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This abstract is focused on the Periodically Aggregated Resource-Constrained Project Scheduling Problem (PARCPSP) (Morin *et. al.* 2017b), that can be seen as a continuous-time variant of a restricted Resource-Constrained Project Scheduling Problem with partially renewable resources (RCPSP/π) (Böttcher *et. al.* 1999). The purpose of this work is to compare an existing compact formulation with a new extended formulation.

The PARCPSP is defined as follows. A set $\mathcal{A}$ of activities, subject to end-to-start precedence relations $E \subset \mathcal{A} \times \mathcal{A}$, and a set $\mathcal{R}$ of renewable resources are given. During its processing (duration p_i), activity $i \in \mathcal{A}$ requires $r_{i,k}$ units of resource $k \in \mathcal{R}$ (capacity b_k). The scheduling horizon is divided uniformly into a set $\mathcal{L}$ of L periods of length Δ. The PARCPSP can be described by the following abstract model:

$$\text{Minimize} \ : \ S_{n+1} - S_0 \tag{1}$$

$$\text{s.t.} \ : \ S_j - S_i \geq p_i \qquad \forall (i,j) \in E \tag{2}$$

$$\sum_{i \in \mathcal{A}} r_{i,k} \frac{d_{i,\ell}(S_i)}{\Delta} \leq b_k \qquad \forall k \in \mathcal{R}\ ,\ \forall \ell \in \mathcal{L} \tag{3}$$

Where S_i is the start date of activity i and $d_{i,\ell}(t)$ is the length of the intersection of the intervals $[(\ell-1)\Delta, \ell\Delta]$ and $[t, t+p_i]$. The objective (1) is to minimize the project duration (activities $0/n{+}1$ are the dummy beginning/end of the project) under precedence constraints (2) and periodically aggregated resource constraints (3): for every resource, in every period, the capacity should not be exceeded on average.

1 Compact model

Two formulations based on mixed (continuous and discrete) time frameworks have been proposed to model the PARCPSP. Although the computation of the values $d_{i,\ell}(S_i)$ can be done by introducing only step binary variables (Morin *et. al.* 2017b), we focus here on an alternative scheme based on period partitionning (Morin *et. al.* 2017a) that requires more continuous variables, but involves less constraints, all big-$\mathcal{M}$-free, thus yielding a better linear relaxation.

Two additional functions are considered. Let $\lambda_{i,\ell}(t)$ be the length of the intersection of the intervals $[(\ell-1)\Delta, \ell\Delta]$ and $(-\infty, t]$; let $\mu_{i,\ell}(t)$ be the length of the intersection of the intervals $[(\ell-1)\Delta, \ell\Delta]$ and $[t+p_i, +\infty)$ (cf. Figure 1).

Notice that it is easier to describe $\lambda_{i,\ell}$ and $\mu_{i,\ell}$ compared to $d_{i,\ell}$. Moreover, the intervals whose lengths are measured by these functions form a partition of period ℓ. Therefore:

$$\lambda_{i,\ell}(t) + d_{i,\ell}(t) + \mu_{i,\ell}(t) = \Delta \qquad \forall i \in \mathcal{A}\ ,\ \forall \ell \in \mathcal{L}\ ,\ \forall t \in \mathbb{R} \tag{4}$$

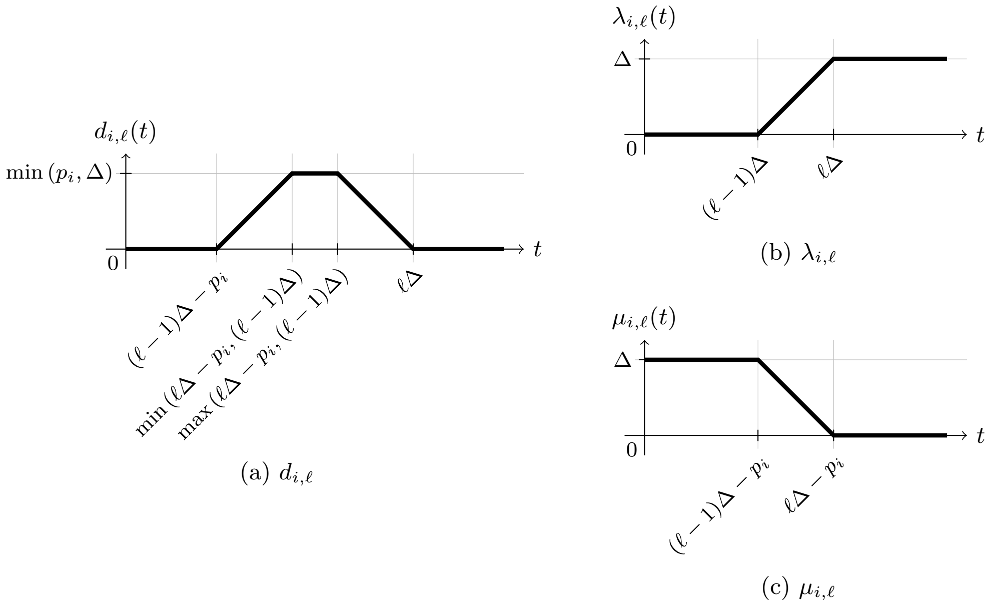

Fig. 1: Piecewise linear functions $d_{i,\ell}$, $\lambda_{i,\ell}$ and $\mu_{i,\ell}$

