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Stability of Discrete-time Control Systems with Uniform and Logarithmic Quantizers

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Abstract: This paper deals with stability analysis of discrete-time linear systems involving finite quantizers on the input of the controlled plant. Two kinds of quantization are analyzed: uniform and logarithmic. Through LMI-based conditions, an attractor of the state trajectories and a set of admissible initial conditions are determined. A method is proposed to compare the performances of the two kinds of quantization in terms of the dimensions of the attractor, considering a scenario of Networked Control Systems (NCS). Computational issues are discussed and a numerical example is presented to validate the work.

Keywords: Quantized control, stability, attractors, Networked Control Systems (NCS)

1. INTRODUCTION

The use of digital controllers became very frequent in control systems in the last decade. It is well known that in digital control, some undesired side effects such as time-delays, asynchronism, saturation and quantization are implicit. For instance, saturation may not only reduce the performance of a system, but also deteriorate stability. Similarly, the effect of quantization in control systems is a known phenomenon, which may lead to limit cycles, undesired equilibria or chaotic behavior, even if the controller is a stabilizing one (Delchamps [1990], Tarbouriech and Gouaisbaut [2012]). Now that Networked Control Systems (NCS) are becoming increasingly popular, researches on quantization regained attention of the control community. Since in NCS the control loop elements exchange information through communication channels with limited bandwidth, the control systems may become more susceptible to quantization side effects.

Then, this kind of communication constraints has attracted the attention of researchers over the last years; see, for example, Brockett and Liberzon [2000], Fridman and Dambrine [2009], Liberzon [2003]. It is also important to note that several studies have been proposed for continuous-time systems. In the context of discrete-time systems, one can cite several works dealing with the controller or observer design in presence of uniform or logarithmic quantizers: see, for example, Picasso and Colaneri [2008] and Xia et al. [2013]. In particular, in Elia and Mitter [2001], it has been shown that, for a quadratically stabilizable system, a logarithmic quantizer is the optimal solution in terms of coarse quantization

density. However, it is also shown that the quantizer must have an infinite number of quantization levels, which is not possible to implement. Following the same idea, Fu and Xie [2005] have introduced the sector bound approach for quantized feedback systems giving simple formulae to the stabilization problem considering state and output feedback controllers. Many other works were carried out as in de Souza et al. [2010], where the sector bound approach is applied to derive LMI based conditions for estimating a set of initial conditions and an attractor, such that all the state trajectories starting in the first set will enter the attractor in a finite time and remain inside it. In that work the controller and a finite logarithmic quantizer are supposed to be given and the stability analysis problem is addressed.

Despite these results obtained for logarithmic quantizers, research on uniform quantizers is not less relevant. In Ferrante et al. [2015], sector conditions are used to find a compact invariant set surrounding the origin, which attracts all the state trajectories, considering two different settings involving uniform quantization in continuous-time linear systems. The controller is designed in order to minimize the dimensions of the attractor. Tarbouriech and Gouaisbaut [2012] also considers the effects of saturation to design a state feedback control law while minimizing the attractor and maximizing the set of admissible initial conditions.

This paper deals with the stability analysis of discrete-time linear systems involving saturation and either a uniform or a logarithmic finite quantizer. For simplicity, only the input quantization case is considered. Considering uniform quantizers, we extend the work of Ferrante et al. [2015] and Tarbouriech and Gouaisbaut [2012] to the discrete-time case. Using sector conditions and the S-procedure, uniform ultimate boundedness stability is analyzed for a given static state feedback controller, while the attractor

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and the set of admissible initial conditions are determined. Then an LMI-based optimization problem is proposed to minimize the attractor and simultaneously maximize the set of admissible initial conditions (which can be considered as an estimate of the region of attraction of the origin). For the logarithmic quantizers, a similar analysis is carried out. After the analysis of both uniform and logarithmic quantization, we propose a method to compare the performance of quantizers, in terms of attractor size, in a NCS scenario. Finally, some simulations are presented to validate the results.

