \\ * & 0 & -M & 0 & 0 \\ * & 0 & 0 & -I_q & \lambda_i D_{zw}^\top \\ * & 0 & 0 & * & -\hat{\gamma}^2 I_\ell \end{bmatrix} < 0, \quad i = 2, N \quad (47)$$

where we have defined:

$$\Gamma := \text{He} \left( \begin{bmatrix} A_p Y + \lambda_i B_{pu} \hat{C} & A_p \\ \hat{A} & W A_p - \hat{B} C_p \end{bmatrix} + \hat{W} H \right), \quad (48)$$

$$\hat{W} := \begin{bmatrix} I_n & 0 \\ 0 & W \end{bmatrix}, \quad \Sigma := \text{He} \left( \begin{bmatrix} 0 & 0 \\ B_{pu} \hat{C} & 0 \end{bmatrix} \right), \quad \hat{\gamma}^2 := \lambda_2^2 \gamma^2, \quad (49)$$

then controller (8) with

$$\begin{aligned} C_k &:= \hat{C} Z^{-1} \\ B_k &:= W^{-1} \hat{B} \\ A_k &:= A_p Y Z^{-1} - B_k C_p Y Z^{-1} + \lambda_c B_{pu} C_k - W^{-1} \hat{A} Z^{-1}, \end{aligned} \quad (50)$$

where  $Z := Y - W^{-1}$  (which is guaranteed to be nonsingular), solves Problem 1.

*Proof.* First let us prove that  $Z := Y - W^{-1}$  is nonsingular. To this end consider (46), which gives  $Z > 0$  after a Schur complement. Clearly positive definiteness implies non-singularity of  $Z$ . To prove the rest of the theorem, we exploit Theorem 1 and show that (47) implies (32) specialized with selection (50). Then the result follows from Theorem 1. Suppose that a solution to (47) exists with

variables  $Y, W, \hat{A}, \hat{B}, \hat{C}, M, H, \lambda_c$ . Applying a Schur complement to (47), we obtain

$$\begin{bmatrix} \Gamma_1 & \begin{bmatrix} B_{pw} \\ WB_{pw} - \hat{B}D_{pw} \end{bmatrix} & \lambda_i \begin{bmatrix} Y \\ I_n \end{bmatrix} C_{zp}^\top \\ * & -I_q & \lambda_i D_{zw}^\top \\ * & * & -\hat{\gamma}^2 I_\ell \end{bmatrix} < 0 \quad (51)$$

with

$$\Gamma_1 := \text{He} \left( \begin{bmatrix} A_p Y + \lambda_i B_{pu} \hat{C} & A_p \\ \hat{A} & W A_p - \hat{B} C_p \end{bmatrix} + \hat{W} H \right) + \hat{W} M \hat{W} + H^\top (M - (\lambda_i - \lambda_c) \Sigma)^{-1} H. \quad (52)$$

Since from (47) we have that  $M - (\lambda_i - \lambda_c) \Sigma > 0$ , we can write

$$\left( (M - (\lambda_i - \lambda_c) \Sigma) \hat{W} + H \right)^\top (M - (\lambda_i - \lambda_c) \Sigma)^{-1} \left( (M - (\lambda_i - \lambda_c) \Sigma) \hat{W} + H \right) \geq 0. \quad (53)$$

From (53), and since (again from (47))  $M$  is positive definite, then  $\Gamma_1$  in (52) satisfies

$$\Gamma_1 \geq \Gamma_2 := \text{He} \left( \begin{bmatrix} A_p Y + \lambda_i B_{pu} \hat{C} & A_p \\ \hat{A} & W A_p - \hat{B} C_p \end{bmatrix} \right) + (\lambda_i - \lambda_c) \hat{W} \Sigma \hat{W}, \quad (54)$$

and from (51) and (54) we obtain

$$\begin{bmatrix} \Gamma_2 & \begin{bmatrix} B_{pw} \\ WB_{pw} - \hat{B}D_{pw} \end{bmatrix} & \lambda_i \begin{bmatrix} Y \\ I_n \end{bmatrix} C_{zp}^\top \\ * & -I_q & \lambda_i D_{zw}^\top \\ * & * & -\hat{\gamma}^2 I_\ell \end{bmatrix} < 0. \quad (55)$$

