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Abstract: This paper deals with the stability analysis of time delay systems based on continuous-time approach. The originality of the present paper relies on the construction of several models for a same time-delay systems using the interconnection of an ordinary differential equation and a transport partial differential equation. The stability analysis is then performed using a Lyapunov functional. These models are constructed in order to first reduce potentially the complexity of the resulting stability conditions. Second several models are build in order to be interpreted as a discretization scheme as the one usually used in the Lyapunov functional. The proposed result can be seen as a generalized $(N-M)$ discretization which consists in both a time-discretization of the delay interval into M sub-intervals, and the projection of the state function within each sub-interval on the Legendre polynomials of degree less than N. The efficiency of this novel approach is illustrated on an academic example.

Keywords: Time-delay systems, Model transformations, Transport equation, Lyapunov stability, Integral inequalities, Linear matrix inequalities.

1. INTRODUCTION

Time-delay systems have been widely investigated in many different areas, as in networked control systems, mechanical transmissions or biological systems. There is indeed a very large literature in time-delay system and we can refer to Xu et al. (2006), Mondie et al. (2005) and Gu et al. (2003) for instance, or the survey from Fridman (2014). Stability of time-delay systems has a crucial, practical and theoretical importance since the delay term can be a source of instability and poor performance of a system. Many stability studies and results have been proposed, as for instance in Chen (1995), Gao and Wang (2004) and Xu et al. (2001) (see also the references therein).

A relevant direction of research that was popular in the 2000's consists in applying several models transformation to a time-delay system in order to derive less conservative results as described for instance in Richard (2003). Among them the reader may look at Gu and Niculescu (2000) or at the descriptor representation introduced in Fridman and Shaked (2002). It is also well-known that these transformations may include some additional dynamics as clearly explained in Gu and Niculescu (2000).

Unlike the usual method based on the application of the Lyapunov-Krasovskii theorem, this paper aims at demonstrating the benefits of employing a model of time-delay systems, which is represented as the interconnection of an ordinary differential equation and a transport equation. This idea is obviously not new (see for instance Krstic (2009)). In fact, the transport equation allows to express the delay term of the time-delay system in a coupled system. More precisely, consider the following time delay system

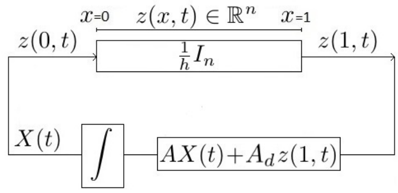

Fig. 1. Illustration of system (3) equivalent to time-delay system (1).

$$\begin{array}{rll} \dot{X}(t) &= AX(t) + A_d X(t-h), & t \geq 0 \\ X(\theta) &= \phi(\theta), & \theta \in [-h,\ 0] \end{array} \quad (1)$$

where $X(t) \in \mathbb{R}^{n\times n}$ is the state of the system and the matrices $A, A_d \in \mathbb{R}^{n\times n}$ are constant and where ϕ represents the initial conditions of the delay system. It is easy to show that the delay term $X(t-h)$ of (1) could be expressed through a transport equation taking the inverse of the delay $\frac{1}{h}$ as the transport velocity. Let us indeed consider the following transport equation over the space domain $(0,1)$:

$$h\partial_t z(x,t) + \partial_x z(x,t) = 0, \quad x \in (0,1), t > 0, \quad (2)$$

which can be illustrated by Figure 1. The left boundary in $x=0$ has the state $X(t)$ of the ODE input in $z(0,t)$ of the transport equation (2), and it takes time before reaching the output $z(1,t)$ at the right boundary in $x=1$. This amount of time corresponds here to the delay h. Therefore, we obtain the delayed term $X(t-h)$ at the output $z(1,t)$ of the tranport equation (2). Thus, system (1) is equivalent to the following coupled ODE-PDE system

$$\begin{cases} \dot{X}(t) = AX(t) + A_d z(1,t) & t > 0, \\ h\partial_t z(x,t) + \partial_x z(x,t) = 0, & x \in (0,1), t > 0, \\ z(0,t) = X(t), & t > 0, \end{cases} \quad (3)$$

Control and stability of this specific kind of systems have been studied in many recent papers in literature in differ-

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ent applicative fields. For example, in hydraulic domain, the paper Coclite et al. (2005) treats a fluidodynamic model for traffic flow using wave front tracking approach and Gugat et al. (2011) designs a feedback laws stabilising a fan-shaped network given by a coupled ODE-PDE system. The works presented in Baudouin et al. (2016) and Safi et al. (2017b) give a hierarchy stability conditions of the coupled system above using linear matrix inequalities depending on different parameters of the coupled system.

