Local Input-to-State Stabilization of 1-D Linear Reaction-Diffusion Equation with Bounded Feedback

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***Abstract*— The problem of robust stabilization with bounded feedback control is considered for a scalar reaction-diffusion system with uncertainties in the dynamics. The maximum value of the control input acting on one of the boundary points has to respect a given bound at each time instant. It is shown that, if the initial condition and the disturbance satisfy the certain bounds (computed as a function of the bound imposed on the control input), then the proposed control respects the desired saturation level and renders the closed-loop system locally input-to-state stable, that is, the trajectories with certain bound on the initial condition converge to a ball parameterized by certain norm of the disturbance.**

***Index Terms*— Boundary controlled PDEs; input-to-state stability; bounded feedback; Lyapunov methods.**

***AMS Classification Codes*— 93C20; 93D05; 93D09.**

I. Introduction

Over the past decade, there has been a growing emphasis on generalizing control-theoretic methods for analysis and design of infinite-dimensional systems modeled by partial differential equations (PDEs). Design of stabilizing feedback controllers is one area where sufficient progress has been made. But, from implementation point of view, it is important that such controllers possess some robustness properties and satisfy the constraints imposed by the physical limitations of the devices. With this motivation, this article addresses the problem of feedback stabilization in scalar reaction-diffusion with uncertainties in dynamics. The feedback control not only robustly stabilizes the system, but must also respect the constraint that at each time its maximum value is bounded.

When dealing with finite-dimensional systems described by ordinary differential equations (ODEs), the notion of *input-to-state stability* (ISS) very aptly captures the robustness due to uncertainties in dynamics. The paper [20], which coined this term, also describes the procedure for constructing robust feedback laws (in ISS sense). Using this classical recipe as a template, we address the question of constructing feedback laws for stabilizing one-dimensional boundary controlled PDE with reaction-diffusion equation. Due to the presence of uncertainties in the dynamics, the global feedback that we design must render the system robust in ISS sense. The notion of ISS for infinite-dimensional systems has indeed gathered some attention for different classes of PDEs. The references [14], [15] treat the disturbances appearing in dynamics of the equation, whereas the more recent papers [7], [8], and [22] deal with ISS with respect to boundary disturbances.

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On the other hand, motivated by the practical implementation, there is an added requirement that we impose on the control: the absolute value of the control input at each time must be less than or equal to a prescribed value. This allows us to address systems where the actuator may saturate and the control effort that can be injected into the system is limited. Intuitively speaking, if the open-loop system is unstable and the feedback is linear, we can only stabilize the system if the initial condition is not *too large* in norm while respecting the prescribed constraint on the maximum value of the control. Thus, it is reasonable to work with the notion of local stability, or bounded region of attractions, when the feedback control is bounded. This is the point of view adopted in several works in finite-dimensional ODEs. Moreover, even if the open-loop system is Lyapunov stable, simply saturating a stabilizing feedback can lead to undesirable behavior for the asymptotic stability of the closed-loop system (see e.g., [4]). In these cases, the feedback-law has to be designed taking into account the nonlinearity (see e.g. the so-called nested saturation solution given in [23] for the chain of integrators and generalized in [21], and [12] for another solution based on optimization). A third viewpoint that exists in dealing systems with saturation is that the nonlinearity introduced by the saturation functions satisfies sector conditions. The sector is a partition of the state-space where the saturation is bounded by a linear function. Hence, when the trajectory of a dynamical system evolves in the sector, one can recover the global asymptotic stability properties.

The goal of this paper is *to design a stabilizing feedback control for a boundary controlled reaction-diffusion equation with uncertainties, so that the resulting control stays bounded and the closed-loop system is ISS with respect to the system uncertainties.* The finite-dimensional approach that we generalize here relates to that of local stability, that is, our system is open-loop unstable and find bounds on the initial conditions and the uncertainties which render the closed-loop system stable in ISS sense.