Notation. Throughout the article, I denotes the identity matrix and 0 denotes the null matrix (equivalently the null vector) of appropriate dimensions. For a matrix $A \in \mathbb{R}^{n \times m}$, A' , $A_{(i)}$, $tr(A)$ denote its transpose, its i th row and its trace respectively. The matrix $diag(A_1, A_2, \dots, A_n)$ is the block-diagonal matrix having A_1, A_2, \dots, A_n as diagonal blocks and in symmetric matrices $*$ stands for symmetric blocks. For a vector $x \in \mathbb{R}^n$, $x_{(i)}$, x' , $|x|$ denote its i th component, its transpose and the componentwise absolute value operator respectively. $sign(x)$ is the componentwise sign function, with $sign(0) = 0$, and $\lfloor x \rfloor$ the componentwise floor operator. For two sets \mathcal{S}_1 and \mathcal{S}_2 , $\mathcal{S}_1 \setminus \mathcal{S}_2$ denotes the set \mathcal{S}_1 deprived of \mathcal{S}_2 .

2. STABILITY WITH UNIFORM QUANTIZERS

2.1 Problem statement

Consider the following discrete-time linear system:

$$\begin{cases} x(k+1) = Ax(k) + Bsat(q(u(k))) \\ x(0) = x_0 \end{cases} \quad (1)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, $x_0 \in \mathbb{R}^n$ are respectively the state, the input of the system and the initial state. A , B are real matrices of suitable dimensions, and $q(\cdot)$ is a componentwise uniform quantizer, which is described by the static nonlinear function defined as

$$q_{(i)}(u) := \Delta sign(u_{(i)}) \left\lfloor \frac{|u_{(i)}|}{\Delta} \right\rfloor \quad (2)$$

where Δ is a positive real scalar representing the quantization step. The saturation map $sat(\cdot)$ represents a symmetric saturation function defined as follows:

$$sat(u_{(i)}) = sign(u_{(i)}) \min\{u_0, |u_{(i)}|\}, \quad i = 1, \dots, p \quad (3)$$

with $u_0 > 0$ being the symmetric bounds on the control input $u_{(i)}$, $i = 1, \dots, p$.

Assuming that the state x is fully accessible, we want to analyze stability of system (1) subject to the following control law $u = Kx$. Therefore, by defining the functions

$$\psi(v) := q(v) - v \quad (4)$$

$$\phi(v) := sat(v) - v \quad (5)$$

the closed-loop system becomes:

$$\begin{cases} x(k+1) = (A + BK)x(k) + B\psi(k) + B\phi(k) \\ x(0) = x_0 \end{cases} \quad (6)$$

with
$$\begin{cases} \psi(k) = \psi(u(k)) \\ \phi(k) = \phi(q(u(k))) = \phi(u(k) + \psi(u(k))) \end{cases}$$

The presence of the uniform quantizer defined in (2) can represent a real obstacle to the asymptotic stabilization of

the closed-loop system, due to its deadzone effect (Ferrante et al. [2015]). If the matrix A is not Schur-Cohn, the asymptotic stabilization of the origin for the closed-loop system cannot be achieved even when the gain K is supposed to be a stabilizing one (that is, when $(A + BK)$ is Schur-Cohn). Moreover, if A is not Schur-stable, under input saturation, only local (regional) stability can be achieved (Tarbouriech et al. [2011]). In this work, we focus on this case. Then the problem we aim to solve can be stated as follows.

Problem 1. Given the matrices A , B and a stabilizing gain K of adequate dimensions and a positive real quantization step Δ , determine a set $\mathcal{S}_0 \subset \mathbb{R}^n$ and a compact set $\mathcal{S}_u \subset \mathbb{R}^n$ containing the origin, such that

- \mathcal{S}_0 and \mathcal{S}_u are invariant sets;
- For every initial condition $x_0 \in \mathcal{S}_0 \setminus \mathcal{S}_u$, the trajectories are bounded and converge in a finite time into \mathcal{S}_u (which is therefore an attractor of the trajectories).

2.2 Main results

To solve Problem 1, we recall the sector conditions used in (Ferrante et al. [2015]) and (Tarbouriech and Gouaisbaut [2012]).

Lemma 1. For every $\psi(u)$ as defined in (4) and every control law $u = Kx$, $u \in \mathbb{R}^p$, the following relations are verified:

$$\psi' S_1 \psi - tr(S_1) \Delta^2 \leq 0 \quad (7)$$

$$\psi' S_2 (\psi + Kx) \leq 0 \quad (8)$$

for any diagonal positive definite matrices $S_1, S_2 \in \mathbb{R}^{p \times p}$.

