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# Unsafe Point Avoidance in Linear State Feedback

Philipp Braun<sup>1,2</sup>, Christopher M. Kellett<sup>1</sup> and Luca Zaccarian<sup>3</sup>

## Abstract

We propose a hybrid solution for the stabilization of the origin of a linear time-invariant stabilizable system with the property that a suitable neighborhood of a pre-defined unsafe point in the state space is avoided by the closed-loop solutions. Hybrid tools are motivated by the fact that the task at hand cannot be solved with continuous feedback, whereas the proposed hybrid solution induces nominal and robust asymptotic stability of the origin. More specifically, we formulate a semiglobal version of the problem at hand and describe a fully constructive approach under the assumption that the unsafe point to be avoided does not belong to the equilibrium subspace induced by the control input on the linear dynamics. The approach is illustrated on a numerical example.

## I. INTRODUCTION

Lyapunov functions [6] provide a well established tool to analyze and characterize stability properties of general dynamical systems and are an important mechanism in the control literature to construct stabilizing feedback laws. While global asymptotic stability/stabilization (GAS) of unconstrained dynamical systems is well understood, stability/stabilization of dynamical systems subject to bounded state constraints, e.g., obstacle avoidance for mobile robots or collision avoidance in the coordination of drones, has yet to be addressed rigorously for general classes of dynamical systems. While in the context of unconstrained stabilization, discontinuous control laws only need to be considered for the class of systems that are asymptotically controllable but not Lipschitz continuous feedback stabilizable (e.g., the nonholonomic integrator [4]), discontinuous feedback laws are necessary in the presence of bounded constraints, independent of the system dynamics (see [2] for an illustrative proof). A similar need for discontinuous feedback laws is discussed in [7] in terms of topological obstructions on manifolds.

When using control Lyapunov functions, the need for discontinuous feedback laws precludes the use of Sontag's universal formula [14], for example, since it leads to a continuous feedback law. Thus, approaches extending classical results on control Lyapunov functions by control barrier functions [18] to include constraints in the state space, are limited to constraints defining unbounded sets. In particular, this impacts approaches in [8], [15], [1], [11], since they rely on the existence of continuous feedback laws.

Additionally, note that the model predictive control literature does not provide a general framework for obstacle avoidance and global stabilization. Even though it is simple to define an optimization problem to iteratively compute a feedback law, proving GAS of the closed loop and recursive feasibility is nontrivial.

One way to define discontinuous feedback laws, and which we will follow in this paper, is to unite local and global controllers. This approach traces back to [16] and was further investigated and established using the formalism of hybrid dynamical systems in [9], [17], [10], [12], [13]. While the results in these works are promising and motivating, the papers address particular applications and do not provide a general tool for controller design subject to bounded state constraints.

In contrast to the approaches discussed above, we propose a constructive method to design a hybrid control law for a controllable linear system that simultaneously guarantees GAS of the origin and avoidance of a neighborhood around a given obstacle described by a single point. While we address the case of a single unsafe point, our approach easily extends to the case of multiple points.

The paper is structured as follows. In Section II the mathematical setting and the problem under consideration are formalized. In Section III the “wipeout” property is introduced, ensuring that solutions getting close to the obstacle are guaranteed to leave a neighborhood around the obstacle in finite time. This result is used in Section IV to define a local obstacle avoidance controller. Section V combines the results to obtain a global hybrid control law. Here, the main result providing GAS while avoiding the obstacle is stated. The results of the hybrid controller are illustrated on a numerical example in Section VI before the paper concludes in Section VII.

Throughout the paper the following notation is used. For  $x \in \mathbb{R}^n$  we use the vector norm  $|x| = \sqrt{\sum_{i=1}^n x_i^2}$ . Similarly, the distance to a point  $y \in \mathbb{R}^n$  is denoted by  $|x|_y = |x - y|$ . For a closed set  $\mathcal{A} \in \mathbb{R}^n$  and  $r > 0$  we define  $\mathcal{B}_r(\mathcal{A}) = \{x \in \mathbb{R}^n \mid \min_{y \in \mathcal{A}} |x - y| \leq r\}$ . The closure, the boundary and the interior of a set are denoted by  $\bar{\mathcal{A}}$ ,  $\partial\mathcal{A}$  and  $\text{int}(\mathcal{A})$ , respectively. The identity matrix of appropriate dimension is denoted by  $I$ .

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## II. SETTING & PROBLEM FORMULATION

In this paper we consider linear dynamical systems

$$\dot{x} = Ax + Bu, \quad x_0 = x(0) \in \mathbb{R}^n \quad (1)$$

with state  $x \in \mathbb{R}^n$ , one dimensional input  $u \in \mathbb{R}$  and matrices  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^n$ . An extension to the multidimensional input case  $u \in \mathbb{R}^m$ ,  $m \in \mathbb{N}$ , is straightforward. As motivated in the introduction, the paper addresses the following general problem and provides a solution under some simplifying assumptions described below.

*Problem 1:* (Semiglobal  $x_a$ -avoidance augmentation with GAS) Given an “unsafe” point  $x_a \in \mathbb{R}^n$  that must be avoided by the controller, and a stabilizing state feedback  $u_s = K_s x$ , for each  $\delta > 0$ , design a feedback selection of  $u$  that guarantees

- (i) (GAS) uniform global asymptotic stability of the origin;
- (ii) (semiglobal preservation) the feedback matches the original stabilizer  $u(x) = K_s x$  in  $\mathbb{R}^n \setminus \mathcal{B}_\delta(x_a)$ ; and
- (iii) (semiglobal  $x_a$ -avoidance) all solutions starting outside the ball  $\mathcal{B}_\delta(x_a)$  never enter a suitable “safety” neighborhood of  $x_a$ . ┘

Problem 1 entails the desirable property that the modifications enforced by the avoidance augmentation are minimally invasive, because semiglobal preservation ensures that the pre-defined stabilizer  $u_s$  is unchanged in an arbitrarily large subset of the state space. Note that this goal is similar but not quite the same as the one of safety region avoidance, addressed in [11] and [1].