To the best of our knowledge, analysis of infinite-dimensional systems subject to saturations started with [19] and [18] and it is only very recently that other works such as [11] and [16] have appeared. In [19] and [18], global asymptotic stability of the closed-loop system is tackled using nonlinear semigroup theory. Moreover, [11] and [16] use similar results, but for a wider class of saturations. In [11], the Korteweg-de Vries equation (a nonlinear PDE) is considered and some Lyapunov arguments are used together with a characterization of the sector condition to establish the global asymptotic stability of the closed-loop system. Some nonlinear open-loop abstract control systems are studied in [10], where an infinite-dimensional version of the LaSalle Invariance Principle is applied to prove global asymptotic stability.

All these papers deal with open-loop stable systems. This explains why these results deal with global asymptotic stability. For results with bounded region of attractions using saturated control, the recent article [6] addresses asymptotic stability for a coupled heat-ODE equation with a feedback law based on the backstepping method modified due to saturation. The aim of our paper is to study the asymptotic stability of a 1-D linear reaction-diffusion equation with bounded feedback and subjected to a distributed disturbance. The controllability properties of this system have been studied in [2] and [3]. The problem of feedback stabilization with delayed boundary control for such systems is studied in [17]. In these papers, it is emphasized that only a finite part of the eigenvalues of the system are unstable so that, in general, one designs a feedback to compensate only for these eigenvalues.

Our point of departure is a system, which in open-loop, can be decomposed into stable and unstable components after a suitable transformation. The number of unstable modes are finitely many, and thus, the unstable behavior can be modeled by a finite-dimensional ODE. We adopt the viewpoint that for unstable ODEs, bounded feedback laws can be designed with bounded region of attractions, and one can compute explicit bounds on the initial conditions which can be driven to the origin asymptotically with bounded feedbacks. This fundamental idea allows us to construct bounded feedback for the unstable subsystem in our PDE and we show that such feedbacks also achieve ISS property. We then combine this feedback stabilized component with the stable component, and analyze the robustness of the whole system in ISS sense.

In the remainder of this paper, we discuss the system class and problem formulation in Section II. The technical development for solving the proposed problem appears in Section III along with the main result appearing at the end of that section. Some concluding remarks and perspective directions are provided in Section IV.

II. Problem Formulation

This section introduces the class of PDEs and the related stability notions, which are studied in this paper. We then formulate the problem which is then solved in the subsequent sections.

A. System Class

The class of PDEs studied in this paper are described by one-dimensional linear reaction-diffusion equation with distributed uncertainties

$$y_t(t,z) = y_{zz}(t,z) + c(z)y(t,z) + d(t,z) \tag{1a}$$

where $y : [0,\infty) \times [0,1] \to \mathbb{R}$ is the state trajectory, and $c \in \mathcal{L}^2((0,1);\mathbb{R})$. The disturbance d appearing in (1a) due to unmodeled dynamics is assumed to be such that $d(t,\cdot) \in \mathcal{L}^2((0,1);\mathbb{R})$ for each $t \geq 0$. The boundary conditions with the control input u are given by

$$y(t,0) = 0, \quad y(t,1) = u(t), \tag{1b}$$

and the initial condition is

$$y(0,z) = y^0(z), \quad z \in (0,1) \tag{1c}$$

with $y^0 \in \mathcal{H}^2((0,1);\mathbb{R})$. For system (1), the objective is to design the control input u such that the closed-loop system is robustly stable with respect to the disturbance d.

B. Stability notions

The stability notions of interest is a a generalization of ISS, which has been extensively studied for ODEs. For the descriptions of notation, or functions used in formulating the definition, we refer the reader to [9, Chapter 4].

Definition 1: For a given input $t \mapsto u(t)$, system (1) is called ISS with respect to d if there exist a class $\mathcal{K}$ function γ and a class $\mathcal{KL}$ function β such that the solution y to the Cauchy problem (1) satisfies

$$\begin{aligned}\|y(t,\cdot)\|_{\mathcal{L}^2((0,L),\mathbb{R})} \leq \beta(\|y^0(\cdot)\|_{\mathcal{L}^2((0,L),\mathbb{R})}, t) \\ + \gamma\left(\|d(t,\cdot)\|_{\mathcal{L}^2((0,L),\mathbb{R})}\right)\end{aligned} \tag{2}$$