By substituting the expressions (50) into (55) and by defining the matrix (as in (Fichera et al., 2012; Masubuchi et al., 1998)),

$$P := P^\top = \begin{bmatrix} Y & Z \\ Z & Z \end{bmatrix}^{-1} = \begin{bmatrix} W & -W \\ -W & W + Z^{-1} \end{bmatrix}, \quad \Pi := \begin{bmatrix} Y & Z \\ I_n & 0 \end{bmatrix}, \quad (56)$$

where  $Z = Z^\top > 0$ ,  $W = (Y - Z)^{-1}$ , which can be shown to imply  $Z = Y - W^{-1}$  and  $\Pi P = \begin{bmatrix} I_n & 0 \\ W & -W \end{bmatrix}$ , we obtain that (55) is equivalent to

$$\begin{bmatrix} \text{He} \left( \Pi P \begin{bmatrix} A_p & \lambda_i B_{pu} C_k \\ B_k C_p & A_k \end{bmatrix} \Pi^\top \right) & \Pi P \begin{bmatrix} B_{pw} \\ B_k D_{pw} \end{bmatrix} & \Pi \begin{bmatrix} \lambda_i C_{zp}^\top \\ 0 \end{bmatrix} \\ * & -I_q & \lambda_i D_{zw}^\top \\ * & * & -\hat{\gamma}^2 I_\ell \end{bmatrix} < 0. \quad (57)$$

Finally, by pre- and post-multiplying (57) by  $\text{diag}(\Pi^{-1}, I_q, I_\ell)$  and its transpose, we obtain (32) with  $P_i = P$  for  $i = 2, N$ . To prove (32) for  $i = 3, \dots, N-1$ , it is sufficient to perform convex combinations of (57), because  $\lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_{N-1} \leq \lambda_N$ .  $\square$

**Remark 2:** In the matrix inequalities formulation (47) we specify  $P_i$  as a common Lyapunov matrix, i.e.,  $P_i = P$  for  $i = 2, \dots, N$ , when the controller is parameterized as in (50). This choice

is conservative with respect to the matrix inequality formulation in (32). However, this choice allows restricting these inequalities to the cases  $i = 2, N$ —namely, the smallest and largest positive eigenvalues of  $L$ .

**Remark 3:** With respect to the approach used in (Y. Liu & Jia, 2010), we do not constrain the Lyapunov matrix  $P$  in (56) to have a block diagonal structure. This choice leads to possibly less conservative solutions to Problem 1. In fact, constraining the multipliers is usually less conservative than constraining the Lyapunov matrix (see, e.g., (de Oliveira & Skelton, 2001)).

**Remark 4:** There is no loss of generality by parameterizing  $P$  as (56), because this particular structure does not lead to any conservatism of the design conditions (see (Yang & Ye, 2010, Lemma 1)).

**Remark 5:** In the nonlinear matrix inequalities formulation (32), the Lyapunov function matrices  $P_i$  involved in (35), are coupled with the controller matrices  $A_k, B_k, C_k$  in (32). On the other hand, in the relaxed formulation (47), the transformed controller matrices  $\hat{A}, \hat{B}, \hat{C}$ , obtained from (50), are decoupled from the Lyapunov matrices  $Y$  and  $W$ . This decoupling technique arises from the completion of squares (52), in which the slack variables  $H$  and  $M$  are introduced, providing an extra degree of freedom and relaxing the structure of the constraints (32). This makes the constraints (47) more tractable than (32) from a numerical standpoint, and they allow designing the iterative algorithm presented in Section 6.

The following result provides a relaxation of conditions (41) for the noiseless case  $w_i = \mathbf{0}_q$ , for all  $i \in \mathcal{N}$ .

