Hierarchical estimation of the region of attraction for systems subject to a state delay and a saturated input

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***Abstract*— This paper addresses the local stability analysis problem for linear systems subject to input saturation and state delay. Thanks to the construction of a Lyapunov-Krasovskii functional associated to Legendre polynomials and the use of generalized sector conditions, sufficient linear matrix inequalities (LMIs) are derived to guarantee the local stability of the origin for the closed-loop system. In addition, an estimate of the basin of attraction of the origin is provided, which does not include the derivative of the initial condition. A convex optimization problem leans on these conditions to maximize the size of the estimate of the basin of attraction. Optimal LMIs form a hierarchy, which is competitive to improve the size criterium of the estimate of the basin of attraction. An example illustrates the potential of the technique.**

I. Introduction

This paper is concerned with the computation of regions of attraction of time-delay systems subject to input saturations. Taking into account these constraints is of crucial interest. Indeed, on the one hand such a nonlinearity might make the resulting closed-loop system unstable (see e.g. [19], which surveys the stability analysis problem of linear finite-dimensional systems subject to saturation). On the other hand, it is well known that the introduction of delays in a control loop may lead to a notable degradation of the performance or even to instability [3], [7], [9], [12], [17]. In the situation of saturated time-delay systems, the standard method follows a two steps design. First, the design is carried out without taking into account the saturation. Second, we add the saturation and provide a stability analysis based on some Lyapunov functions.

In general, there is no hope to have a global asymptotic result, which makes crucial the problem of computing regions of attraction or at least estimate of them. Note that tackling this particular nonlinearity in the case of finite-dimensional system is already a difficult problem. However, nowadays, numerous techniques are available to provide such an analysis (see e.g., [19], [22], etc.). Most of them are based on the Lyapunov function for the system without saturation and some *sector condition* property satisfied naturally by the saturation. Roughly speaking, this sector condition describes some partitions of the state space where the saturation can be approached by a linear function. In one of these partitions, one can retrieve the stability of the system without saturation.

As mentioned above, Lyapunov functionals for the system without saturation (which resumes to a linear time-delay system) are crucial. While these functionals can be computed explicitely with LMIs in the case of linear systems without delay, it is harder in the case with delay to obtain such an explicit form. Indeed, even if [6, Theorem 5.9.] ensures the existence of a Lyapunov-Krasovkii functional for stable systems, this latter is in general difficult to compute numerically. Therefore, in general, we provide an approximation of this functional so that computations become tractable. Numerous numerical approximations have been proposed in recent decades (see e.g., [6] for the discretization method, [2], [18], for the delay-partitionning method, or [11], for sum of squares methods.)

In order to compute estimate of the region of attraction of time-delay systems subject to saturation, most of the existing results in the literature uses classical Lyapunov functionals, based on the quadratic form coming from finite-dimensional systems and integral terms (see e.g., [1], [5], [4], [10], or [20]). A generalized sector condition is also added in order to achieve the Lyapunov stability analysis. Unfortunately, most of these approximations induces some conservatism. In this paper, our approximation is based on a Legendre polynomial setting, borrowed from [14], which leads to a less conservative Lyapunov functional.

The latter Lyapunov functional is defined with an augmented state, that is the projection of state functions to a finite number of Legendre polynomials of degree at most equal to $N \in \mathbb{N}$. One of the interests is that it is possible to built efficient functionals without including double integral terms that depends on the derivative of the states. This indeed leads to many complications when it comes to the estimation of the basin of attraction, which has then to be defined using the derivative of the initial condition (see for instance in [5]). Another interest of this particular Lyapunov functional is that we may increase this degree (our approximation is therefore indexed by N), to reduce the conservatism of the results thanks to the hierarchical structure of the LMIs. Moreover, this implies that, increasing the degree, we may improve at each step of the approximation the size of the region of attraction.

The paper is organized as follows. Section II presents the system under consideration and the problem we intend to solve. Section III proposes preliminary results on Legendre polynomials and the modified sector conditions. Section IV is dedicated to provide the sufficient conditions for Local stability. In section V, some optimization issues are discussed. Section VI develops a hierarchical estimation of the basin of attraction. The technique developed in the paper is illustrated in Section VII. Finally, Section VIII ends the

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paper with some concluding remarks.