Lemma 2. For every $\phi(u)$ as defined in (5) and every matrix $G \in \mathbb{R}^{p \times n}$, the following relation is verified:

$$\phi' S_3 (sat(q(Kx)) + Gx) \leq 0 \quad (9)$$

for any diagonal positive matrix $S_3 \in \mathbb{R}^{p \times p}$, provided that $x \in S(u_0)$ with

$$S(u_0) = \{x \in \mathbb{R}^n; -u_0 \leq G_{(i)}x \leq u_0, \forall i \in \{1, \dots, p\}\} \quad (10)$$

Note that condition (9), combined with (4) and (5), reads:

$$\phi' S_3 (\phi + \psi + Kx + Gx) \leq 0 \quad (11)$$

By using Lemmas 1 and 2, the following proposition to solve Problem 1 can be stated.

Proposition 1. If there exist a symmetric positive definite matrix $W \in \mathbb{R}^{n \times n}$, three diagonal positive matrices $S_1, S_2, R_3 \in \mathbb{R}^{p \times p}$, a matrix $Z \in \mathbb{R}^{p \times n}$, two positive scalars τ_1, τ_2 and a scalar α , $0 \leq \alpha \leq 1$, satisfying the conditions (12)–(14):

$$\begin{bmatrix} W(\tau_2 - \alpha\tau_1 - 1) & -WK'S_2 & -WK' - Z' & W(A + BK)' \\ * & -S_1 - 2S_2 & -I & B' \\ * & * & -2R_3 & R_3B' \\ * & * & * & -W \end{bmatrix} < 0 \quad (12)$$

$$tr(S_1)\Delta^2 + \tau_1 - \tau_2 \leq 0 \quad (13)$$

$$\begin{bmatrix} \alpha W & Z'_{(i)} \\ Z_{(i)} & u_0^2 \end{bmatrix} > 0, \quad i = 1, \dots, p \quad (14)$$

then, it follows that $\forall x(0) \in \mathcal{S}_0 \setminus \mathcal{S}_u$ the trajectories of system (6) converge in a finite time to the set \mathcal{S}_u , with

$$\mathcal{S}_u = \varepsilon(P) = \{x \in \mathbb{R}^n : x'Px \leq 1\}, \quad P = W^{-1} \quad (15)$$

$$\mathcal{S}_0 = \varepsilon(\alpha P) = \{x \in \mathbb{R}^n : x'\alpha Px \leq 1\} \quad (16)$$

Moreover, \mathcal{S}_0 and \mathcal{S}_u are positively invariant sets with respect to (6).

Proof. Considering the quadratic Lyapunov function $V(x) = x'Px$, we want to prove that

$$\Delta V(x) = V(x(k+1)) - V(x(k)) < 0, \quad \forall x \in \mathcal{S}_0 \setminus \mathcal{S}_u. \quad (18)$$

In this case, it follows that \mathcal{S}_0 is an invariant set and that $\forall x(0) \in \mathcal{S}_0$, the corresponding trajectory converge to \mathcal{S}_u . Note that

$$\mathcal{S}_0 \setminus \mathcal{S}_u = \{x \in \mathbb{R}^n : (x'\alpha Px - 1) \leq 0 \text{ and } (1 - x'Px) \leq 0\}$$

Now applying S-procedure along with Lemmas 1 and 2, if

$$\Delta V(x) - \tau_1(x'\alpha Px - 1) - \tau_2(1 - x'Px) - \psi'S_1\psi + \text{tr}(S_1)\Delta^2 - 2\psi'S_2(\psi + Kx) - 2\phi'S_3(\phi + \psi + Kx + Gx) < 0 \quad (19)$$

it follows that (18) is satisfied, provided that $x \in S(u_0)$.

Developing (19) by using equation (6), it is possible to re-write (19) as follows:

$$\begin{bmatrix} x' & \psi' & \phi' \end{bmatrix} \mathcal{N}_u \begin{bmatrix} x \\ \psi \\ \phi \end{bmatrix} + \text{tr}(S_1)\Delta^2 + \tau_1 - \tau_2 < 0 \quad (20)$$

where \mathcal{N}_u is given at the bottom of the page in (17).

It is straightforward to verify that (20) will be satisfied if we guarantee (13) and $\mathcal{N}_u < 0$.