**Corollary 2:** Assume that there exist positive definite matrices  $Y, W \in \mathbb{R}^{n \times n}$ , matrices  $\hat{A} \in \mathbb{R}^{n \times n}$ ,  $\hat{B} \in \mathbb{R}^{n \times p}$ ,  $\hat{C} \in \mathbb{R}^{m \times n}$ , matrices  $M \in \mathbb{R}^{2n \times 2n}$  and  $H \in \mathbb{R}^{2n \times 2n}$ , and a scalar  $\lambda_c$  such that (46) is satisfied and

$$\Upsilon_i := \begin{bmatrix} \text{He} \left( \begin{bmatrix} A_p Y + \lambda_i B_{pu} \hat{C} & A_p \\ \hat{A} & W A_p - \hat{B} C_p \end{bmatrix} + \hat{W} H \right) & H & \hat{W} M \\ * & (\lambda_i - \lambda_c) \Sigma - M & 0 \\ * & * & -M \end{bmatrix} < 0, \quad (58)$$

for  $i = 2, N$ , where  $\hat{W}, \Sigma$  are defined in (49). Then, for any initial condition  $x_i(0) = [x_{pi}(0)^\top \ x_{ci}(0)^\top]^\top \in \mathbb{R}^{2n}$ , the resulting trajectories of the closed-loop system (1), (8), (50), with  $w_i = 0$  for all  $i \in \mathcal{N}$  asymptotically synchronize.

*Proof.* Following the proof of Theorem 2 we obtain that (58) is a sufficient condition for (41).  $\square$

## 6. ILMI Dynamic Output Feedback Design

In this section we address the problem of designing suitable matrices  $A_k, B_k, C_k$  of the distributed dynamic output feedback compensator (8) that solve Problem 1. As shown in the previous sections, the proposed output feedback design problem inherently leads to a nonconvex formulation.

The controller synthesis is obtained based on feasible solutions to the relaxed BMI conditions (47). In fact, according to Theorem 2, any solution to (47) provides suitable controller matrices to solve Problem 1, via selection (50). Furthermore, we investigate through a numerical example the gap between the solution to the relaxed feasibility problem (47), and the original nonlinear one (32).

Of course, we would like to go one step further asking whether it is possible to optimize the distributed controller (8) for better disturbance rejection, which amounts to minimizing the perfor-

mance bound  $\gamma$  in (7). Moreover, taking  $\gamma$  as an additional decision variable allows us to increase the number of feasible solutions to (47), making it easier to find the solution to Problem 1.

More precisely, the problem we want to solve in this section is the following optimization problem

$$\begin{aligned}
 (\gamma^*)^2 := & \min_{W, Y, \hat{A}, \hat{B}, \hat{C}, H, M, \lambda_c, \gamma^2} \gamma^2, \\
 \text{s.t.} & \quad (46), (47), W > 0, Y > 0, \gamma^2 > 0.
 \end{aligned} \tag{59}$$

If the solution to (59) gives a value  $(\gamma^*)^2$  less or equal to the bound prescribed in (7), then, the corresponding controller solves Problem 1. Note that the controller matrices  $A_k, B_k, C_k$  in (8) are obtained from the solution  $\hat{A}, \hat{B}, \hat{C}$  to (59) based on selection (50).

It is well known that BMI problems, like the one we are considering, are NP-hard, and so far there is no polynomial algorithm to compute the optimal solutions (Toker & Özbay, 1995). Since in (47) there are product terms (also known as *complicating variables*) between the Lyapunov parameters and the slack variables (i.e.,  $\hat{W}H$  and  $\hat{W}M$ ), and the controller matrices and the slack variables (i.e.,  $\lambda_c \hat{C}$ ), our approach to solve the non-convex optimization problem (59) is based on an iterative LMI (ILMI) procedure. The proposed algorithm alternates between two different LMI problems: (59) fixing the set of variables  $\{\lambda_c, H, M\}$ , and (59) fixing the set of variables  $\{W, \hat{C}\}$ . At each iteration the value of  $\gamma^2$  is minimized, and a controller with possibly better disturbance rejection is determined.

The procedure of alternating between the LMI problems is an iterative approach allowing to solve nonconvex problems, without a clear guarantee of convergence. Moreover, using the proposed relaxation technique, we can provide only sub-optimal solutions, and consequently a sub-optimal controller. However, this algorithm has been tested in several examples and it is effective in practice in determining a controller solving Problem 1.

The proposed method requires an initially feasible solution from which the suboptimal process starts. In the next section we will provide a method for choosing the values of a set of variables to initialize the algorithm, and we give a detailed description of the algorithm.