for each $t \geq 0$. We say that the system (1) is *locally ISS* if there exists $M > 0$ such that (2) holds only for y^0 satisfying

$$\|y^0(\cdot)\|_{\mathcal{L}^2((0,L),\mathbb{R})} \leq M \tag{3}$$

and for each $t \geq 0$,

$$\|d(t,\cdot)\|_{\mathcal{L}^2((0,L),\mathbb{R})} \leq M. \tag{4}$$

As already mentioned, the notion of ISS in infinite dimensional systems is a generalization of definitions developed in the case of ODEs, and several research directions have been pursued in the literature to study ISS for PDEs. However, the notion of local ISS has attracted relatively less attention and certain results in this direction can be found in [13]. For our purposes, local ISS comes up rather naturally because we are dealing with open-loop unstable systems and the feedback control has to respect some specific bounds. For this reason, the trajectories must be confined to a bounded region of attraction which requires us to work with bounded initial conditions and bounded disturbances.

C. Problem Statement

Without the control input, that is, with $u \equiv 0$, the system described in (1) is not necessarily stable if one doesn't impose any sign restrictions on the functions $c(\cdot)$. Our basic goal is to design a boundary control feedback law such that the resulting system is stable in the sense of Definition 1. For the case of bounded feedback, we are only interested in local stability notions.

Problem: Let $\sigma > 0$ be given. Does there exist an operator

$$k : \mathcal{H}^2((0,1);\mathbb{R}) \to \mathbb{R}$$

and a constant $M(\sigma) > 0$ such that the feedback control u defined as

$$u(t) := k(w(t, \cdot))$$

has the property that $|u(t)| \le \sigma$, for each $t \ge 0$, and for each initial condition y^0 and the disturbance d satisfying

$$\|y^0\|_{\mathcal{L}^2((0,1),\mathbb{R})} \le M(\sigma),$$
$$\|d(t, \cdot)\|_{\mathcal{L}^2((0,L),\mathbb{R})} \le M(\sigma), \quad \forall\, t \ge 0,$$

the closed-loop system (1) is locally ISS?

A solution to this problem is developed in the next section where we arrive at the main result after working out a series of intermediate steps. The kind of feedback we construct is linear and static, and we provide explicit bounds on the constant $M(\sigma)$, which increase linearly with σ, implying that the volume of the region of attractions increases as the allowed saturation margin gets large. Moreover, the robustness margin of the closed-loop system increases with σ. These observations are indeed consistent with what is observed in the literature on finite-dimensional ODEs with bounded feedbacks.

III. Solution Methodology

This section develops the solution to the problem stated in the previous section by following a sequence of steps: we first rewrite the system in coordinates with unforced boundary conditions; then by introducing a spectral decomposition, we represent the unstable modes of the system by an ODE; a bounded feedback law is defined for this finite-dimensional system which is locally ISS; and finally we study the stability properties of the entire system using Lyapunov-based techniques. Similar roadmap is adopted in [17] for studying stabilization problem with delayed feedback control.

A. System Transformation

As a first step in developing the solution to the stabilization problem, introduce the transformation

$$w(t, z) = y(t, z) - z\, u(t)$$

which results in

$$w_t(t, z) = w_{zz}(z) + c(z)w(z) + c(z)zu(t) - z\dot{u}(t) + d(t, z) \tag{5a}$$

with the boundary condition

$$w(t, 0) = w(t, 1) = 0, \tag{5b}$$

and the initial condition

$$w(0, z) = y(0, z) - zu(0). \tag{5c}$$

By defining the operator $\mathcal{A}$ to be

$$(\mathcal{A}w)(z) := w_{zz}(z) + c(z)w(z)$$

with the domain $\mathrm{dom}(\mathcal{A}) := \mathcal{H}^2((0, 1); \mathbb{R}) \cap \mathcal{H}_0^1((0, 1); \mathbb{R})$, and introducing the functions $a, b \in \mathcal{L}^2((0, 1); R)$ as

