Global Asymptotic Stability of a PID Control System with Coulomb Friction

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***Abstract*—For a point mass subject to Coulomb friction in feedback with a PID controller, we consider a model based on a differential inclusion comprising all the possible magnitudes of static friction during the stick phase and having unique solutions. We study the set of all equilibria and we establish its global asymptotic stability using a discontinuous Lyapunov-like function, and a suitable LaSalle's invariance principle. The well-posedness of the proposed model allows to establish useful robustness results, including an ISS property from a suitable input in a perturbed context. Simulation results are also given to illustrate our statements.**

I. Introduction

WITHIN the control community, the interest in the dynamical properties of friction had its peak in the 1990's, and the control engineering reasons for this interest are lucidly argued in [15, §1]. These properties have been studied along a modeling direction in the Dahl model [9], the LuGre model [4], [7], the models by Bliman and Sorine [6] and the Leuven model [21]. In particular, when a mass moves with steady velocity and the corresponding friction force is measured, there is a small interval of velocities near zero where the friction force decreases before increasing again due to viscous friction and this behaviour is given the name of *Stribeck effect*. In [6], considering friction dependent only on the path, allows using the theory of hysteresis operators [23] and the LuGre model itself proved to be amenable to theoretical analysis, as [5] presents necessary and sufficient conditions for the passivity of its underlying operator from velocity to friction force.

In this work, we consider a point mass under Coulomb friction and actuated by a proportional-integral-derivative (PID) controller, which is a classical problem in the (control-oriented) friction literature. However, we characterize Coulomb friction as set-valued, obtaining a differential inclusion for the whole system. Starting from proving uniqueness of solutions, we prove through a Lyapunov-like function the global asymptotic stability of the attractor having zero velocity, zero position and a bounded integral error. The use of a set-valued mapping for the friction force can be seen as quite natural and is taken into consideration in [6], [16], [22]: in [22] it is applied to uncontrolled multi-degree-of-freedom mechanical systems with unique solutions, in [16] to a PD controlled 1 degree-of-freedom system. The combination of set-valued friction laws and Lyapunov tools is also the subject of [14, Chap. 5-6]. The mathematical challenges associated with Coulomb friction are also extensively illustrated in [20] and references therein.

Global asymptotic stability was proven in the Russian literature, as highlighted in [24], which exploits solution properties like dichotomy. Additionally, it was proved (see [1, Thm. 1] and the related works [2], [3]) that in our same setting of PID control there exists no stick-slip limit cycle (the so-called hunting phenomenon, see [1, p. 679] and [7, §V-A.]; see [1] and Remark 1 for the definitions of stick and slip). On the other hand, we provide here an alternative proof of global asymptotic stability, which we complement with various robustness results. Our proof is based on the construction of an intuitive Lyapunov-like function discussed in Section III, which has an explicit expression, is not built from solutions and can lead to compensation schemes recovering exponential stability, or can be used for nonlinear control design in a control Lyapunov function approach.

The properties established through Lyapunov tools for differential inclusions and the regularity of our model imply robustness of asymptotic stability. This, in turn, allows us to prove an input-to-state stability (ISS) property for the perturbed dynamics, establishing that more general friction phenomena (including the Stribeck effect) cause a gradual deterioration of the response, in an ISS sense. We regard this work as a stepping stone to static friction larger than Coulomb and to its description through hybrid systems [11], and to proposing compensation schemes using hybrid friction laws.

The paper is structured as follows. We present the proposed model and the main results in Section II. The novel Lyapunov-like function that we introduce in this work, together with its relevant properties, is presented in Section III. Section IV illustrates our main results by simulation. The proofs are reported at the end of the paper in Section V.

Notation. The sign function is defined as: $\mathrm{sign}(x) := 1$ if $x > 0$, $\mathrm{sign}(0) := 0$, $\mathrm{sign}(x) := -1$ if $x < 0$. The saturation function is defined as: $\mathrm{sat}(x) := \mathrm{sign}(x)$ if $|x| > 1$, $\mathrm{sat}(x) := x$ if $|x| \leq 1$. For $c \neq 0$, the deadzone function is defined as $\mathrm{dz}_c(x) := x - c\,\mathrm{sat}\left(\frac{x}{c}\right)$. $|x|$ denotes the Euclidean norm of vector x. The distance of a vector $x \in \mathbb{R}^n$ to a closed set $\mathcal{A} \subset \mathbb{R}^n$ is defined as $|x|_{\mathcal{A}} := \inf_{y \in \mathcal{A}} |x - y|$. $\langle \cdot, \cdot \rangle$ defines the inner product between its two vector arguments.

Research supported in part by AFOSR grant FA9550-15-1-0155 and NSF grant ECCS-1508757, by the ANR project LimICoS contract number 12 BS03 005 01, and by grant OptHySYS funded by the University of Trento.

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A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is lower semicontinuous (lsc) if $\liminf_{x\to x_0} f(x) \geq f(x_0)$ for each point x_0 in its domain.

II. Proposed model and main results

A. Derivation of the model

Consider a point mass m described by position s and velocity v. The overall friction force f_f acting on the mass comprises both Coulomb and viscous friction and is parametrized (see [2, Eq. (3)], or similarly [15, Eq. (5)]) by a Coulomb friction constant $\bar{f}_c > 0$ and by the viscous friction constant $\alpha_v > 0$. The expression of f_f reads then

$$f_f(f_r, v) := \begin{cases} \bar{f}_c \operatorname{sign}(v) + \alpha_v v, & \text{if } v \neq 0 \\ f_r, & \text{if } v = 0,\ |f_r| < \bar{f}_c \\ \bar{f}_c \operatorname{sign}(f_r), & \text{if } v = 0,\ |f_r| \geq \bar{f}_c \end{cases} \tag{1}$$

where f_r is the resultant tangential force. The mass is actuated by the PID control u_{PID}

$$\begin{aligned} u_{\text{PID}}(t) &:= -\bar{k}_p s(t) - \bar{k}_i \int_0^t s(\tau) d\tau - \bar{k}_d \frac{ds(t)}{dt} \\ &= -\bar{k}_p s(t) - \bar{k}_i e_i(t) - \bar{k}_d v(t), \end{aligned} \tag{2}$$

where e_i is defined to be the integral of the position error and is the state of the controller, satisfying $\dot{e}_i = s$ and $e_i(0) = 0$.

By Newton's law, the mechanical dynamics is $\dot{s} = v$ and $m\dot{v} = u_{\text{PID}} - f_f(u_{\text{PID}}, v)$. The convenient definitions $u := \frac{u_{\text{PID}} - \alpha_v v}{m}$, $(k_p, k_v, k_i) := (\frac{\bar{k}_p}{m}, \frac{\bar{k}_d + \alpha_v}{m}, \frac{\bar{k}_i}{m})$ and $f_c := \frac{\bar{f}_c}{m}$ yield

$$\dot{e}_i = s \tag{3a}$$

$$\dot{s} = v \tag{3b}$$

$$\dot{v} = \begin{cases} u - f_c & \text{if } v > 0 \text{ or } (v = 0, u \geq f_c) \\ 0 & \text{if } (v = 0, |u| < f_c) \\ u + f_c & \text{if } v < 0 \text{ or } (v = 0, u \leq -f_c) \end{cases} \tag{3c}$$

$$u = -k_p s - k_v v - k_i e_i, \tag{3d}$$

where we used that $u_{\text{PID}} = m\, u$ for $v = 0$.