In contrast to those works, we do not assume the existence of a control Lyapunov function and a control barrier function avoiding an a priori fixed neighborhood around  $x_a$ , characterized in item (iii). Instead we provide a constructive method to design the control and provide a corresponding bound on the size of the neighborhood that can be avoided. To keep the discussion simple, we only address one point  $x_a$  but our construction carries over trivially to the case of multiple unsafe points, always providing a constructive solution to the corresponding semiglobal avoidance design. We emphasize again that Problem 1 cannot be solved by a continuous feedback, as motivated in the introduction, and thus we provide a hybrid solution to the problem here. For our construction, we will enforce the following assumption on the system data.

*Assumption 1:* Basic assumptions:

- (a) Matrix  $A_s := A + BK_s$  is Hurwitz.
- (b) Vectors  $Ax_a$  and  $B$  are linearly independent.
- (c) Vector  $B$  is a unit vector (namely  $|B| = 1$ ).
- (d) The norm  $x \mapsto |x|^2$  is contractive under the stabilizer  $u_s = K_s x$  (equivalently,  $A_s + A_s^T < 0$ ). ┘

We note that Assumption 1(a) is a necessary condition for Problem 1(i,ii), whereas Assumptions 1(c) and 1(d) are simplifying assumptions that can be easily removed by suitable input and state transformations, respectively. In particular, Assumption 1(c) can be achieved through the definition  $\dot{x} = Ax + B_\circ u_\circ$ , where  $u_\circ := |B|u$  and  $B_\circ := B/|B|$ . With respect to Assumption 1(d), if  $V(x) = x^T S x$  is a Lyapunov function for the closed-loop system  $\dot{x} = A_s x$ , then  $V(\tilde{x}) = |\tilde{x}|^2$  is a Lyapunov function in the coordinates  $\tilde{x} = S_F^T x$ , where  $S_F^T S_F = S$  denotes the Cholesky factorization of  $S$ .

Assumption 1(b) is the only substantial restriction that we make in this paper and will be addressed in future work. Even under this simplifying assumption it appears that Problem 1 requires a sufficient amount of sophistication. Assumption 1(b) enables us to exploit the convenient property that solutions transit through any small enough neighborhood of  $x_a$  independently of the input  $u$ . This property, that we call the “wipeout” property, is characterized in Section III hereafter, and is one of the two main ingredients of our solution. The other ingredient corresponds to a suitable repulsive control design, characterized in Section IV, ensuring that solutions that approach  $x_a$  are suitably modified to avoid entering a peculiar “shell” corresponding to the above characterized “safety neighborhood” of  $x_a$ . We emphasize that the two above mentioned ingredients, developed in Sections III and IV below, are independent of each other, which establishes a desirable modularity in our design, prone for future developments of this research direction.

## III. $\eta$ -NEIGHBORHOOD AND WIPEOUT PROPERTY

In this section we provide a thorough characterization of the implications of Assumption 1(b) to ensure that local equilibria around  $x_a$  cannot be created by whatever feedback solution  $u$  we may design to solve Problem 1. We first provide a few equivalences.

*Lemma 1:* The following items are equivalent:

- (i) Assumption 1(b) holds.
- (ii) The point  $x_a$  cannot be an induced equilibrium of the linear dynamics, namely

$$x_a \notin \mathcal{E} := \{y \in \mathbb{R}^n : \exists u^*, Ay + Bu^* = 0\}. \quad (2)$$

- (iii) It holds that

$$A_B x_a := (I - BB^T)Ax_a \neq 0. \quad (3)$$

*Proof:* The equivalence between (i) and (ii) is a trivial consequence of the definition of linear independence.

“(ii)  $\Rightarrow$  (iii):” If  $(I - BB^T)Ax_a = 0$ , then selecting  $u^* = -B^T Ax_a$  leads to  $Ax_a + Bu^* = 0$ .

“(iii)  $\Rightarrow$  (ii):” If  $\exists u^*$  such that  $Ax_a + Bu^* = 0$ , then, using  $B^T B = 1$  and  $Ax_a = -Bu^*$  implies  $(I - BB^T)Ax_a = Ax_a + BB^T Bu^*$ .  $\blacksquare$

In light of the property in (2), an important parameter in the control design proposed here is the (positive) distance between  $x_a$  and the subspace  $\mathcal{E}$ , defined as

$$\eta^2 := \min_{y \in \mathcal{E}} |x_a - y|^2. \quad (4)$$

The parameter  $\eta$  is a positive scalar under Assumption 1 (by virtue of Lemma 1) and its positivity is essential for establishing that there exists a linear function of the state that monotonically increases in the interior of  $\mathcal{B}_\eta(x_a)$ , regardless of the choice of the input  $u$ . This property, called “wipeout” henceforth, is useful to establish that any solution flowing in  $\mathcal{B}_\eta(x_a)$  must approach its boundary and leave any compact subset of its interior, in finite time. This wipeout feature helps in the analysis of the evolution of solutions within  $\mathcal{B}_\eta(x_a)$ , because solutions naturally drift away from small enough neighborhoods of  $x_a$ , regardless of the input  $u$ .

*Proposition 1: (Wipeout Property).* Let Assumption 1 hold. Consider the function  $H(x) := x_a^T A_B^T x$ , where  $A_B$  is defined in (3), and the scalar  $\eta > 0$  is defined in (4). For each  $x \in \mathcal{B}_\eta(x_a)$  we have  $\langle \nabla H(x), Ax + Bu \rangle \geq 0$  for all  $u \in \mathbb{R}$ . Moreover, for each  $\bar{\eta} < \eta$ , there exists  $\underline{h} > 0$  such that

$$\langle \nabla H(x), Ax + Bu \rangle \geq \underline{h}, \quad \forall u \in \mathbb{R}, \forall x \in \mathcal{B}_{\bar{\eta}}(x_a). \quad (5)$$