$$a(z) := c(z)z, \quad b(z) := z, \tag{6}$$

we can rewrite (5a) as

$$w_t(t, z) = (\mathcal{A}w)(z) + a(z)u(t) - b(z)\dot{u}(t) + d(t, z). \tag{7}$$

The operator $\mathcal{A}$ in (7) is self-adjoint and has a compact inverse. We, therefore, consider a Hilbert basis $\{e_j\}_{j\in\mathbb{N}}$ of $\mathcal{L}^2((0, 1); \mathbb{R})$ consisting of eigenfunctions of $\mathcal{A}$, associated with the sequence of distinct eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}}$. The eigenvalues are such that $\lambda_1 > \lambda_2 > \cdots$ and $\lim_{j\to\infty} \lambda_j = -\infty$, and for each $j \ge 1$, the corresponding eigenfunction $e_j \in \mathcal{H}_0^1((0, 1); \mathbb{R}) \cap \mathcal{C}^2([0, 1]; \mathbb{R})$.

It is now possible to write the solution $w(t, \cdot)$ as a convergent series in eigenfunctions $\{e_j\}_{j\in\mathbb{N}}$, so that

$$w(t, \cdot) := \sum_{j=1}^{\infty} w_j(t)e_j,$$

where

$$w_j(t) := \langle w(t, \cdot), e_j(\cdot)\rangle_{\mathcal{L}^2(0,1)}, \quad j \in \mathbb{N}.$$

Similarly, by assuming that $d(t, \cdot)$ is sufficiently regular, we can write

$$d(t, \cdot) := \sum_{j=1}^{\infty} d_j(t)e_j, \quad d_j(t) := \langle d(t, \cdot), e_j(\cdot)\rangle_{\mathcal{L}^2(0,1)}$$

Also, by letting

$$a_j := \langle a, e_j\rangle_{\mathcal{L}^2(0,1)}, \quad b_j := -\langle b, e_j\rangle_{\mathcal{L}^2(0,1)}$$

for each $j \in \mathbb{N}$, we can rewrite (7) as

$$\dot{w}_j(t) = \lambda_j w_j(t) + a_j u(t) + b_j \dot{u}(t) + d_j(t), \quad j \in \mathbb{N}. \tag{8}$$

By considering u as a state, and letting $v := \dot{u}$, we can think of (8) as an infinite-dimensional linear system controlled by v. Let $n \in \mathbb{N}$ be the number of non-negative eigenvalues associated to the operator $\mathcal{A}$ so that

$$\lambda_1 > \lambda_2 > \dots \lambda_n \ge 0$$

and $\lambda_k < 0$ for $k \ge n + 1$. We now introduce $\mathbb{R}^{n+1}$-valued vector x as follows:

$$x(t) := \mathrm{col}(u(t), w_1(t), \cdots, w_n(t))$$

and focus our attention on the following finite dimensional system:

$$\begin{aligned}
\dot{u}(t) &= v(t)\\
\dot{w}_1(t) &= \lambda_1 w_1(t) + a_1 u(t) + b_1 v(t) + d_1(t)\\
&\ \ \vdots\\
\dot{w}_n(t) &= \lambda_n w_n(t) + a_n u(t) + b_n v(t) + d_n(t)
\end{aligned}$$

where we only took the first n equations of system (8). This finite-dimensional system can be written in a more compact form as

$$\dot{x}(t) = Ax(t) + Bv(t) + \widehat{d}(t) \tag{9}$$

where the matrices A, B are

$$A := \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ a_1 & \lambda_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & 0 & 0 & \cdots & \lambda_n \end{bmatrix}, \quad B := \begin{bmatrix} 1 \\ b_1 \\ \vdots \\ b_n \end{bmatrix} \tag{10}$$

and the vector $\widehat{d}$ is defined as

$$\widehat{d}(t) := \mathrm{col}(d_1(t), d_2(t), \cdots, d_n(t)).$$

Realizing that (8), along with (5b) and (5c), is an equivalent representation of (1), we now proceed to find a stabilizing control for the unstable modes of (8), which are represented by the ODE (9).

B. Bounded Stabilizing Control

We now design a bounded feedback control for system (9). While this system is controlled by v, it must be noted that the *real control* u is a part of the vector x. Thus, it suffices to design v such that x satisfies specific bounds. This will ensure that a bounded u stabilizes the system (9) and the closed-loop system (9) is locally ISS.