Model (3) arises from a relatively intuitive mechanical description of the forces acting on the point mass. Despite the discontinuous right hand side of (3), existence of solutions for any initial condition can be proven through similar reasonings to those in the proof of the subsequent Claim 1. For establishing stability properties we use the monotone set-valued friction law [14, Eq. 5.36], for which existence of solutions is structurally guaranteed. By defining the overall state $z := (e_i, s, v)$, this is equivalent to applying the Filippov [10] or Krasovskii regularization to the discontinuous dynamics (3) and obtaining

$$\dot{z} \in \begin{bmatrix} s \\ v \\ -k_i e_i - k_p s - k_v v \end{bmatrix} - f_c \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \operatorname{SGN}(v) =: \tilde{F}(z) \tag{4a}$$

where the function SGN is a set-valued mapping defined as

$$\operatorname{SGN}(v) := \begin{cases} \operatorname{sign}(v), & \text{if } v \neq 0 \\ [-1, 1], & \text{if } v = 0. \end{cases} \tag{4b}$$

Note that model (4) recognizes that the Coulomb friction can be selected as any force in the set $[-\bar{f}_c, \bar{f}_c]$ when v is zero and has magnitude $\bar{f}_c$ and direction opposite to v whenever $v \neq 0$. The following Lemma 1 establishes that *no* artificial solution is introduced by such an enriched dynamics. (Relative to (3) or (4), we consider a *solution* to be any locally absolutely continuous function $\mathbf{x}$ that satisfies respectively $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t))$ or $\dot{\mathbf{x}}(t) \in \mathbf{F}(\mathbf{x}(t))$ for almost all t in its domain.) Indeed, once existence of solutions is proven for (3) (by similar reasonings to those in Claim 1), uniqueness of solutions for (4) implies that the unique solution to (4) must necessarily be the unique solution to (3) because (3) allows for only *some* selections of $\dot{v}$ compared to those allowed by (4), so that any solution to (3) is also a solution to (4).

Lemma 1: For any initial condition $z(0) \in \mathbb{R}^3$, system (4) has a unique solution defined for all $t \geq 0$. ⌟

Proof. Existence of solutions follows from [10, §7, Thm. 1] because the mapping in (4) is outer semicontinuous and locally bounded with nonempty compact convex values (see also [11, Prop. 6.10]). Completeness of maximal solutions follows from local existence and no finite escape times, as (4) can be regarded as a linear system forced by a bounded input. To prove uniqueness, consider two solutions $z_1 = (z_{1,e_i}, z_{1,s}, z_{1,v})$, z_2 both starting at z_0 and define $\delta = (\delta_{e_i}, \delta_s, \delta_v) := z_1 - z_2$. Then, $\delta(0) = 0$ and, for almost all $t \geq 0$,

$$\begin{aligned} \dot{\delta}(t) \in & A_\delta \delta(t) - f_c \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \big(\operatorname{SGN}(z_{1,v}(t)) - \operatorname{SGN}(z_{1,v}(t) - \delta_v(t)) \big) \\ & A_\delta := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k_i & -k_p & -k_v \end{bmatrix}. \end{aligned} \tag{5}$$

Denote λ_δ the maximum singular value of A_δ. Therefore we can write for almost all $t \geq 0$

$$\begin{aligned} &\tfrac{d}{dt} \tfrac{|\delta(t)|^2}{2} = \delta(t)^T \dot{\delta}(t) \leq \lambda_\delta |\delta(t)|^2 + M(t) \\ &M(t) := \max_{\substack{f_1 \in f_c \operatorname{SGN}(z_{1,v}(t)) \\ f_2 \in f_c \operatorname{SGN}(z_{1,v}(t) - \delta_v(t))}} \delta_v(t)(f_2 - f_1). \end{aligned}$$

Whether $z_{1,v}(t)$ and $z_{1,v}(t) - \delta_v(t)$ are positive, zero or negative, by trivial inspection of all the cases it can be shown that $M(t) \leq 0$ for all $t \geq 0$. Therefore,

$$\tfrac{d}{dt} \tfrac{|\delta(t)|^2}{2} \leq \lambda_\delta |\delta(t)|^2 \text{ for almost all } t \geq 0,$$

where $\delta(0) = 0$ implies $\delta(t) = 0$ (i.e., $z_1(t) = z_2(t)$) for all $t \geq 0$ due to comparison theorems (like [13, Lem. 3.4]). □

B. Main results

The advantage in the use of the compact dynamics (4) is that we may adopt Lyapunov tools to study the asymptotic stability properties of the rest position under the following standard assumption (see, e.g., [1]).

Assumption 1: The parameters in (3d) are such that

$$k_i > 0,\ k_p > 0,\ k_v k_p > k_i. \qquad \text{⌟}$$

According to the Routh stability test, Assumption 1 holds if and only if the origin of the dynamics in (4) with $f_c = 0$ is globally exponentially stable.

Under Assumption 1, one readily sees that all possible equilibria of (4) correspond to $(e_i, s, v) = (\bar{e}_i, 0, 0)$ with $|\bar{e}_i| \leq \frac{f_c}{k_i}$, that is, whenever the mass is at rest at zero position and the size of the integral error e_i is bounded by the threshold $\frac{f_c}{k_i}$. Any of these points is an equilibrium for (4) because

in (4) a value can be selected from $f_c\,\mathrm{SGN}(0)$ such that the (unique) solution maintains $\dot{z}$ identically zero. Note that here we consider the problem of tracking a position setpoint $s^o = 0$, but constant position setpoints s^o can be tracked because we can shift the position coordinate to $s - s^o$ and our results are global. Denote then the set of the equilibria as

$$\mathcal{A} := \left\{ (e_i, s, v) : \ s = 0, \ v = 0, e_i \in \left[-\frac{f_c}{k_i}, \frac{f_c}{k_i} \right] \right\}. \quad (6)$$

Proposition 1: Under Assumption 1, the attractor $\mathcal{A}$ in (6) is 1) globally attractive and 2) Lyapunov stable for (4). ⌟

The global attractivity of $\mathcal{A}$ is proven in Section V-B (by a suitable discontinuous Lyapunov-like function and an integral version of LaSalle's invariance principle) and its stability in Section V-C. Note that no smaller set could be proven to be globally attractive because $\mathcal{A}$ is a union of equilibria. Our results are presented for a symmetric Coulomb friction $f_c\,\mathrm{SGN}(v)$ in (4a), but *all* of them hold for a translated attractor in the case of *asymmetric* Coulomb friction $f_c\,\mathrm{SGN}(v) - f_0$, for any $f_0 \in \mathbb{R}$. This fact can be proven by shifting by f_0 the coordinate ϕ introduced in Section III.

With $\mathbb{B}$ denoting the closed unit ball, $\overline{\mathrm{co}}$ the closed convex hull of a set, and $\rho : \mathbb{R}^3 \to \mathbb{R}_{\geq 0}$ a suitable continuous perturbation function satisfying $z \notin \mathcal{A} \Rightarrow \rho(z) > 0$ and vanishing in $\mathcal{A}$, we have the following perturbation of dynamics $\tilde{F}$ in (4a):

$$\dot{z} \in \overline{\mathrm{co}}\tilde{F}(z + \rho(z)\mathbb{B}) + \rho(z)\mathbb{B}. \quad (7)$$

Our main result in Theorem 1 establishes the following two relevant stability properties involving the solutions to (7). *Robust global $\mathcal{KL}$ asymptotic stability of $\mathcal{A}$* corresponds to the existence of $\beta_0 \in \mathcal{KL}$ such that all solutions to (7) satisfy $|z(t)|_{\mathcal{A}} \leq \beta_0(|z(0)|_{\mathcal{A}}, t)$ for all $t \geq 0$ [11, Def. 7.18(a)], and is equivalent to *robust uniform global asymptotic stability of $\mathcal{A}$*, namely the property that $\mathcal{A}$ is uniformly globally stable and attractive for (7) (see [11, Def. 3.6]), thanks to [11, Thm. 3.40].