*Proof:* Consider the identities, where we use the fact that the projection  $\Pi_B := (I - BB^T)$  satisfies  $\Pi_B^2 = \Pi_B$ :

$$\begin{aligned} \dot{H}(x) &= x_a^T A_B^T \dot{x} = x_a^T A^T (I - BB^T)(Ax + Bu) \\ &= x_a^T A^T (I - BB^T)Ax = (A_B x_a)^T (A_B x). \end{aligned} \quad (6)$$

By definition of  $\eta$ , and the left expression in (6), we know that  $\dot{H}(x) \neq 0$  in  $\mathcal{B}_\eta(x_a)$ . By the right expression in (6), we know that  $\dot{H}(x_a) > 0$  and from continuity we obtain  $\dot{H}(x) > 0$  for all  $x \in \text{int}(\mathcal{B}_\eta(x_a))$ . Then (5) follows from  $\mathcal{B}_{\bar{\eta}}(x_a) \subset \text{int}(\mathcal{B}_\eta(x_a))$ , for all  $\bar{\eta} < \eta$ . Finally,  $\langle \nabla H(x), Ax + Bu \rangle \geq 0$  in  $\mathcal{B}_\eta(x_a)$  follows from continuity of  $\dot{H}(\cdot)$ .  $\blacksquare$

#### IV. UNSAFE SHELL AND AVOIDANCE CONTROLLER

##### A. The eye-shaped shell $\mathcal{S}$

A second ingredient used in this paper, whose construction is parallel to, and independent of the wipeout function  $H$  introduced in the previous section, is the safety or avoidance controller  $u_a$ , acting in a neighborhood of the unsafe point  $x_a$ . The neighborhood is a nonsmooth compact set, having the shape of an eye, as visualized in Figure 1, defined based on two geometrical parameters:

- 1) the size  $\delta \in \mathbb{R}_{>0}$  of the shell;
- 2) the aspect ratio  $\mu \in (0, 2)$  of the shell.

Based on these two parameters, the shell  $\mathcal{S}$  is the following intersection between two balls centered at some shifted versions of the unsafe point  $x_a$ :

$$\delta_\mu := \delta \left( \frac{1}{\mu} - \frac{\mu}{4} \right), \quad (7a)$$

$$\mathcal{O}_q := \mathcal{B}_{\left(\frac{\mu\delta}{2} + \delta_\mu\right)}(x_a - q\delta_\mu B), \quad q \in \{1, -1\}, \quad (7b)$$

$$\mathcal{S}(\delta) := \mathcal{O}_1 \cap \mathcal{O}_{-1}. \quad (7c)$$

Figure 1 represents a few possible shapes of these sets together with the distances that go with them. Note that  $\mu \in (0, 2)$  fixes the aspect ratio of the shell, whose height corresponds to  $\mu\delta$ , which resembles an eye that is increasingly closed as  $\mu$  approaches its lower limit 0. Conversely, as  $\mu$  approaches its upper limit 2, the eye is increasingly open and converges to a circle. In our construction, we will assume that a certain desired aspect ratio  $\mu$  is fixed a priori, and we will establish suitable results by exploiting the fact that the shell  $\mathcal{S}(\delta)$  can be made arbitrarily large and arbitrarily small, by adjusting the positive parameter  $\delta$ . In particular, the following fact will be used throughout our constructions. A proof of the statement can be found in the preprint [3].

*Lemma 2:* Given an aspect ratio  $\mu \in (0, 2)$ , for each  $\delta > 0$ , the following inclusions hold for the shell  $\mathcal{S}(\delta)$  defined in (7):

$$\mathcal{B}_{\frac{\mu\delta}{2}}(x_a) \subset \mathcal{S}(\delta) \subset \mathcal{B}_\delta(x_a). \quad (8)$$

##### B. Avoidance Controller

The shell  $\mathcal{S}(\delta)$  introduced in the previous section is intrinsically composed of two separate boundaries, thereby simplifying the design of a hybrid-based avoidance controller that depends on a logical state  $q \in \{1, -1\}$ . The value of  $q$  indicates whether the avoidance controller should cause sliding of the solution “under” the shell (so to speak, based on the “up” direction of the unit vector  $B$ ) if  $q = -1$ , or over the shell if  $q = 1$ .

