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Stabilisation of discrete-time systems with finite-level uniform and logarithmic quantisers

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Abstract: This study deals with the stabilisation of discrete-time linear systems subject to static finite-level quantisation on the control inputs. Two kinds of quantisers are considered: uniform and logarithmic. The modelling of the finite-level quantisation is obtained by the application of deadzone and saturation maps to an infinite-level quantiser. From this model, conditions for the synthesis of state feedback control laws guaranteeing the convergence of the trajectories to an attractor around the origin provided that the initial state belongs to a certain admissible set are proposed. These conditions can thus be incorporated in linear matrix inequality-based optimisation schemes to compute the stabilising gain while minimising the size of the attractor.

1 Introduction

Quantisation of signals is inherent to a digital implementation of control systems and can be considered as a hard non-linearity in the loop. Its effects may lead to undesirable phenomena such as limit cycles, multiple equilibria or chaotic behaviour, even if the controller is supposed to be a stabilising one [1, 2]. Considering the increasing implementation of Networked Control Systems (NCS), researches on quantisation regained the attention of the control community. In particular, since in NCS the control loop elements exchange information through communication channels with limited bandwidth, quantisation plays an important role in reducing the data traffic. However, the use of coarse quantisation may considerably degrade the behaviour of the controlled system.

With this motivation, the study of the effects of quantisation on control systems has attracted the attention of several researchers over the last years; see, e.g. [3–6]. It is also important to note that many of these studies have been proposed for continuous-time systems. In the context of discrete-time systems, one can cite some works dealing with the controller or observer design in the presence of uniform or logarithmic quantisers: see, e.g. [7, 8]. In particular, in [9], it has been shown that, for a quadratically stabilisable single-input system, a logarithmic quantiser is the optimal solution in terms of coarse quantisation density. However, it is also shown that the quantiser must have an infinite number of quantisation levels, which is not feasible in practice. Considering also infinite-level logarithmic quantisers, in [10] the authors have introduced the sector bound approach for quantised feedback systems giving simple formulae to the stabilisation problem considering state and output feedback controllers. The problem of finite-level quantisation can be addressed by using dynamic quantisers (see e.g. [11–13]). However, the implementation of such quantisers is in general much more complex than static ones and may not be possible in standard digital platforms.

When static finite-level quantisers are considered, an important issue regards the fact that the asymptotic convergence of the trajectories to the origin cannot be ensured for open-loop unstable systems. In this case, due to an implicit input deadzone, the trajectories will converge to either periodic or chaotic orbits around the origin. The idea, in this case, is to guarantee that the trajectories converge asymptotically to a set called attractor. Considering logarithmic quantisers, this problem is addressed in [9] for single-input systems. In that reference, for a given optimal control law, the authors focus on providing bounds on the quantiser parameters

to guarantee the convergence of the trajectories to a positively invariant set. On the other hand, quantiser overflow can be seen as an input saturation. It is well known that the global attractivity of the trajectories in the presence of bounded controls cannot be achieved if the open-loop system is exponentially unstable [14]. Hence, in this case, it is important to characterise sets of admissible initial conditions, which are included in the basin of attraction of the attractor. Considering single-input systems and a finite-level logarithmic quantiser, these problems are addressed in [15, 16], where a sector bound approach is applied to derive linear matrix inequality (LMI)-based conditions for estimating a set of initial conditions and an attractor, such that all the state trajectories starting in the first set will converge to the second in a finite time. In those works, the control law is supposed to be given, i.e. only the stability analysis problem is addressed. Moreover, the set of admissible states is constrained to be included in the region where the saturation of the quantiser is not active. Finite-level uniform quantisers are considered for instance in [2, 17]. In these works, based on sector-based relations, conditions to design stabilising state feedback control laws aiming at the minimisation of the attractor set are derived. In [2] the set of admissible initial states, for which the convergence to the attractor is ensured, is also characterised. It should, however, be pointed out that these papers deal only with continuous-time systems.

This work addresses the synthesis of stabilising control laws for multi-input discrete-time linear systems subject to finite-level static uniform and logarithmic quantisers. It can be seen as a comprehensive version of the conference paper [18], where only the stability analysis has been addressed. Furthermore, we extend the approach to deal with different quantisation parameters in each input channel. Based on the modelling of the finite-quantisation through deadzone and saturation maps, a unified framework to deal with both uniform and logarithmic quantisations is proposed. Although the approach also applies to stable systems, we particularly focus on the case in which the plant is exponentially unstable. In this paper, considering explicitly the non-linear effects induced by the quantisation, the deadzone as well as the saturation, conditions to compute a state feedback gain to ensure that the trajectories of the non-linear closed-loop system converge asymptotically to an attractor around the origin, provided that the initial condition belongs to an admissible set, are proposed. Differently from references [15, 16], we consider the effective non-linear effect of the saturation and a formal proof regarding the

positive invariance of the attractor is provided. The proposed conditions are cast in LMI-based optimisation problems to compute the controller gain in order to minimise the size of the attractor while guaranteeing the set of admissible initial states includes some pre-specified region in the state space. A numerical example to illustrate the methodology and to compare the solutions with uniform and logarithmic quantisers is provided.

The paper is organised as follows. In Section 2, the generic framework and the problem statement are presented. Section 3 addresses the case of uniform quantisation, while Section 4 deals with the logarithmic case. Optimisation and computational issues are discussed in Section 5. A numerical example and some concluding remarks end the paper.

Notation: Throughout the paper, I denotes the identity matrix and 0 denotes the null matrix (equivalently the null vector) of appropriate dimensions. For a matrix $A \in \mathbb{R}^{n \times n}$, A' , $A_{(i)}$, $\text{tr}(A)$ denote its transpose, its i th row and its trace, respectively. The matrix $\text{diag}(A_1, A_2, \dots, A_n)$ is the block-diagonal matrix having A_1, A_2, \dots, A_n as diagonal blocks. In symmetric matrices, $*$ stands for symmetric blocks. For a vector $x \in \mathbb{R}^n$, $x_{(i)}$, x' , $|x|$ denote its i th component, its transpose and the component-wise absolute value operator, respectively. $\text{sign}(x)$ is the component-wise sign function, with $\text{sign}(0) = 0$, and $\lfloor x \rfloor$ the component-wise floor operator. For two sets \mathcal{S}_1 and \mathcal{S}_2 , $\mathcal{S}_1 \setminus \mathcal{S}_2$ denotes the set \mathcal{S}_1 deprived of \mathcal{S}_2 .