Applying Schur complement, $\mathcal{N}_u < 0$ is equivalent to:

$$\begin{bmatrix} P(\tau_2 - \alpha\tau_1 - 1) & -K'S_2 & -K'S_3 - G'S_3 & (A + BK) \\ * & -S_1 - 2S_2 & -S_3 & B' \\ * & * & -2S_3 & B' \\ * & * & * & -P^{-1} \end{bmatrix} < 0 \quad (21)$$

Then by pre- and post-multiplying relation (21) by $\text{diag}(P^{-1}, I, S_3^{-1}, I)$ and with the change of variables $P^{-1} = W$, $GP^{-1} = Z$, $S_3^{-1} = R_3$, we get relation (12).

Furthermore, one has to prove that the set \mathcal{S}_u is positively invariant. This is proven if we guarantee that $x(k+1)'Px(k+1) \leq 1$ whenever $x(k)'Px(k) \leq 1$ (i.e., when $x \in \mathcal{S}_u$). It is possible to do that by verifying the following inequality, for some positive scalar τ_3 ,

$$x'(k+1)Px(k+1) - 1 - \tau_3(x'(k)Px(k) - 1) \leq 0 \quad (22)$$

The inequality (22) is true if we are able to verify that

$$\text{tr}(S_1)\Delta^2 + \tau_3 - 1 \leq 0 \quad (23)$$

$$\mathcal{N}_u = \begin{bmatrix} P(\tau_2 - \alpha\tau_1 - 1) & -K'S_2 & -K'S_3 - G'S_3 \\ * & -S_1 - 2S_2 & -S_3 \\ * & * & -2S_3 \end{bmatrix} + \begin{bmatrix} (A + BK) \\ B' \\ B' \end{bmatrix} P \begin{bmatrix} (A + BK) & B & B \end{bmatrix} \quad (17)$$

and

$$x'(k+1)Px(k+1) - \tau_3x'(k)Px(k) - \psi'S_1\psi - 2\psi'S_2(\psi + Kx) - 2\phi'S_3(\phi + \psi + Kx + Gx) \leq 0 \quad (24)$$

Now by choosing $-\tau_3 = \tau_2 - \alpha\tau_1 - 1$, it is straightforward to verify that (24) is implied by (12). From (23), since $\alpha \leq 1$ and $\tau_1, \tau_2 > 0$, it follows that

$$\text{tr}(S_1)\Delta^2 + \alpha\tau_1 - \tau_2 \leq \text{tr}(S_1)\Delta^2 + \tau_1 - \tau_2 \quad (25)$$

Therefore, (13) implies (23). Then, if (12) and (13) are satisfied, (22) is implicitly verified and thus \mathcal{S}_u is positively invariant.

Finally, the satisfaction of relation (14) implies that the ellipsoid $\mathcal{S}_0 = \varepsilon(\alpha P)$ is included in the polyhedral set $S(u_0)$ defined in (10), which ensures the validity of condition (11) in \mathcal{S}_0 . That concludes the proof. \square

3. STABILITY WITH LOGARITHMIC QUANTIZERS

3.1 Problem statement

Consider the following discrete-time linear system:

$$\begin{cases} x(k+1) = Ax(k) + B \cdot dz\left(\text{sat}\left(q(u(k))\right)\right) \\ x(0) = x_0 \end{cases} \quad (26)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, $x_0 \in \mathbb{R}^n$ are respectively the state, the input of the system and the initial state. A , B are real matrices of suitable dimensions, and $q(\cdot)$ is the logarithmic quantizer, which is described by the static nonlinear function defined as

$$q_{(i)}(u) := \begin{cases} \frac{\lambda}{\rho^j}, & \text{if } \frac{\lambda}{\rho^j(1+\delta)} \leq u_{(i)} < \frac{\lambda}{\rho^j(1-\delta)}, \quad j \in \mathbb{Z} \\ -q(-u_{(i)}), & \text{if } u_{(i)} < 0 \end{cases} \quad (27)$$

$$\delta = \frac{1-\rho}{1+\rho}, \quad \rho = \frac{1-\delta}{1+\delta}, \quad 0 < \rho < 1 \quad (28)$$

The parameter ρ is said to be the quantization density (instead of quantization step as in the uniform quantization), and λ is the least positive quantization level outside the deadzone. λ is related to the deadzone limit by $\lambda = \epsilon(1+\delta)$, i.e., the deadzone is defined by its limit ϵ , $\epsilon > 0$, and the nonlinear function given by

$$dz_{(i)}(u) := \begin{cases} 0, & \text{if } -\epsilon < u_{(i)} < \epsilon \\ u_{(i)}, & \text{if } |u_{(i)}| \geq \epsilon \end{cases} \quad (29)$$

The saturation map $\text{sat}(\cdot) \in \mathbb{R}^p$ is defined from the symmetric saturation function having as level the positive scalar $\mu = \frac{\lambda}{\rho^{N-1}}$, $\forall i = 1, \dots, p$:

$$\text{sat}(u_{(i)}) = \text{sign}(u_{(i)}) \min\{\mu, |u_{(i)}|\}, \quad i = 1, \dots, p \quad (30)$$

N is the number of positive levels between μ and ϵ .