The purpose of this paper consists in exploiting the stability result of Safi et al. (2017b) and applying a discretizetion scheme on the delayed term of system (1). In other words, we will divide the delay interval $[-h, 0]$ of system (1) in M sub-intervals $\left([-h, \frac{-(M-1)h}{M}], ..., [\frac{-2h}{M}, \frac{-h}{M}], [\frac{-h}{M}, 0]\right)$, and we will project the delayed state corresponding to each sub-interval on the first $N+1$ Legendre polynomials. Applying a M-discretization of the delay interval and N-sized projection on Legendre polynomials will allow to compare the efficiency of each method and could lead at evaluate an optimal value pair (M, N) giving the best results for a given compromise between calculability and efficiency.

The document is presented as follows. The next section introduces briefly the tools used in this work. Section 3 formulates the problem, provides some preliminaries and exposes the main results of the paper Safi et al. (2017b) on the stability analysis of (1) using the coupled system approach. Section 4 itemizes the different models of the coupled ODE-PDE system allowing to express the time-delay system (1). Afterwards, Section 5 . These results are applied on academic example in section 6. Finally, the last section draws some conclusions and perspectives.

Notations: $\mathbb{N}$ is the set of positive integer, $\mathbb{R}^n$ is the n-dimensional Euclidean space with vector norm $|\cdot|_n$. I_n is the identity matrix in $\mathbb{R}^{n\times n}$, $0_{n,m}$ the null matrix $\in \mathbb{R}^{n\times m}$, $\left[\begin{smallmatrix} A & B \\ \star & C \end{smallmatrix}\right]$ replaces the symmetric matrix $\left[\begin{smallmatrix} A & B \\ B^\top & C \end{smallmatrix}\right]$. We denote $\mathbb{S}^n \subset \mathbb{R}^{n\times n}$ (resp. $\mathbb{S}^n_+ = \{P \in \mathbb{S}^n, P \succ 0\}$, and $\mathbb{D}^n_+$) the set of symmetric (resp. symmetric positive definite and diagonal positive definite) matrices and $\mathrm{diag}(A, B)$ is a bloc diagonal matrix. For any square matrix A, we define $\mathrm{He}(A) = A + A^\top$. Finally, $L^2(0, 1; \mathbb{R}^m)$ represents the space of square integrable functions over the interval $[0, 1] \subset \mathbb{R}$ with values in $\mathbb{R}^m$ and the partial derivative in time and space are denoted ∂_t and ∂_x, while the classical derivative are $\dot{X} = \frac{d}{dt}X$ and $\mathcal{L}' = \frac{d}{dx}\mathcal{L}$. We first define the following set of matrices commuting with a matrix $\Lambda \in \mathbb{D}^m_+$ as follows $\mathcal{M}^m_\Lambda := \{M \in \mathbb{S}^m_+, M\Lambda = \Lambda M\}$. The notation X_t stands for $X_t(\theta) = X(t+\theta)$ for all $\theta \in [-h, 0]$ where h is a positive scalar.

2. PRELIMINARIES

2.1 Legendre polynomials

Using a formulation of time-delay system (1) as a coupled ODE-transport PDE system like (3), one will need to work in the functional space $L^2(0, 1)$. To conduct this stability study, we choose the shifted Legendre polynomials since they form an orthogonal base of $L^2(0, 1)$ with the canonical scalar product

$$< \mathcal{L}_j, \mathcal{L}_k >= \int_0^1 \mathcal{L}_j(x)\mathcal{L}_k(x)\, dx = \frac{1}{2k+1}\delta_{jk},$$

where δ_{jk} is the Kronecker's coefficient, equal to 1 if $j = k$ and 0 otherwise. Legendre polynomials are denoted $\{\mathcal{L}_k\}_{k\in\mathbb{N}}$ and they are characterised by the boundary values $\mathcal{L}_k(0) = (-1)^k, \mathcal{L}_k(1) = 1$ and their differentiation for all $k \geq 1$, $\mathcal{L}'_k(x) = \sum_{j=0}^{k-1} \ell_{kj}\mathcal{L}_j(x)$ where

$$\ell_{kj} = \begin{cases} (2j+1)(1-(-1)^{k+j}), & \text{if} \quad j \leq k-1, \\ 0, & \text{if} \quad j \geq k. \end{cases}$$

The shifted Legendre polynomials will provide basis of polynomials that will be used to project the state z of the transport equation for the purpose of the stability study.