2 Problem statement

Consider the following discrete-time system:

$$\begin{cases} x(k+1) = Ax(k) + B \text{dz}(\text{sat}(q(u(k)))) \\ x(0) = x_0 \end{cases} \quad (1)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, $x_0 \in \mathbb{R}^n$ are, respectively, the state, the input of the system and the initial state. A , B are real matrices of suitable dimensions and $q: \mathbb{R}^p \rightarrow \mathbb{R}^p$ is a component-wise infinite level quantiser. In order to model the finite-level quantisation, we consider a component-wise saturation function, $\text{sat}: \mathbb{R}^p \rightarrow \mathbb{R}^p$ and a component-wise deadzone function $\text{dz}: \mathbb{R}^p \rightarrow \mathbb{R}^p$, i.e. for $v \in \mathbb{R}^p$ it follows that:

$$\text{sat}_{(i)}(v) := \text{sign}(v_{(i)}) \min\{u_{0(i)}, |v_{(i)}|\}, \quad i = 1, \dots, p \quad (2)$$

$$\text{dz}_{(i)}(u) := \begin{cases} 0 & \text{if } -\epsilon_{(i)} < v_{(i)} < \epsilon_{(i)} \\ v_{(i)} & \text{if } |v_{(i)}| \geq \epsilon_{(i)} \end{cases}, \quad i = 1, \dots, p \quad (3)$$

In this case, $\pm u_{0(i)}$ represents the maximum/minimum quantisation levels for the i th input channel, while $\pm \epsilon_{(i)}$ denote the limits for which the i th input signal is coded as zero.

In order to stabilise system (1), we consider a state feedback control law

$$u(k) = Kx(k) \quad (4)$$

By defining now the functions ψ , ϕ and θ as follows:

$$\psi(v) := q(v) - v \quad (5)$$

$$\phi(v) := \text{sat}(v) - v \quad (6)$$

$$\theta(v) := \text{dz}(v) - v \quad (7)$$

the closed-loop dynamics is given by the non-linear system

$$x(k+1) = (A + BK)x(k) + B\psi(k) + B\phi(k) + B\theta(k) \quad (8)$$

where for simplicity we denote $\psi(k) = \psi(u(k))$, $\phi(k) = \phi(q(u(k))) = \phi(u(k) + \psi(u(k)))$ and $\theta(k) = \theta(\text{sat}(q(u(k)))) = \theta(u(k) + \psi(k) + \phi(k))$.

From (2)–(4), two sets of interest in the state space can be defined

$$\mathcal{S}_{\text{ns}} = \{x \in \mathbb{R}^n : |K_{(i)}x| \leq u_{0(i)}, i = 1, \dots, p\} \quad (9)$$

$$\mathcal{S}_{\text{dz}} = \{x \in \mathbb{R}^n : |K_{(i)}x| \leq \epsilon_{(i)}, i = 1, \dots, p\} \quad (10)$$

The set \mathcal{S}_{ns} corresponds to the region where none of the inputs is saturated in the maximal or minimal quantisation level, while \mathcal{S}_{dz} corresponds to the region where the actual control input injected into the system is zero.

If $x(k) \in \mathcal{S}_{\text{ns}} \setminus \mathcal{S}_{\text{dz}}$, the closed-loop dynamics reads

$$x(k+1) = Ax(k) + q(u(k)) = (A + BK)x(k) + B\psi(k) \quad (11)$$

i.e. only the non-linear effect of the infinite-level quantiser $q(k)$ is present.

On the other hand, note that if $x(k) \in \mathcal{S}_{\text{dz}}$, the system is in open loop. Hence, if the matrix A is not Schur–Cohn, the asymptotic stabilisation of the origin for the closed-loop system cannot be achieved even when the gain K is supposed to be a stabilising one (i.e. when $(A + BK)$ is Schur–Cohn). Actually, in this case, either a limit cycle or a chaotic behaviour will be generated around the origin. Hence the convergence to the origin should be replaced by the convergence to an attractor set around the origin, as small as possible (i.e. referred as *practical stability* in [9]). Moreover, if A is not Schur–Cohn, under input saturation only local (regional) stability can be achieved [14, 19].

In this work, we focus on the case in which matrix A is not Schur–Cohn, i.e. the open-loop system is exponentially unstable. Then, the problem we aim to solve can be stated as follows.

Problem 1: Given system (1), with saturation and deadzone limits given, respectively, by vectors $u_0 \in \mathbb{R}^p$ and $\epsilon \in \mathbb{R}^p$, and the control law (4), determine a gain K , a set $\mathcal{S}_0 \subset \mathbb{R}^n$ and a compact set $\mathcal{S}_u \subset \mathbb{R}^n$ containing the origin, such that

- \mathcal{S}_0 and \mathcal{S}_u are positively invariant sets with respect to the closed-loop system (1)–(4) (or equivalently system (8)).
- For every initial condition $x_0 \in \mathcal{S}_0 \setminus \mathcal{S}_u$, the trajectories of the closed-loop system (1)–(4) are bounded and converge in finite time to \mathcal{S}_u (which is therefore an attractor of the trajectories).
- \mathcal{S}_u is as smaller as possible.

The sets in Problem 1 are depicted in Fig. 1. Note that the set \mathcal{S}_0 is implicitly included in the region of attraction of the attractor set \mathcal{S}_u . As aforementioned, since A is assumed not to be Schur–Cohn, the global convergence to the attractor is not possible to achieve with bounded controls. On the other hand, the set \mathcal{S}_u is the region where the trajectories are ultimately bounded and, if minimised can be seen as an estimate of the periodic or chaotic orbit around the origin.

In the sequel, we address Problem 1 considering uniform and logarithmic finite-level quantisations.