Notation. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with Euclidian norm $|\cdot|$, $\mathbb{R}^{n\times m}$ is the set of all $n\times m$ real matrices. The notation $P\succ 0$, for $P\in\mathbb{R}^{n\times n}$, means that P is symmetric and positive definite. The sets $\mathbb{S}_n$ and $\mathbb{S}_n^+$ represent, the set of symmetric and symmetric positive definite matrices of $\mathbb{R}^{n\times n}$, respectively. The set of continuous functions from an interval $[-h,\ 0]\subset\mathbb{R}$ to $\mathbb{R}^n$ which are, consequently, square integrable is demoted as space $\mathcal{L}_2([-h,0],\mathbb{R}^n)=\mathcal{L}_2$. Let us also define the set $\mathbb{R}^n_M[u]\subset\mathcal{L}_2$ as the set of polynomials in $\mathbb{R}^n$ of degree less than M. For any function $f\in\mathcal{L}_2$, the norm $|f|_h$ refers to $\sup_{\theta\in[-h,\ 0]}|f(\theta)|$. The symmetric matrix $\left[\begin{smallmatrix}A & B\\ * & C\end{smallmatrix}\right]$ stands for $\left[\begin{smallmatrix}A & B\\ B^\top & C\end{smallmatrix}\right]$. $diag(A,B)$ stands for the diagonal matrix $\left[\begin{smallmatrix}A & 0\\ 0 & B\end{smallmatrix}\right]$. For any positive integer $j\le n$ any vector $x\in\mathbb{R}^n$ and any matrix $A\in\mathbb{R}^{n\times n}$, the notation A_j and x_j refer to the j^{th} line of matrix A and the j^{th} component of vector x, respectively. Moreover, for any square matrix $A\in\mathbb{R}^{n\times n}$, we define $\mathrm{He}(A)=A+A^\top$. The matrix I represents the identity matrix of appropriate dimensions. The notation $0_{n,m}$ stands for the matrix in $\mathbb{R}^{n\times m}$ whose entries are zero and, when no confusion is possible, the subscript will be ommited. For any function $x:\ [-h,\ +\infty)\to\mathbb{R}^n$, the notation $x_t(\theta)$ stands for $x(t+\theta)$, for all $t\ge 0$ and all $\theta\in[-h,\ 0]$. The notation $\binom{k}{l}$ refers to the binomial coefficients given by $\frac{k!}{(k-l)!l!}$.

II. Problem formulation

Consider a linear time-delay system described by:

$$\begin{cases}\dot{x}(t)=Ax(t)+A_dx(t-h)+Bu(t), & \forall t\ge 0,\\ x(t)=\phi(t), & \forall t\in[-h,0],\end{cases}\tag{1}$$

where $x(t)\in\mathbb{R}^n$ is the instantaneous state vector, ϕ is the initial conditions and A, A_d and B, are constant matrices of appropriate dimensions. The delay h is assumed to be constant and known. Furthermore, the input is limited in magnitude as follows:

$$|u_i|\le u_{0i},\quad u_{0i}>0,\ i=1,\ldots,m.\tag{2}$$

Then, the effective control signal to be applied to the system is given by

$$u(t)=sat(Kx(t)),\tag{3}$$

with $K\in\mathbb{R}^{m\times n}$ and

$$u_i(t)=sat(K_ix(t))=sign(K_ix(t)))\min\{u_{0i},|K_ix(t)|\},$$

for all $i=1,\ldots,m$. Hence, the closed-loop system reads

$$\dot{x}(t)=Ax(t)+A_dx(t-h)+Bsat(Kx(t)).\tag{4}$$

Due to the saturation, system (4) is a nonlinear system. Therefore, the resulting stability properties have to be regarded with precaution. In particular, if the open-loop system (i.e. $u=0$) is unstable, global asymptotic stability cannot be guaranteed for the closed-loop system, and one has to look at for regional (local) asymptotic stability criteria. One of the possible control objectives consists in the maximization of the estimate of the basin of attraction of the origin for the closed-loop system. While, in the context of finite-dimensional systems, i.e. without delay, the estimation of the region of attraction is simply characterized by a vector representing the possible finite-dimensional initial condition, the infinite-dimensional case of time-delay systems is more complicated to formulate. Indeed, it usually imposes to relate the region of attraction to the supremum norm of the vectorial function that represents the initial conditions. This formulation has the drawback of being over conservative but it is unfortunately necessary in order to provide a tractable estimation of the basin of attraction of the origin

Note that system (4) is not affected by the problem of the first delay interval [8], since the delay only affect the state of the system but not the control input.

The problem we intend to solve can be summarized as follows.

Problem 1: Given a state feedback gain K such that the matrix $A+A_d+BK$ is Hurwitz, characterize an estimate of the basin of attraction of the origin for closed-loop system (4).

The novelty of this paper relies on the recent developments regarding the stability analysis of time-delay systems based on polynomial approximation and the derivation of efficient integral inequalities. The basic idea of this paper is to restrict the set of allowable initial condition functions to a finite-dimensional truncation of $\mathcal{L}_2$, consisting of polynomials of limited order N. This restriction allows deriving in a direct and less conservative manner the estimate of region of attraction, for any initial condition in this truncated set.

III. Preliminary lemma

A. Bessel-Legendre inequality

Let us first recall the framework introduced in [14], [15] based on the use of sequence of Legendre polynomials, which is orthogonal with respect to the inner product associated to the $\mathcal{L}_2$ norm.

Definition 2: The Legendre polynomials considered over the interval $[-h,\ 0]$ are defined by $L_k(u)=(-1)^k\sum_{l=0}^k p_l^k\left(\frac{u+h}{h}\right)^l$, for all $k\in\mathbb{N}$, where $p_l^k=(-1)^l\binom{k}{l}\binom{k+l}{l}$.

The set of Legendre polynomials $\{L_k\}_{k\in\mathbb{N}}$ forms an orthogonal sequence with respect to the inner product:

$$\langle f,g\rangle=\int_{-h}^0 f(t)g(t)\mathrm{d}t,\quad \forall f,g\in\mathcal{C}.\tag{5}$$

Hence, the Legendre polynomials described in Definition 2 satisfy the following properties:

Property 3: The Legendre polynomials satisfy the following properties:

P1 Orthogonality:

$$\forall(k,l)\in\mathbb{N}^2,\quad \int_{-h}^0 L_k(u)L_l(u)\mathrm{d}u\ =\ \frac{h\delta_{kl}}{2k+1}\tag{6}$$

where δ_{kl} is the Kronecker index, such that $\delta_{kl}=1$ if $k=l$ and 0 otherwise.