2.2 Bessel-Legendre inequality

The Bessel inequality based on Legendre polynomials is characterised by the positivity of the error between the complete norm of an $L^2(0, 1)$ vector and its projection on Legendre polynomials. The result is given by

Lemma 1. Let $u \in L^2(0, 1; \mathbb{R}^m)$ and $R \in \mathbb{S}^m_+$. Then the following inequality holds for all $N \in \mathbb{N}$

$$\int_0^1 u^\top(x)Ru(x)dx \geq U_N^\top \begin{bmatrix} R & & \\ & \ddots & \\ & & (2N+1)R \end{bmatrix} U_N, \tag{4}$$

where

$$U_N = \begin{bmatrix} \int_0^1 u(x)\mathcal{L}_0(x)\, dx \\ \vdots \\ \int_0^1 u(x)\mathcal{L}_N(x)\, dx \end{bmatrix} \in \mathbb{R}^{m(N+1)}.$$

In our context, this basic lemma will allow to prove that a computable stability criteria based on linear matrix inequalities (LMis) proves actually the exponential stability of the whole ODE-PDE system under consideration.

2.3 Definition : exponential stability of system (3)

Considering the finite dimensional system in $X(t)$ coupled to the transport equation in the variable $z(x, t)$, the expression of $E(t)$ the total energy of this coupled ODE-transport PDE system is $E(t) = |X(t)|^2_n + ||z(t)||^2_{L^2(0,1;\mathbb{R}^m)}$. System (3) is exponentially stable if there exist constant $K > 1$ and $\delta > 0$, depending on different parameters of the system, such that $E(t) \leq Ke^{-\delta t}E(0)$. Thus, the total energy $E(t)$ of system (3) admits a decreasing upper bound and the system's state converges exponentially. These tools and their proofs are more detailed in Safi et al. (2017b).

3. PRELIMINARY ON STABILITY OF COUPLED TRANSPORT ODE-PDE SYSTEMS

Consider now the following more general coupled ODE-transport PDE system

$$\begin{cases} \dot{X}(t) = AX(t) + Bz(1,t) & t > 0, \\ \partial_t z(x,t) + \Lambda\partial_x z(x,t) = 0, & x \in (0,1), t > 0, \\ z(0,t) = CX(t) + Dz(1,t), & t > 0, \\ z(x,0) = z^0(x), & x \in (0,1) \\ X(0) = X^0. \end{cases} \tag{5}$$

where the state is composed by the finite variable $X(t) \in \mathbb{R}^n$ of the ODE and the infinite dimensional state $z(\cdot, t) \in L^2(0, 1; \mathbb{R}^m)$ of the transport PDE. The matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^{m\times m}$ are constant. The transport matrix $\Lambda \in \mathbb{D}_+^m$ of the PDE is constant and diagonal.

In Safi et al. (2017b), sufficient conditions were provided to guarantee the L^2 stability of system (5). This result uses Lyapunov method based on Bessel-Legendre inequality and Legendre polynomials $\{\mathcal{L}_k\}_{k=0\ldots N}$ to approximate the infinite dimensional state $z(x,t)$ of the PDE. This method provides LMIs guaranteeing stability of the coupled system led by the choice of an appropriate Lyapunov functional. This Lyapunov functional is indexed by the order of approximation N and is given by

$$V_N(t) = V_{N,1}(t) + V_{N,2}(t), \tag{6}$$

where

$$V_{N,1}(t) = \begin{bmatrix} X(t) \\ Z_N(t) \end{bmatrix}^\top \begin{bmatrix} P & Q_N \\ * & T_N \end{bmatrix} \begin{bmatrix} X(t) \\ Z_N(t) \end{bmatrix} \quad \text{and}$$

$$V_{N,2}(t) = \int_0^1 z^\top(x,t) e^{-2\delta x \Lambda^{-1}} (S + (1-x)R) z(x,t) dx.$$

The parameter $\delta > 0$ is the decay rate of the energy $E(t)$ of the coupled system (5) and the vector $Z_N(t) = \left[\int_0^1 z^\top(x,t)\mathcal{L}_0(x)dx, \ldots, \int_0^1 z^\top(x,t)\mathcal{L}_N(x)dx\right]^\top$ in $\mathbb{R}^{m(N+1)}$ contains the projections of the state $z(x,t)$ over the $(N+1)$ first Legendre polynomials. In Safi et al. (2017b), the following sufficient exponential stability theorem was provided based on the Lyapunov functionals in the sense of Definition 2.3 and is stated below.