The values $d_{i,\ell}(S_i)$, $\lambda_{i,\ell}(S_i)$ and $\mu_{i,\ell}(S_i)$ are represented as continuous variables $D_{i,\ell}$, $\Lambda_{i,\ell}$ and $M_{i,\ell}$, respectively. To model the piecewise linear functions $\lambda_{i,\ell}$ and $\mu_{i,\ell}$, auxiliary binary variables are introduced; more precisely, to ensure a non-increasing (resp. non-decreasing) step behavior of the variables $\Lambda_{i,\ell}$ (resp. $M_{i,\ell}$), step binary variables $z^{\lambda}_{i,\ell}$ (resp. $z^{\mu}_{i,\ell}$) are required.

$$\text{Minimize} \; : \; S_{n+1} - S_0 \tag{5}$$

$$\text{s.t.} \; : \; S_j - S_i \geq p_i \qquad \forall (i,j) \in E \tag{6}$$

$$\sum_{i \in \mathcal{A}} r_{i,k} D_{i,\ell} \leq b_k \Delta \qquad \forall k \in \mathcal{R} \, , \; \forall \ell \in \mathcal{L} \tag{7}$$

$$S_i = \sum_{\ell \in \mathcal{L}} \Lambda_{i,\ell} \qquad \forall i \in \mathcal{A} \tag{8}$$

$$D_{i,\ell} = \Delta - \Lambda_{i,\ell} - M_{i,\ell} \qquad \forall i \in \mathcal{A} \, , \; \forall \ell \in \mathcal{L} \tag{9}$$

$$D_{i,\ell} \geq 0 \qquad \forall i \in \mathcal{A} \, , \; \forall \ell \in \mathcal{L} \tag{10}$$

$$\sum_{\ell \in \mathcal{L}} D_{i,\ell} = p_i \qquad \forall i \in \mathcal{A} \tag{11}$$

$$z^{\lambda}_{i,\ell+1} \leq \frac{\Lambda_{i,\ell}}{\Delta} \leq z^{\lambda}_{i,\ell} \qquad \forall i \in \mathcal{A} \, , \; \forall \ell \in \mathcal{L} \tag{12}$$

$$z^{\mu}_{i,\ell-1} \leq \frac{M_{i,\ell}}{\Delta} \leq z^{\mu}_{i,\ell} \qquad \forall i \in \mathcal{A} \, , \; \forall \ell \in \mathcal{L} \tag{13}$$

$$z^{\lambda}_{i,\ell} \in \{0,1\} \qquad \forall i \in \mathcal{A} \, , \; \forall \ell \in \mathcal{L} \tag{14}$$

$$z^{\mu}_{i,\ell} \in \{0,1\} \qquad \forall i \in \mathcal{A} \, , \; \forall \ell \in \mathcal{L} \tag{15}$$

The objective (5) is to minimize the project duration, under both precedence constraints (6) and periodically aggregated resource constraints (7). Constraints (8) enable the computation of start dates S_i directly from $\Lambda_{i,\ell}$ variables, while constraints (9), derived from the partition relation (4), enable the computation of $D_{i,\ell}$ values that cannot be negative [constraints (10)]. Constraints (11) permit to balance the values of $\Lambda_{i,\ell_i^\lambda}$ and M_{i,ℓ_i^μ}, where ℓ_i^λ (resp. ℓ_i^μ) is the period activity i starts (resp. completes) in. Finally, constraints (12) [resp. (13)] enforce an interdependent non-increasing (resp. non-decreasing) step behavior of variables $\Lambda_{i,\ell}$ and $z_{i,\ell}^\lambda$ (resp. $M_{i,\ell}$ and $z_{i,\ell}^\mu$) using binary variables [constraints (14) and (15)]. Therefore, every variable $\Lambda_{i,\ell}$ (resp. $M_{i,\ell}$) with $\ell \neq \ell_i^\lambda$ (resp. $\ell \neq \ell_i^\mu$) is bound either to 0 or Δ, as shown in Figure 2.