Notice that we obtain a finite-level quantizer by applying saturation and deadzone effects over an infinite-level quantizer.

The functions ψ and ϕ related to the quantization and saturation are defined as in (4)–(5). By defining now the function

$$\theta(v) := dz(v) - v \quad (31)$$

the closed-loop system becomes:

$$\begin{cases} x(k+1) = (A + BK)x(k) + B\psi(k) + B\phi(k) + B\theta(k) \\ x(0) = x_0 \end{cases} \quad (32)$$

Due to the deadzone, when approaching the origin, the quantizer output will be null if its input is inside $(-\epsilon, \epsilon)$. Thus, asymptotic stabilization of the origin cannot be achieved if A is not Schur-Cohn. Again we analyze ultimate boundedness stability rather than stability in the conventional sense. The following problem is addressed in this case.

Problem 2. Given the matrices A , B and a stabilizing gain K of appropriate dimensions, the quantization density ρ , the deadzone limit ϵ and the saturation limit μ , determine a set $\mathcal{S}_0 \subset \mathbb{R}^n$ and a compact set $\mathcal{S}_u \subset \mathbb{R}^n$ containing the origin, such that

- \mathcal{S}_0 and \mathcal{S}_u are invariant sets;
- For every initial condition $x_0 \in \mathcal{S}_0 \setminus \mathcal{S}_u$, the trajectories are bounded and converge in a finite time into \mathcal{S}_u (which is an attractor of the trajectories).

3.2 Main results

To solve Problem 2, we recall a sector condition verified by the nonlinearities ψ and θ (de Souza et al. [2010]):

Lemma 3. For $\psi = q(u) - u$ with $q(u)$ defined in (27)–(28) and $u = Kx \in \mathbb{R}^p$, the following relation is verified:

$$(\psi + \delta Kx)' S_1 (\psi - \delta Kx) \leq 0 \quad (33)$$

for any diagonal positive definite matrix $S_1 \in \mathbb{R}^{p \times p}$.

For the sector conditions originated by the saturation, we use Lemma 2 (rewritten with $S_2 \in \mathbb{R}^{p \times p}$ instead of S_3 for clarity purposes):

$$\phi' S_2 (\phi + \psi + Kx + Gx) \leq 0 \quad (34)$$

Lemma 4. For every θ defined in (31), and every control law $u = Kx \in \mathbb{R}^p$, the following relation is verified:

$$\theta' S_3 \theta - \text{tr}(S_3) \epsilon^2 \leq 0 \quad (36)$$

for any diagonal positive definite matrix $S_3 \in \mathbb{R}^{p \times p}$.

By using Lemmas 2–4, the following proposition to solve Problem 2 can be stated.

Proposition 2. If there exist a symmetric positive definite matrix $W \in \mathbb{R}^{n \times n}$, three diagonal positive matrices R_1 ,

R_2 , $S_3 \in \mathbb{R}^{p \times p}$, a matrix $Z \in \mathbb{R}^{p \times n}$, two positive scalars τ_1, τ_2 and a scalar α , $0 \leq \alpha \leq 1$, satisfying the conditions (37)–(39):

$$\begin{bmatrix} \gamma W & 0 & -WK' - Z' & 0 & W(A + BK)' & \delta WK' \\ * & -R_1 & -R_1 & 0 & R_1 B' & 0 \\ * & * & -2R_2 & 0 & R_2 B' & 0 \\ * & * & * & -S_3 & B' & 0 \\ * & * & * & * & -W & 0 \\ * & * & * & * & * & -R_1 \end{bmatrix} < 0, \quad (37)$$

$$\text{where } \gamma = \tau_2 - \alpha\tau_1 - 1 \quad (37)$$

$$\text{tr}(S_3) \epsilon^2 + \tau_1 - \tau_2 \leq 0 \quad (38)$$

$$\begin{bmatrix} \alpha W & Z_{(i)}' \\ Z_{(i)} & \mu^2 \end{bmatrix} > 0, \quad i = 1, \dots, p \quad (39)$$

then, it follows that $\forall x(0) \in \mathcal{S}_0 \setminus \mathcal{S}_u$ the trajectories of system (32) converge in a finite time to the set \mathcal{S}_u , with \mathcal{S}_u and \mathcal{S}_0 as in (15)–(16). Moreover, \mathcal{S}_0 and \mathcal{S}_u are positively invariant sets with respect to (32).