Theorem 1: Under Assumption 1, the attractor $\mathcal{A}$ in (6) is robustly uniformly globally asymptotically stable and robustly globally $\mathcal{KL}$ asymptotically stable. ⌟

Proof. Theorem 1 follows from Proposition 1 thanks to [11, Thm. 7.21], which applies because (4) is well-posed from the regularity of $\tilde{F}$ [11, Thm. 6.30] and $\mathcal{A}$ is compact. □

From Theorem 1, arbitrarily small discrepancy in the friction parameters do not destroy the established stability property.

A specific perturbation arises when selecting a constant scalar $\rho_v \in \mathbb{R}$ and perturbing the friction effect as follows:

$$\dot{z} \in \begin{bmatrix} s \\ v \\ -k_i e_i - k_p s - k_v v \end{bmatrix} - f_c \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mathrm{SGN}_{\rho_v}(v) \quad (8)$$

$$\mathrm{SGN}_{\rho_v}(v) := \begin{cases} [\mathrm{sign}(v) - |\rho_v|, \mathrm{sign}(v) + |\rho_v|], & \text{if } |v| > |\rho_v| \\ [-1 - |\rho_v|, 1 + |\rho_v|], & \text{if } |v| \leq |\rho_v|. \end{cases}$$

Note that the inflation of the set-valued mapping SGN in (4b) is $\mathrm{SGN}(v + |\rho_v|\mathbb{B}) + |\rho_v|\mathbb{B}$, from (7). This inflation coincides with SGN_{ρ_v} in (8), and in the special case $\rho_v = 0$, SGN_0 clearly coincides with SGN. This perturbation is of interest because it comprises the Stribeck effect, as discussed after the statement of Proposition 2. Its proof, reported in Section V-D to avoid breaking the flow of the exposition, exploits an interesting consequence of the robustness result established in Theorem 1, namely the semiglobal practical robust asymptotic stability of attractor $\mathcal{A}$ [11, Def. 7.18(b)] established in [11, Lemma 7.20].

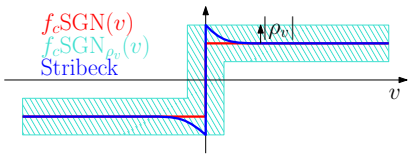

Fig. 1. Stribeck effect is included in the perturbation (8).

Proposition 2: Under Assumption 1, $\mathcal{A}$ in (6) is globally input-to-state stable for dynamics (8) from input ρ_v. ⌟

A consequence of Proposition 2 is that the Stribeck effect, which is known to lead to persistent oscillations (the so-called *hunting* phenomenon), produces solutions that are graceful degradations in the ISS sense of the asymptotically stable solutions to the unperturbed dynamics because Stribeck deformations lead to graphs included in the graph of $f_c\,\mathrm{SGN}_{\rho_v}(v)$, as shown in Figure 1.

III. A DISCONTINUOUS LYAPUNOV-LIKE FUNCTION

To prove Proposition 1, we adopt a specific change of coordinates $x := (\sigma, \phi, v)$ for (4), that is,

$$\sigma := -k_i s, \quad \phi := -k_i e_i - k_p s, \quad v := v. \quad (9)$$

The change of coordinates is nonsingular thanks to Assumption 1 (k_i, k_p strictly positive) and it rewrites (4) as

$$\begin{aligned} \dot{x} := \begin{bmatrix} \dot{\sigma} \\ \dot{\phi} \\ \dot{v} \end{bmatrix} &\in \begin{bmatrix} -k_i v \\ \sigma - k_p v \\ \phi - k_v v - f_c\,\mathrm{SGN}(v) \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} 0 & 0 & -k_i \\ 1 & 0 & -k_p \\ 0 & 1 & -k_v \end{bmatrix}}_{=:A} \begin{bmatrix} \sigma \\ \phi \\ v \end{bmatrix} - \underbrace{\begin{bmatrix} 0 \\ 0 \\ f_c \end{bmatrix}}_{=:b} \mathrm{SGN}(v) \\ &= Ax - b\,\mathrm{SGN}(v) =: F(x). \end{aligned} \quad (10)$$

In the coordinates x introduced in (9), the attractor $\mathcal{A}$ in (6) can be expressed as

$$\mathcal{A} = \{(\sigma, \phi, v) : |\phi| \leq f_c, \sigma = 0, v = 0\}. \quad (11)$$

Among other things, the simple expression of $\mathcal{A}$ in (11) allows writing explicitly the distance of a point x to $\mathcal{A}$ as

$$|x|_{\mathcal{A}}^2 := \left(\inf_{y \in \mathcal{A}} |x - y| \right)^2 = \sigma^2 + v^2 + \mathrm{dz}_{f_c}(\phi)^2 \quad (12)$$

where $\mathrm{dz}_{f_c}(\phi) := \phi - f_c\,\mathrm{sat}(\phi/f_c)$ is the symmetric scalar deadzone function returning zero when $\phi \in [-f_c, f_c]$. Indeed, the rightmost expression in (12) follows from separating the cases $\phi < -f_c$, $|\phi| \leq f_c$, $\phi > f_c$ and applying the definition given by the middle expression in (12).

For dynamics (10), we introduce the discontinuous Lyapunov-like function:

$$\begin{aligned} V(x) &:= \begin{bmatrix} \sigma \\ v \end{bmatrix}^T \begin{bmatrix} \frac{k_v}{k_i} & -1 \\ -1 & k_p \end{bmatrix} \begin{bmatrix} \sigma \\ v \end{bmatrix} + \min_{f \in f_c\,\mathrm{SGN}(v)} |\phi - f|^2 \\ &= \min_{f \in f_c\,\mathrm{SGN}(v)} \begin{bmatrix} \phi - f \\ \sigma \\ v \end{bmatrix}^T P \begin{bmatrix} \phi - f \\ \sigma \\ v \end{bmatrix} \end{aligned} \quad (13a)$$