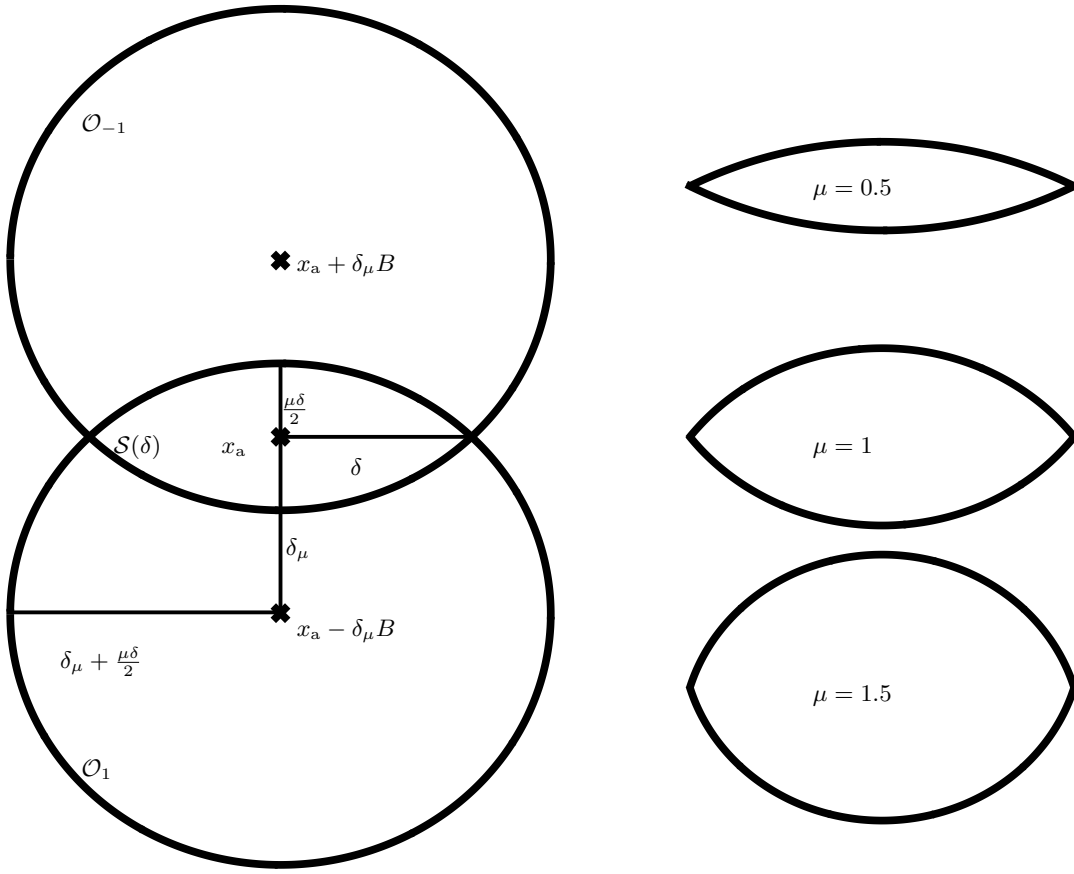


Fig. 1. The construction of the eye-shaped shell  $S(\delta)$  around the unsafe point  $x_a$ , based on the size  $\delta \in \mathbb{R}_{>0}$  and the aspect ratio  $\mu \in (0, 2)$ .

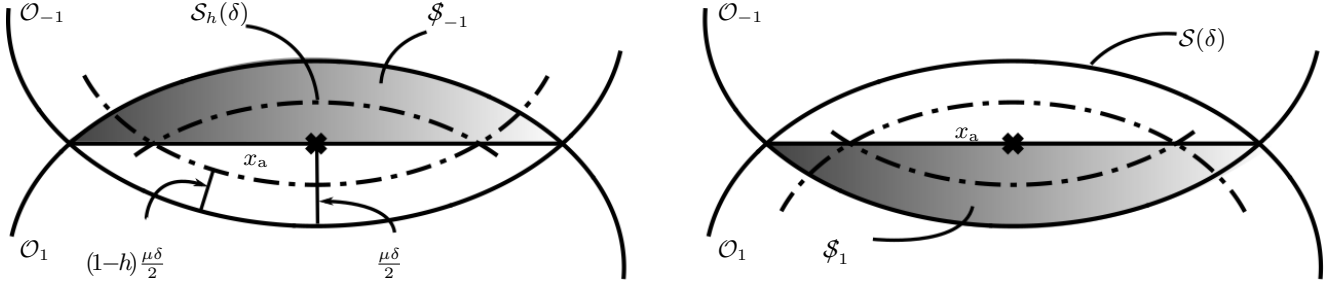


Fig. 2. The shrunken shell  $S_h(\delta)$  and the half shells  $S_1$  and  $S_{-1}$  considered in Proposition 2.

We define such a “binary” avoidance controller as a parametric state feedback defined for  $q \in \{1, -1\}$  as

$$u_a(x, q) := -\frac{\langle x - (x_a - q\delta_\mu B), Ax \rangle}{\langle x - (x_a - q\delta_\mu B), B \rangle}. \quad (9)$$

The avoidance control law (9) is activated by some hybrid logic in the solution proposed in Section V, wherein a suitable  $h$ -hysteresis switching is enforced, based on a region  $S_h(\delta)$  obtained by shrinking  $S(\delta)$  by a factor  $h \in (0, 1)$  as follows, and according to the pictorial representation in Figure 2:

$$\mathcal{O}_{h,q} := \mathcal{B}_{h\frac{\mu\delta}{2} + \delta_\mu}(x_a - q\delta_\mu B), \quad q \in \{1, -1\}, \quad (10)$$

$$S_h(\delta) := \mathcal{O}_{h,1} \cap \mathcal{O}_{h,-1}. \quad (11)$$

It is clear that for each  $q \in \{1, -1\}$  the set  $\mathcal{O}_{h,q}$  is a ball sharing the same center as  $\mathcal{O}_q$  but having a smaller radius that approaches  $\delta_\mu$  as  $h$  approaches 0. As a consequence,  $S_h(\delta)$  is a smaller eye-shaped set, with the same aspect ratio as  $S(\delta)$  (see Figure 2).

The desirable features of the avoidance controller (9) is that it enforces sliding of the solution above or below the shell  $S_h(\delta)$  because it does enforce a constant distance from the upper and the lower balls  $\mathcal{O}_{h,1}$ ,  $\mathcal{O}_{h,-1}$  involved in the definition

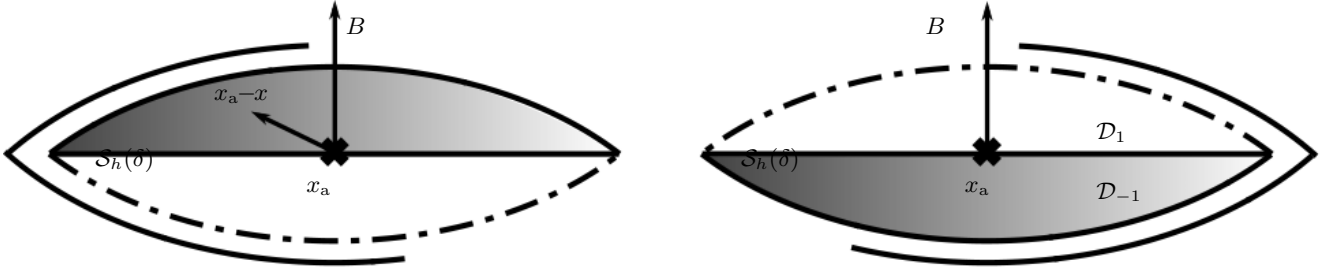


Fig. 3. The upper and lower half-shells associated to  $\mathcal{D}_1$  and  $\mathcal{D}_{-1}$ , respectively, in (14).

of  $\mathcal{S}_h(\delta)$ . Such a desirable sliding mechanism is well understood in terms of the following closed half shells

$$\mathcal{S}_q := \mathcal{S}(\delta) \cap \{x \in \mathbb{R}^n : qB^T(x - x_a) \geq 0\}, \quad q \in \{-1, 1\}, \quad (12)$$

represented in Figure 2.