Proof. Applying S-procedure along with Lemmas 2–4, if

$$\begin{aligned} \Delta V(x) - \tau_1(x' \alpha P x - 1) - \tau_2(1 - x' P x) \\ - (\psi + \delta Kx)' S_1 (\psi - \delta Kx) \\ - 2\phi' S_2 (\phi + \psi + Kx + Gx) - \theta' S_3 \theta + \text{tr}(S_3) \epsilon^2 < 0 \end{aligned} \quad (40)$$

then it follows that $\Delta V(x) < 0$, $\forall x \in \mathcal{S}_0 \setminus \mathcal{S}_u$ is satisfied, provided that $x \in S(u_0)$.

Developing (40) by using equation (32) and considering $\xi = [x' \ \psi' \ \phi' \ \theta']'$, it is possible to re-write (40) as follows:

$$\xi' \mathcal{N}_1 \xi + \text{tr}(S_3) \epsilon^2 + \tau_1 - \tau_2 < 0 \quad (41)$$

where \mathcal{N}_1 is given at the bottom of the page in (35). It is straightforward to verify that (41) will be satisfied if we guarantee (38) and $\mathcal{N}_1 < 0$. By applying Schur complement twice, then by pre- and post-multiplying by $\text{diag}(P^{-1}, S_1^{-1}, S_2^{-1}, I, I, I)$ and with the change of variables $P^{-1} = W$, $GP^{-1} = Z$, $S_1^{-1} = R_1$, $S_2^{-1} = R_2$, we get relation (37).

To prove that the set \mathcal{S}_u is positively invariant, it suffices to verify that, for some positive scalar τ_3 ,

$$x'(k+1)Px(k+1) - 1 - \tau_3(x'(k)Px(k) - 1) \leq 0 \quad (42)$$

Then, by choosing $-\tau_3 = \tau_2 - \alpha\tau_1 - 1$, following the same reasoning done in the proof of Proposition 1, it follows that (37)–(38) imply (42).

Finally, the satisfaction of relation (39) implies that the ellipsoid $\mathcal{S}_0 = \varepsilon(\alpha P)$ is included in the polyhedral set $S(\mu)$ defined in (10). That concludes the proof. \square

4. COMPUTATIONAL ISSUES

Regarding Problems 1 and 2, the objective is to obtain a set \mathcal{S}_0 as large as possible and a set \mathcal{S}_u as small as

$$\mathcal{N}_1 = \begin{bmatrix} P(\tau_2 - \alpha\tau_1 - 1) + \delta^2 K' S_1 K & 0 & -K' S_2 - G' S_2 & 0 \\ * & -S_1 & -S_2 & 0 \\ * & * & -2S_2 & 0 \\ * & * & * & -S_3 \end{bmatrix} + \begin{bmatrix} (A + BK)' \\ B' \\ B' \\ B' \end{bmatrix} P [(A + BK) \ B \ B \ B] \quad (35)$$

possible. To achieve this objective, one needs to find a suitable measure for the set \mathcal{S}_u and minimize it. The largest solution for \mathcal{S}_0 will be determined by taking the smallest value of α for which the problem remains feasible. It is possible to implicitly minimize the volume of \mathcal{S}_u by minimizing $tr(P^{-1})$, or equivalently, $tr(W)$.