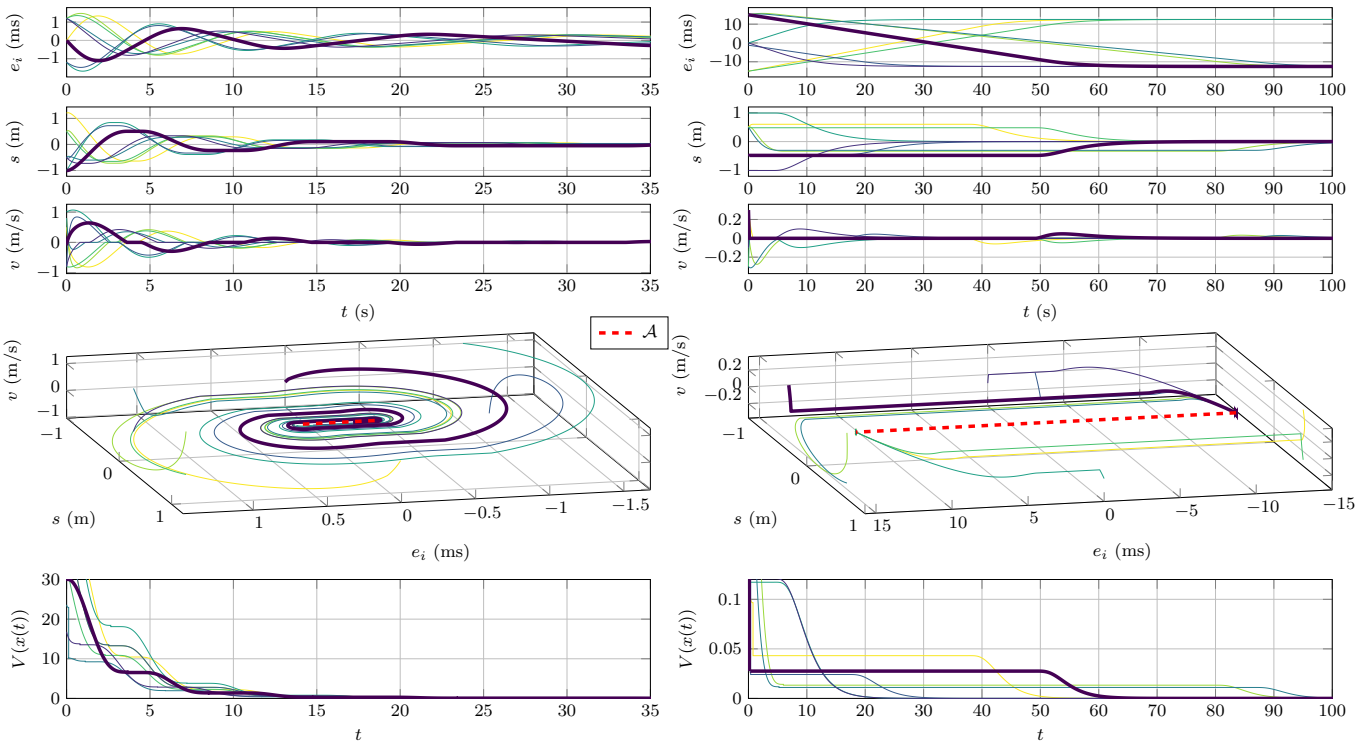

Fig. 2. Top: solutions to (4) for different initial conditions. Center: phase portraits for the same solutions. Bottom: Lyapunov-like function V in (13) evaluated along the same solutions. The left (resp., right) column corresponds to $(k_v, k_p, k_i) = (6.4, 3, 4)$ (resp., $(1.5, 0.66, 0.08)$), and both share $f_c = 1$ m/s^2.

where the matrix P is given by

$$P := \begin{bmatrix} \frac{k_v}{k_i} & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & k_p \end{bmatrix}. \tag{13b}$$

Function (13a) is rather intuitive because P in (13b) is a solution to $A^T P + PA \leq 0$ for A defined in (10) and V corresponds to the minimum quadratic form induced by P when accounting for all possible values allowed by the set-valued friction model. Note that for $v \neq 0$ the minimization in (13a) becomes trivial because f can take only the value $f_c \operatorname{sign}(v)$. It is emphasized that function V is discontinuous. For example, if we evaluate V along the sequence of points $(\sigma_k, \phi_k, v_k) = (0, 0, \varepsilon_k)$ for $\varepsilon_k \in (0, 1)$ converging to zero, V converges to f_c^2, even though its value at zero is zero. Nevertheless, function V enjoys a number of useful properties established in the next lemma, which is a key step for proving Proposition 1. Its proof is reported in Section V-A.

Lemma 2: The Lyapunov-like function in (13) is lower semicontinuous and enjoys the following properties:

1) $V(x) = 0$ for all $x \in \mathcal{A}$ and there exists $c_1 > 0$ such that $c_1 |x|_{\mathcal{A}}^2 \leq V(x)$ for all $x \in \mathbb{R}^3$,
2) there exists $c > 0$ such that each solution $x = (\sigma, \phi, v)$ to (10) satisfies for all $t_2 \geq t_1 \geq 0$

$$V(x(t_2)) - V(x(t_1)) \leq -c \int_{t_1}^{t_2} v(t)^2 dt. \tag{14}$$

Remark 1: In [1] it is proven that if a solution is in a *slip phase* in the nonempty time interval (t_i, t_{i+1}) (namely, for all $t \in (t_i, t_{i+1})$, $v(t) \neq 0$) and the slip phase is preceded and followed by a *stick phase* (namely, there exist $\delta > 0$ such that, for all $t \in [t_i - \delta, t_i] \cup [t_{i+1}, t_{i+1} + \delta]$, $v(t) = 0$ and $|\phi(t)| \leq f_c$), then

$$|\sigma(t_{i+1})| < |\sigma(t_i)|. \tag{15}$$

Instead of using the explicit form of solutions as [1, Lemma L2] depending on the nature of the eigenvalues of A, (15) is easily concluded from (14), the definition (13a), and $|\phi(t_i)| \leq f_c$, $|\phi(t_{i+1})| \leq f_c$. ⌟

Remark 2: Function V in (13) may lead to convenient hybrid friction compensation schemes where the integral error state is suitably reinitialized to reduce the length of the stick phase, thereby possibly recovering exponential convergence (see the discussion in Remark 3). This specific feature should be enabled by the decoupled structure of V in the first line of (13a), where only the last term depends on the state ϕ. Notably, also the Lyapunov-like function previously proposed in the Russian literature [24, Eq. (3.86)] is discontinuous, however it is defined in terms of solutions and it is not clear how to exploit it in control design. ⌟

IV. Illustration by Simulation

Before we prove the results, we illustrate by simulation the typical behaviour of solutions to (4) and their convergence to the attractor. Simulations capture, for each initial condition, the unique solution to (4) because of Lemma 1. When $f_c = 0$, (4) reduces to a linear system with characteristic polynomial $s^3 + k_v s^2 + k_p s + k_i = 0$, whose roots have negative real part from Assumption 1. Although our subsequent proofs do not differentiate anyhow among the possible locations of these

roots in the complex plane, we present our simulations for two representative cases, complex conjugate and three distinct real roots. Specifically, roots $\{-6.01, -0.19 \pm i0.79\}$ and $\{-0.8, -0.5, -0.2\}$ are obtained for parameters $(k_v, k_p, k_i) = (6.4, 3, 4)$ and $(k_v, k_p, k_i) = (1.5, 0.66, 0.08)$, respectively. $f_c = 1$ m/s^2 is common to all simulations.

First, we present the solutions to (4) for different sets of initial conditions for the complex conjugate and real root cases, respectively in the left and right top plots of Figure 2. In the solution represented by a heavier dark violet line, two different phases are visible: the mass is in motion (called *slip phase* in the friction literature), or the mass is at rest (called *stick phase*) and the velocity is zero on a nonzero time interval. Whenever the mass is in a slip phase, the PID control acts in the direction of getting the mass closer to the position setpoint at zero. During a stick phase starting at t_i, only the error integral builds up linearly in time as $e_i(t) = e_i(t_i) + s(t_i)(t - t_i)$ until the control action u overcomes the Coulomb friction, that is, $|u| = |-k_i e_i - k_p s| = f_c$. So, the closer the mass is to the zero position (smaller $s(t_i)$), the longer it takes the error to build up and exit a stick phase. Moreover, position and velocity converge to zero, but the error integral does not in general: it continues to oscillate and enters asymptotically the set $\left[-\frac{f_c}{k_i}, \frac{f_c}{k_i}\right]$ as the position approaches zero for complex conjugate roots (top, left); it approaches the equilibria $\frac{f_c}{k_i}$ or $-\frac{f_c}{k_i}$ for distinct real roots (top, right) because after a stick phase the position and the velocity converge to zero exponentially, so that v remains always nonzero.

Second, we present in the left and right center plots of Figure 2 the phase portraits for the same solutions above.

Third, for the same initial conditions and parameters, we present the evolution along solutions of the Lyapunov-like function introduced in (13). In particular, this function is nonincreasing along solutions, it can be discontinuous (e.g., the left, bottom, dark blue curve at $t = 0.123$ s), and remains constant during stick.