### 6.1 Algorithm Initialization and Description

In this section we provide a preliminary procedure to compute initial values of the variables to initialize the design algorithm. Since the size of the optimization problem (59) is large, and the random choice of the initial variables might lead to infeasible solutions to (47), we introduce a preliminary initialization problem, that makes it more likely to find an initial feasible solution to (59). This preliminary procedure consists in finding solutions to the following optimization problem

$$\begin{aligned}
 t^* := & \min_{W, Y, \hat{A}, \hat{B}, \hat{C}, H, M, \lambda_c, t} t, \\
 \text{s.t.} & \quad (46), \Upsilon_i - tI_{6n} < 0, i = 2, N, W > 0, Y > 0,
 \end{aligned} \tag{60}$$

where  $\Upsilon_i$  is defined in (58), and corresponds to the first  $6n \times 6n$  diagonal block in (47). More specifically, we are finding a solution to Problem 1 in the noiseless case  $w_i = \mathbf{0}_q$ . Problem (60) is bilinear in the decision variables and can be solved with the ILMI procedure described above. Note that problem (60) performs the minimization of the variable  $t$ , playing the role of a slack variable with the goal of ensuring feasibility of each step of the iterative algorithm. In fact, for any choice of the parameters, constraint  $\Upsilon_i - tI < 0$ , for  $i = 2, N$ , is always satisfied for  $t$  sufficiently large.

If a solution to (60) exists with  $t^* < 0$ , then  $\Upsilon_i$ , for  $i = 2, \dots, N$  are negative definite and the resulting controller guarantees synchronization of the controlled multi-agent system (1) and (8), according to Corollary 2. We will call such a controller a *synchronizing* controller. The detailed algorithm to solve (60) is described in Algorithm 1. A few useful guarantees stemming from this

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**Algorithm 1** Dynamic output feedback controller design for state synchronization

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**Input:** Matrices  $A_p, B_{pu}, C_p$ , Laplacian  $L$ , and a tolerance  $\tau > 0$ .

**Initialization:** Set  $\lambda_c = 0, H = 0$  and  $M = 0$ .

**Iteration**

*Step 1:* Given  $M, H$  and  $\lambda_c$  from the previous step, solve the convex optimization problem

$$\begin{aligned} & \min_{W, Y, \hat{A}, \hat{B}, \hat{C}, t} t, \\ \text{s.t. } & (46), \Upsilon_i - tI_{6n} < 0, i = 2, N, W = W^\top > 0, Y = Y^\top > 0. \end{aligned} \quad (61)$$

Pick the optimal solution  $\hat{C}$  and  $W$  corresponding to the minimum value of  $t$  for the next step.

*Step 2:* Given  $\hat{C}$  and  $W$  from the previous step, solve the convex optimization problem

$$\begin{aligned} & \min_{Y, \hat{A}, \hat{B}, H, M, \lambda_c, t} t, \\ \text{s.t. } & (46), \Upsilon_i - tI_{6n} < 0, i = 2, N, Y = Y^\top > 0. \end{aligned} \quad (62)$$

Pick the optimal solution  $M, H, \lambda_c$  corresponding to the minimum value of  $t$  for the next step.

**until**  $t$  does not decrease more than  $\tau$  over three consecutive steps.

**Output:**  $\hat{A}, \hat{B}, \hat{C}, W, Y, H, M, \lambda_c$  and  $t^* = t$ .

---

algorithm are formalized next.

**Proposition 1:** *Algorithm 1 enjoys the following properties: 1) Both LMIs at steps 1 and 2 are feasible for any value of the system parameters; 2) The optimal value of  $t$  can never increase when going from step 1 to step 2 and vice-versa.*

*Proof.* Item 1 follows from the fact that for any selection of  $W, Y, \hat{A}, \hat{B}, \hat{C}, H, M, \lambda_c$ , there exists a large enough value of  $t$  satisfying the constraints in (61) and (62). Item 2 instead follows from the fact that the optimal solution of the previous step remains a feasible solution of the subsequent step.  $\square$

The preliminary procedure of Algorithm 1 is convenient because the synchronizing controller design requires the iterative solution to a set of BMIs of smaller size as compared to conditions (47). Moreover, the existence of a synchronizing controller is necessary for the existence of a sub-optimal controller. The algorithm for the sub-optimal controller design to solve (59) is presented in Algorithm 2. The solution to (60) coming from Algorithm 1 is taken as a starting point. Algorithm 2 only works if this solution is associated to  $t^* < 0$ —that is, it corresponds to a synchronizing controller.