P2 Boundary conditions:

$$\forall k \in \mathbb{N}, \quad L_k(0) = 1, \quad L_k(-h) = (-1)^k.$$

P3 Differentiation:

$$\forall k \in \mathbb{N}, \quad \frac{d}{du} L_k(u) = \sum_{i=0}^{k} \frac{(2i+1)(1-(-1)^{k+i})}{h} L_i(u).$$

We are now in position to recall the Bessel-Legendre inequality, which aims at providing a lower bound of a function x in $\mathcal{L}_2$ expressed in terms of a quadratic form of its projections over a truncated sequence of Legendre polynomials.

Lemma 4: Let h, x and R be a strictly positive number, a function in $\mathcal{L}_2$ and a matrix in $\mathbb{S}_n^+$, respectively. Then, the inequality

$$\int_{-h}^{0} x^\top(s) R x(s) ds \geq \frac{1}{h} \tilde{x}_N^\top \begin{bmatrix} R & & & \\ & 3R & & \\ & & \ddots & \\ & & & (2N+1)R \end{bmatrix} \tilde{x}_N, \quad (7)$$

holds for all $N \in \mathbb{N}$, where the projection vector $\tilde{x}_N$ is given, for any positive integer N by

$$\tilde{x}_N = \begin{bmatrix} \int_{-h}^{0} L_0(s) x(s) \mathrm{d}s \\ \int_{-h}^{0} L_1(s) x(s) \mathrm{d}s \\ \vdots \\ \int_{-h}^{0} L_N(s) x(s) \mathrm{d}s \end{bmatrix}.$$

Proof: Various proofs can be found in [14], [15]. It is based on the calculation of the $\mathcal{L}_2$ norm of the error variable $z_N(u)$ given by

$$z_N(u) = x(u) - \sum_{k=0}^{N} \frac{2k+1}{h} L_k(u) \int_{-h}^{0} L_k(s) x(s) \mathrm{d}s.$$

■

Remark 1: The components of the vector $\tilde{x}_N$ can be interpreted as the collection of the projections of the function x over the $N+1$ first Legendre polynomials. This notion of projection will be used in the sequel.

B. Modified sector conditions

Let us introduce the dead-zone function ψ as follows

$$\psi(Kx(t)) = Kx(t) - sat(Kx(t)). \quad (8)$$

Note that, $\psi(Kx(t))$ corresponds to a decentralized dead-zone nonlinearity. Considering the function $\psi(Kx(t))$, the closed-loop system can be re-written as

$$\dot{x}(t) = (A + BK)x(t) + A_d x(t-h) - B\psi(Kx(t)). \quad (9)$$

Let us define the polyhedral set

$$\mathcal{S}(u_0) = \{v, w \in \mathbb{R}^m \times \mathbb{R}^m, |(v_i - w_i)| \leq u_{0i}, \; \forall i = 1, ..., m\}. \quad (10)$$

The next Lemma, borrowed from [19] concerning the dead-zone nonlinearity $\psi(v)$ is recalled.

Lemma 5: Consider the function ψ defined in (8). If $(v, w) \in \mathcal{S}(u_0)$ then the relation

$$\psi^\top(v) U[\psi(v) - w] \leq 0, \quad (11)$$

is verified for any matrix $U \in \mathbb{R}^{m \times m}$ diagonal and positive definite.

The result in Lemma 5 can be seen as an extension of the classical sector condition (used for instance in [21]), leading to less conservative stability conditions.

In the sequel, Lemma 5 is used by considering:

$$v = Kx \text{ and } w = Xx + Y_N \tilde{x}_N. \quad (12)$$

For the sake of simplicity of the presentation, the notation $\psi(t)$ will stand in the sequel for $\psi(Kx(t))$.

IV. Local stability analysis of delayed and saturated systems

This section provides a solution to Problem 1, that is providing a local asymptotic stability result for system (4), or equivalently system (9).

Theorem 6: For a given integer N, a constant delay h, a given controller gain K and a given diagonal positive matrix U, assume that there exist matrices $P_N \in \mathbb{S}_{n(N+2)}^+$, $S, R, W \in \mathbb{S}_n^+$, $X \in \mathbb{R}^{m \times n}$ and $Y_N \in \mathbb{R}^{m \times (N+1)n}$ such that the set of LMIs (with $i = 1, \ldots, m$)

$$\begin{array}{rcl} \Theta_N(h) & = & \begin{bmatrix} P_N + \frac{1}{h} S_N & \begin{bmatrix} (K-X)_i^\top \\ Y_{Ni}^\top \end{bmatrix} \\ * & u_{0i}^2 \end{bmatrix} \succ 0, \\ \Psi_N(h) & = & \begin{bmatrix} \Psi_N^0(h) & \Psi_N^1(h) \\ * & -2U \end{bmatrix} \prec 0, \\ \Phi_N & = & \begin{bmatrix} W & 0 \\ * & W_N \end{bmatrix} - P_N \succeq 0 \end{array} \quad (13)$$