3 Uniform quantisation

The function $q(u)$ corresponding to a uniform quantisation is defined as follows:

$$q_{(i)}(u) := \epsilon_{(i)} \text{sign}(u_{(i)}) \left\lfloor \frac{|u_{(i)}|}{\epsilon_{(i)}} \right\rfloor, \quad i = 1, \dots, p \quad (12)$$

where $\epsilon_{(i)}$ is a positive real-scalar representing the quantisation step of the i th control input. In this case, $u_{0(i)} = N_i \epsilon_{(i)}$, where N_i is the number of positive levels between $u_{0(i)}$ and zero. Moreover, as it can be seen in Fig. 2, the deadzone is implicitly included in the definition of q . Hence, the equation describing the closed-loop system (8), considering a finite-level uniform quantisation can be simplified to the following one:

$$x(k+1) = (A + BK)x(k) + B\psi(k) + B\phi(k) \quad (13)$$

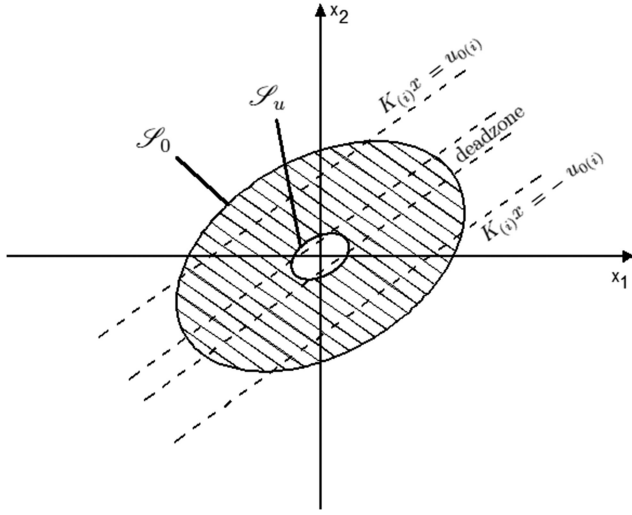


Fig. 1 Sets \mathcal{S}_0 and \mathcal{S}_u

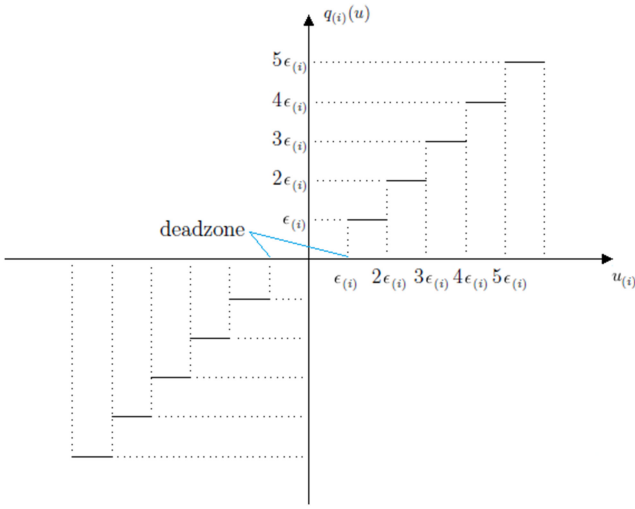


Fig. 2 Uniform quantisation

with $\psi(k) = \psi(u(k))$, $\phi(k) = \phi(q(u(k))) = \phi(u(k) + \psi(u(k)))$.

Before, proposing a result to address Problem 1, we recall some auxiliary conditions regarding the sector bounded functions ψ and ϕ .

Lemma 1: (see [17] and [19] for details) For $\psi(u)$ as defined in (5), with $q(u)$ defined by (12), and considering $\epsilon = [\epsilon_{(1)} \dots \epsilon_{(p)}]'$, the relations

$$\psi(u)' S_1 \psi(u) - \epsilon' S_1 \epsilon \leq 0, \quad (14)$$

$$\psi(u)' S_2 (\psi(u) + u) \leq 0 \quad (15)$$

are verified for any diagonal positive definite matrices $S_1, S_2 \in \mathbb{R}^{p \times p}$.

Actually, (14) corresponds to a generalisation of the condition in [17] to cope with different quantisation parameters in each input.

Lemma 2: (see [17] and [19] for details) For $\phi(v)$ as defined in (6) and a vector $w \in \mathbb{R}^p$, the relation

$$\phi(v)' S_3 (\phi(v) + v + w) \leq 0 \quad (16)$$

is verified for any diagonal positive matrix $S_3 \in \mathbb{R}^{p \times p}$, provided that $w \in S(u_0)$ with

$$S(u_0) = \{w \in \mathbb{R}^p; |w_{(i)}| \leq u_{0(i)}, i = 1, \dots, p\} \quad (17)$$

Note that considering $v = q(u)$, from (5), condition (16) reads

$$\phi(v)' S_3 (\phi(v) + \psi(u) + u + w) \leq 0 \quad (18)$$

By using Lemmas 1 and 2, the following proposition to solve Problem 1 can be stated.

Proposition 1: If there exists a symmetric positive definite matrix $W \in \mathbb{R}^{n \times n}$, diagonal positive matrices $S_1, S_2, R_3 \in \mathbb{R}^{p \times p}$, matrices $Y, Z \in \mathbb{R}^{p \times n}$, positive scalars τ_1, τ_2 and a scalar $\alpha, 0 \leq \alpha \leq 1$, satisfying the following conditions:

$$\begin{bmatrix} \beta W & -Y' S_2 & -Y' - Z' & WA' + Y' B' \\ * & -S_1 - 2S_2 & -I & B' \\ * & * & -2R_3 & R_3 B' \\ * & * & * & -W \end{bmatrix} < 0 \quad (19)$$

$$\text{with } \beta = (\tau_2 - \alpha \tau_1 - 1)$$

$$\epsilon' S_1 \epsilon + \tau_1 - \tau_2 \leq 0 \quad (20)$$

$$\begin{bmatrix} \alpha W & Z'_{(i)} \\ Z_{(i)} & u_{0(i)}^2 \end{bmatrix} > 0, \quad i = 1, \dots, p \quad (21)$$

then, for $K = YW^{-1}$, it follows that $\forall x(0) \in \mathcal{S}_0 \setminus \mathcal{S}_u$ the trajectories of the system (13) converge in finite time to the set \mathcal{S}_u with

$$\mathcal{S}_u = \mathcal{L}(P) = \{x \in \mathbb{R}^n; x' P x \leq 1\}, \quad P = W^{-1} \quad (22)$$

$$\mathcal{S}_0 = \mathcal{L}(\alpha P) = \{x \in \mathbb{R}^n; x' \alpha P x \leq 1\} \quad (23)$$

Moreover, \mathcal{S}_0 and \mathcal{S}_u are positively invariant sets with respect to (13).

Proof: Considering the quadratic Lyapunov function $V(x) = x' P x$, with $P = P' > 0$, we want to prove that

$$\Delta V(x) = V(x(k+1)) - V(x(k)) < -\mu \|x(k)\|^2, \quad (24)$$

with μ being a positive scalar, $\forall x(k) \in \mathcal{S}_0 \setminus \mathcal{S}_u$.