We may once again establish useful properties of Algorithm 2 in the following proposition whose proof is omitted because it parallels that of Proposition 1.

**Proposition 2:** *Algorithm 2 enjoys the following properties: 1) Both LMIs at steps 1 and 2 are feasible if the algorithm is initialized with a solution  $\hat{A}, \hat{B}, \hat{C}, W, t^*$  from Algorithm 1 such that  $t^* < 0$ ; 2) The optimal value of  $\gamma^2$  can never increase when going from step 1 to step 2 and vice-versa.*

Note that if no synchronizing controller is found by Algorithm 1, no sub-optimal controller can be found either, because  $\Upsilon_i$  in (41) would not be negative definite for some  $i$ , and (47) is violated.

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**Algorithm 2** Dynamic output feedback controller design with  $\mathcal{H}_\infty$  performance

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**Input:** Matrices  $A_p, B_{pu}, C_p, B_{pw}, D_{pw}, C_{zp}, D_{zw}$ , the output parameters of Algorithm 1, Laplacian  $L$ , and a tolerance  $\tau > 0$ . A solution  $\hat{A}, \hat{B}, \hat{C}, W, t^*$  from Algorithm 1 such that  $t^* < 0$ .

**Iteration**

*Step 1:* Given  $\hat{C}$  and  $W$  from the previous step, solve the convex optimization problem

$$\begin{aligned} & \min_{Y, \hat{A}, \hat{B}, H, M, \lambda_c, \gamma^2} \gamma^2, \\ \text{s.t.} \quad & (46), (47), Y = Y^\top > 0, \gamma^2 > 0. \end{aligned} \tag{63}$$

Pick the optimal solution  $M, H, \lambda_c$  corresponding to the minimum value of  $\gamma^2$  for the next step.

*Step 2:* Given  $M, H$  and  $\lambda_c$  from the previous step, solve the convex optimization problem

$$\begin{aligned} & \min_{W, Y, \hat{A}, \hat{B}, \hat{C}, \gamma^2} \gamma^2, \\ \text{s.t.} \quad & (46), (47), W = W^\top > 0, Y = Y^\top > 0, \gamma^2 > 0. \end{aligned} \tag{64}$$

Pick the optimal solution  $\hat{C}$  and  $W$  corresponding to the minimum value of  $\gamma^2$  for the next step. **until**  $\gamma^2$  does not decrease more than  $\tau$  over three consecutive steps.

Compute  $A_k, B_k, C_k$  from (50).

**Output:**  $A_k, B_k, C_k$  and  $(\gamma^*)^2 = \gamma^2$ .

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## 7. Simulations

We provide an illustrative example to show the effectiveness of the controller design presented in Section 6. Consider a multi-agent system composed by  $N = 6$  agents, each of them described by (1), and the following data

$$\begin{aligned} A_p &= \begin{bmatrix} 0.05 & 0.9 \\ -0.9 & 0.05 \end{bmatrix}, B_{pu} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, B_{pw} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C_p = [0 \quad 1], \\ C_{zp} &= [1 \quad 1], D_{pw} = D_{zw} = 0, \end{aligned} \tag{65}$$

which has been selected to obtain an oscillatory response with exponential divergence. The interconnection graph depicted in Figure 2 represents the communications among the agents in the network.

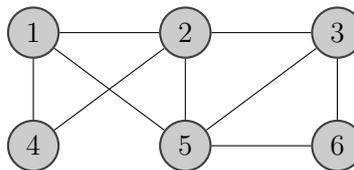


Figure 2. Network interconnections of the multi-agent system (1) with data (65) in Example 7.

Algorithm 1 and Algorithm 2 are implemented in MATLAB and solved using the YALMIP toolbox (Löfberg, 2004), and the MOSEK solver (Mosek, 2010).