holds where

$$\begin{array}{rcl} \Psi_N^0(h) & = & \mathrm{He}\left(G_N^\top(h) P_N H_N\right) - hR_N, \\ & & +diag\{S + hR, -S, 0_{(N+1)n}\}, \\ \Psi_N^1(h) & = & G_N^\top(h) \left(\begin{bmatrix} X^\top \\ Y_N^\top \end{bmatrix} U - P_N \begin{bmatrix} B \\ 0 \end{bmatrix} \right), \\ G_N(h) & = & \begin{bmatrix} I_n & 0_n & 0_{n,n(N+1)} \\ 0_{n(N+1),n} & 0_{n(N+1),n} & hI_{n(N+1)} \end{bmatrix}, \\ H_N & = & \begin{bmatrix} F_N^\top & \Gamma_N^\top(0) & \Gamma_N^\top(1) & \ldots & \Gamma_N^\top(N) \end{bmatrix}^\top, \\ W_N & = & diag(W, 3W, \ldots, (2N+1)W), \\ S_N & = & diag(0, S, 3S, \ldots, (2N+1)S) \\ R_N & = & diag(0, 0, R, 3R, \ldots, (2N+1)R), \end{array} \quad (14)$$

and where

$$\begin{array}{rcl} F_N & = & \begin{bmatrix} A + BK & A_d & 0_{n,n(N+1)} \end{bmatrix}, \\ \Gamma_N(k) & = & \begin{bmatrix} I & (-1)^{k+1} I & \gamma_k^0 I & \ldots & \gamma_k^N I \end{bmatrix}, \\ \gamma_k^i & = & \begin{cases} -(2i+1)(1-(-1)^{k+i}), & \text{if } i \leq k, \\ 0, & \text{if } i \geq k+1. \end{cases} \end{array} \quad (15)$$

Then time-delay system (1) with the saturated static state feedback controller $u(t) = sat(Kx(t))$ is locally asymptotically stable for any initial conditions ϕ in $\mathcal{E}_N(h, W)$, where the latter set is defined as follows

$$\mathcal{E}_N(h, W) := \left\{ \begin{array}{l} \phi \in \mathcal{L}_2 \;, \; \phi^\top(0) W \phi(0) \\ + \int_{-h}^{0} \phi^\top(s)(S + h(R+W))\phi(s) \mathrm{d}s \leq 1, \end{array} \right\}. \quad (16)$$

Proof: Guided by the Bessel-Legendre inequality (7) and the projections vectors involved therein, a similar Lyapunov functional as the one presented in [13] is considered and requires the following augmented state $\tilde{x}_N(t)$, for a prescribed integer $N \geq 0$ defined by:

$$\tilde{x}_N(t) = \begin{bmatrix} \int_{-h}^{0} L_0(s)x_t(s)\mathrm{d}s \\ \vdots \\ \int_{-h}^{0} L_N(s)x_t(s)\mathrm{d}s \end{bmatrix}.$$

This augmented vector $\tilde{x}_N$ gathers the projections of the state function x_t to the $N+1$ first Legendre polynomials. The Lyapunov functional is defined as follows:

$$\begin{aligned} V_N(x_t) &= \begin{bmatrix} x(t) \\ \tilde{x}_N(t) \end{bmatrix}^\top P_N \begin{bmatrix} x(t) \\ \tilde{x}_N(t) \end{bmatrix} \\ &\quad + \int_{-h}^{0} x_t^\top(s)(S+(h+s)R)x_t(s)\mathrm{d}s \end{aligned} \tag{17}$$

In order to ensure the positive definiteness of the functional, let us note that the positive definiteness of R and the application of the Bessel-Legendre inequality in Lemma 4 ensure that, for all $N \geq 0$

$$\int_{-h}^{0} x^\top(s)(S+(h+s)R)x(s)\mathrm{d}s \geq \frac{1}{h} \begin{bmatrix} x(t) \\ \tilde{x}_N(t) \end{bmatrix}^\top S_N \begin{bmatrix} x(t) \\ \tilde{x}_N(t) \end{bmatrix}$$

which leads to

$$V_N(x_t) \geq \begin{bmatrix} x(t) \\ \tilde{x}_N(t) \end{bmatrix}^\top \left(P_N + \frac{1}{h}S_N\right) \begin{bmatrix} x(t) \\ \tilde{x}_N(t) \end{bmatrix}. \tag{18}$$

Hence, if matrix $\Theta_N(h)$ is positive definite, functional V_N is positive definite.