Theorem 1. Consider System (5) with matrices A, B, C and D. If there exists an integer $N > 0$ such that there exist $\delta > 0$, $P \in \mathbb{S}_+^n$, $Q_N \in \mathbb{R}^{n,(N+1)m}$, $T_N \in \mathbb{S}^{(N+1)m}$, S and $R \in \mathcal{M}_\Lambda^m$, satisfying the following LMIs

$$\Phi_N(\Lambda,\delta) = \begin{bmatrix} P & Q_N \\ * & T_N + S^N(\Lambda) \end{bmatrix} \succ 0, \tag{7}$$

$$\Psi_N(\Lambda,\delta) = \begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} \\ * & \Psi_{22} & \Psi_{23} \\ * & * & \Psi_{33} \end{bmatrix} \prec 0, \tag{8}$$

where

$$\begin{aligned}
\Psi_{11} &= \mathrm{He}(PA + Q_N \mathbb{1}_N^*(\Lambda)C) + C^\top \Lambda(R+S)C + 2\delta P, \\
\Psi_{12} &= PB + Q_N(\mathbb{1}_N^*(\Lambda)D - \mathbb{1}_N(\Lambda)) + C^\top \Lambda(R+S)D, \\
\Psi_{13} &= A^\top Q_N + C^\top \mathbb{1}_N^{*\top}(\Lambda)T_N + Q_N L_N(\Lambda) + 2\delta Q_N, \\
\Psi_{22} &= -e^{-2\delta\Lambda^{-1}}\Lambda S + D^\top \Lambda(R+S)D, \\
\Psi_{23} &= B^\top Q_N + (\mathbb{1}_N^*(\Lambda)D - \mathbb{1}_N(\Lambda))^\top T_N, \\
\Psi_{33} &= \mathrm{He}(T_N L_N(\Lambda)) - R^N(\Lambda) + 2\delta T_N,
\end{aligned}$$

and where

$$\begin{aligned}
\mathbb{1}_N(\Lambda) &= [\Lambda \ \ldots \ \Lambda]^\top \in \mathbb{R}^{(N+1)m,m}, \\
\mathbb{1}_N^*(\Lambda) &= \left[\Lambda \ -\Lambda \ \ldots \ (-1)^N\Lambda\right]^\top \in \mathbb{R}^{(N+1)m,m}, \\
L_N(\Lambda) &= [\ell_{kj}\Lambda]_{j,k=0..N} \in \mathbb{R}^{(N+1)m,(N+1)m},
\end{aligned} \tag{9}$$

then, system (5) is exponentially stable. Moreover, for a given transport speed matrix $\Lambda \in \mathbb{D}_+^m$, there exists a constant $K > 0$ and a guaranteed decay rate $\delta^* > \delta$ such that the energy of the system verifies, $\forall t > 0$,

$$E(t) \le K e^{-2\delta^* t}\left(|z^0(0)|_m^2 + \|z^0\|_{L^2(0,1;\mathbb{R}^m)}^2\right). \tag{10}$$

Proof : The proof of this stability result is fully detailed in Safi et al. (2017b) and is not recalled in the present paper. Nevertheless a sketch of the proof is included to latter illustrate the relationship between this result and the Lyapunov-Krasovskii stability.

First of all, the proof is based on the proposed Lyapunov functional (6) and the use of Lemma 1. More precisely, the objective is to prove that the Lyapunov functional V_N is an equivalent norm of the total energy $E(X(t), z(t))$ and decreases exponentially to zero. This is formalized by the following inequalities

$$\begin{aligned}
&\varepsilon_1 E(t) \le V_N(t) \le \varepsilon_2 E(t), \\
&\dot{V}_N(t) + 2\delta V_N(t) \le -\varepsilon_3 E(t),
\end{aligned} \tag{11}$$

for some positive scalars ε_1, ε_2, which are guaranteed by $\Phi_N(\Lambda,\delta) \succ 0$) and ε_3, by the $\Psi_N(\Lambda,\delta) \prec 0$. □

Remark 1. The stability condition of Theorem 1 are valid of any integer N; As in Seuret and Gouaisbaut (2015), it was also proven in Safi et al. (2017b) that these conditions form a hierarchy. In other words, increasing N can only reduce the conservatism of the condition.

Remark 2. It is worth to note that there exists a relation between the delayed version of ODE state X and the functional state z. Indeed solving the partial differential equation, one easily obtains that $z(x,t) = X(t-hx)$. This also means that the Lyapunov functional presented in (6) can be easily translated as a usual Lyapunov-Krasovskii functionals.

3.1 L^2-norm stability of system (5)

We note that Theorem 1 ensures the L^2 stability of the ODE-PDE system. This stability is weaker the Lyapunov-Krasovskii stability, which relies on the supremum norm. However, L^2 stability for system (5) implies stability in the sense of Lyapunov-Krasovskii Theorem. This can be done by recalling that $X(t-\theta) = z(t, \theta/h)$ and by noting that

$$|X(t)|_n^2 \le E(t) \le \eta \sup_{s\in[-h,0]} |X(s)|_n^2, \quad \forall t > 0,$$

and the inequalities (11) also imply

$$\varepsilon_1 |X(t)|_n^2 \le V_N(X,z) \le \varepsilon_2 \eta \sup_{s\in[-h,0]} |X(s)|_n^2,$$

$$\dot{V}_N(X,z) + 2\delta V_N(X,z) \le -\varepsilon_3 |X(t)|_n^2.$$

This shows that this formulation does not bring any restriction with the usual Lyapunov-Krasovskii Theorem.