In this case, it follows that \mathcal{S}_0 is a positively invariant set and $\forall x(0) \in \mathcal{S}_0$, the corresponding trajectory converges in finite time to \mathcal{S}_u . Note that

$$\mathcal{S}_0 \setminus \mathcal{S}_u = \{x \in \mathbb{R}^n; (x' \alpha P x - 1) \leq 0 \text{ and } (1 - x' P x) \leq 0\}$$

Now applying S -procedure along with Lemmas 1 and 2, considering $u = Kx$ and $w = Gx$, with G being a free matrix variable, if

$$\begin{aligned} \Delta V(x) - \tau_1 (x' \alpha P x - 1) - \tau_2 (1 - x' P x) - \psi' S_1 \psi + \epsilon' S_1 \epsilon \\ - 2\psi' S_2 (\psi + Kx) - 2\phi' S_3 (\phi + \psi + Kx + Gx) < -\mu \|x(k)\|^2 \end{aligned} \quad (25)$$

it follows that (24) is satisfied, provided that $x \in \tilde{\mathcal{S}}(u_0)$, with

$$\tilde{\mathcal{S}}(u_0) = \{x \in \mathbb{R}^n; |G_{(i)} x| \leq u_{0(i)}, i = 1, \dots, p\} \quad (26)$$

By using (13), it is possible to re-write (25) as follows:

$$[x' \quad \psi' \quad \phi'] \mathcal{N}_u \begin{bmatrix} x \\ \psi \\ \phi \end{bmatrix} + \epsilon' S_1 \epsilon + \tau_1 - \tau_2 < -\mu x' x \quad (27)$$

with \mathcal{N}_u defined as follows:

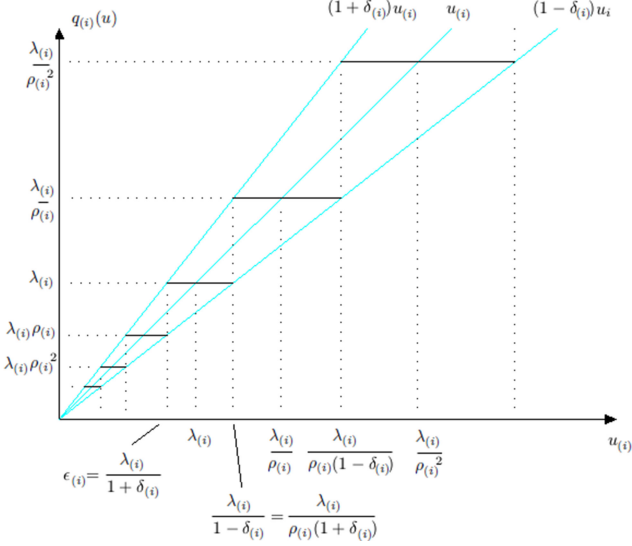


Fig. 3 Logarithmic quantisation

$$\mathcal{N}_u = \begin{bmatrix} (\tau_2 - \alpha\tau_1 - 1)P & -K'S_2 & -K'S_3 - G'S_3 \\ * & -S_1 - 2S_2 & -S_3 \\ * & * & -2S_3 \end{bmatrix} + \begin{bmatrix} (A+BK)' \\ B' \\ B' \end{bmatrix} P \begin{bmatrix} (A+BK) & B & B \end{bmatrix} \quad (28)$$

It is straightforward to verify that (27) will be satisfied for some $\mu > 0$ if we guarantee (20) and $\mathcal{N}_u < 0$. Applying Schur complement, $\mathcal{N}_u < 0$ is equivalent to

$$\begin{bmatrix} \beta P & -K'S_2 & -K'S_3 - G'S_3 & (A+BK)' \\ * & -S_1 - 2S_2 & -S_3 & B' \\ * & * & -2S_3 & B' \\ * & * & * & -P^{-1} \end{bmatrix} < 0 \quad (29)$$

Then by pre- and post-multiplying (29) by $\text{diag}(P^{-1}, I, S_3^{-1}, I)$ and with the change of variables $P^{-1} = W$, $KW = Y$, $GP^{-1} = Z$, $S_3^{-1} = R_3$, we obtain relation (19).

Furthermore, one has to prove that the set \mathcal{S}_u is positively invariant. This is proven if we guarantee that $x(k+1)'Px(k+1) \leq 1$ whenever $x(k)'Px(k) \leq 1$ (i.e. when $x(k) \in \mathcal{S}_u$). It is possible to do that by verifying the following inequality, for some positive scalar τ_3 :

$$x(k+1)'Px(k+1) - 1 - \tau_3(x(k)'Px(k) - 1) \leq 0 \quad (30)$$

On the other hand, inequality (30) is true if we are able to verify that

$$\tau_3 - 1 \leq -\epsilon'S_1\epsilon \quad (31)$$

and

$$\begin{aligned} & x(k+1)'Px(k+1) - \tau_3x(k)'Px(k) - \psi(k)'S_1\psi(k) \\ & - 2\psi(k)'S_2(\psi(k) + Kx(k)) \\ & - 2\phi(k)'S_3(\phi(k) + \psi(k) + Kx(k) + Gx(k)) \leq 0 \end{aligned} \quad (32)$$

provided that $x \in \tilde{\mathcal{S}}(u_0)$.

Now by choosing $-\tau_3 = \tau_2 - \alpha\tau_1 - 1$, it is straightforward to verify that (32) is implied by (19). Furthermore, since $\alpha \leq 1$ and $\tau_1, \tau_2 > 0$, it follows that:

$$\epsilon'S_1\epsilon + \tau_3 - 1 = \epsilon'S_1\epsilon + \alpha\tau_1 - \tau_2 \leq \epsilon'S_1\epsilon + \tau_1 - \tau_2 \quad (33)$$

Hence (20) implies (31). Then, if (19) and (20) are satisfied, (30) is implicitly verified and thus \mathcal{S}_u is positively invariant with respect to system (13).