Let us consider now the derivative of V_N along the trajectories of the closed-loop system (9). To do so, let us introduce the following augmented vector $\xi_N(t)$ given by

$$\xi_N(t) = \begin{bmatrix} x(t) \\ x_t(-h) \\ \frac{1}{h}\tilde{x}_N(t) \end{bmatrix}, \quad N \geq 0,$$

The objective of the next developments is to derive an upper bound of the derivative of functional V_N, which is expressed as a quadratic term of the augmented vector $\xi_N(t)$ and the dead-zone function $\psi(t)$. The computation of this derivative refers to classical manipulations on Lyapunov-Krasovskii functionals and yields

$$\begin{aligned} \dot{V}_N(x_t, \dot{x}_t) &= 2\begin{bmatrix} x(t) \\ \tilde{x}_N(t) \end{bmatrix}^\top P_N \begin{bmatrix} \dot{x}(t) \\ \dot{\tilde{x}}_N(t) \end{bmatrix} \\ &\quad + x^\top(t)(S+hR)x(t) - x^\top(t-h)Sx(t-h) \\ &\quad - \int_{-h}^{0} x_t^\top(s)Rx_t(s)\mathrm{d}s. \end{aligned} \tag{19}$$

The next step of the proof consists in expressing $\dot{x}(t)$ and $\dot{\tilde{x}}_N(t)$ thanks to the augmented vector $\xi_N(t)$. On the one hand, we have

$$\begin{aligned} \dot{x}(t) &= (A+BK)x(t) + A_d x(t-h) - B\psi(t) \\ &= F_N\xi_N(t) - B\psi(t), \end{aligned}$$

where matrix F_N is given in (15). On the other hand, for any positive integer $k \leq N$, an integration by parts ensures that

$$\begin{aligned} \int_{-h}^{0} L_k(s)\dot{x}_t(s)\mathrm{d}s &= L_k(0)x_t(0) - L_k(-h)x_t(-h) \\ &\quad - \int_{-h}^{0} \dot{L}_k(u)x_t(u)\mathrm{d}u. \end{aligned}$$

Thanks to properties **P2** and **P3** of the Legendre polynomials, the following expression is derived

$$\begin{aligned} \int_{-h}^{0} L_k(s)\dot{x}_t(s)\mathrm{d}s &= x_t(0) - (-1)^k x_t(-h) \\ &\quad - \sum_{i=0}^{k-1} \gamma_{Nk}^i \frac{1}{h} \int_{-h}^{0} L_i(u)x(u)\mathrm{d}u \\ &= \Gamma_N(k)\xi_N(t) \end{aligned}$$

where matrices $\Gamma_N(k)$ are defined in (15). Then, by putting together all the components of $\dot{\tilde{x}}_N(t)$, we obtain

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\tilde{x}}_N(t) \end{bmatrix} = H_N\xi_N(t) - \begin{bmatrix} B \\ 0 \end{bmatrix}\psi(t)$$

with H_N defined in (14). Finally, noticing that $\tilde{x}_N(t) = G_N(h)\xi_N(t)$, where matrix $G_N(h)$ is given in (14), it yields

$$\begin{aligned} \dot{V}_N(x_t) &= \xi_N^\top(t)\Psi_N^0(h)\xi_N(t) - 2\xi_N^\top(t)G_N^\top(h)P_N \begin{bmatrix} B \\ 0 \end{bmatrix}\psi(t) \\ &\quad - \int_{-h}^{0} x_t^\top(s)Rx_t(s)\mathrm{d}s + \frac{1}{h}\tilde{x}_N^\top(t) \begin{bmatrix} R & & \\ & 3R & \\ & & \ddots \\ & & & (2N{+}1)R \end{bmatrix} \tilde{x}_N(t). \end{aligned} \tag{20}$$

where we have introduced the matrix Ψ_N^0 defined in (14). This manipulation imposes the introduction of the last term of the previous expression. The Bessel-Legendre inequality in Lemma 4 ensures that the sum of the two last terms is negative definite, so that we have

$$\begin{aligned} \dot{V}_N(x_t, \dot{x}_t) &\leq \xi_N^\top(t)\Psi_N^0(h)\xi_N(t) \\ &\quad -2\xi_N^\top(t)G_N^\top(h)P_N \begin{bmatrix} B \\ 0 \end{bmatrix}\psi(t). \end{aligned} \tag{21}$$

Introducing matrix $\Psi_N(h)$ in the previous expression yields

$$\begin{aligned} \dot{V}_N(x_t) &\leq \begin{bmatrix} \xi_N(t) \\ \psi(t) \end{bmatrix}^\top \Psi_N(h) \begin{bmatrix} \xi_N(t) \\ \psi(t) \end{bmatrix} \\ &\quad +2\psi^\top(t)U\left(\psi(t) - Xx(t) - Y_N\tilde{x}_N(t)\right). \end{aligned}$$

by using Lemma 5 with (12).

Hence, if the matrix $\Psi_N(h)$ is negative definite, then there exists a sufficiently small scalar $\varepsilon > 0$, such that previous inequality reduces to

$$\dot{V}_N(x_t) \leq -\varepsilon|x(t)|^2 + 2\psi^\top(t)U\left(\psi(t) - Xx(t) - Y_N\tilde{x}_N(t)\right)$$

The final step of the proof consists then in ensuring the negativity of the right-hand-side of the previous inequality. To do so, let us first apply the Schur Complement to the last row and column of $\Theta_N(h)$ and pre- and post-multiply by the vector $[x_t^\top(t)\ \tilde{x}_N^\top(t)]^\top$ and its transpose, respectively. This leads to

$$\begin{aligned}\left|K_i x(t) - [\,X_i\ Y_{Ni}\,]\begin{bmatrix} x(t) \\ \tilde{x}_N(t)\end{bmatrix}\right|^2 &\\ \leq u_{0i}^2 \begin{bmatrix} x(t) \\ \tilde{x}_N(t)\end{bmatrix}^\top \left(P_N + \tfrac{1}{h}S_N\right)\begin{bmatrix} x(t) \\ \tilde{x}_N(t)\end{bmatrix}&\\ \leq u_{0i}^2 V_N(x_t),&\end{aligned}$$

where the last inequality has been obtained from inequality (18). Therefore, we have

$$|(K-X)_i x(t) - Y_{Ni}\tilde{x}_N(t)|^2 \leq u_{0i}^2 V_N(x_t) \leq u_{0i}^2 V_N(\phi)$$

where ϕ is the initial conditions of the closed-loop system.