The following proposition ensures that whenever using the avoidance controller (9) with a suitable value of  $q$ , the ensuing solution does not enter the shell  $\mathcal{S}_h(\delta)$  and actually remains at a constant distance from the corresponding ball containing  $\mathcal{S}_h(\delta)$ .

*Proposition 2:* Let  $\mu \in (0, 2/\sqrt{3})$ ,  $\delta > 0$  and  $h \in (0, 1)$  be given. For each  $q \in \{-1, 1\}$  and any point  $x_0 \in \mathcal{S}_h(\delta) \subset \mathcal{S}(\delta)$ , the local controller  $u = u_a(x_0, q)$ , in (9), is well defined. Moreover, the solution to (1) with  $u = u_a(x, q)$  starting at  $x_0$  remains at a constant (non-negative) distance from the center  $x_a - q\delta_\mu B$  of the ball  $\mathcal{O}_q$  until it remains in  $\mathcal{S}(\delta)$ .  $\lrcorner$

*Proof:* We show the assertion of the lemma for all  $x_0 \in \mathcal{S}(\delta)$ , which includes the results for  $x_0 \in \mathcal{S}_h(\delta)$  due to the set inclusion  $\mathcal{S}_h(\delta) \subset \mathcal{S}(\delta)$  for all  $h \in (0, 1)$ . To simplify the notation we define the points

$$p_q = x_a - q\delta_\mu B, \quad q \in \{-1, 1\},$$

as the centers of  $\mathcal{O}_q$ . As a first step, we show that the local control law (9) is well defined under the condition  $\mu \in (0, 2/\sqrt{3})$ , i.e., we show that  $\langle x - p_q, B \rangle \neq 0$  for all  $x \in \mathcal{S}(\delta)$ ,  $q \in \{-1, 1\}$ . Due to the definition of  $\delta_\mu$  in (7a) and  $\mu$  satisfying  $0 < \mu < \mu^* := 2/\sqrt{3}$ , it holds that

$$\delta_\mu = \frac{\delta}{\mu} \left(1 - \frac{\mu^2}{4}\right) > \frac{\delta}{\mu^*} \left(1 - \frac{(\mu^*)^2}{4}\right) = \frac{\delta}{\sqrt{3}} = \frac{\delta\mu^*}{2} > \frac{\delta\mu}{2},$$

which particularly implies that  $p_q \notin \mathcal{O}_{-q}$ ,  $q \in \{-1, 1\}$ . Thus, every  $x_0 \in \mathcal{S}(\delta)$  can be represented as  $x_0 = p_q + q\alpha B + \beta B^\perp$ , where  $\alpha > 0$ ,  $\beta \in \mathbb{R}$  and  $B^\perp \in \mathbb{R}^n$  satisfies  $\langle B, B^\perp \rangle = 0$ . Due to this definition, it holds that

$$\begin{aligned} \langle x_0 - p_q, B \rangle &= \langle p_q + q\alpha B + \beta B^\perp - p_q, B \rangle \\ &= q\alpha |B| = q\alpha \neq 0 \end{aligned}$$

and the local controller (9) is well defined for all  $x_0 \in \mathcal{S}(\delta)$ .

To show the second statement of the proposition, which means  $|x(t)|_{p_q}$  is constant for the closed-loop system using the feedback law  $u_a(x, q)$ , we show that  $\frac{d}{dt}|x(t) - p_q|^2 = 0$  is satisfied. Due to the definition of the control law (9), it follows immediately that

$$\langle x - p_q, Ax + Bu_a(x, q) \rangle = 0$$

holds for all  $x \in \mathcal{S}(\delta)$ .  $\blacksquare$

The avoidance controller  $u_a$  provides a tool to ensure that the closed-loop solution does not enter the inner shell  $\mathcal{S}_h(\delta)$ . In the next section we show how the avoidance controller can be combined with the stabilizing controller  $u_s$  to ensure asymptotic stability of the origin.

*Remark 1:* Note that the non-smoothness of the boundary of the shell  $\mathcal{S}(\delta)$  in  $\partial\mathcal{O}_{-1} \cap \partial\mathcal{O}_1$  is an essential property for the avoidance controller (9). The idea of sliding along the boundary  $\mathcal{S}(\delta)$  cannot be replaced by sliding along the boundary of a set with a smooth boundary, e.g., a ball centered around  $x_a$ . In the case of a smooth boundary there always exists at least one point  $x$  on the boundary such that  $\langle x - (x_a + \alpha B), B \rangle = 0$  for some  $\alpha \in \mathbb{R}$ , i.e., the tangent of the boundary in  $x$  is aligned with the direction of the vector  $B$ . Thus, a finite input  $u$  does not keep the closed-loop solution on the boundary.  $\lrcorner$

## V. A HYBRID CONTROL SOLUTION

### A. Hybrid dynamics selection

To ensure global asymptotic stability of the origin for the closed loop, we need to patch the two feedback laws  $u_s(x)$  (the stabilizing controller), and  $u_a(x, q)$  (the avoidance controller). Such a patching operation is done here using a hybrid switching strategy exploiting the  $h$ -hysteresis margin between  $\mathcal{S}_h(\delta)$  and  $\mathcal{S}(\delta)$ . Hybrid feedback is a natural choice in light of the discussion above that no continuous feedback can simultaneously ensure GAS of the origin and avoidance of  $x_a$ .