Finally, the satisfaction of relation (21) implies that the ellipsoid $\mathcal{S}_0 = \mathcal{E}(\alpha P)$ is included in the polyhedral set $\tilde{\mathcal{S}}(u_0)$ defined in (26), which ensures the validity of condition (18), with $w = Gx$, $\forall x \in \mathcal{S}_0$, which concludes the proof. \square

4 Logarithmic quantisation

In the case of the logarithmic quantisation, each component of the decentralised vector $q(u)$ is defined as

$$q_{(i)}(u) :=$$

$$\begin{cases} \frac{\lambda_{(i)}}{\rho_{(i)}^j}, & \text{if } u_{(i)} > 0 \text{ and } \frac{\lambda_{(i)}}{\rho_{(i)}^j(1+\delta_{(i)})} \leq u_{(i)} < \frac{\lambda_{(i)}}{\rho_{(i)}^j(1-\delta_{(i)})} \\ -\frac{\lambda_{(i)}}{\rho_{(i)}^j}, & \text{if } u_{(i)} < 0 \text{ and } \frac{\lambda_{(i)}}{\rho_{(i)}^j(1+\delta_{(i)})} \leq -u_{(i)} < \frac{\lambda_{(i)}}{\rho_{(i)}^j(1-\delta_{(i)})} \end{cases}, \quad (34)$$

$$i = 1, \dots, p, j \in \mathbb{Z}$$

$$\delta_{(i)} = \frac{1 - \rho_{(i)}}{1 + \rho_{(i)}}, \quad \rho_{(i)} = \frac{1 - \delta_{(i)}}{1 + \delta_{(i)}}, \quad 0 < \rho_{(i)} < 1 \quad (35)$$

The parameter $\rho_{(i)}$ is said to be the quantisation density (instead of quantisation step as in the uniform quantisation). The function $q_{(i)}(u)$ corresponding to an infinite-level quantiser is depicted in Fig. 3.

Recall now that, to obtain a finite-level quantiser, saturation and deadzone effects are applied over the infinite-level quantiser $q(u)$. In this case, $\lambda_{(i)}$ is considered as the least positive quantisation level. Actually, $\lambda_{(i)}$ is related to the deadzone limit through the equation $\lambda_{(i)} = \epsilon_{(i)}(1 + \delta_{(i)})$. Moreover, the saturation limit of each quantised input is given by $u_{0(i)} = \lambda_{(i)}/\rho_{(i)}^{N_i-1}$, where N_i corresponds to the number of quantisation levels between $u_{0(i)}$ and zero.

To solve Problem 1, we recall a sector condition verified by the nonlinearity ψ (it corresponds to a multi-input version of the one in [15])

Lemma 3: For $\psi(u) = q(u) - u$ with $q(u)$ defined in (34) and (35), and $D = \text{diag}(\delta_{(1)}, \dots, \delta_{(p)})$, the relation

$$(\psi(u) + Du)'S_1(\psi(u) - Du) \leq 0 \quad (36)$$

is verified for any diagonal positive definite matrix $S_1 \in \mathbb{R}^{p \times p}$.

For the non-linearity θ , one can use a similar condition to that one stated in Lemma 1, as follows:

Lemma 4: For every $\theta(v)$ defined in (7), the relation

$$\theta(v)'S_3\theta(v) - \epsilon'S_3\epsilon \leq 0 \quad (37)$$

is verified for any diagonal positive definite matrix $S_3 \in \mathbb{R}^{p \times p}$.

By using Lemmas 2–4, the following proposition to solve Problem 1 can be stated.

Proposition 2: If there exists a symmetric positive definite matrix $W \in \mathbb{R}^{n \times n}$, diagonal positive matrices $R_1, R_2, S_3 \in \mathbb{R}^{p \times p}$, matrices $Y, Z \in \mathbb{R}^{p \times n}$, positive scalars τ_1, τ_2 and a scalar $\alpha, 0 \leq \alpha \leq 1$, satisfying the following conditions:

$$\begin{bmatrix} \beta W & 0 & -Y' - Z' & 0 & WA' + Y'B' & Y'D \\ * & -R_1 & -R_1 & 0 & R_1 B' & 0 \\ * & * & -2R_2 & 0 & R_2 B' & 0 \\ * & * & * & -S_3 & B' & 0 \\ * & * & * & * & -W & 0 \\ * & * & * & * & * & -R_1 \end{bmatrix} < 0, \quad (38)$$

$$\text{with } \beta = (\tau_2 - \alpha\tau_1 - 1)$$

$$e'S_3\epsilon + \tau_1 - \tau_2 \leq 0 \quad (39)$$

$$\begin{bmatrix} \alpha W & Z'_{(i)} \\ Z_{(i)} & u_{0(i)}^2 \end{bmatrix} > 0, \quad i = 1, \dots, p \quad (40)$$

then, for $K = YW^{-1}$, it follows that $\forall x(0) \in \mathcal{S}_0 \setminus \mathcal{S}_u$ the trajectories of the system (8) converge in finite time to the set \mathcal{S}_u , with \mathcal{S}_u and \mathcal{S}_0 as defined in (22) and (23). Moreover, \mathcal{S}_0 and \mathcal{S}_u are positively invariant sets with respect to (8).

Proof: Applying S -procedure along with Lemmas 2–4, if

$$\begin{aligned} \Delta V(x) - \tau_1(x'\alpha Px - 1) - \tau_2(1 - x'Px) \\ - (\psi + DKx)'S_1(\psi - DKx) - 2\phi'S_2(\phi + \psi + Kx + Gx) \\ - \theta'S_3\theta + e'S_3\epsilon < -\mu\|x\|^2 \end{aligned} \quad (41)$$

with μ being a positive scalar, then it follows that $\Delta V(x) < -\mu\|x\|^2$, $\forall x \in \mathcal{S}_0 \setminus \mathcal{S}_u$ is satisfied, provided that $x \in \tilde{\mathcal{S}}(u_0)$.

By using (8) and considering $\xi = [x' \ \psi' \ \phi' \ \theta']'$, it is possible to re-write (41) as follows:

$$\xi' \mathcal{N}_l \xi + e'S_3\epsilon + \tau_1 - \tau_2 < -\mu x'x \quad (42)$$

with \mathcal{N}_l defined as follows:

$$\begin{aligned} \mathcal{N}_l = & \begin{bmatrix} P(\tau_2 - \alpha\tau_1 - 1) + K'DS_1DK & 0 & -K'S_2 - G'S_2 & 0 \\ * & -S_1 & -S_2 & 0 \\ * & * & -2S_2 & 0 \\ * & * & * & -S_3 \end{bmatrix} \\ & + \begin{bmatrix} (A+BK)' \\ B' \\ B' \\ B' \end{bmatrix} P \begin{bmatrix} (A+BK) & B & B & B \end{bmatrix} \end{aligned} \quad (43)$$

It is straightforward to verify that (42) will be satisfied for some $\mu > 0$ if we guarantee (39) and $\mathcal{N}_l < 0$. By applying Schur complement twice, then by pre- and post-multiplying by $\text{diag}(P^{-1}, S_1^{-1}, S_2^{-1}, I, I, I)$ and with the change of variables $P^{-1} = W$, $KP^{-1} = Y$, $GP^{-1} = Z$, $S_1^{-1} = R_1$, $S_2^{-1} = R_2$, we obtain relation (38).