Provided that condition $\Phi_N \succeq 0$ holds, an upper bound of the V_N can be obtained as follows

$$\begin{aligned} V_N(\phi) \ &\leq\ \begin{bmatrix}\phi(0)\\ \tilde{\phi}_N\end{bmatrix}^\top \begin{bmatrix} W & 0 \\ * & W_N\end{bmatrix}\begin{bmatrix}\phi(0)\\ \tilde{\phi}_N\end{bmatrix} \\ &\quad + \int_{-h}^{0} \phi^\top(s)(S+(h+s)R)\phi(s)\mathrm{d}s \\ &=\ \phi^\top(0)W\phi(0) + \tilde{\phi}_N^\top(t)W_N\tilde{\phi}_N(t) \\ &\quad + \int_{-h}^{0}\phi^\top(s)(S+hR)\phi(s)\mathrm{d}s \end{aligned}$$

with

$$\tilde{\phi}_N = \begin{bmatrix} \int_{-h}^0 L_0(s)\phi(s)\mathrm{d}s \\ \int_{-h}^0 L_1(s)\phi(s)\mathrm{d}s \\ \vdots \\ \int_{-h}^0 L_N(s)\phi(s)\mathrm{d}s \end{bmatrix}.$$

Then, by application of the Bessel-Legendre inequality to the term $\tilde{\phi}_N^\top W_N \tilde{\phi}_N$, the functionals can be upper bounded as follows

$$V_N(\phi) \leq \phi^\top(0)W\phi(0) + \int_{-h}^{0}\phi^\top(s)(S+h(R+W))\phi(s)\mathrm{d}s$$

Therefore, for any initial conditions $\phi \in \mathcal{E}_N(h,W)$, the inequality $V_N(\phi) \leq 1$ holds and the assumption of Lemma 5 is satisfied and the regional stability of system (1) with the saturated control law (3) is ensured. ■

Remark 2: It is worth noting that the estimation of the basin of attraction provided in Theorem 6 only depends on the norm of the vector $\phi(0)$ and on the $\mathcal{L}_2$ norm of the initial condition ϕ, while in most of the results in the literature provides such an estimation, which also contains the $\mathcal{L}_2$ norm of the derivative of the initial condition, i.e. $\dot{\phi}$. This constitutes one of the major contribution of this paper.

V. Optimization of the approximation of the basin of attraction

Theorem 6 exposes some local stability conditions for time-delay systems subject to a saturated input. These conditions provide, for a given controller gain K, an estimation of the basin of attraction defined by any initial conditions $\phi \in L_2(-h,0,\mathbb{R}^n)$ verifying (16). Compared to the case without delays, the estimation of region of attraction is defined over an infinite-dimensional subset of $L_2(-h,0,\mathbb{R}^n)$. In order to measure the size of the estimation, one has to include some restrictions to this set. Among the numerous possibilities to do so, the usual assumption on the allowable set of initial conditions consists in introducing the supremum norm of the initial condition as follows

$$\mathcal{E}_{sup}(c) := \left\{\phi \in \mathcal{L}_2(-h,0),\ |\phi|_h = \sup_{s\in[-h,\ 0]}\|\phi(s)\| \leq c\right\}. \tag{22}$$

for a positive scalar $c > 0$ to be determined.

This leads to the following optimization scheme:

Optimization 7: For a given integer N, the minimization problem given by

$$\begin{gathered}\lambda_N^*(h) = \min_{\{W,P_N,R,S,X,Y_N\}} \quad \lambda \\ \text{subject to} \\ \Theta_N(h) \succ 0,\ \Psi_N(h) \prec 0, \\ \Phi_N(h) \succ 0 \text{ and } S + h(R+W) \leq \frac{\lambda}{1+h}I\end{gathered}$$

ensures that for an optimal selection of estimation of the region of attraction is given by

$$\phi \in \mathcal{E}_{sup}\left((\lambda_N^*(h))^{-1/2}\right).$$

Proof: Since matrices S and R are positive definite, the proof consists in noting that, from Theorem 6, we have,

$$\begin{aligned} V_N(\phi) &\leq \phi(0)^T W\phi(0) + \int_{-h}^{0}\phi^T(s)(S+h(R+W)\phi(s)\mathrm{d}s \\ &\leq \frac{\lambda_N^*(h)}{1+h}\left(\|\phi(0)\|^2 + \int_{-h}^{0}\|\phi(s)\|^2 ds\right) \\ &\leq \lambda_N^*(h)|\phi(s)|_h^2. \end{aligned}$$

Therefore, if the initial condition belongs to $\mathcal{E}_{sup}\left((\lambda_N^*(h))^{-1/2}\right)$, it implies that $V_N(\phi) \leq 1$, which concludes the proof. ■

VI. Hierarchical estimation of basin of attraction

One of the main advantages of using the Bessel-Legendre framework was demonstrated in [14], [15], [16], [23], in which a hierarchical structure of the LMI conditions with respect to the order N of the conditions was proven. While in the papers cited above, the hierarchical structure aims at demonstrating that the solutions to the LMI, for a given delay h, obtained at a given order N, are included in the solutions of the same condition at higher orders, this section aims at proving that the same hierarchical structure applies to the estimation of the basin of attraction. This is formulated in the following theorem based on the stability conditions of Theorem 6.