To suitably orchestrate the choice of the active controller, we define an augmented state  $\xi = (x, q)$  for the hybrid dynamics, comprising the plant state  $x$  and the quantity  $q \in \{1, 0, -1\}$  already discussed in the previous section and responsible for whether solutions should slide above ( $q = 1$ ) or below ( $q = -1$ ) the shell when using the avoidance feedback. The value  $q = 0$  is associated to the activation of the stabilizing feedback  $u_s$ . The control selection is summarized by the feedback law

$$u = \gamma(x, q) := (1 - |q|)u_s(x) + |q|u_a(x, q). \quad (13)$$

The overall idea of the controller is to use the feedback law  $u_s$  until solutions enter the shell  $\mathcal{S}(\delta)$ . To ensure a robust switching between the local and global controllers, we exploit the  $h$ -hysteresis and orchestrate the switching of the logic variable  $q$  as follows:

$$\xi^+ = \begin{bmatrix} x^+ \\ q^+ \end{bmatrix} \in \begin{bmatrix} x \\ G_q(\xi) \end{bmatrix}, \quad \xi \in \mathcal{D}_1 \cup \mathcal{D}_{-1} \cup \mathcal{D}_0 \quad (14)$$

$$\mathcal{D}_q := (\mathcal{S}_h(\delta) \cap \mathcal{S}_q) \times \{0\}, \quad q \in \{1, -1\}$$

$$\mathcal{D}_0 := \overline{\mathbb{R}^n \setminus \mathcal{S}(\delta)} \times \{1, -1\}$$

$$G_q(\xi) := \begin{cases} 1, & \text{if } \xi \in \mathcal{D}_1 \setminus \mathcal{D}_{-1} \\ -1, & \text{if } \xi \in \mathcal{D}_{-1} \setminus \mathcal{D}_1 \\ \{1, -1\} & \text{if } \xi \in \mathcal{D}_1 \cap \mathcal{D}_{-1} \\ 0 & \text{if } \xi \in \mathcal{D}_0, \end{cases} \quad (15)$$

where, according to the representation in Figure 3, the two sets  $\mathcal{D}_1$  and  $\mathcal{D}_{-1}$  correspond to the upper and lower halves of the shell  $\mathcal{S}_h(\delta)$ . Note that these sets have a nonzero intersection, associated to the equator plane of the shell. To ensure suitable regularity properties of the jump map  $G$  in (15), we perform a set-valued selection, which allows for either  $q^+ = 1$  or  $q^+ = -1$ . Note that this does not generate multiple simultaneous jumps because we impose  $q = 0$  in the jump sets  $\mathcal{D}_1 \cup \mathcal{D}_{-1}$ , so that, once a decision has been made about whether sliding above or below the shell, this decision cannot be changed.

The hybrid closed-loop behavior is completed by the following flow dynamics, emerging from (1) and (13),

$$\dot{\xi} = \begin{bmatrix} \dot{x} \\ \dot{q} \end{bmatrix} = F(\xi) = \begin{bmatrix} Ax + B\gamma(x, q) \\ 0 \end{bmatrix}, \quad \xi \in \mathcal{C}, \quad (16)$$

where the flow set  $\mathcal{C}$ , is defined as the closed complement of the union of the jump sets defined above. In particular, using  $\Xi := \mathbb{R}^n \times \{-1, 0, 1\}$ , we select

$$\mathcal{C} := \overline{\Xi \setminus (\mathcal{D}_1 \cup \mathcal{D}_{-1} \cup \mathcal{D}_0)}, \quad (17)$$

which, using the fact that  $\mathcal{S}_1 \cup \mathcal{S}_{-1} = \mathcal{S}(\delta) \supset \mathcal{S}_h(\delta)$ , can also be expressed as

$$\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_0 \quad (18)$$

$$:= \{\xi : |q| = 1 \wedge x \in \mathcal{S}(\delta)\} \cup \{\xi : q = 0 \wedge x \in \overline{\mathbb{R}^n \setminus \mathcal{S}_h(\delta)}\}.$$

The selection above for the proposed jump sets has the important advantage that immediately after a jump the solution is in the interior of the flow set at a distance of at least  $(1 - h)\mu\delta/2$  from the jump set  $\mathcal{D}$ . Before our main result is given in the next section, we note that the following structural regularity conditions of the dynamical system are satisfied, whose proof is straightforward and therefore omitted.

*Lemma 3:* The closed-loop dynamics (13)–(17) satisfies the hybrid basic conditions in [5, Assumption 6.5] and all maximal solutions are complete.  $\square$

### B. Main result: GAS and local preservation

We now prove that the hybrid architecture proposed in the previous section provides a solution to Problem 1 discussed in Section II. In particular, we provide quantitative information about a maximal size  $\delta^*$  of the shell  $\mathcal{S}(\delta)$ , such that the hybrid control solution in (13)–(17) solves Problem 1 for any  $\delta < \delta^*$ . A trivial corollary of our result is that regardless of all the parameters, there always exists a small enough  $\delta$  for which our solution is guaranteed to solve Problem 1.

To the end of providing the value  $\delta^*$ , we need the following quantity

$$\zeta := -\frac{2|A_s|}{\lambda_{\max}(A_s^T + A_s)} > 0, \quad (19)$$

which is positive due to Assumption 1(d), ensuring that  $A_s^T + A_s$  is negative definite. Then, we define  $\delta^*$  as

$$\delta^* := \frac{1}{2} \left( |x_a| + \eta + \zeta - \sqrt{(|x_a| + \eta + \zeta)^2 - 4|x_a|\eta} \right) > 0, \quad (20)$$

which is notably independent of  $\mu$  and is well characterized in the next lemma.