Remark 3: The above observation that initial positions closer to $\mathcal{A}$ imply longer time for the integral error to build up and exit a stick phase, entails that $\mathcal{A}$ is not locally exponentially stable. Indeed, consider an initial condition $(e_i(0), s(0), v(0)) = (0, \epsilon_k, 0)$ with $\epsilon_k \in \left(0, \frac{f_c}{k_p}\right)$. Then $|z(0)|^2_{\mathcal{A}} = \epsilon_k^2$. Using similar reasonings as in Claim 1 and its proof, we can establish that $e_i(t) = \epsilon_k t$, $s(t) = \epsilon_k$, $v(t) = 0$ for all $t \in [0, T_k] := \left[0, \frac{1}{k_i}\left(\frac{f_c}{\epsilon_k} - k_p\right)\right]$. Then for a sequence $\{\epsilon_k\}_{k=1}^{\infty}$ with $\epsilon_k \to 0$, $|z_k(t)|_{\mathcal{A}} = |z_k(0)|_{\mathcal{A}} \neq 0$ for all $t \leq T_k$ and $\lim_{k\to\infty} T_k = +\infty$. Such a sequence clearly excludes exponential convergence. ⌟

V. Proofs

A. Proof of Lemma 2

To the end of proving Lemma 2, we note that model (10) and function (13) suggest that there are three relevant affine systems and smooth functions associated to the three cases in (3c) that are worth considering (and will be used in our proofs). They correspond to

$$\dot{\xi} = f_1(\xi) := A\xi - b, \qquad \xi(0) = \xi_1, \tag{16a}$$

$$\dot{\xi} = f_0(\xi) := \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xi, \qquad \xi(0) = \xi_0, \tag{16b}$$

$$\dot{\xi} = f_{-1}(\xi) := A\xi + b, \qquad \xi(0) = \xi_{-1}, \tag{16c}$$

and for $\xi = (\xi_\sigma, \xi_\phi, \xi_v)$ and the definition $|\xi|^2_P := \xi^T P \xi$,

$$V_1(\xi) := \left|\begin{bmatrix} \xi_\sigma \\ \xi_\phi - f_c \\ \xi_v \end{bmatrix}\right|^2_P, \; V_0(\xi) := \left|\begin{bmatrix} \xi_\sigma \\ 0 \\ 0 \end{bmatrix}\right|^2_P, \; V_{-1}(\xi) := \left|\begin{bmatrix} \xi_\sigma \\ \xi_\phi + f_c \\ \xi_v \end{bmatrix}\right|^2_P. \tag{16d}$$

Based on the description above, we can state the following claim relating (16) to solutions of (10) and to V in (13). Its proof mostly relies on straightforward inspection of the various cases and is given in Appendix A.

Claim 1: There exists $c > 0$ such that, for each initial condition $(\bar{\sigma}, \bar{\phi}, \bar{v})$, one can select $k \in \{-1, 0, 1\}$ and $T > 0$ satisfying the following:

1) the unique solution $\xi = (\xi_\sigma, \xi_\phi, \xi_v)$ to the k-th initial value problem among (16a)-(16c) with initial condition $\xi_k = (\bar{\sigma}, \bar{\phi}, \bar{v})$ coincides in $[0, T]$ with the unique solution to (10);
2) the above solution ξ satisfies for all $t \in [0, T]$

$$V(\xi(t)) = V_k(\xi(t)), \tag{17a}$$

$$\tfrac{d}{dt} V_k(\xi(t)) \leq -c|\xi_v(t)|^2. \tag{17b}$$

Additionally, we restate a fact from [12] that is beneficial to proving Lemma 2. Specifically, we use [12, Theorem 9] together with the variant in [12, Section 5 (point a.)]. We also specialize the statement, using the fact that when the function g is integrable, the standard integral can replace the upper integral (as noted after [12, Definition 8]). The lower right Dini derivative D_+h of h is defined as $D_+h(t) := \liminf_{\epsilon\to 0^+} \frac{h(t+\epsilon) - h(t)}{\epsilon}$.

Fact 1: [12] Given $t_2 > t_1 \geq 0$, suppose that h is lower semicontinuous and that l is locally integrable in $[t_1, t_2]$. If $D_+h(\tau) \leq l(\tau)$ for all $\tau \in [t_1, t_2]$, then

$$h(t_2) - h(t_1) \leq \int_{t_1}^{t_2} l(\tau) d\tau.$$

⌟

Building on Claim 1 and Fact 1 we can prove Lemma 2.

Proof of Lemma 2. We show first that V is lower semicontinuous (lsc). Define the set-valued mapping

$$G(x) := \bigcup_{f \in f_c\,\mathrm{SGN}(v)} g(\sigma, \phi, v, f), \;\; g(\sigma, \phi, v, f) := \begin{bmatrix} \sigma \\ \phi \\ v \end{bmatrix}^T P \begin{bmatrix} \sigma \\ \phi \\ f \end{bmatrix},$$

and consider the additional set-valued mapping $(\sigma, \phi, v) \rightrightarrows H(\sigma, \phi, v) := (\sigma, \phi, v, f_c\,\mathrm{SGN}(v))$. By the very definition of set-valued mapping, we can write $G = g \circ H$ (the composition of g and H), that is, $(\sigma, \phi, v) \rightrightarrows g(\sigma, \phi, v, f_c\,\mathrm{SGN}(v)) = G(x)$. Then, G is outer semicontinuous (osc) by [17, Proposition 5.52, item (b)] because both g and H are osc and H is locally bounded. Finally, by the definition of distance $d(\mathrm{u}, \mathrm{S})$ between a point u and a closed set S, we can write $V(x) = d(0, G(x))$. Then, V is lsc by [17, Proposition 5.11, item (a)] because G was proven to be osc.

We prove now the properties of V item by item.

Item 1). There exists $\mathfrak{g} > 0$ such that $\left[\begin{smallmatrix}\sigma\\ v\end{smallmatrix}\right]^T \left[\begin{smallmatrix}\frac{k_v}{k_i} & -1\\ -1 & k_p\end{smallmatrix}\right] \left[\begin{smallmatrix}\sigma\\ v\end{smallmatrix}\right] \geq \mathfrak{g}(\sigma^2 + v^2)$ because the inner matrix is positive definite by Assumption 1. Moreover, from (13a), $\min_{f \in f_c\,\mathrm{SGN}(v)} \big(\phi - f\big)^2 \geq \min_{f \in [-f_c, f_c]} \big(\phi - f\big)^2 = \mathrm{dz}_{f_c}(\phi)^2$. Therefore, (12) yields $V(x) \geq c_1 |x|_{\mathcal{A}}^2$ with $c_1 := \min\{\mathfrak{g}, 1\}$.

Item 2). Equation (14) is a mere application of Fact 1 for $h(\cdot) = V(x(\cdot))$ and $l(\cdot) = -c(v(\cdot))^2$ where $x = (\sigma, \phi, v)$ is a solution to (10) and c is from Claim 1. So, we need to check that the assumptions of Fact 1 are verified.

We already established above that V is lsc. Solutions x to (10) are absolutely continuous functions by definition. Then, because the composition of a lsc and a continuous function is lsc (see [17, Exercise 1.40]), the Lyapunov-like function (13a) evaluated along the solutions of (10) is lsc in t. Since solutions are absolutely continuous, $-cv^2$ is locally integrable.

Finally, it was proven in Claim 1, item 1 that for each initial condition, the unique solution to (10) coincides with the solution to one of the three affine systems in (16) (numbered k) on a finite time interval T. Moreover, from Claim 1, item 2 V coincides in $[0, T]$ with the function V_k in (17), which is differentiable, hence $V(x(\cdot))$ is at least differentiable from the right at $t = 0$ and the lower right Dini derivative coincides with the right derivative. In particular, we established in (17) that this right derivative is upper bounded by $-cv^2$. □

B. Proof of item 1) of Proposition 1 (global attractivity)

We can now prove the first item of Proposition 1 based on Lemma 2 and a generalized version for differential inclusions of the invariance principle in [13, §4.2]. The following fact comes indeed from specializing the result in [18, Thm. 2.10] to our case, where the differential inclusion (4) has actually unique solutions defined for all nonnegative times (as established in Lemma 1). We also select $G = \mathbb{R}^3$, $U = \mathbb{R}^3$ in the original result of [18].