To proceed with the design of v, it is of course fundamental to analyze the existence of stabilizing feedback laws. This existence is indeed established by the following statement.

Lemma 1: The pair (A, B) in (9) is controllable and hence, there exists $K \in \mathbb{R}^{1\times\mathbb{R}^{n+1}}$ such that the matrix $(A + BK)$ is Hurwitz.

This lemma allows us to work with the input $v = Kx$ and we can show that it has the desired robustness properties with respect to the disturbance d.

Proposition 1: Let K be such that $(A+BK)$ is Hurwitz and P be the symmetric positive definite matrix satisfying

$$(A + BK)^\top P + P(A + BK) \leq -Q \tag{11}$$

for some symmetric positive definite matrix Q. Let $\overline{d} := \|\widehat{d}\|_{\mathcal{L}^\infty([0,\infty);\mathbb{R})}$. The solution to system (9), with $v = Kx$, has the property that

$$|x(t)| \leq \max\left\{\beta(|x(0)|, t), \frac{2\lambda_{\max}(P)}{\lambda_{\min}(Q)}\overline{d}\right\}, \tag{12}$$

for some class $\mathcal{KL}$ function β. Furthermore, for a given $\sigma > 0$, consider the ellipsoid

$$\Omega_0 = \left\{x \in \mathbb{R}^n \middle| x^\top P x \leq \sigma^2 \lambda_{\min}(P)\right\},$$

and assume that

$$\overline{d} \leq \gamma \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)} \sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}\, \sigma \tag{13}$$

for some $0 < \gamma < 1$. The solution x to system (9) with $x(0) \in \Omega_0$, and $\overline{d}$ satisfying (13) is such that

$$|x(t)| \leq \max\left\{\exp(-\alpha_1 t)\sigma, \sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}\sigma\right\} \tag{14}$$

for some $\alpha_1 > 0$, and in particular, the control u satisfies the bound

$$|u(t)| \leq \sigma$$

for each $t \geq 0$.

Proof: Consider the function $W : \mathbb{R}^{n+1} \to \mathbb{R}$ defined as $W(x) = x^\top P x$, where P is the symmetric positive definite matrix satisfying (11). By substituting $v = Kx$ in (9), it is then seen that

$$\begin{aligned} \frac{d}{dt}W(x(t)) &= -x(t)^\top Q x(t) + 2x^\top P D\widehat{d}(t) \\ &\leq -\lambda_{\min}(Q)|x(t)|^2 + 2\lambda_{\max}(P)|x| \cdot |D\widehat{d}(t)| \\ &\leq -\lambda_{\min}(Q)|x(t)| \left(|x(t)| - \frac{2\lambda_{\max}(P)}{\lambda_{\min}(Q)}\overline{d}\right). \end{aligned}$$

The bound in (12) follows by observing that the expression on the right-hand side is strictly negative for

$$|x(t)| > \frac{2\lambda_{\max}(P)}{\lambda_{\min}(Q)}\overline{d}.$$

Next, let us introduce the ball $\mathcal{B}_\gamma \subset \mathbb{R}^{n+1}$ as

$$\mathcal{B}_\gamma := \left\{x \in \mathbb{R}^{n+1} \,\middle|\, |x|^2 \leq \frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}\gamma^2\sigma^2\right\},$$

then it is immediately seen that $\mathcal{B}_\gamma \subset \Omega_0$ and the inclusion is strict since $\gamma < 1$.

From the expression of $\frac{d}{dt}W(x(t))$, it follows that

$$\frac{d}{dt}W(x(t)) \leq -\lambda_{\min}(Q)|x(t)| \left(|x(t)| - \gamma\sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}\sigma\right)$$

and thus for each $x \in \Omega_0 \setminus \mathcal{B}_{\gamma=1}$

$$\frac{d}{dt}W(x(t)) \leq -\lambda_{\min}(Q)(1-\gamma)\sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}\sigma|x(t)|.$$