*Lemma 4:* Under Assumption 1, given  $\eta$  in (4), the scalar  $\delta^*$  in (20) is a positive real number, and for any value of  $\delta$  satisfying  $\delta < \delta^*$ , we have  $\delta < \eta$ .  $\square$

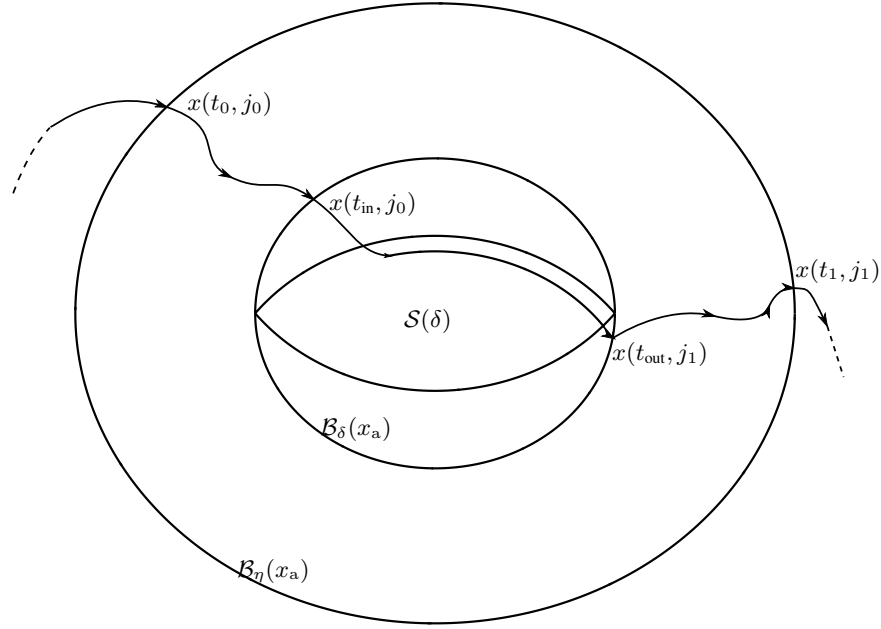


Fig. 4. The intuition behind the two statements of Proposition 3 and the hybrid times  $(t_0, j_0) \leq (t_{in}, j_0) \leq (t_{out}, j_1) \leq (t_1, j_1)$ , characterized in its statement and its proof.

*Proof:* Since  $\eta, \zeta > 0$  and  $\eta < |x_a|$ , by expanding the squared terms, it is straightforward to verify the inequalities.

$$0 < (|x_a| - \eta + \zeta)^2 < (|x_a| + \eta + \zeta)^2 - 4|x_a|\eta. \quad (21)$$

Taking the square root and adding  $2\eta$  on both sides provides

$$|x_a| - \eta + \zeta + 2\eta < \sqrt{(|x_a| + \eta + \zeta)^2 - 4|x_a|\eta} + 2\eta.$$

Finally, moving the square root to the left leads to the estimate

$$2\delta^* = |x_a| + \eta + \zeta - \sqrt{(|x_a| + \eta + \zeta)^2 - 4|x_a|\eta} < 2\eta,$$

which shows the assertion  $\delta^* < \eta$ .

The proof is complete since  $\delta^* \in \mathbb{R}_{>0}$  follows from (21), showing that the square root in (20) is positive.  $\blacksquare$

The selection of  $\delta^*$  in (20) is used in the following proposition.

*Proposition 3:* Let Assumption 1 hold for the hybrid system (13)–(17) and let  $\delta \in (0, \min\{\frac{\eta}{1+\zeta}, \delta^*\})$ . Then the following properties hold for solutions  $\xi(\cdot, \cdot)$  starting at  $\xi_0$ .

- (i) (*Wipeout property*) Let  $\xi_0 \in \mathcal{B}_\delta(x_a) \times \{-1, 0, 1\}$ . Then there exists a time  $(t^*, j^*) \in \text{dom}(\xi)$  such that either  $\xi(t^*, j^*) \in \partial\mathcal{B}_\eta(x_a) \times \{-1, 0, 1\}$  or  $\xi(t, j) \notin \mathcal{B}_\delta(x_a) \times \{-1, 0, 1\}$  for all  $(t, j) \geq (t^*, j^*)$ .
- (ii) (*Decrease property*) Let  $\xi_0 \in \mathbb{R}^n \setminus \mathcal{S}(\delta) \times \{-1, 0, 1\}$ . Additionally, consider any four times in the domain of  $\xi(\cdot, \cdot)$ , such that

$$(t_0, j_0) \leq (t_{in}, j_0) \leq (t_{out}, j_1) \leq (t_1, j_1), \quad (22)$$

$$\text{and } \begin{aligned} \xi(t_0, j_0), \xi(t_1, j_1) &\in \partial\mathcal{B}_\eta(x_a) \times \{0\}, \\ \xi(t_{in}, j_0), \xi(t_{out}, j_1) &\in \partial\mathcal{B}_\delta(x_a) \times \{0\}. \end{aligned} \quad (23)$$

Then either

$$|x(t_1, j_1)| < \min_{z \in \mathcal{S}(\delta)} |z| \quad \text{or} \quad |x(t_1, j_1)| \leq |x(t_0, j_0)| - \varepsilon \quad (24)$$

for  $\varepsilon > 0$ , is satisfied.  $\lrcorner$

The intuition behind the two items of Proposition 3 is illustrated in Figure 4. Item (i) ensures that any solution evolving with the avoidance controller  $u_a$  will switch to the stabilizing controller  $u_s$  and will not switch back to  $u_a$  unless its  $x$  component first reaches the set  $\partial\mathcal{B}_\eta(x_a)$ . Item (ii) ensures that any solution crossing  $\partial\mathcal{B}_\eta(x_a) \times \{0\}$  at some time  $(t_0, j_0)$  and then switching to the avoidance controller  $u_a$ , if crossing again  $\partial\mathcal{B}_\eta(x_a) \times \{0\}$  at some later time  $(t_1, j_1)$ , must satisfy (24), compensating for the increase in  $|x|^2$  due to the avoidance controller. The two cases in (24) are helpful to prove asymptotic stability. If a solution enters and leaves the ball  $\mathcal{B}_\eta(x_a)$  a decrease of at least  $\varepsilon$  in the Lyapunov function  $V(x) = |x|$  is guaranteed. Otherwise,  $x(t_1, j_1) < \min_{z \in \mathcal{S}(\delta)} |z|$  implies  $q(t_1, j_1) = 0$  and  $u = u_s$  which leads to the fact that  $\mathcal{B}_{|x(t_1, j_1)|}(0)$  is forward invariant and thus  $x(t, j) \notin \mathcal{S}(\delta)$  for all  $(t, j) \geq (t_1, j_1)$ . A proof of Proposition 3 and of Theorem 1, which is given next, can be found in the preprint [3].