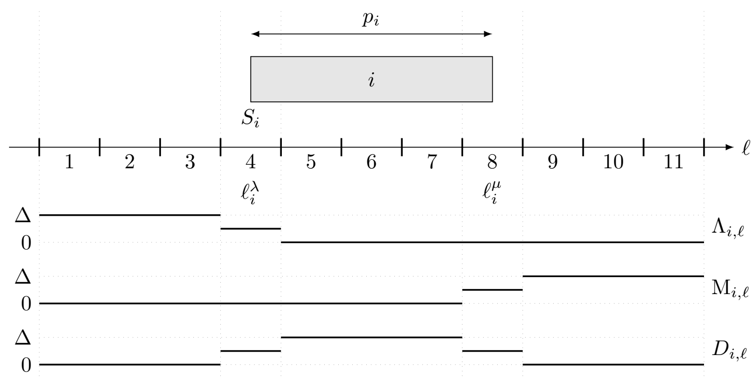

Fig. 2: Partition-based mixed time framework

2 Dantzig-Wolfe decomposition

We now introduce a new extended formulation that enhances and exploits the combinatorial structure of the PARCPSP. On the one hand, the (restricted) master problem consists in selecting start dates $t \in \mathcal{T}_i$ for every activity $i \in \mathcal{A}$ (binary decision variables $x_{i,t}$ such that $S_i = \sum_{t \in \mathcal{T}_i} t\, x_{i,t}$, $\forall i \in \mathcal{A}$) in such a way that all constraints are satisfied, while minimizing the project duration. On the other hand, the sub-problem consists in finding time points t to insert into sets $\mathcal{T}_i$. Notice that, although the start date of an activity can be any (real) time point in the (continuous) interval $[0, L\Delta - p_i]$, only a finite number of them need to be considered, since optimal solutions match extreme points of a polytope described by a finite number of constraints.

2.1 Master problem

$$\text{Minimize} \;:\; \sum_{t \in \mathcal{T}_{n+1}} t\, x_{n+1,t} - \sum_{t \in \mathcal{T}_0} t\, x_{0,t} \tag{16}$$

$$\alpha_i \;:\; \sum_{t \in \mathcal{T}_i} x_{i,t} = 1 \qquad \forall i \in \mathcal{A} \tag{17}$$

$$\beta_{i,j} \;:\; -\sum_{t \in \mathcal{T}_j} t\, x_{j,t} + \sum_{t \in \mathcal{T}_i} t\, x_{i,t} \leq -p_i \qquad \forall (i,j) \in E \tag{18}$$

$$\gamma_{k,\ell} \;:\; \sum_{i \in \mathcal{A}} \sum_{t \in \mathcal{T}_i} r_{i,k}\, d_{i,\ell}(t)\, x_{i,t} \leq b_k\, \Delta \qquad \forall k \in \mathcal{R},\ \forall \ell \in \mathcal{L} \tag{19}$$

$$x_{i,t} \in \{0,1\} \qquad \forall i \in \mathcal{A},\ \forall t \in \mathcal{T}_i \tag{20}$$

The objective (16) is to minimize the project duration, assigning a unique start date to each activity [constraints (17)], under both precedence constraints (18) and periodically aggregated resource constraints (19), using binary variables [constraints (20)].

Notice that dual variables $\beta_{i,j}$ and $\gamma_{k,\ell}$ are non-negative. The linear relaxation of the master problem is obtained by replacing constraints (20) with "$x_{i,t} \geq 0$"; notice that constraints α_i imply "$x_{i,t} \leq 1$".

2.2 Sub-problem

$$\text{Minimize} \ : \ \alpha_i + \sum_{j \in E_i^{\oplus}} \beta_{i,j}\, t - \sum_{j \in E_i^{\ominus}} \beta_{j,i}\, t + \sum_{k \in \mathcal{R}} \sum_{\ell \in \mathcal{L}} \gamma_{k,\ell}\, r_{i,k}\, d_{i,\ell}(t) \tag{21}$$

$$ES_i \leq t \leq LS_i \tag{22}$$

Where, for each activity $i \in \mathcal{A}$: $E_i^{\oplus} = \{j \in \mathcal{A} \ : \ (i,j) \in E\}$ (set of direct successors of i), $E_i^{\ominus} = \{j \in \mathcal{A} \ : \ (j,i) \in E\}$ (set of direct predecessors of i), ES_i and LS_i are respectively the earliest and latest starting time of i (those input values are typically obtained by computing longest paths in the activity precedence graph).

Given an activity $i \in \mathcal{A}$, the sub-problem returns a candidate start t within the horizon [constraint (22)] such that the new variable $x_{i,t}$ has the least reduced cost [objective (21)]. This returned date t will be inserted in $\mathcal{T}_i$ in the restricted master problem only if needed, i.e., if the reduced cost of $x_{i,t}$ is negative.

Notice that, after the partition relation (4), the reduced cost of $x_{i,t}$ can be transformed into a sum of continuous monotonic piecewise linear functions of t. Therefore, the sub-problem can be solved by a forward algorithm, linear in the number of breakpoints, hence linear in the number of periods.

Computational experiments will be provided by time of the conference. Depending on the results, it could be interesting to additionally separate either precedence or periodically aggregated resource constraints. For instance, the framework proposed by Mingozzi *et. al.* (1998) for the standard Resource-Constrained Project Scheduling Problem (RCPSP) could be adapted to the case of the PARCPSP. The precedence constraints are managed by the master problem, while the resource constraints are managed by the sub-problem. Instead of using vector columns with binary components indicating whether an activity is processed in a unit time period, these components should be replaced with real values in the interval $[0, \Delta]$ indicating how much each activity is processed in a period of length Δ.

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