In particular, the set Ω_0 is forward invariant and we observe that $|x| \leq \sigma$, for each $x \in \Omega_0$. This fact establishes the desired bound on $|x(t)|$ in (14). Since control u is a part of the vector x, we obtain

$$|u(t)| = |x_1(t)| \leq |x(t)| \leq \sigma$$

for each $t \geq 0$. ∎

C. Robustness of the Closed-loop system

Having designed a stabilizing control for the finite-dimensional unstable component, we now analyze the closed-loop system (7) by substituting $\dot{u} = Kx$, and $u(0) = 0$. Toward this end, we introduce the Lyapunov function $V : \mathbb{R}^{n+1} \times \mathcal{L}^2((0,1);\mathbb{R}) \to \mathbb{R}_{\geq 0}$ given by

$$\begin{aligned} V(x, w) &:= \mu\, x^\top P x - \frac{1}{2}\langle w, \mathcal{A}w\rangle_{\mathcal{L}^2((0,1),\mathbb{R})} \\ &= \mu x^\top P x - \frac{1}{2}\sum_{j=1}^{\infty}\lambda_j w_j^2 \end{aligned}$$

for some constant $\mu > 0$ (large enough) to be specified in the sequel.

Because of the quadratic term with the negative sign in the Lyapunov function V, we first make sure that V is positive definite. This is indeed the case as shown in the following lemma:

Lemma 2: There exist constants C_1 and C_2 such that, for each $t \geq 0$,

$$C_1\left(|u(t)|^2 + \|w(t,\cdot)\|^2_{\mathcal{L}^2((0,1);\mathbb{R})}\right) \leq V(x(t),w(t)) \leq C_2\left(|u(t)|^2 + \|w(t,\cdot)\|^2_{\mathcal{L}^2((0,1);\mathbb{R})}\right). \quad (15)$$

Proof: Let us rewrite the vector x as

$$x = \operatorname{col}(u, w_{1,\dots,n}).$$

Since P is symmetric and positive definite, there are constants c_1 and c_2 such that

$$c_1(|u|^2 + |w_{1,\dots,n}|^2) \leq x^\top P x \leq c_2(|u|^2 + |w_{1,\dots,n}|^2).$$

On the other hand, we can write

$$-\frac{1}{2}\sum_{j=1}^{\infty}\lambda_j w_j^2 = -\frac{1}{2}\sum_{j=1}^{n}\lambda_j w_j^2 - \frac{1}{2}\sum_{j=n+1}^{\infty}\lambda_j w_j^2$$

where the first term on the right-hand side is negative, but the second term is strictly positive since $\lambda_j < 0$ for $j \geq n+1$. We can find $\mu > 0$ large enough such that

$$\mu\, c_1|w_{1,\dots,n}|^2 - \frac{1}{2}\sum_{j=1}^{n}\lambda_j w_j^2 \geq c_3\sum_{j=1}^{n}\lambda_j w_j^2$$

for some $c_3 > 0$, and at the same time there exists $c_4 > 0$ such that

$$\mu\, c_2|w_{1,\dots,n}|^2 - \frac{1}{2}\sum_{j=1}^{n}\lambda_j w_j^2 \leq c_4\sum_{j=1}^{n}\lambda_j w_j^2.$$

Combining the last two inequalities, we get

$$c_1|u|^2 + c_3\sum_{j=1}^{n}\lambda_j w_j^2 + \sum_{j=1}^{n}\lambda_j w_j^2 \leq V(x,w) \leq c_2|u|^2 + c_4\sum_{j=1}^{n}\lambda_j w_j^2 + \sum_{j=1}^{n}\lambda_j w_j^2$$

from where one can find constants C_1, C_2 such that (15) holds. ∎

The next step is to look at the derivative of the function $V(x,w)$ and analyze it to show that the desired stability result indeed holds.

Lemma 3: Consider system (7) and (9) with $v = Kx$ and $u(0) = 0$. If $x(0) \in \Omega_0$ and $\widehat{d}$ is such that (13) holds, then there exists $\mu > 0$ and $\alpha_2 > 0$, such that

$$\frac{d}{dt}V(x(t),w(t)) \leq -\alpha_2 V(x(t),w(t)) + \|d(t,\cdot)\|^2_{\mathcal{L}^2((0,1);\mathbb{R})}$$

for each $t \geq 0$, and $|x(t)| \geq \sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}\sigma$.