Theorem 8: For any time-delay system (1) subject to a saturated control input (3), and any delay h, the solutions $\lambda_N^*(h)$ to Optimization 7, obtained for various values of N verify the inequality $\lambda_{N+1}^*(h) \leq \lambda_N^*(h)$, for any positive integer N. In other words, the inclusion

$$\mathcal{E}_{sup}\left((\lambda_N^*(h))^{-1/2}\right) \subseteq \mathcal{E}_{sup}\left((\lambda_{N+1}^*(h))^{-1/2}\right)$$

holds, for any positive integer N.

Theorem 6	$N=0$	$N=1$	$N=2$	$N=3$	$N=4$	$N=5$	$N=6$	$N=7$	$N=8$	$N=9$	$N=10$
h=0.5	2.4356	2.7362	2.9602	3.0111	3.0267	3.0354	3.0407	3.0437	3.0454	3.0468	3.0479
h=1	1.4637	1.5705	1.7387	1.7912	1.8125	1.8247	1.8305	1.8343	1.8365	1.8383	1.8396
h=1.5	0	1.0858	1.1854	1.2574	1.2833	1.2974	1.3024	1.3063	1.3089	1.3109	1.3122

TABLE I

SOLUTION, $(\lambda_N^*(h))^{-1/2}$ OF OPTIMIZATION 7 FOR SEVERAL VALUES OF h AND N

Proof: Consider a given integer $N \in \mathbb{N}$ and the solution to Optimization 7, $\lambda_N^*(h)$, associated to the decision variables P_N^*, S^*, R^*, W^*, X^* and Y_N^*, which verify the conditions $\Theta_N(h) \succ 0$, $\Psi_N(h) \prec 0$ and $\Phi_N(h) \succ 0$, for this value of N. Let us introduce now, the decision variables

$$P_{N+1} = \begin{bmatrix} P_N^* & 0 \\ 0 & 0 \end{bmatrix}, \qquad Y_{N+1} = \begin{bmatrix} Y_N^* \\ 0 \end{bmatrix}$$

Let us verify that these two matrices together with the same matrices S^*, R^*, W^* and X^* fulfilled the LMI conditions at the order $N+1$. It is easy to see that, by a simple permutation

$$\Theta_{N+1}(h) = \begin{bmatrix} P_N^* + \frac{1}{h} S_N^* & 0 & \begin{bmatrix} (K-X^*)_i^\top \\ {Y_{Ni}^*}^\top \end{bmatrix} \\ * & \frac{2N+1}{h} S^* & 0 \\ * & * & u_{0i}^2 \end{bmatrix},$$

or equivalently,

$$\Theta_{N+1}(h) \succ 0 \qquad \Leftrightarrow \qquad \begin{bmatrix} \Theta_N(h) & 0 \\ * & (2N+3)S^* \end{bmatrix} \succ 0,$$

which is positive definite, if and only if $\Theta_N(h) \succeq 0$ and $S^* \succ 0$. Following the same procedure, it is possible to show that

$$\Phi_{N+1} = \begin{bmatrix} \Phi_N & 0 \\ * & (2N+3)W^* \end{bmatrix},$$

which is positive if and only if $\Phi_N \succ 0$ and $W^* \succ 0$, which are necessary conditions for the conditions to hold at the order N.

In order to verify that $\Psi_{N+1}(h) \prec 0$ for this particular set of decision variables, let us first note that the construction of the matrices G_N, H_N, F_N and $\tilde{S}_N$ imposes the following structures

$$\begin{aligned} H_{N+1} &= \begin{bmatrix} H_N & 0_{(N+1)n,n} \\ \multicolumn{2}{c}{\Gamma_{N+1}(h)} \end{bmatrix}, \\ G_{N+1}(h) &= \begin{bmatrix} G_N(h) & 0_{(N+1)n,n} \\ 0_{n,(N+1)n} & hI \end{bmatrix}, \end{aligned}$$

From these expressions, matrix $\Psi_{N+1}(h)$ can be expressed using the matrix $\Phi_N(h)$ as follows

$$\Psi_{N+1}(h) = \begin{bmatrix} \Psi_N^0(h) & 0 & \Psi_N^1(h) \\ * & -(2N+3)R^* & 0 \\ * & * & -2U \end{bmatrix}.$$

where $\Psi_N^0(h)$ and $\Psi_N^1(h)$ are given in (14). Thanks to permutations of the second and third rows and columns of the previous matrix, the following statements are equivalent

$$\Psi_{N+1}(h) \prec 0 \qquad \Leftrightarrow \qquad \begin{bmatrix} \Psi_N(h) & 0 \\ * & -(2N+3)R^* \end{bmatrix} \prec 0,$$

which ensures that $\Psi_{N+1}(h)$ is negative definite for this particular selection of P_{N+1} and Y_{N+1}.