Algorithm 1 is run with tolerance  $\tau = 10^{-4}$ , and gives  $t^* = -0.61$ , see Figure 3 (top). Algorithm 2 can therefore be initialized, and the minimization of  $\gamma^2$  is shown in Figure 3 (bottom). The resulting

sub-optimal controller matrices are:

$$A_k = \begin{bmatrix} -0.52 & 5.38 \\ 0.72 & -25.78 \end{bmatrix}, \quad B_k = \begin{bmatrix} -0.054 \\ 2.686 \end{bmatrix}, \quad C_k = [0.37 \quad -1.37] \quad (66)$$

and  $(\gamma^*)^2 = 39.4$ . We consider the following piecewise constant disturbance  $w_i \in \mathcal{L}_2$ , for all  $i \in \mathcal{N}$

$$w_i = \begin{cases} w_{0i} & \text{if } 20s \leq t \leq 25s \\ 0 & \text{otherwise,} \end{cases} \quad i \in \mathcal{N}, \quad (67)$$

with  $w_{0i}$  constant values randomly chosen in the interval  $[-1, 1]$ .

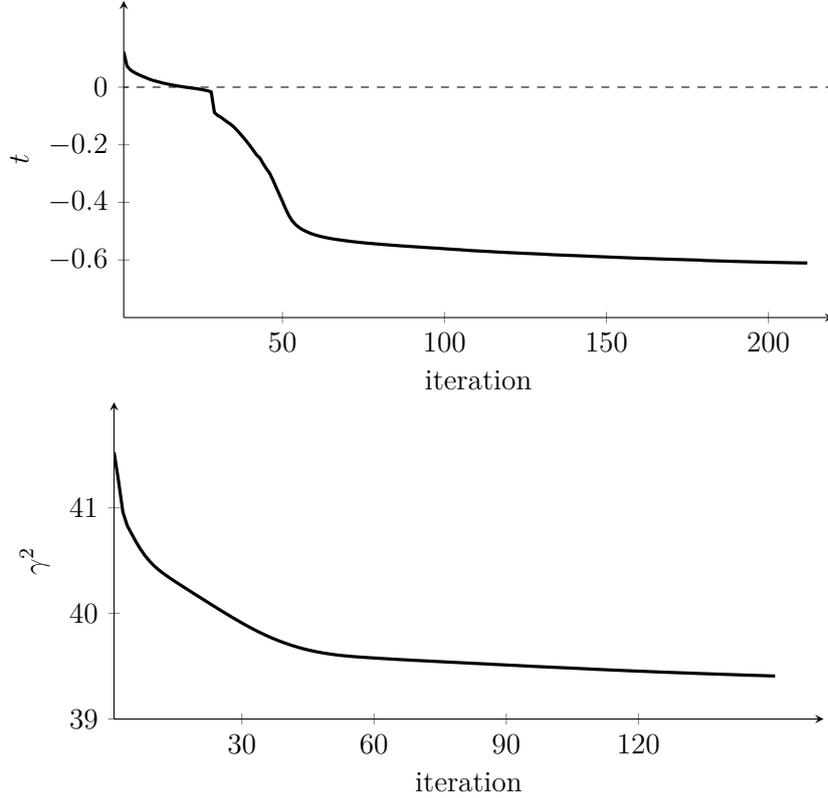


Figure 3. Minimization of the value of  $t$  (left) and  $\gamma$  (right) performed by Algorithm 1 and Algorithm 2, respectively. The minimum values of  $t$  and  $\gamma$  are  $t^* = -0.61$  and  $(\gamma^*)^2 = 39.4$ , respectively.

The time responses of the closed-loop multi-agent system (1) with data (65), and (8) with data (66), and  $w_i = \mathbf{0}_q$  for all  $i \in \mathcal{N}$ , are depicted in Figure 4. Each plot represents the time evolution of the components  $x_{pi}^{(1)}$  and  $x_{pi}^{(2)}$  of the states  $x_{pi} \in \mathfrak{R}^2$ , for all the agents  $i = 1, \dots, 6$ . We observe that the agents reach state synchronization, and from Corollary 1, we know that they synchronize to the solution to

$$\dot{\bar{x}}_p = A_p \bar{x}_p, \quad \bar{x}_p(0) = \frac{1}{N} \sum_{i=1}^6 x_i(0), \quad (68)$$

which is,

$$\bar{x}_p(t) = e^{0.05t} \left( k_1 \begin{bmatrix} \frac{1}{\sqrt{2}} \cos(0.9t) \\ \frac{1}{\sqrt{2}} \sin(0.9t) \end{bmatrix} + k_2 \begin{bmatrix} \frac{1}{\sqrt{2}} \sin(0.9t) \\ -\frac{1}{\sqrt{2}} \cos(0.9t) \end{bmatrix} \right), \quad (69)$$

where  $k_1, k_2$  are constants that depend on the initial conditions  $x_{pi}(0)$ , with  $i \in \mathcal{N}$ . We observe that the synchronization trajectory grows unbounded.

