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# Stabilization of the linear Kuramoto-Sivashinsky equation with a delayed boundary control<sup>★</sup>

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**Abstract:** In this paper we stabilize the linear Kuramoto-Sivashinsky equation by means of a delayed boundary control. From the spectral decomposition of the spatial operator associated to the equation, we find that there is a finite number of unstable eigenvalues. After applying the Artstein transform to deal with the delay phenomenon, we design a feedback law based on the pole-shifting theorem to exponential stabilize the finite-dimensional system associated to the unstable eigenvalues. Then, thanks to the inversion of the Artstein transform and the use of a Lyapunov function, we obtain a delayed feedback law that exponential stabilize the original unstable infinite-dimensional system.

*Keywords:* Partial differential equations; time delay; global stability; feedback stabilization; transformations; pole assignment; Lyapunov function.

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## 1. INTRODUCTION

The Kuramoto-Sivashinsky equation

$$z_t + z_{xxxx} + \lambda z_{xx} + z z_x = 0, \quad (1)$$

where  $\lambda > 0$  is known as the anti-diffusion parameter, was originally derived by Kuramoto and Tsuzuki (1975) as a model of phase turbulence in the context of a reaction-diffusion system, and by Sivashinsky (1977) in the context of flame front propagation. In the later case  $z(t, x)$  represents the perturbation of a flame front which propagates in a combustible mixture.

In this paper we stabilize the linear Kuramoto-Sivashinsky equation

$$z_t + z_{xxxx} + \lambda z_{xx} = 0, \quad (2)$$

by means of a delayed boundary control. Let  $L > 0$  be the length of the domain and  $D \geq 0$  be the delay. In this paper we consider

$$\begin{cases} z_t + z_{xxxx} + \lambda z_{xx} = 0, & (t, x) \in (0, \infty) \times (0, L), \\ z(t, 0) = u(t - D), & t \in (0, \infty), \\ z(t, L) = z_x(t, 0) = z_x(t, L) = 0, & t \in (0, \infty), \\ z(0, x) = z_0(x), & x \in (0, L), \end{cases} \quad (3)$$

where  $u : [-D, \infty) \rightarrow \mathbb{R}$  is the delayed boundary control, which we ask it to satisfy

$$u(t - D) = 0 \text{ when } t \in [0, D). \quad (4)$$

Let us note that we are dealing with a unstable partial differential equation (PDE). Indeed, it is known from (Liu and Krstić, 2001, Remark 1) that uncontrolled (3) is unstable if  $\lambda > 4\pi^2/L^2$ .

Only in the undelayed case, which is when  $D = 0$ , there are boundary stabilization results for (1) or (2); as can be seen in Liu and Krstić (2001), Cerpa (2010), and Coron and Lü (2015). To our knowledge, this is the first paper including delay in the boundary stabilization of (2).

In Datko et al. (1986) and Datko (1988) there are examples of one-dimensional hyperbolic PDEs stabilized by means of boundary feedback controls that are unrobust with respect to delays, in the sense that the inclusion of delay in the boundary feedback control might cause the apparition of a unstable solution. This behavior has also been observed in Nicaise and Pignotti (2006) for the multi-dimensional wave equation and in Nicaise and Rebiai (2011) for the multi-dimensional Schrödinger equation. Nevertheless, let us note that the delay not always causes destabilization. Indeed, there are examples of boundary feedback controls that are robust with respect to delays, as can be seen in Xu and Wang (2013) for the Timoshenko beam system, in Baudouin et al. (2018) for the Korteweg-de Vries equation, and in Prieur and Trélat (2018) for a reaction-diffusion equation.

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The stabilization of abstract second order evolution equations by means of unbounded feedback controls with delay has been addressed in Fridman et al. (2010), thus in particular solving a large class of boundary stabilization problems for hyperbolic PDEs. Of course, (3) cannot be put in terms of their formalism.

The main contribution of this paper is the exponential stabilization of (3) in  $H_0^2(0, L)$  by means of a feedback control that is designed from a finite-dimensional system with input delay  $D \geq 0$ .

The rest of this paper is organized as follows. Our main result is presented, along with some comments on its proof, in Section 2. Then, its proof is given in Section 3. Finally, in Section 4 we give some concluding remarks.