To conclude the proof, it suffices to note that a solution to Optimization 7 at order $N+1$ is built based on the solution obtained at order N. This means that Optimization 7 recovers at least the solution obtained at previous orders and that $\lambda_{N+1}^*(h) \leq \lambda_N^*(h)$ and the inclusion of the sets $\mathcal{E}_{sup}(^{-1/2}((\lambda_N^*(h))^{-1/2})$ straightforwardly follows from the previous inequality. ∎

VII. NUMERICAL APPLICATION AND ILLUSTRATIONS

Let us consider the time-delay system driven by (1) and with the saturated static state feedback controller $u(t) = sat(Kx(t))$, with the following matrices

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0.1 \end{bmatrix},\ A_d = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ K = -\begin{bmatrix} 2 \\ 1 \end{bmatrix}^\top,$$

and $U = 1$. It is easy to note that matrices A and $A + A_d$ are not Hurwitz. This ensures that, when $h = 0$, only a local stability guarantee can be obtained. The gain K has been chosen so that matrix $A + A_d + BK$ is Hurwitz. This ensures that, at least, when the delay is zero or sufficiently small, the system can be regionally stabilized to the origin.

Table I shows the numerical values of $(\lambda_N^*(h))^{-1/2}$ obtained for $h = 0.5,\ 1$ and 1.5 for $N = 0$ to 10. One can see from this table that increasing the delay leads to a decrease of the size of the estimation of the basin of attraction. This behavior corresponds to the expected effect of delays. The table also shows that increasing the order of the LMI provides, for this example, an increase of estimation of the domain of attraction. Even though these improvements are quite small when N is large, the augmentation of $(\lambda_N^*(h))^{-1/2}$ is relatively notable. This illustrates the hierarchical result provided in Theorem 8.

Another way to understand the hierarchical structure of the contributions of this paper is presented in Figures 1 and 2. These figures present the evolution of $(\lambda_N^*(h))^{-1/2}$ with respect to h in $[0,\ 1.6]$, for $N = 0$ to 6. Figure 2 shows a zoom of Figure 1, where the vertical axis is constrained to the interval $[0,\ 5]$. One can see in Figure 1 that the size of the domain of attraction increases notably, when the delay tends to zero. Figure 2 allows us to see that increasing N for all values of h can only provide a larger estimation of the domain of attraction. Alternatively, Figure 2 illustrates the reduction of the conservatism of the stability conditions of Theorem 6 by noting that the maximal allowable delay for which a solution is obtained increases with N. One can also see that the size of the domain has a for any N.

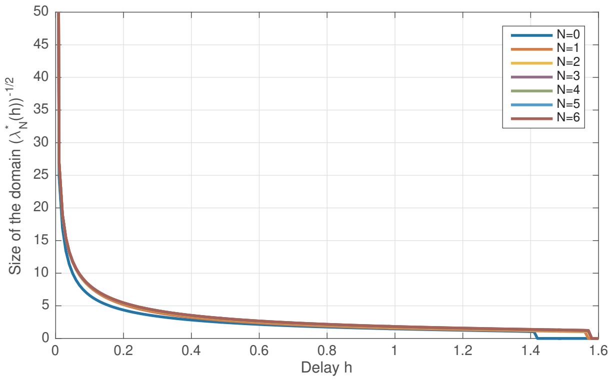

Fig. 1. Evolution of $(\lambda_N^*(h))^{-1/2}$ with respect to h in $[0,\ 1.6]$, for $N = 0$ to 6.

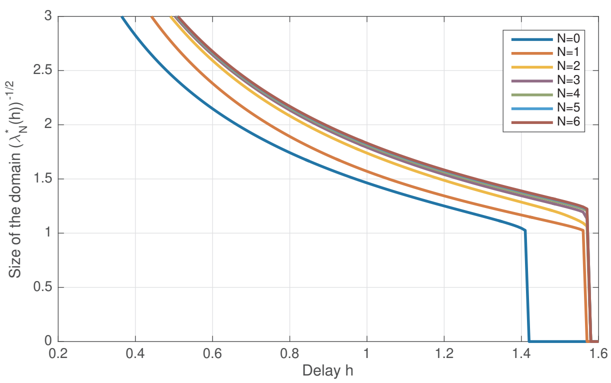

Fig. 2. Zoom of Figure 1.

VIII. Conclusions

In this paper, the stability analysis problem for linear systems subject to input saturation and time-delayed state have been addressed. Considering a Lyapunov-Krasovskii functional associated to Legendre polynomials and generalized sector conditions, sufficient conditions based on a set of LMIs were derived to guarantee the regional stability of the origin for the closed-loop system. An estimate of the basin of attraction of the origin was also characterized. A convex optimization problem associated to these conditions allowed to maximize the size of the estimate of the basin of attraction. We have then showed that the optimal LMIs form a hierarchy, which is competitive to improve the size criterium of the estimate of the basin of attraction. An example finally illustrated the potential of the technique.

This paper paves the way for future works. In particular, it should be interesting to extend the conditions, and therefore the hierarchical expansion, to the context of stabilization, that is to be able to determine in the same step the controller and the estimation of the basin of attraction. Furthermore, the study of the influence of the hierarchical conditions to some performance level purposes could be carried out.

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