As presented in the introduction, this paper aims at presented several ODE-PDE models that are equivalent to the original time-delay system (1). Then, based on the stability conditions of Theorem 1, a simple procedure to design discretized Lyapunov-Krasovskii functionals mixed with the framework of the Bessel-Legendre inequality.

4. MODEL TRANSFORMATIONS

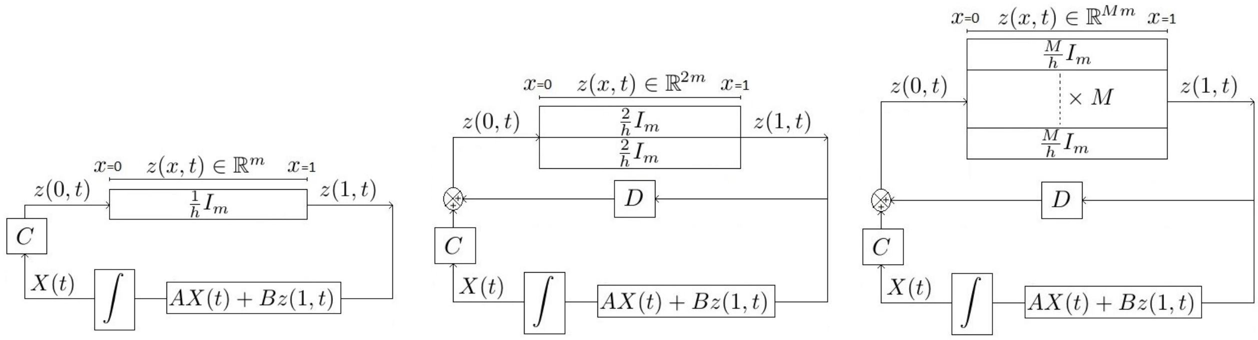
Let us first recall the model provided in the introduction. It was mentioned there that the system (1) writes as the ODE-PDE system (5) with the following matrices, besides A which is unchanged

$$B = A_d, \quad C = I_n, \quad D = 0_n.$$

This is obviously not the only way to model (1) as (5). Some relevant ones are presented below.

4.1 First modelling

In the situation where matrix A_d is not full rank, it is possible to reduce the complexity of the resulting stability conditions by noting that matrix A_d can be rewritten as



(a) Illustration of a reduced model.

(b) Illustration of the discretized model at the order 2.

(c) Illustration of the general discretized model at order M.

Fig. 2. Three model transformations illustrating the discretization scheme though the ODE-PDE system (5).

$A_d = B_d C_d$ where B_d and C_d are two matrices in $\mathbb{R}^{n\times m}$ and $\mathbb{R}^{m\times n}$ respectively, where m is the rank of A_d. Hence, In this situation, the ODE-PDE model (5) with

$$B = B_d, \quad C = C_d, \quad D = 0_m.$$

and again with the same matrix A, is still able to represent the time-delay system (1). This modification only implies a light modification in the model as depicted in Figure 2(a). However, this modification has a relevant impact on the complexity of Theorem 1 since the dimension of the state function z reduces from n to m.

4.2 Discretization model

The idea of this paragraph is to propose a discretization scheme of the Lyapunov functional by a simple transformation of the ODE-PDE model. To do so, this system (5) with

$$\Lambda = \frac{2}{h} I_{2m},\ B = [0_{n,m}\ B_d]\,,\ C = \begin{bmatrix} C_d \\ 0_{m,n} \end{bmatrix},\ D = \begin{bmatrix} 0_m & 0_m \\ I_m & 0_m \end{bmatrix}.$$

In this model, illustrated in Figure 2(b), the matrices B, C and D have been defined such that $BDC = B_d C_d = A_d$. We note that the dimension of the variable z and the transport speed have been doubled. The boundary conditions impose the following constraints

$$\begin{bmatrix} z_1(0,t) \\ z_2(0,t) \end{bmatrix} = \begin{bmatrix} C_d X(t) \\ z_1(1,t) \end{bmatrix}$$

Moreover, by computing the solution of the transport equation, we can see that $z_1(1,t) = C_d X(t-h/2)$ and $z_2(1,t) = z_1(1, t-h/2) = C_d X(t-h)$. Hence, compared to the previous model, the formulation introduce a additional information on the system state $z_1(1,t) = C_d X(t-h/2)$, which can be understood as a discretization process of the delay interval.