Proof: Recalling that $V(x,w) = \mu W(x) - \frac{1}{2}\langle w, \mathcal{A}w\rangle_{\mathcal{L}^2((0,1);\mathbb{R})}$, we observe that

$$\begin{aligned}\frac{d}{dt}V(x(t),w(t)) = \mu\frac{d}{dt}W(x(t)) &- \langle \mathcal{A}w(t), \mathcal{A}w(t)\rangle_{\mathcal{L}^2((0,1),\mathbb{R})} \\ &- \langle a(\cdot)u(t) + b(\cdot)\dot{u}(t), \mathcal{A}w(t)\rangle_{\mathcal{L}^2((0,1),\mathbb{R})} \\ &- \langle d(t,\cdot), \mathcal{A}w(t)\rangle_{\mathcal{L}^2((0,1),\mathbb{R})}. \quad (16)\end{aligned}$$

We have already analyzed the first term on right-hand side in the proof of Proposition 1. We analyze the remaining terms using the Young's inequality, and the fact that $|u(t)| \leq |x(t)|$ and $\dot{u}(t) = Kx(t)$. We thus obtain

$$\left|\langle a(\cdot)u(t), \mathcal{A}w(t,\cdot)\rangle_{\mathcal{L}^2((0,1);\mathbb{R})}\right| \leq \|a\|^2_{\mathcal{L}^2((0,1);\mathbb{R})}|x(t)|^2 + \frac{1}{4}\|\mathcal{A}w(t,\cdot)\|^2_{\mathcal{L}^2((0,1);\mathbb{R})}$$

and similarly,

$$\left|\langle b(\cdot)\dot{u}(t), \mathcal{A}w(t,\cdot)\rangle_{\mathcal{L}^2((0,1);\mathbb{R})}\right| \leq \frac{1}{4}\|\mathcal{A}w(t,\cdot)\|^2_{\mathcal{L}^2((0,1);\mathbb{R})} + \|b\|^2_{\mathcal{L}^2((0,1);\mathbb{R})}\|K\|\,|x(t)|^2.$$

Finally, the term involving disturbance d can be bounded as

$$\left|\langle d(t,\cdot), \mathcal{A}w(t,\cdot)\rangle_{\mathcal{L}^2((0,1);\mathbb{R})}\right| \leq \frac{1}{4}\|\mathcal{A}w(t,\cdot)\|^2_{\mathcal{L}^2((0,1);\mathbb{R})} + \|d(t,\cdot)\|^2_{\mathcal{L}^2((0,1);\mathbb{R})}.$$

Substituting these expressions in (16), we get

$$\begin{aligned}\frac{d}{dt}V(x(t),w(t)) \leq \mu\frac{d}{dt}W(x(t)) &- \frac{1}{4}\|\mathcal{A}w(t,\cdot)\|^2_{\mathcal{L}^2((0,1);\mathbb{R})} \\ &+ \|a\|^2_{\mathcal{L}^2((0,1);\mathbb{R})} + \|K\|\,\|b\|^2_{\mathcal{L}^2((0,1);\mathbb{R})} \\ &+ \|d(t,\cdot)\|^2_{\mathcal{L}^2((0,1);\mathbb{R})}.\end{aligned}$$

From the proof of Proposition 1, we established that $x(t) \in \Omega_0$ for each $t \geq 0$ which gives $|x(t)| \leq \sigma$, and for $|x(t)| \geq \sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}\sigma$, we have

$$\frac{d}{dt}W(x(t)) \leq -\lambda_{\min}(Q)(1-\gamma)\sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}\sigma|x(t)|$$

Choose μ in the definition of V such that

$$\mu\lambda_{\min}(Q)(1-\gamma)\sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}} > \|a\|^2_{\mathcal{L}^2((0,1);\mathbb{R})} + \|b\|^2_{\mathcal{L}^2((0,1);\mathbb{R})}\|K\| \quad (17)$$

then it follows that, there exists $C_3 > 0$ such that for $|x(t)| \geq \sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}\sigma$, we have

$$\frac{d}{dt}V(x(t),w(t)) \leq -C_3 W(x(t)) - \frac{1}{4}\|\mathcal{A}w(t,\cdot)\|^2_{\mathcal{L}^2((0,1);\mathbb{R})} + \|d(t,\cdot)\|^2_{\mathcal{L}^2((0,1);\mathbb{R})}.$$