Fact 2: [18] Let $\ell : \mathbb{R}^3 \to \mathbb{R}_{\geq 0}$ be lower semicontinuous and such that $\ell(x) \geq 0$, for all $x \in \mathbb{R}^3$. If x is a complete and bounded solution to (10) satisfying $\int_0^{+\infty} \ell(x(t))dt < +\infty$, then x converges to the largest forward invariant subset $\mathcal{M}$ of $\Sigma := \{x \in \mathbb{R}^3 \colon \ell(x) = 0\}$. ⌟

Proof of item 1) of Proposition 1 (global attractivity of $\mathcal{A}$). The proof exploits Fact 2, where we take $\ell(x) = v^2$. From Lemma 2, $V(x(t)) \leq V(x(0))$ (item 2) and $c_1|x(t)|_{\mathcal{A}}^2 \leq V(x(t))$ (item 1), so that $c_1|x(t)|_{\mathcal{A}}^2 \leq V(x(0))$ and consequently all solutions to (10) are bounded (their completeness is established in Lemma 1). Apply (14) from 0 to t, and obtain $c\int_0^t v^2(\tau)d\tau \leq V(x(0)) - V(x(t)) \leq V(x(0))$ because $V(x(t)) \geq 0$ from Lemma 2, item 1. Then we have $\int_0^t v^2(\tau)d\tau \leq \frac{V(x(0))}{c}$, and if $t \to +\infty$ we get the required boundedness of the integral of $\ell(x(\cdot))$. Then Fact 2 guarantees that each solution converges to the largest forward invariant subset $\mathcal{M}$ of $\Sigma = \{x \colon v = 0\}$. We claim that such a subset is $\mathcal{A}$. Indeed, $\mathcal{M} \subset \Sigma$ implies $v = 0$ in $\mathcal{M}$. Moreover, $\sigma = 0$ in $\mathcal{M}$ because each solution starting from $v = 0$ and $\sigma \neq 0$ causes a ramp of ϕ that eventually reaches $|\phi| > f_c$ and drives v away from zero (therefore out of Σ). Finally, in $\mathcal{M}$ we must have $|\phi| \leq f_c$ otherwise v would become nonzero again. Then the largest forward invariant set $\mathcal{M}$ in Σ is $\mathcal{A}$. □

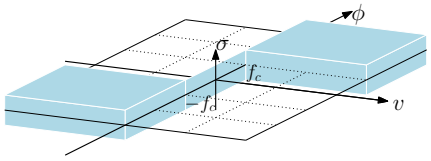

Fig. 3. R is the (closed) blue region in Lemma 3, $\hat{R}$ is its complement.

C. Proof of item 2) of Proposition 1 (stability)

The Lyapunov-like function introduced in (13) of the previous section is not enough to prove stability. Indeed, its discontinuity on the attractor $\mathcal{A}$ prevents us from obtaining a uniform continuous upper bound depending on the distance from $\mathcal{A}$. However, a stability bound can be constructed through an auxiliary function $\hat{V}$ defined for $x := (\sigma, \phi, v)$ as

$$\hat{V}(x) := \tfrac{1}{2}k_1\sigma^2 + \tfrac{1}{2}k_2\big(\mathrm{dz}_{f_c}(\phi)\big)^2 + k_3|\sigma||v| + \tfrac{1}{2}k_4 v^2. \quad (18)$$

Function $\hat{V}$ allows establishing bounds in the directions of discontinuity of V. In particular, we define the two subsets

$$R := \{x \in \mathbb{R}^3 \colon v(\phi - \mathrm{sign}(v)f_c) \geq 0\}$$
$$\hat{R} := \mathbb{R}^3 \backslash R$$

represented in Figure 3. The following holds.

Lemma 3: For suitable positive scalars $k_1, \ldots, k_4$ in (18), there exist positive scalars c_1, c_2, $\hat{c}_1$, $\hat{c}_2$ such that

$$c_1|x|_{\mathcal{A}}^2 \leq V(x) \leq c_2|x|_{\mathcal{A}}^2, \qquad \forall x \in R, \quad (19a)$$
$$\hat{c}_1|x|_{\mathcal{A}}^2 \leq \hat{V}(x) \leq \hat{c}_2|x|_{\mathcal{A}}^2, \qquad \forall x \in \hat{R}, \quad (19b)$$
$$\hat{V}^\circ(x) := \max_{\mathfrak{v} \in \partial\hat{V}(x), \mathfrak{f} \in F(x)} \langle \mathfrak{v}, \mathfrak{f} \rangle \leq 0, \qquad \forall x \in \hat{R}, \quad (19c)$$

where $\partial\hat{V}(x)$ denotes the generalized gradient of $\hat{V}$ at x (see [8, §1.2]) and F is the set-valued mapping in (10). ⌟

Proof. Note that $\min_{f \in f_c\,\mathrm{SGN}(v)} \big(\phi - f\big)^2 = \mathrm{dz}_{f_c}(\phi)^2$ whenever $x \in R$. Since P in (13b) is positive definite and $V(x) = \left[\begin{smallmatrix}\sigma\\ \mathrm{dz}_{f_c}(\phi)\\ v\end{smallmatrix}\right]^T P \left[\begin{smallmatrix}\sigma\\ \mathrm{dz}_{f_c}(\phi)\\ v\end{smallmatrix}\right]$ in R, positive c_1 and c_2 can be chosen to satisfy (19a), using the definition (12). (The lower bound in (19a) was already established for *all* $x \in \mathbb{R}^3$ in Lemma 2, item 1.) For positive $k_1, \ldots, k_4$ and $k_1k_4 > k_3^2$, the inner matrix in $\hat{V}(x) = \frac{1}{2}\left[\begin{smallmatrix}|\sigma|\\ |\mathrm{dz}_{f_c}(\phi)|\\ |v|\end{smallmatrix}\right]^T \left[\begin{smallmatrix}k_1 & 0 & k_3\\ 0 & k_2 & 0\\ k_3 & 0 & k_4\end{smallmatrix}\right] \left[\begin{smallmatrix}|\sigma|\\ |\mathrm{dz}_{f_c}(\phi)|\\ |v|\end{smallmatrix}\right]$ is positive definite and (19b) can be satisfied for the same reason.

To prove (19c), we consider only the set $\hat{R}_> := \hat{R} \cap \{x \colon v > 0\}$ because a parallel reasoning can be followed in $\hat{R} \cap \{x \colon v < 0\}$. For $x \in \hat{R}_>$, we have $v > 0$, $\phi < f_c$ and (10) reduces to the differential equation

$$\begin{aligned}\dot{\sigma} &= -k_i v =: \mathfrak{f}_\sigma(x)\\ \dot{\phi} &= \sigma - k_p v =: \mathfrak{f}_\phi(x)\\ \dot{v} &= -k_v v + \phi - f_c =: \mathfrak{f}_v(x) \leq -k_v|v| - |\mathrm{dz}_{f_c}(\phi)|.\end{aligned} \quad (20a)$$