Therefore, applying the stability conditions of Theorem 1 on this particular model can be interpreted as a mixture of the Bessel-Legendre inequality and a discretization process of the delay interval. Indeed, one may look at the structure of the associated Lyapunov functional leading to this stability conditions.

Again, the computation of the solution of the transport PDE in (5) with $\Lambda = \frac{2}{h} I_{2m}$ yields

$$z(x,t) = \begin{bmatrix} C_d X(t - \frac{h}{2}x) \\ C_d X(t - \frac{h}{2} - \frac{h}{2}x) \end{bmatrix}. \tag{12}$$

Therefore, re-injecting the expression of z in the definition of the Lyapunov functional leads to

$$V_N(X(t), z(\cdot,t)) = \begin{bmatrix} X(t) \\ \mathcal{X}_N(t) \end{bmatrix}^\top \begin{bmatrix} P & Q_N \\ * & T_N \end{bmatrix} \begin{bmatrix} X(t) \\ \mathcal{X}_N(t) \end{bmatrix}$$
$$+ \int_0^1 \begin{bmatrix} X(t - \frac{h}{2}x) \\ X(t - \frac{h}{2} - \frac{h}{2}x) \end{bmatrix}^\top e^{-\delta h x}(S + (1-x)R) \begin{bmatrix} X(t - \frac{h}{2}x) \\ X(t - \frac{h}{2} - \frac{h}{2}x) \end{bmatrix} dx.$$

where the augmented state $\mathcal{X}_N$ corresponds to the projection of the state z given in (12) on the Legendre polynomials of degree less than N. More especially, an expression of this vector is given, after a change of variable, by

$$\mathcal{X}_N = \begin{bmatrix} \int_0^1 \begin{bmatrix} C_d X(t - \frac{h}{2}x) \\ C_d X(t - \frac{h}{2} - \frac{h}{2}x) \end{bmatrix} \mathcal{L}_0(x)\, ds \\ \vdots \\ \int_0^1 \begin{bmatrix} C_d X(t - \frac{h}{2}x) \\ C_d X(t - \frac{h}{2} - \frac{h}{2}x) \end{bmatrix} \mathcal{L}_N(x)\, ds \end{bmatrix}$$

Applying the changes of variable $s = t - \frac{h}{2}x$ and $s = t - \frac{h}{2} - \frac{h}{2}x$ to the previous expression, the augmented vector $\mathcal{X}_N$ can be rewritten as follows

$$\mathcal{X}_N(t) = \frac{2}{h} \begin{bmatrix} \int_{t-\frac{h}{2}}^{t} C_d X(s) \mathcal{L}_0(2\frac{t-s}{h})\, ds \\ \int_{t-h}^{t-\frac{h}{2}} C_d X(s) \mathcal{L}_0(2\frac{t-s}{h})\, ds \\ \vdots \\ \int_{t-\frac{h}{2}}^{t} C_d X(s) \mathcal{L}_N(2\frac{t-s}{h})\, ds \\ \int_{t-h}^{t-\frac{h}{2}} C_d X(s) \mathcal{L}_N(2\frac{t-s}{h})\, ds \end{bmatrix}$$

and apply the change of variable $s = t - \frac{h}{2}x$ to the last integral term of V_N, the following expression of the functional is obtained and is consistent with the definition of a Lyapunov-Krasovskii functional

$$V_N(X_t) = \begin{bmatrix} X(t) \\ \mathcal{X}_N(t) \end{bmatrix}^\top \begin{bmatrix} P & Q_N \\ * & T_N \end{bmatrix} \begin{bmatrix} X(t) \\ \mathcal{X}_N(t) \end{bmatrix}$$
$$+ \frac{2}{h} \int_{t-\frac{h}{2}}^{t} \begin{bmatrix} X(s) \\ X(s - \frac{h}{2}) \end{bmatrix}^\top e^{-\delta h x}(S + \frac{2}{h}(s - t + \frac{h}{2})R) \begin{bmatrix} X(s) \\ X(s - \frac{h}{2}) \end{bmatrix} ds,$$

These comments demonstrate the potential of the ODE-PDE modeling of time-delay delay systems. Indeed, it has been shown that the simple and understandable modifications have a strong impact on the interpretation of the stability conditions developed in Theorem (1). A first aspect deals with the reduction of complexity of the LMI (due to both a reduction of the size and of the number of decision vairables) if the delay matrix A_d is not full rank. Second, It has been showed that if is possible to construct an augmented model which must verify the constraint $BDC = A_d$ in order to provide an equivalent formulation of the time-delay system. It can also be understood that this decomposition (A, B, C, D, Λ) is not unique and many other models can be generated. In the next section, a general formulation of the discretization is provided to extend the discretization process to any order $M \in \mathbb{N}$.