To prove that the set \mathcal{S}_u is positively invariant, it suffices to verify that, for some positive scalar τ_3 ,

$$x(k+1)'Px(k+1) - 1 - \tau_3(x(k)'Px(k) - 1) \leq 0 \quad (44)$$

Then, by choosing $-\tau_3 = \tau_2 - \alpha\tau_1 - 1$, following the same reasoning is done in the proof of Proposition 1, it follows that (38) and (39) imply (44).

Finally, the satisfaction of relation (40) implies that the ellipsoid $\mathcal{S}_0 = \mathcal{E}(\alpha P)$ is included in the polyhedral set $\tilde{\mathcal{S}}(u_0)$ defined in (26), which concludes the proof. \square

5 Computational issues

In this section, we discuss how to compute the stabilising control law by using the results stated in Propositions 1 and 2.

Recalling Problem 1, the goal is to compute K in order to obtain an attractor set \mathcal{S}_u as small as possible. To achieve this objective, one needs to find a suitable measure for the set \mathcal{S}_u and minimise it subject to conditions of Propositions 1 and 2. In particular, it is possible to implicitly minimise the volume of \mathcal{S}_u by minimising $\text{tr}(P^{-1})$, or equivalently, $\text{tr}(W)$.

On the other hand, it is also important to guarantee that the set \mathcal{S}_0 has a reasonable size or that it covers a given set of admissible initial conditions \mathcal{X}_0 . With this aim, we can consider \mathcal{X}_0 as a polytope in the state space described by the convex hull of its vertices, i.e. $\mathcal{X}_0 = \text{Conv}\{v_1, \dots, v_r\}$. In this case, an extra condition to ensure that $\mathcal{X}_0 \subset \mathcal{S}_0$ is given as follows:

$$\begin{bmatrix} 1 & \alpha v_l' \\ * & \alpha W \end{bmatrix} > 0, \quad l = 1, \dots, r \quad (45)$$

Hence, in order to compute K to minimise the attractor while ensuring a guaranteed region of attractivity, i.e. regional stability, the following optimisation problems can be considered.

- Uniform quantisation

$$\begin{aligned} \min \text{tr}(W) \\ \text{s. t. :} \end{aligned} \quad (46)$$

$$S_1 > 0; S_2 > 0; (19) - (21); (45)$$

- Logarithmic quantisation

$$\begin{aligned} \min \text{tr}(W) \\ \text{s. t. :} \end{aligned} \quad (47)$$

$$(38) - (40); (45)$$

Notice that conditions $W > 0$ and $R_3 > 0$ are implicitly ensured in (46) by (19). On the other hand, conditions $W > 0$, $R_1 > 0$, $R_2 > 0$, $S_3 > 0$ are implicitly ensured in (47) by (38).

It is important to note that conditions (19) and (21), in the uniform quantisation case, and conditions (38) and (40), in the logarithmic quantisation case, are non-linear in the decision variables (i.e. they are not LMIs), which prevents from solving directly a convex optimisation problem. Nevertheless, it is possible to overcome this problem by considering the variables τ_1 , τ_2 and α as tuning parameters. In the uniform quantisation case, S_2 must be fixed as well [20]. Moreover, to guarantee feasibility, it is necessary that the following conditions are respected:

$$\tau_2 > \tau_1; \tau_2 < 1 + \alpha\tau_1 \quad (48)$$

Thus, the optimal value of (46) or (47) can be obtained by solving LMI-based optimisation problems on a grid with respect to α , τ_1 , τ_2 and S_2 (in the uniform case). This can be efficiently addressed by using, for instance, the Nelder–Mead simplex method (implemented in MATLAB by the *fminsearch* function).

As aforementioned, in problems (46) and (47) we ensure that \mathcal{S}_0 is sufficiently large by imposing that $\mathcal{X}_0 \subset \mathcal{S}_0$. If we seek a larger \mathcal{S}_0 , we can enlarge the shape set \mathcal{X}_0 . This can be done, for instance, by using a scaling factor η for \mathcal{X}_0 in order to ensure $\eta\mathcal{X}_0 \subset \mathcal{S}_0$, which can be translated in the LMI constraint

$$\begin{bmatrix} \bar{\eta} & \alpha v_l' \\ * & \alpha W \end{bmatrix} > 0, \quad l = 1, \dots, r \quad (49)$$

where $\bar{\eta} = \eta^{-2}$. In this case, while $\text{tr}(W)$ is minimised, the scalar α is implicitly adjusted to fulfil (49). However, by increasing η a side effect is that the optimal \mathcal{S}_u will also be probably increased since both sets are defined from the same matrix P . There is, in fact, a trade-off to manage. Moreover, note that for η too large the conditions may be not feasible. This comes from the fact that, since the open-loop system is exponentially unstable (by assumption), the global stabilisation is indeed not possible.

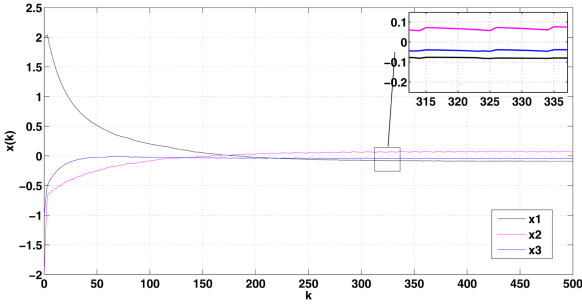


Fig. 4 States evolution in the unif. quantisation case

On the other hand, we can formulate a converse problem in which the main objective is the maximisation of \mathcal{S}_0 , while keeping \mathcal{S}_u sufficiently small. For this, we can consider for instance a polyhedral shape set \mathcal{X}_u represented as

$$\mathcal{X}_u = \{x \in \mathbb{R}^n : H_{(i)}x \leq 1, \quad i = 1, \dots, l\} \quad (50)$$

where $H_{(i)} \in \mathbb{R}^n$. Thus, in order to ensure that \mathcal{S}_u is sufficiently small we can force $\mathcal{S}_u \subset \mathcal{X}_u$, which is ensured by the following set of constraints:

$$\begin{bmatrix} W & WH'_{(i)} \\ H_{(i)}W & 1 \end{bmatrix} > 0, \quad i = 1, \dots, l \quad (51)$$