One can now introduce a constant $\alpha_2 > 0$ such that

$$\frac{d}{dt}V(x(t),w(t)) \leq -\alpha_2 V(x(t),w(t)) + \|d(t,\cdot)\|^2_{\mathcal{L}^2((0,1);\mathbb{R})}$$

which is the desired inequality. ∎

The lemma thus establishes that for bounded initial conditions and bounded disturbance, the derivative of V is the sum of a negative definite term and a positive definite function of the pointwise $\mathcal{L}^2$-norm of the disturbance at each time instant. Drawing analogy from the literature on ISS for ODEs, we call V the ISS-Lyapunov function.

D. Main Result

The development of this section can now be put together to present the solution to the problem posed in Section II. To state the main result, we first introduce the map $\pi : \mathcal{L}^2((0,1)) \to \mathbb{R}^n$ which denotes the vector of coefficients obtained by orthogonal projection of a function along the basis vectors $e_1, \cdots, e_n$ of $\mathcal{L}^2((0,1);\mathbb{R})$, that is,

$$\pi_1(w) = \mathrm{col}(w_1, \cdots, w_n), \quad w_j = \langle w, e_j \rangle_{\mathcal{L}^2((0,1);\mathbb{R}^n)}.$$

Next, it is observed that, by choosing, $u(0) = 0$, we have

$$w(0,z) = y(0,z) = y^0(z).$$

Also, the vector x appearing in (9) can be written in terms of y as

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n+1}(t) \end{pmatrix} = \begin{pmatrix} u(t) \\ \langle y(t,\cdot) - b(\cdot)u(t), e_1(\cdot)\rangle_{\mathcal{L}^2((0,1);\mathbb{R})} \\ \vdots \\ \langle y(t,\cdot) - b(\cdot)u(t), e_1(\cdot)\rangle_{\mathcal{L}^2((0,1);\mathbb{R})} \end{pmatrix}. \tag{18}$$

The main result of this paper can now be stated as follows:

Theorem 1: Consider system (1) and let $\sigma > 0$ be fixed. With A, B given in (10), let P be a symmetric positive definite matrix, and K be such that (11) holds. Choose the control input u such that

$$u(0) = 0, \quad \dot{u}(t) = Kx(t)$$

where $x(t)$ is given in (18). Assume that the initial condition y^0 satisfies

$$(\pi_1(y^0))^\top P(\pi_1(y^0)) \le \sigma^2 \lambda_{\min}(P),$$

and for each $t \ge 0$, $d(t,\cdot)$ is bounded in the sense that

$$|\pi_1(d(t,\cdot))| \le \gamma \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)} \sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}\, \sigma$$

for some $0 \le \gamma < 1$. Then, the closed-loop system (1) is locally ISS with respect to the disturbance d.

IV. Conclusion

In this paper, it has been proved that, for bounded inputs and disturbances, the 1D reaction-diffusion equation is locally ISS with a bounded feedback law. This result has been obtained by synthesizing the control law to stabilize the finite-dimensional unstable part of the system. Then, by analyzing an infinite-dimensional Lyapunov function, it is shown that the entire system is locally ISS with the proposed feedback law.

For future research lines, we would like to study in a more precise way the basin of attraction of this system. Adapting the techniques from the literature on finite dimensional systems, it might be possible to obtain more refined estimates for the basin of attraction. This potentially allows the possibility for the control to be saturated. It is also of interest to look into optimization tools that have been developed for approximating basin of attractions as it is possible to transform this problem into a polynomial optimization, see [5] for finite-dimensional counterparts. Also, as an extension to the framework developed here, it is reviewed in [1] that there exists a finite-dimensional unstable part for the linearized Kuramoto-Sivashinsky equation. Adapting the methodology presented here could lead to similar results for this equation.

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