It is important to note that conditions (12) and (14), in the uniform quantization case, and conditions (37) and (39), in the logarithmic quantization case, are nonlinear in the decision variables, which prevents from solving directly a convex optimization problem. Nevertheless, it is possible to overcome this problem by considering the variables τ_1 , τ_2 and α as tuning parameters. In the uniform quantization case, S_2 must be fixed as well. To be feasible, it is strictly necessary that the following conditions are respected:

$$\tau_2 > \tau_1 \ ; \ \tau_2 < 1 + \alpha\tau_1 \quad (43)$$

Thus, the optimal value for the trace minimization can be performed by solving a convex-optimization problem on a grid. For simulation purposes, it is also interesting to require the set \mathcal{S}_0 to be large enough to contain the initial conditions of interest. One can impose x_0 to be inside the ellipsoid of \mathcal{S}_0 by writing $x_0' \alpha P x_0 < 1$. To write it in an LMI form in terms of W , one can rearrange that statement as follows:

$$\begin{bmatrix} 1 & \alpha x_0' \\ * & \alpha W \end{bmatrix} > 0 \quad (44)$$

After choosing the initial conditions to be considered and fixing τ_1 , τ_2 , α , S_2 at feasible values, the convex optimization to address Problem 1 is stated as follows:

$$\min tr(W), \text{ s.t.: } S_1 > 0; S_2 > 0; (12)-(14); (44) \quad (45)$$

Similarly, the convex optimization to address Problem 2 is stated as follows:

$$\min tr(W), \text{ subject to: } (37)-(39); (44) \quad (46)$$

Notice that conditions $W > 0$, $R_1 > 0$, $R_2 > 0$, $R_3 > 0$, $S_3 > 0$ are ensured by (12) and (37).

5. COMPARISON OF THE QUANTIZERS

Now we consider a scenario of a networked control system where the only element that need to be designed is the quantizer. In such scenario, the sampling period is already fixed. Thus, so are the matrices A, B of the discrete-time system. The number of bits used for each variable is also a parameter of the network, and it determines the number of positive levels of the quantizer. Considering that the most significant bit is reserved for the sign, the relation between the number of bits N_{bits} and the number of positive levels of the quantizer N is given by

$$N = 2^{(N_{bits}-1)} - 1 \quad (47)$$

The level of saturation is also considered as being the same when comparing an uniform and a logarithmic quantizer: $u_0 = \mu$.

Fixing all these parameters is already enough to determine an uniform quantizer. The quantization step can be calculated by $\Delta = \frac{u_0}{N}$. In the logarithmic quantization case, the quantization density ρ is a degree of freedom, which

will affect the smallest positive quantized value λ by the following relation:

$$\lambda = \mu\rho^{(N-1)} \quad (48)$$

The performance of the system in the logarithmic quantizer case depends directly on a good choice of ρ . If ρ is too close to 1, λ will be not small. Thus the attractor set \mathcal{S}_0 risks of being too large (because the deadzone will have large bounds). On the other hand, a too small ρ can deteriorate the dynamics of the control system because of a poor resolution. Furthermore, ρ should be carefully chosen even in order to keep feasibility of the optimization problem. Note that a small quantization density can lead to closed-loop instability if the system is open-loop unstable.

Finally, the analysis of the two cases taking into account the scenario described should enable a comparison of the quantizers' performance in terms of the smallest attractor around the origin. Nevertheless, simulation is always necessary to confirm the results obtained, since the proposed methods can only estimate the attractor.

6. NUMERICAL EXAMPLE

Consider the following discrete-time system derived from the continuous-time system in (Amato et al. [2007]), with a sampling period $T_s = 0.01s$:

$$A = \begin{bmatrix} 0.9960 & 0.0169 & 0.0418 \\ 0.0451 & 1.0640 & 0.0544 \\ 0.0347 & 0.0730 & 1.0772 \end{bmatrix}, B = \begin{bmatrix} -0.0068 & -0.0136 \\ 0.0001 & -0.0450 \\ 0.0082 & -0.0173 \end{bmatrix} \quad (49)$$

Let us assume the static state feedback controller is given by $K = \begin{bmatrix} 0.6137 & -0.3853 & -10.4675 \\ 9.2395 & 17.0548 & 14.6553 \end{bmatrix}$, which would be a stabilizing gain if we had no quantization.

Assuming $N_{bits} = 6$ (which yields $N = 31$), the saturation level $u_0 (= \mu) = 12.4$ and the considered initial condition as $x_0 = [2 \ 2 \ 2]'$, both quantization cases are analyzed in the sequel.