4.3 General discretization process

Here, we define a new parameter M, which corresponds to the number of subintervals of $[0,1]$ to be considered. Following the previous section, let us consider the following model transformation, which allows to express the time-delay system (1) as in the ODE-PDE system (5) with the following matrices .

$$\begin{aligned} B &= \begin{bmatrix} 0_{n,(M-1)m} & B_d \end{bmatrix}, & C &= \begin{bmatrix} C_d \\ 0_{(M-1)m,n} \end{bmatrix}, \\ D &= \begin{bmatrix} 0_{m,(M-1)m} & 0_m \\ I_{(M-1)m} & 0_{(M-1)m,m} \end{bmatrix}, & \Lambda &= \tfrac{M}{h} I_{Mm}. \end{aligned} \tag{13}$$

Again, it can be easily seen that $BDC = B_d C_d = A_d$. A graphical interpretation of this model is given in Figure 2(c). It can be seen there that the boundary conditions impose the following constraints, that expend the one provided in the previous model

$$\begin{bmatrix} z_1(0,t) \\ z_2(0,t) \\ \vdots \\ z_M(0,t) \end{bmatrix} = \begin{bmatrix} C_d X(t) \\ z_1(1,t) \\ \vdots \\ z_{M-1}(1,t) \end{bmatrix}$$

Following the same arguments as in the previous section, we get that

$$\forall i = 1, \ldots, M-1, \qquad z_{i+1}(0,t) = z_i(1,t) = z_i(0, t - \frac{h}{M}),$$

where we used the solutions of the transport equation. Hence, re-injecting these expressions and by injection, we get that

$$\forall i = 1, \ldots, M, \qquad z_i(0,t) = C_d X(t - \frac{(i-1)h}{M})$$

and $z_M(1,t) = C_d X(t-h)$.
Hence, this model transformation on the ODE-PDE system allows to include some intermediate values of the state function X_t. It is also worth noting that the associated Lyapunov functionals can also be rewritten as a Lyapunov-Krasovskii functionals as presented in the previous section. The computation leading to this functional follows exactly the same procedure and is not therefore not presented here.

A last comment deals with the complexity of the resulting stability conditions. Increasing both N and M notably increase the size of the LMI as well as the number of decision variables. The number of decision variables DV involved in the discretization with respect to any integer M at any order N of the Legendre polynomials is given by

$$DV = \frac{1}{2}(n^2 + n + Mm(3N+5) + (Mm)^2((N+1)^2+2) \tag{14}$$

5. NUMERICAL EXAMPLES

Consider the time-delay system (1) with the following matrices

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}.$$

Equivalent representations of the time-delay system above using the general discretized model of the coupled system (5) are

$$\begin{aligned} B &= \begin{bmatrix} 0_{2,2(M-1)} & A_d \end{bmatrix}, & C &= \begin{bmatrix} I_2 \\ 0_{2(M-1),2} \end{bmatrix}, \\ D &= \begin{bmatrix} 0_{2,2(M-1)} & 0_2 \\ I_{2(M-1)} & 0_{2(M-1),2} \end{bmatrix}, & \Lambda &= \tfrac{M}{h} I_{2M}. \end{aligned}$$

For this system, it has been proved that the maximum decay rate of the energy is $\delta_{max} = 4.35$ for a delay $h = \frac{1}{3.5}$ (see Safi et al. (2017a)). Using Theorem 1, we have performed several simulations which are reported in Table 1. It gives the estimation of δ for several values of the pair (M, N) and for the delay $h = \frac{1}{3.5}$.

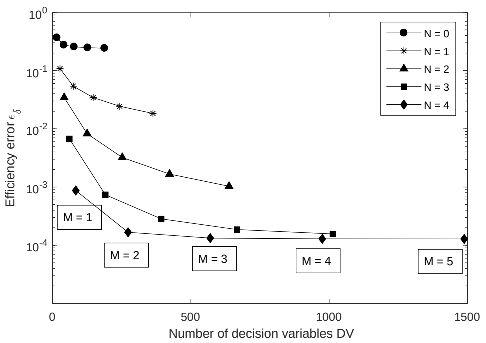
For $h = \frac{1}{3.5}$	$N = 0$	$N = 1$	$N = 2$
$M = 1$	$\delta = 1.3404$	$\delta = 2.6038$	$\delta = 3.4402$
$M = 2$	$\delta = 1.3729$	$\delta = 3.1009$	$\delta = 3.9252$
$M = 3$	$\delta = 1.3841$	$\delta = 3.3478$	$\delta = 4.1134$

Table 1. The decay rate δ depending on (M, N).