Consistently, the max in (19c) is to be checked only for the singleton $\mathfrak{f} = (\mathfrak{f}_\sigma(x), \mathfrak{f}_\phi(x), \mathfrak{f}_v(x))$ which $F(x)$ reduces to for all $x \in \hat{R}_>$. Moreover,

$$\tfrac{d}{d\phi}\left(\tfrac{1}{2}\big(\mathrm{dz}_{f_c}(\phi)\big)^2\right) = \mathrm{dz}_{f_c}(\phi),\ \partial\left(|\sigma|\right) = \mathrm{SGN}(\sigma), \quad (20b)$$

where $\partial\left(|\sigma|\right)$ denotes the generalized gradient of $\sigma \mapsto |\sigma|$. We need then to find suitable positive constants $k_1, \ldots, k_4$ satisfying $k_1 k_4 > k_3^2$ and such that $\hat{V}^\circ(x)$ is negative semidefinite in $\hat{R}_>$. Since in $\hat{R}_>$we have $v = |v|$ and $\mathrm{dz}_{f_c}(\phi) = -|\mathrm{dz}_{f_c}(\phi)|$, then we get $\max_{\zeta \in \partial|\sigma|}(-k_i k_3 |v|^2 \zeta) = k_i k_3 |v|^2$ for all $x \in \hat{R}_>$, which gives in turn:

$$\begin{aligned}\hat{V}^\circ(x) \leq & [k_i k_3 |v|^2 - k_4 k_v |v|^2] + [k_2 \sigma \mathrm{dz}_{f_c}(\phi) - k_3 |\sigma| |\mathrm{dz}_{f_c}(\phi)|] \\ & + [-k_1 k_i \sigma v - k_3 k_v |v| |\sigma|] + [k_2 k_p |v| |\mathrm{dz}_{f_c}(\phi)| - k_4 |v| |\mathrm{dz}_{f_c}(\phi)|].\end{aligned}$$

Since $k_1, \ldots, k_4$ are positive by assumption, in each pair in brackets the second term is negative semidefinite and dominates the first (sign-indefinite or nonnegative) term as long as $k_3 > \max\left\{\frac{k_i}{k_v} k_1, k_2\right\}$ and $k_4 > \max\left\{\frac{k_i}{k_v} k_3, k_p k_2, \frac{k_3^2}{k_1}\right\}$. With this selection, (19b) and (19c) are simultaneously satisfied. □

Proof of item 2) of Proposition 1 (stability of $\mathcal{A}$). Based on the constants c_1, c_2, $\hat{c}_1$, $\hat{c}_2$ introduced in Lemma 3, the following stability bound for each solution x to (10)

$$|x(t)|_{\mathcal{A}} \leq \sqrt{\tfrac{c_2 \hat{c}_2}{c_1 \hat{c}_1}} |x(0)|_{\mathcal{A}},\ \forall t \geq 0 \quad (21a)$$

is proven by splitting the analysis in two cases.
Case (i): $x(t) \notin R$, $\forall t \geq 0$. Since $R \cup \hat{R} = \mathbb{R}^3$, $x(t) \in \hat{R}$ for all $t \geq 0$ and from (19c)

$$\hat{V}^\circ(x(t)) \leq 0,\ \forall t \geq 0 \Rightarrow \hat{V}(x(t)) \leq \hat{V}(x(0)),\ \forall t \geq 0.$$

Using bound (19b) we obtain

$$\hat{c}_1 |x(t)|_{\mathcal{A}}^2 \leq \hat{V}(x(t)) \leq \hat{V}(x(0)) \leq \hat{c}_2 |x(0)|_{\mathcal{A}}^2,\ \forall t \geq 0,$$

which implies (21a) because $1 \leq \sqrt{c_2/c_1}$ from (19a).
Case (ii): $\exists t_1 \geq 0$ such that $x(t_1) \in R$. Consider the smallest $t_1 \geq 0$ such that $x(t_1) \in R$ (the existence of such a *smallest* time follows from R being closed). Then, following the analysis of *Case (i)* for the (possibly empty) time interval $[0, t_1)$ and using continuity of solutions, we obtain

$$\hat{c}_1 |x(t)|_{\mathcal{A}}^2 \leq \hat{c}_2 |x(0)|_{\mathcal{A}}^2,\ \forall t \in [0, t_1]. \quad (21b)$$

At t_1 we apply (19a) (because $x(t_1) \in R$) and (21b) to obtain $V(x(t_1)) \leq c_2\big(\frac{\hat{c}_2}{\hat{c}_1} |x(0)|_{\mathcal{A}}^2\big)$. Finally, by the bounds in items 1 and 2 of Lemma 2,

$$c_1 |x(t)|_{\mathcal{A}}^2 \leq V(x(t)) \leq V(x(t_1)) \leq c_2 \tfrac{\hat{c}_2}{\hat{c}_1} |x(0)|_{\mathcal{A}}^2,\ \forall t \geq t_1. \quad (21c)$$

Since $\sqrt{\frac{c_2}{c_1}} \geq 1$, (21b) implies $c_1 |x(t)|_{\mathcal{A}}^2 \leq c_2 \frac{\hat{c}_2}{\hat{c}_1} |x(0)|_{\mathcal{A}}^2$, $\forall t \in [0, t_1]$, which proves (21a) when combined with (21c). □

D. Proof of Proposition 2

The solutions to (8) are a subset of the solutions to $\dot{z} = A_\delta z - f_c \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mu$, where: A_δ in (5) is Hurwitz from Assumption 1, and μ is a locally integrable signal satisfying $\mu(t) \leq 1 + |\rho_v|$ for all t because, for the constant scalar ρ_v, $\mathrm{SGN}_{\rho_v}(v(t)) \leq 1 + |\rho_v|$ for all t. From BIBO stability of exponentially stable linear systems, there exist positive η and λ such that all solutions satisfy

$$|z(t)| \leq \eta e^{-\lambda t} |z(0)| + \eta(1 + |\rho_v|). \quad (22)$$

For the two distances $|z|_{\mathcal{A}}^2 := (\inf_{y \in \mathcal{A}} |z - y|)^2 = s^2 + v^2 + \big(\mathrm{dz}_{f_c/k_i}(e_i)\big)^2$ and $|z|^2 = s^2 + v^2 + e_i^2$, we have $|z|_{\mathcal{A}} \leq |z|$ and $|z|^2 \leq 2|z|_{\mathcal{A}}^2 + 2\big(\frac{f_c}{k_i}\big)^2$ (by splitting into the cases $|e_i| \geq \frac{f_c}{k_i}$ and $|e_i| < \frac{f_c}{k_i}$), which implies $|z| \leq \sqrt{2}\big(|z|_{\mathcal{A}} + \frac{f_c}{k_i}\big)$. These relationships between $|z|_{\mathcal{A}}$ and $|z|$, and (22), imply that there exist positive constants κ_1, κ_2, κ_3 such that all solutions satisfy

$$\begin{aligned}|z(t)|_{\mathcal{A}} \leq |z(t)| &\leq \eta e^{-\lambda t} |z(0)| + \eta(1 + |\rho_v|) \\ &\leq \kappa_1 e^{-\lambda t} |z(0)|_{\mathcal{A}} + \kappa_2 + \kappa_3 |\rho_v|,\ \forall t \geq 0.\end{aligned} \quad (23)$$

Using Theorem 1 and the semiglobal practical robustness of $\mathcal{KL}$ asymptotic stability established in [11, Lemma 7.20], one can transform the δ-ϵ argument of [11, Lemma 7.20] into a class $\mathcal{K}$ function γ_ℓ by following similar steps to [13, Lemma 4.5]. Moreover, using a similar approach to [19, Thm. 2] relating the size of the initial condition and of the input, we obtain the following:

$$\begin{aligned}&|z(0)|_{\mathcal{A}} \leq \tfrac{1}{\delta_\ell}, |\rho_v| \leq \delta_\ell \Rightarrow \\ &\qquad |z(t)|_{\mathcal{A}} \leq \beta_\ell(|z(0)|_{\mathcal{A}}, t) + \gamma_\ell(|\rho_v|),\ \forall t \geq 0,\end{aligned} \quad (24)$$

for some suitable class $\mathcal{KL}$ and class $\mathcal{K}$ functions β_ℓ and γ_ℓ, and for a small enough scalar $\delta_\ell > 0$. Without loss of generality, consider now using in (24) a small enough δ_ℓ such that $(2\delta_\ell)^{-1} \geq \kappa_2 + \kappa_3 \delta_\ell$. Introduce function $T^\star \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $T^\star(s) := \max\{0, \lambda^{-1} \log(2\delta_\ell \kappa_1 s)\}$, which satisfies:

$$\kappa_1 e^{-\lambda T^\star(s)}\ s + \kappa_2 + \kappa_3 \delta_\ell \leq \delta_\ell^{-1}, \quad \forall s \geq 0. \quad (25)$$