In this case, in order to maximise \mathcal{S}_0 the following optimisation problems can be considered:

- Uniform quantisation

$$\begin{aligned} \min \quad & \text{tr}(-M) \\ \text{s. t. :} \quad & \end{aligned} \quad (52)$$

$$W > \alpha M; S_1 > 0; S_2 > 0; (19) - (21); (51)$$

- Logarithmic quantisation

$$\begin{aligned} \min \quad & \text{tr}(-M) \\ \text{s. t. :} \quad & \end{aligned} \quad (53)$$

$$W > \alpha M; (38) - (40); (51)$$

Note that the constraint $W > \alpha M$ ensures that $\text{tr}(\alpha^{-1}W) > \text{tr}(M)$, i.e. the minimisation of $\text{tr}(-M)$ implies the maximisation of the $\text{tr}(\alpha^{-1}W)$.

6 Numerical example

Consider the following discrete-time system derived from the continuous-time system treated in [21], with a sampling period $T_s = 0.01$ s:

$$A = \begin{bmatrix} 0.9960 & 0.0169 & 0.0418 \\ 0.0451 & 1.0640 & 0.0544 \\ 0.0347 & 0.0730 & 1.0772 \end{bmatrix}, \quad B = \begin{bmatrix} -0.0068 & -0.0136 \\ 0.0001 & -0.0450 \\ 0.0082 & -0.0173 \end{bmatrix} \quad (54)$$

Our objective is to compare a uniform and a logarithmic quantiser when designing a static state feedback controller that minimises the size of the attractor. With this aim, we consider that both quantisers use the same number of bits (i.e. presents the same number of quantisation levels). Considering that the most significant bit is reserved for the sign, the relation between the number of bits N_{bits} and the number of positive levels of the quantiser N_i , for each input channel, is given by

$$N_i = 2^{(N_{\text{bits}}, i-1)} - 1, \quad i = 1, \dots, p \quad (55)$$

The levels of saturation in the control channels are also considered as being the same for both quantisers and are given by vector $u_0 = [12.4 \quad 12.4]'$.

Fixing all these parameters, in the uniform case, the quantisation step in each channel can be calculated by $\epsilon_{(i)} = u_{0(i)}/N_i$, $i = 1, \dots, p$, with N_i being the number of quantisation levels in control input i . In the logarithmic case, the quantisation density $\rho_{(i)}$ is a degree of freedom, which will affect the smallest positive quantised value $\lambda_{(i)}$ through the following relation:

$$\lambda_{(i)} = u_{0(i)}\rho_{(i)}^{(N_i-1)} \quad (56)$$

Assuming $N_{\text{bits}} = 6$ for both channels (which yields $N_1 = N_2 = N = 31$), the two quantisation cases are analysed in the sequel. For this, we apply the optimisation problems (46) or (47) with $\mathcal{X}_0 = \{v_1\}$, with $v_1 = [2 \quad 2 \quad 2]'$.

Considering the uniform quantiser, for $N = 31$ and u_0 given above, we obtain $\epsilon_{(1)} = \epsilon_{(2)} = 0.4$. A grid search allows to find the best feasible values for the tuning parameters, respectively, $\alpha = 0.0026$, $\tau_1 = 5.3059 \times 10^{-11}$, $\tau_2 = 0.0198$ and $S_2 = \text{diag}(6.3644 \times 10^{-4}, 0.0015)$. The solution to the optimisation problem (46) in this case yields

$$K = \begin{bmatrix} 0.6137 & -0.3853 & -10.4675 \\ 9.2395 & 17.0548 & 14.6553 \end{bmatrix}$$

and

$$P = \begin{bmatrix} 15.6182 & 7.5034 & -1.4831 \\ 7.5034 & 19.6851 & -1.3810 \\ -1.4831 & -1.3810 & 51.5721 \end{bmatrix}$$

which has its trace equal to 86.88.

In the logarithmic quantiser case, also for $N = 31$ and u_0 given above, we choose $\rho_{(1)} = \rho_{(2)} = 0.7391$. Recalling that $\lambda_{(i)} = \epsilon_{(i)}(1 + \delta_{(i)})$, we obtain $\epsilon_{(1)} = \epsilon_{(2)} = 0.0012$. Comparing with the deadzone limits of the uniform quantiser case, one can expect the attractor size to be considerably smaller in the logarithmic case. A grid search allows to find the best feasible values for the tuning parameters, respectively, $\alpha = 4.2786 \times 10^{-5}$, $\tau_1 = 1.21 \times 10^{-10}$ and $\tau_2 = 0.0014$. The solution to the optimisation problem (47) in this case yields

$$K = \begin{bmatrix} 0.6779 & -0.5418 & -7.0948 \\ 3.0219 & 5.7212 & 5.5884 \end{bmatrix}$$

and

$$P = 10^3 \times \begin{bmatrix} 2.2124 & 0.7294 & -1.5139 \\ 0.7294 & 1.5591 & -0.6808 \\ -1.5139 & -0.6808 & 5.0021 \end{bmatrix}$$

which has traced its equal to 8.7736×10^3 .

Figs. 4–6 show, respectively, the evolution of the state for an initial condition of $x_0 = [2 \quad -2 \quad -1]'$ in the uniform and logarithmic cases, and the ellipsoids \mathcal{S}_u obtained in both cases.