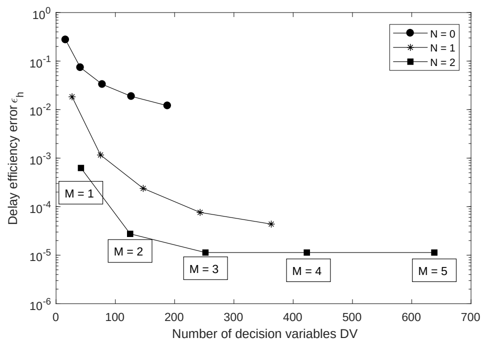
One can note in table 1 that increasing the order M of the time-discretization of the delay, for a given Legendre polynomial's degree N, improves slightly the estimation of the decay rate δ. On the other side, if we keep the same value of M, we get better results by increasing only the order N of Legendre polynomials, but it's still far from the maximum value. However, using both M-discretization and N-projections over Legendre polynomials allows to get better results near to the maximum value $\delta_{max} = 4.35$.
Now, to better evaluate the evolution of the decay rate δ, we define the δ-efficiency error as follows $\epsilon_\delta = 1 - \frac{\delta_{\mathrm{Th1}}}{\delta_{\mathrm{freq}}}$, which represents the missing distance to reach the perfect case $\frac{\delta_{\mathrm{Th1}}}{\delta_{\mathrm{freq}}} = 1$, and compares the decay rate δ_{Th1} found using Theorem 1 and δ_{freq} given by frequency analysis (see Breda et al. (2015)) which are more precise. Figure 3.(a) gives the improvement of ϵ_δ according to the number of decision variables DV which depends on the pair (M, N).

To explain how to read Figure 3.(a), we can give an example of an appropriate pair (M, N) allowing to achieve a fixed worst error efficiency ϵ_δ.
If we decide for example to set the interval of the efficiency error to $\epsilon_\delta \leq 10^{-3}$, the best compromise between the efficiency goal and the number of decision variables DV is clearly $(M, N) = (1, 4)$ (much better than $(2, 3)$), since we achieve the objective with less complexity (less number of DV).

In fact, increasing the pair (M, N) allows to improve results, but it makes the problem more complex by increasing the number of decision variables DV. Table 2 details the evolution of the problem's complexity, expressed by the number of decision variables DV, which depends on the pair (M, N).
One can remark in Table 2 that the LMI problem becomes rapidly more complex by increasing the value M of the discretization, while it becomes slightly complex by increasing the order N of Legendre polynomials. For example, if we consider the pair $(M, N) = (1, 2)$ as a reference, we note



(a) The efficiency error ϵ_δ of the estimation of the decay rate δ_{Th1} compared to frequency result δ_{freq} depending on the number of decision variables DV.



(b) The delay efficiency error ϵ_h of the maximal allowed delay h_{max}(Th1) compared to frequency result h_{max}(freq) depending on the number of decision variables DV.

Fig. 3. The efficiency of Theorem 1 in term of the decay rate δ and the maximal allowed delay h_{max}.

Number of decision variables DV	$N=0$	$N=1$	$N=2$	$N=3$	$N=4$
$M=1$	16	27	42	61	84
$M=2$	41	75	125	191	273
$M=3$	78	147	252	393	570
$M=4$	127	243	423	667	975
$M=5$	188	363	638	1013	1488

Table 2. The number of decision variables DV depending on (M, N).

that the number of decision variables DV increases only by 19 variables at the next order of N ($(M,N) = (1,3)$), while it increases by 83 variables at the next order of M ($(M,N) = (2,2)$).

Now, to evaluate the efficiency of Theorem 1 in term of the maximum delay h_{max} for which the system remains stable, we define the h-efficiency error $\epsilon_h = 1 - \frac{h_{max_{\text{Th1}}}}{h_{max_{\text{freq}}}}$, comparing the maximum delay $h_{max_{\text{Th1}}}$ found by Theorem 1 and $h_{max_{\text{freq}}}$ given by frequency analysis (see Gu et al. (2003)). Figure 3.(b) gives the evolution of ϵ_h depending on the number of decision variables DV.

One can note in Figure 3.(b) that the pair $(M,N) = (2,2)$ gives the better result of the maximum delay $h_{max_{\text{Th1}}}$ for which the system remains stable, in the interval $\epsilon_h \leq 10^{-4}$ with less number of decision variables DV.

6. CONCLUSION

This paper provides a novel approach for stability of time-delay system (1) using the coupled ODE-PDE model (5) and exploiting the stability result obtained in Safi et al. (2017b) based on Lyapunov method. In addition to the projection of the state $z(x,t)$ of the PDE on N Legendre polynomials, the discretization process is also applied through discretizing the delay interval $[-h, 0]$ on M sub-intervals. One can conclude from Figure 3 that using both M-discretization and N-projection over Legendre polynomials allows to achieve quickly the objective with less complexity.

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