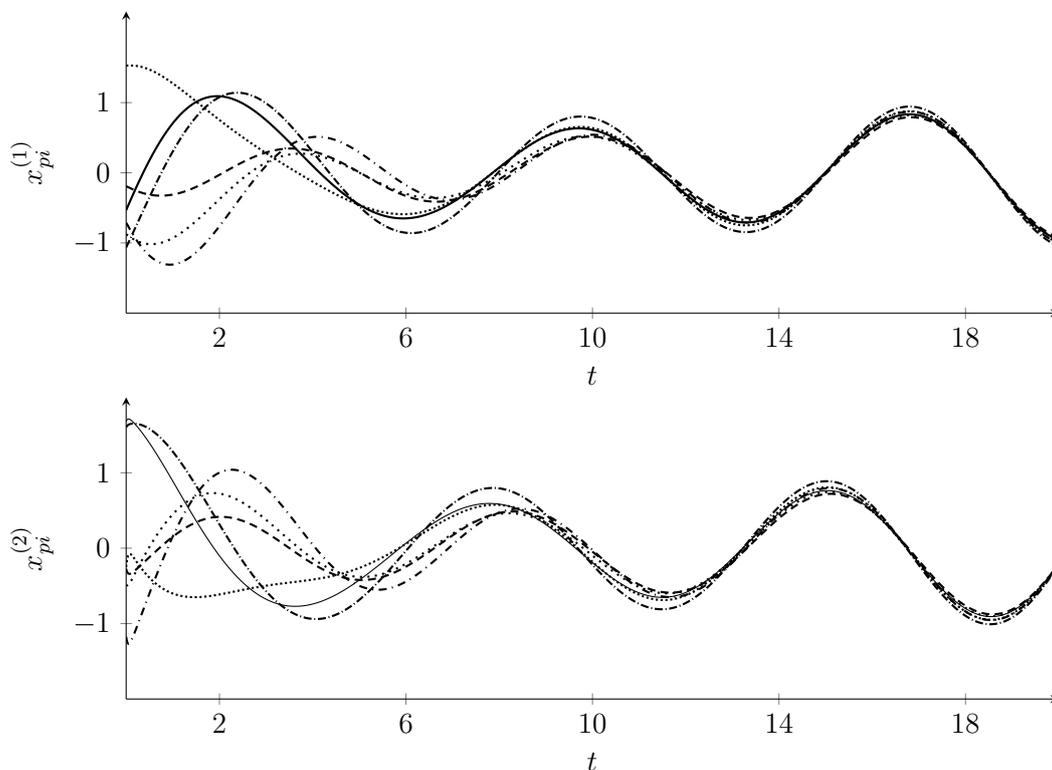


Figure 4. Time evolution of the agent state components  $x_{pi}^{(1)}$  (top) and  $x_{pi}^{(2)}$  (bottom), for all the agents  $i \in \mathcal{N}$ . The agents synchronize to the trajectory characterized in (69).

The same example is revisited with the addition of noise  $w_i$  as in (67), as per (1) with data (65), with dynamic output feedback compensator (8) with data (66). Figure 5 shows the time responses of the state components of the vector  $x_p$ . We can see that the agents initially reach state synchronization. As the perturbation is applied at  $t = 20s$ , the agent trajectories drift away from the desired synchronization trajectory (69), due to the noise term. When the disturbances vanish, that is, for  $t > 25s$ , state synchronization is achieved again.

From both scenarios presented in Figures 4 and 5, we conclude that the proposed controller guarantees state synchronization.