*Theorem 1:* Let Assumption 1 be satisfied. Given any scalar  $\delta \in (0, \min\{\delta^*, \frac{\eta}{1+\zeta}\})$ , according to (20), any  $\mu \in (0, 2/\sqrt{3})$ ,



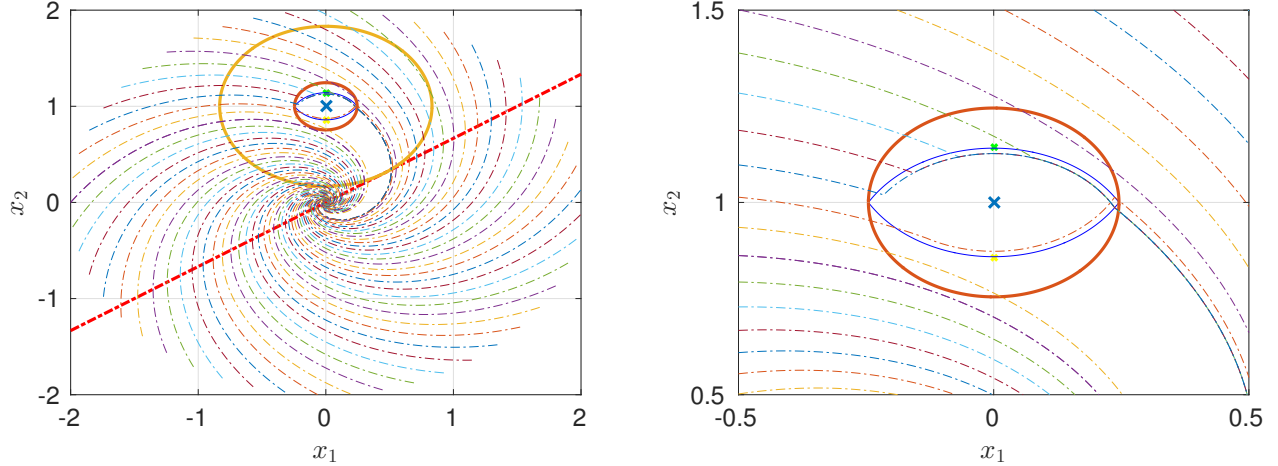


Fig. 5. Avoidance of the unsafe point  $x_a = [0 \ 1]^T$  and closed-loop solutions corresponding to initial values with  $|x_0| = 2$  (and  $q_0 = 0$ ). Additionally, the  $\mathcal{S}(\delta)$ , the  $\eta$ - and  $\delta$ -ball, and the subspace  $\mathcal{E}$  are shown.

and  $h \in (0, 1)$ , the hybrid controller (13)–(17) guarantees that

- (i) the origin  $\xi = (x, q) = (0, 0)$  is uniformly globally asymptotically stable from  $\Xi$ ;
- (ii) for any initial condition  $\xi(0, 0) \in (\mathbb{R}^n \setminus \mathcal{S}(\delta)) \times \{-1, 0, 1\}$ , all the arising solutions satisfy  $|x(t, j)|_{x_a} \geq h \frac{\mu \delta}{2}$  for all  $(t, j) \in \text{dom}(\xi)$ .
- (iii) for any initial condition  $\xi(0, 0) \in (\mathbb{R}^n \setminus \{x_a\}) \times \{0\}$ , all the arising solutions satisfy  $x(t, j) \neq x_a$  for all  $(t, j) \in \text{dom}(\xi)$ .

□

Note that Theorem 1 in particular provides a solution to Problem 1. Due to the selection  $\delta < \min\{\delta^*, \frac{\eta}{1+\zeta}\}$ , it is possible to make the set  $\mathcal{S}(\delta)$  arbitrarily small, which in turn implies: (a) semiglobal preservation, because no solution can flow with  $q \neq 0$  outside  $\mathcal{S}(\delta)$ , (b) semiglobal  $x_a$  avoidance, because Theorem 1(ii) implies that solutions never enter the “safety neighborhood”  $\mathcal{B}_{h \frac{\mu \delta}{2}}(x_a)$ , and (c) GAS, which is guaranteed directly by Theorem 1(i).

## VI. NUMERICAL EXAMPLES

To illustrate our results we simulate the controller for the simple two-dimensional system defined by

$$A = \begin{bmatrix} -1.0 & 1.5 \\ -1.5 & -1.0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (25)$$

and one obstacle  $x_a = [0 \ 1]^T$ , which does not belong to the subspace  $\mathcal{E}$  in (2). The eigenvalues of matrix  $A$  are given by  $\sigma(A) = \{-1 + 1.5i, -1 - 1.5i\}$  and matrix  $A + A^T$  satisfies  $\sigma(A + A^T) = \{-2, -2\}$ , which implies that for  $u_s = 0$  and  $A_s = A$  the origin of the closed-loop system is asymptotically stable and  $V(x) = |x|^2$  is a Lyapunov function. The optimization problem (4) provides a value of  $\eta = 0.8321$ , we set  $\mu = 1.15 < 2/\sqrt{3}$  leading to  $\zeta = 1.8028$  and  $\delta^* = 0.2455$ . For the hysteresis we use a value of  $h = 0.9$  and  $\delta = \delta^*$  (even though the condition  $\delta < \delta^*$  is not satisfied). The simulation results for 50 initial conditions with  $|x_0| = 2$  (and  $q_0 = 0$ ) are shown in Figure 5, where the subspace  $\mathcal{E}$  is shown as a red line. As one might expect from the theoretical results, all simulated solutions asymptotically approach the origin while avoiding the neighborhood around the unsafe point.

## VII. CONCLUSIONS

In this paper we proposed a hybrid controller ensuring GAS of the origin and avoidance of a neighborhood around a given point  $x_a \neq 0$  representing an obstacle. In this respect, an explicit formula for the control law as well as for the size of the neighborhood are given. Even though the results are conservative with respect to the size of the neighborhood and only a single obstacle is considered, the results are presented in such a way that an extension to multiple obstacles and more general system dynamics is straightforward.

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