6.1 Uniform quantizer case

Fixing N and u_0 yields $\Delta = 0.4$. A grid search allows to find the best feasible values for the tuning parameters, respectively, $\alpha = 0.0026$, $\tau_1 = 5.3059 \cdot 10^{-11}$, $\tau_2 = 0.0198$ and $S_2 = \text{diag}(6.3644 \cdot 10^{-4}, 0.0014)$. The solution of the optimization problem (45) in this case yields

$$P = \begin{bmatrix} 15.6182 & 7.5034 & -1.4831 \\ 7.5034 & 19.6851 & -1.3810 \\ -1.4831 & -1.3810 & 51.5721 \end{bmatrix}, \text{ which has trace equal to } 86.88.$$

6.2 Logarithmic quantizer case

In the logarithmic quantizer case, we choose to fix $\rho = 0.7391$ which yields $\epsilon = 0.0012$. Comparing ϵ with $\Delta = 0.4$ of the uniform quantizer case, one can expect the attractor size to be considerably smaller in the logarithmic quantizer case. A grid search allows to find the best feasible values for the tuning parameters, respectively, $\alpha = 9.9218 \cdot 10^{-6}$, $\tau_1 = 1.6985 \cdot 10^{-15}$ and $\tau_2 = 0.0083$. The solution of the optimization problem (46) in this case yields

$P = 10^4 \cdot \begin{bmatrix} 0.3031 & 0.2758 & -0.0925 \\ 0.2758 & 0.4886 & 0.1009 \\ -0.0925 & 0.1009 & 1.1593 \end{bmatrix}$, which has trace equal to $1.9510 \cdot 10^4$.

Figures (1)-(3) show, respectively, the states evolution for a initial condition of $x_0 = [2 \ -2 \ -1]'$ in the uniform and logarithmic cases, and the ellipsoids of \mathcal{S}_u obtained in the examples.

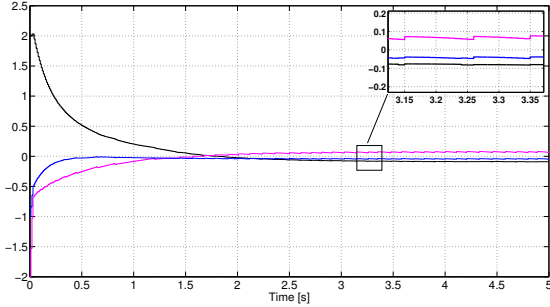


Fig. 1. States evolution in the log. quantization case

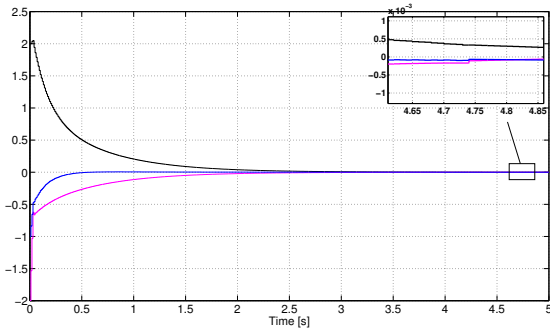


Fig. 2. States evolution in the log. quantization case

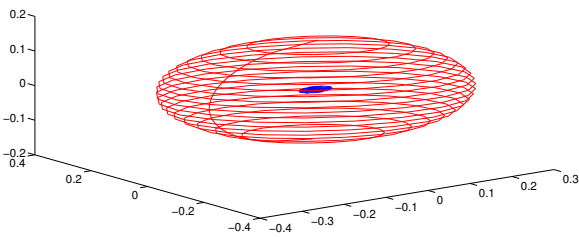


Fig. 3. Ellipsoid of \mathcal{S}_u in the uniform (red) and logarithmic (blue) quantization cases with same number of bits

Notice that both analysis and simulation show the superiority of logarithmic quantizers in terms of attractor size.

7. CONCLUSION

This paper has tackled the stability analysis of discrete-time linear systems involving input finite quantization (uniform or logarithmic). In both cases, the attractor of the states is minimized through a quasi-LMI optimization problem and the results were compared, confirming the superiority of logarithmic quantizers. The performance of

systems involving both kinds of quantizers were compared. A simulation was carried out to confirm the results. In this work we considered only the minimization of the attractor set \mathcal{S}_0 . The simultaneous maximization of \mathcal{S}_u and also considering \mathcal{S}_0 and \mathcal{S}_u of different shapes (i.e. defined with different matrices P) as done in de Souza et al. [2010], are possible extensions of this approach. Furthermore, the stabilization problem and extension of the approach to consider also quantization on the measured states is an ongoing work.

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