We want to conclude this example by providing further insights on the conservatism introduced by the relaxed conditions (47) for the  $\mathcal{H}_\infty$  design. To this end we want to compare the sub-optimal solution given by (59) with the analysis conditions given in (32). More precisely, we plug the controller matrices  $A_k, B_k, C_k$  into conditions (32), and we compute the minimum value of  $\gamma^2$ , such that (32) is satisfied. This procedure gives an optimal value of  $\gamma^2 = 37.4$ , which is reasonably close to the one obtained from Algorithm 2 (see Figure 3) and a common quadratic Lyapunov function. Although the numerical gap between the nonlinear conditions (32) and (47) cannot be mathematically quantified, this qualitative analysis suggests that the relaxation performed in Section 6 is a good alternative to

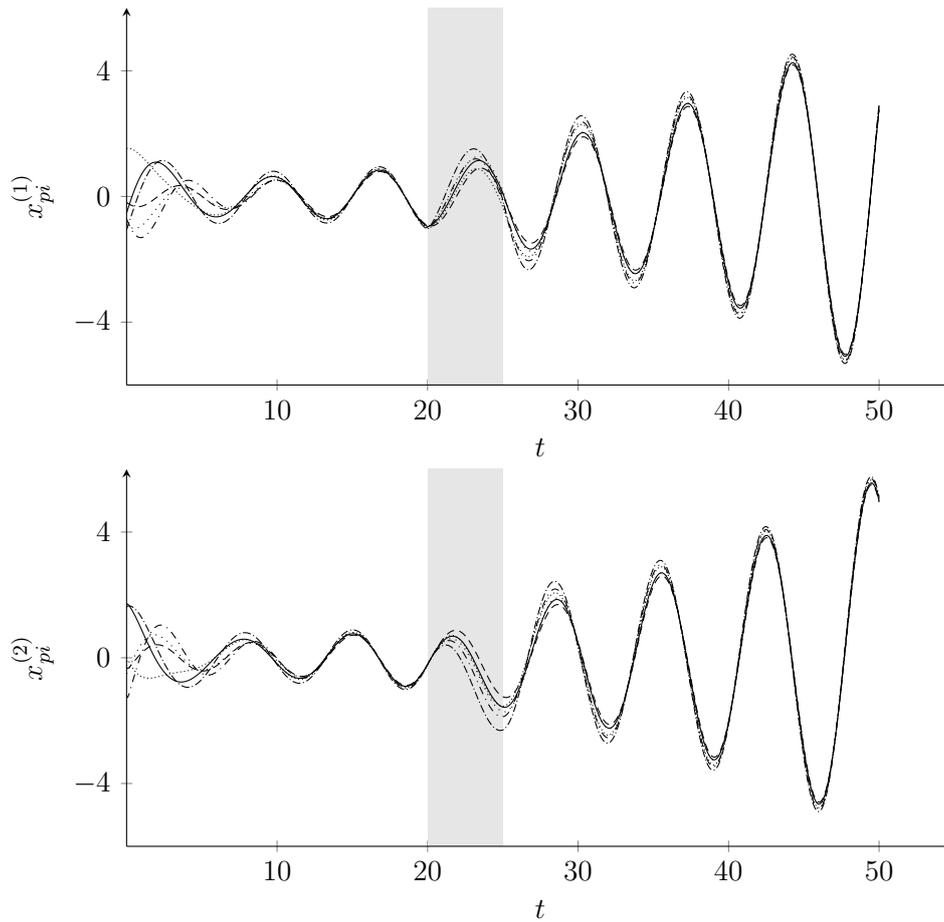


Figure 5. Time evolution of the agent state components. The plots show the time evolution of the components of the state  $x_p$ . The gray area delimits the time interval  $20s \leq t \leq 25s$ , in which the disturbances  $w_i$  in (67) are nonzero.

the nonlinear formulation (32).

## 8. Conclusions

In this paper we have analysed and discussed the effects of external disturbances on the synchronization of identical linear time-invariant multi-agent systems. Sufficient conditions for the existence of a dynamic output feedback protocol that ensures synchronization of the considered multi-agent system with a prescribed  $\mathcal{L}_2$  gain have been given. The protocol can be implemented by each agent in a decentralised fashion. The controller matrices are computed using an algorithm based on the solution of an iterative LMI problem. As a possible topic for future research we intend to investigate the  $\mathcal{H}_\infty$  synchronization problem of multi-agent systems subject to limited informations, such as input saturation constraints and time delays.

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