Notice that in this case both analysis and simulation show the superiority of logarithmic quantisers in terms of attractor size, considering the same number of quantisation levels and the same saturation limits. Nevertheless, as mentioned previously in this section, the quantisation density $\rho_{(i)}$ is a degree of freedom of the logarithmic quantiser and a good choice of this parameter must be made. One would expect that the closer to one $\rho_{(i)}$ is, a smaller \mathcal{S}_u would be obtained since quantisation would become denser and the quantisation non-linearity $q_{(i)}(u)$ (see Fig. 3) would lie in a closer sector (i.e. $\delta_{(i)}$ would be close to zero). However, considering u_0 and the number of quantisation levels N_i fixed, relation (56) shows that there is a trade-off between $\rho_{(i)}$ and $\lambda_{(i)}$ (which is proportional to $\epsilon_{(i)}$,

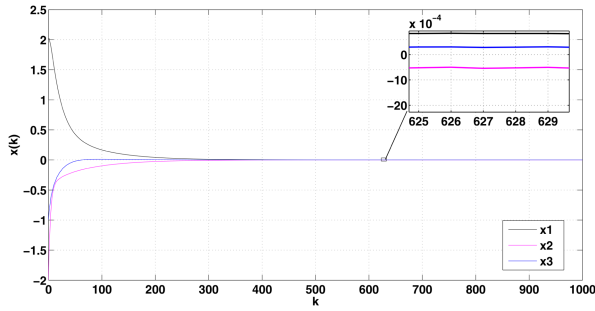


Fig. 5 States evolution in the log. quantisation case

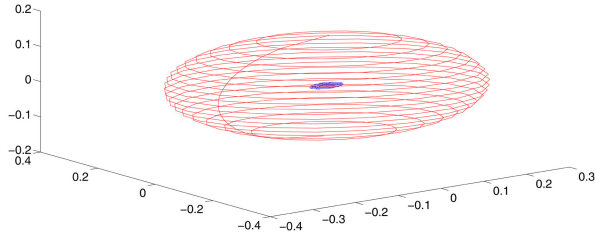


Fig. 6 Ellipsoid \mathcal{S}_u in the uniform (outer one - in red) and logarithmic (inner one - in blue) quantisation cases with the same number of bits

Table 1 Results obtained for different values of $\rho_{(i)}$ – logarithmic quantisation

$\rho_{(i)}$	$\delta_{(i)}$	$\lambda_{(i)}$	ϵ_{ci}	$\text{tr}(P)$
0.4706	0.36	1.87×10^{-9}	1.38×10^{-9}	2569.9
0.5385	0.3	1.07×10^{-7}	8.22×10^{-8}	5830.8
0.7391	0.15	1.43×10^{-3}	1.24×10^{-3}	8773.6
0.7857	0.12	8.94×10^{-3}	7.98×10^{-3}	2958.0
0.8182	0.10	3.01×10^{-2}	2.74×10^{-2}	289.0
0.8349	0.09	5.52×10^{-2}	5.06×10^{-2}	83.7
0.8935	0.056	4.22×10^{-1}	4.00×10^{-1}	1.45

Table 2 Influence of η over the sizes of \mathcal{S}_u and \mathcal{S}_0 – uniform quantisation

η	$\text{tr}(W)$	$\text{tr}(P)$	$\text{tr}(\alpha P)$
0.5	0.0299	1369.70	3.3106
0.8	0.0685	231.77	0.6230
1.0	0.1602	86.88	0.2259
1.2	21.1349	2.34	0.0744
1.3	N.F.	N.F.	N.F.

Table 3 Influence of η over the sizes of \mathcal{S}_u and \mathcal{S}_0 – logarithmic quantisation

η	$\text{tr}(W)$	$\text{tr}(P)$	$\text{tr}(\alpha P)$
0.5	1.31×10^{-4}	2.88×10^5	5.5860
0.8	3.77×10^{-4}	5.22×10^4	1.0100
1.0	0.0017	8.77×10^3	0.3754
1.17	2.9110	36.72	0.0705
1.18	N.F.	N.F.	N.F.

another parameter that influences the size of \mathcal{S}_u). In this example, the best choice for $\rho = \rho_{(1)} = \rho_{(2)}$ was found numerically. Table 1 shows the results of the optimisation problem (47) that would have been obtained with different values of $\rho_{(i)} = \rho$, $i = 1, 2$. Notice that, when $\rho_{(i)} > 0.7391$, ϵ_{ci} becomes larger and, as expected, \mathcal{S}_u becomes larger too (i.e. $\text{tr}(P)$ is smaller). On the other hand, when $\rho_{(i)} < 0.7391$, ϵ_{ci} becomes smaller but, since the quality of quantisation is worse (actually the sector where the $q_{(i)}(u)$ lies is more opened, i.e. more uncertainty is considered), \mathcal{S}_u grows.

Table 1 also shows that for $\rho_{(i)} = 0.8349$, we obtain \mathcal{S}_u which is similar in size to the one obtained in the uniform case. Finally, for $\rho_{(i)} = 0.8935$ the deadzone limit ϵ_{ci} is approximately the same as that one of the uniform quantisation, but \mathcal{S}_u is several times larger than the one obtained in the uniform case. This indicates that, depending on the quantisation density chosen for the logarithmic quantiser, if we fix the saturation and the deadzone limits as well as the number of quantisation levels, the uniform quantiser may provide better results.

In order to illustrate the trade-off regarding the size of \mathcal{S}_0 and \mathcal{S}_u , as discussed in Section 5, we replace now (45) by (49) in optimisation problems (46) and (47). Tables 2 and 3 show the results obtained for different values of the scaling factor $\eta = 1/\sqrt{\eta}$ for the uniform and logarithmic cases, respectively. Note that as η increases, i.e. the larger is the shape set that must be included by \mathcal{S}_0 , the smaller is the value of $\text{tr}(\alpha P)$, which indicates that \mathcal{S}_0 increases. On the other hand, it can be observed that $\text{tr}(P)$ decreases, meaning that \mathcal{S}_u also increases as we increase η . Moreover, as expected, for η too large ($\eta > 1.3$ in the uniform case and $\eta > 1.18$ in the logarithmic one) no feasible solution (N.F.) can be found since the global stability is not achievable because the open-loop system is exponentially unstable.

7 Conclusion

This paper has tackled the stabilisation of (unstable) multi-input discrete-time linear systems involving input static finite quantisation. A unified framework to handle uniform and logarithmic quantisers has been proposed. The approach is based on the modelling of the finite quantisation through the appropriate application of deadzone and saturation operators. Based on the properties of the three types of non-linearities (i.e. infinite quantisation, deadzone and saturation), conditions to design a static state feedback controller that guarantees the convergence to an attractor set \mathcal{S}_u , provided the initial state belongs to a set \mathcal{S}_0 , are developed. These conditions are incorporated in optimisation problems aiming at determining the stabilising feedback gain while minimising the size of the attractor set and guaranteeing that the set \mathcal{S}_0 covers a pre-defined region in the state space.

Although this work considered only input quantisation, the approach can be extended to cope with state quantisation and also be used to optimally tune the quantisers parameters when the gain is given.

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