We can then conclude the proof by establishing the following (global) ISS bound from ρ_v:

$$|z(t)|_{\mathcal{A}} \leq \beta(|z(0)|_{\mathcal{A}}, t) + \gamma(|\rho_v|), \forall z(0), \forall \rho_v, \forall t \geq 0, \quad (26)$$

where functions β and γ of class $\mathcal{KL}$ and class $\mathcal{K}$, respectively, are built starting from the following inequalities:

$$\beta(s,t) \geq \begin{cases} \kappa_1 e^{-\lambda t} s + \kappa_2 + \kappa_3 \delta_\ell, \text{ if } s \geq \frac{1}{\delta_\ell},\ t \leq T^\star(s) & (27a) \\ \mathfrak{b}(s,t), \qquad \text{otherwise} & (27b) \end{cases}$$

$$\mathfrak{b}(s,t) := \max\left\{\beta_\ell\left(s, \max\{0, t - T^\star(s)\}\right), \kappa_1 e^{-\lambda t} s\right\} \quad (27c)$$

$$\gamma(s) \geq \begin{cases} \kappa_2 + \kappa_3 s, \text{ if } s \geq \delta_\ell & (27d) \\ \gamma_\ell(s), \qquad \text{if } s \leq \delta_\ell. & (27e) \end{cases}$$

The effectiveness of selections (27) for establishing the ISS bound (26) can be verified case by case.
Case 1 ($|\rho_v| \geq \delta_\ell$): use (23), (27d), and the bound $\kappa_1 e^{-\lambda t} s$ in (27a)-(27c).
Case 2 ($|\rho_v| \leq \delta_\ell$ and $|z(0)|_{\mathcal{A}} \leq \delta_\ell^{-1}$): use (24), (27e), and the bound $\beta_\ell\left(s, \max\{0, t - T^\star(s)\}\right)$ in (27b)-(27c).

Case 3 ($|\rho_v| \leq \delta_\ell$ and $|z(0)|_{\mathcal{A}} \geq \delta_\ell^{-1}$): for $t \leq T^\star(|z(0)|_{\mathcal{A}})$ use (27a) and nonnegativity of γ, whereas for $t \geq T^\star(|z(0)|_{\mathcal{A}})$ use $|z(T^\star(|z(0)|_{\mathcal{A}}))|_{\mathcal{A}} \leq {\delta_\ell}^{-1}$ (from (23) and (25)) and the semigroup property of solutions to fall again into *Case 2*.

VI. Conclusions and future work

In this work we characterized the properties of a differential inclusion model of the feedback interconnection of a sliding mass with a PID controller under Coulomb friction. We proved global asymptotic stability of the largest set of closed-loop equilibria using a novel Lyapunov-like function enjoying several useful properties. Due to regularity of the differential inclusion model, global asymptotic stability is intrinsically robust. Additionally, taking as input the size of the inflation of a perturbed model, the dynamics is input-to-state stable, and this perturbation includes the well-known Stribeck effect. As future work we will consider extensions to three-dimensional objects where the friction forces are described by a cone. Moreover, we will propose compensation schemes relying on our Lyapunov-like function to recover exponential stability for Coulomb friction, and will address further the case of static friction force larger than the Coulomb one.

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Appendix A
Proof of Claim 1

Proof of Claim 1. For each possible initial condition $(\bar\sigma, \bar\phi, \bar v)$, items 1) and 2) are satisfied by choosing the suitable k as in the table below ($\vee$, $\wedge$ are respectively the logical OR, AND).

Initial condition	k
$(\bar v > 0) \vee (\bar v = 0 \wedge \bar\phi > f_c) \vee (\bar v = 0 \wedge \bar\phi = f_c \wedge \bar\sigma > 0)$	1
$(\bar v = 0 \wedge \bar\phi = f_c \wedge \bar\sigma = 0) \vee (\bar v = 0 \wedge \bar\phi = f_c \wedge \bar\sigma < 0)$ $\vee (\bar v = 0 \wedge \lvert\bar\phi\rvert < f_c) \vee (\bar v = 0 \wedge \bar\phi = -f_c \wedge \bar\sigma > 0)$ $\vee (\bar v = 0 \wedge \bar\phi = -f_c \wedge \bar\sigma = 0)$	0
$(\bar v = 0 \wedge \bar\phi = -f_c \wedge \bar\sigma < 0) \vee (\bar v = 0 \wedge \bar\phi < -f_c) \vee (\bar v < 0)$	-1

The proof of item 1) consists in showing that for each possible initial condition, the k in the table is such that the solution $\xi = (\xi_\sigma, \xi_\phi, \xi_v)$ to the affine system $\dot\xi = f_k(\xi)$ among (16a)-(16c) is also solution to (10) on the interval $[0, T]$. To verify (17a), we evaluate V and V_k along ξ.

We address only the case $\bar v = 0 \wedge \bar\phi > f_c$ because all other cases rely on similar reasonings. The third state equation of (16a) reads $\dot\xi_v = \xi_\phi - k_v\xi_v - f_c$ with $\xi_v(0) = 0$, $\xi_\phi(0) > f_c$, so that $\dot\xi_v(0) > 0$. Then there exists $T > 0$ such that $\xi_v(t) > 0$ for all $t \in (0, T]$. Substitute the solution ξ to (16a) into (10). Because $-f_c\,\mathrm{SGN}(\xi_v(t)) = \{-f_c\}$ for all $t \in (0, T]$, (10) becomes $\dot\xi(t) = A\xi(t) - b$, holding true for all $t \in (0, T]$ since ξ arises from (16a) ($k = 1$). Then the solution ξ is also a solution to (10) for $t \in [0, T]$ because they have the same initial conditions and $\dot\xi(t) \in F(\xi(t))$. For the same case, we prove $V(\xi(t)) = V_1(\xi(t))$ for all $t \in [0, T]$: at $t = 0$, $\xi_\phi(0) > f_c$ and the minimizer in (13a) is $f = f_c$; for $t \in (0, T]$, $\xi_v(t) > 0$ and $f = f_c$ is the only possible selection in (13a).

To verify (17b), for each initial condition and the corresponding k, select $c = 2(k_v k_p - k_i) > 0$ (by Assumption 1). For $k = 1$, $\frac{d}{dt}V_1(\xi(t)) = \frac{d}{dt}\left(\begin{bmatrix}\xi_\sigma\\ \xi_\phi - f_c\\ \xi_v\end{bmatrix}^T P \begin{bmatrix}\xi_\sigma\\ \xi_\phi - f_c\\ \xi_v\end{bmatrix}\right) = (A\xi - b)^T P \begin{bmatrix}\xi_\sigma\\ \xi_\phi - f_c\\ \xi_v\end{bmatrix} + \begin{bmatrix}\xi_\sigma\\ \xi_\phi - f_c\\ \xi_v\end{bmatrix}^T P(A\xi - b) = -c\xi_v(t)^2$, which satisfies (17b) in $[0, T]$. Parallel computations hold for $k = -1$. For $k = 0$, $V_0(\xi(t)) = \frac{k_v}{k_i}\xi_\sigma^2$ so that $\frac{d}{dt}V_0(\xi(t)) = 2\frac{k_v}{k_i}\xi_\sigma\dot\xi_\sigma = 0 \leq 0 = -c\xi_v(t)^2$ in $[0, T]$. □