## 2. MAIN RESULT

In order to present our main result we need to introduce the anti-diffusion set of critical parameters

$$\mathcal{AD} = \{(j^2 + k^2)\pi^2 L^{-2} / (j, k) \in \mathbb{N}^2 \text{ with the same parity and } j < k\} \cup \{4l^2\pi^2 L^{-2} / l \in \mathbb{N}\}, \quad (5)$$

which has been found in Cerpa et al. (2017) for the study of the null controllability of (3) when  $D = 0$ . Our main result is the following one.

*Theorem 1.* Let  $\lambda \in (0, \infty) \setminus \mathcal{AD}$  and  $z_0 \in H_0^2(0, L)$ . Then, (3) is exponentially stabilizable with a feedback control that is designed from a finite-dimensional system with input delay  $D \geq 0$ . Furthermore, there exist  $C \geq 1$  and  $\omega > 0$  such that

$$\begin{aligned} |u(t - D)| + \|z(t, \cdot)\|_{H_0^2(0, L)} \\ \leq C e^{-\omega t} \|z_0\|_{H_0^2(0, L)}, \quad t \geq 0. \end{aligned} \quad (6)$$

*Remark 2.* The expression for the feedback control  $u(t - D)$ , which satisfies (4), can be obtained from (14), (26), and (57). Thus, its value depends on  $X_1(s)$  over  $0 < s < t - D$  (see Remark 12), where  $X_1(t)$  solves (16).

*Remark 3.* The rate of decay is  $\omega = (1/2)C_6(D)^{-1}$ , where  $C_6(D) > 0$  is defined in (43).

To prove our main result we follow the approach developed in Prieur and Trélat (2018). Thus, we first consider the spectral decomposition of the spatial operator associated to (3) with the purpose to decompose the unstable infinite-dimensional system into two parts. The first part consists in a finite-dimensional system containing all the unstable eigenvalues, whereas the second part consists in a infinite-dimensional system containing the remaining eigenvalues, which are stable. Then, after applying the Artstein transform to deal with the delay phenomenon, we design a feedback law based on the pole-shifting theorem to exponentially stabilize the unstable finite-dimensional system. Finally, from the inversion of the Artstein transform and the use of a Lyapunov function, we obtain a delayed feedback law that exponentially stabilize the unstable infinite-dimensional system.

*Remark 4.* The assumption  $\lambda \in (0, \infty) \setminus \mathcal{AD}$  is required to check the Kalman condition on the pair of matrices associated to the unstable finite-dimensional system mentioned above, so that later we can stabilize it by applying the pole-shifting theorem. Accordingly, another approach is needed to stabilize (3) if  $\lambda \in \mathcal{AD}$ .

## 3. FEEDBACK DESIGN

### 3.1 Spectral decomposition

From now on  $F_D(t) = F(t - D)$ , where  $F$  denotes a function. In order to get rid of the non-homogeneous boundary conditions in (3), let us introduce the lifting function

$$y(t, x) = z(t, x) - u_D(t)d(x), \quad (7)$$

where  $d(x) = 2L^{-3}x^3 - 3L^{-2}x^2 + 1$ . Since  $d(0) = 1$  and  $d(L) = d'(0) = d'(L) = 0$ , and also  $u_D(0) = 0$  by (4), we get from (3) that  $y(t, x)$  satisfies

$$\begin{cases} y_t = -y_{xxxx} - \lambda y_{xx} - du'_D - \lambda d''u_D, \\ y(t, 0) = y(t, L) = y_x(t, 0) = y_x(t, L) = 0, \\ y(0, x) = z_0(x), \end{cases} \quad (8)$$

Let us view (8) as an infinite-dimensional system. To this end, let us introduce the operator

$$\begin{cases} A : D(A) \subset L^2(0, L) \rightarrow L^2(0, L), \\ A\phi = -\phi'''' - \lambda\phi'', \\ D(A) = H^4(0, L) \cap H_0^2(0, L). \end{cases} \quad (9)$$

Therefore, setting  $a(x) = -\lambda d''(x)$  and  $b(x) = -d(x)$ , we can view (8) as

$$y_t(t, \cdot) = Ay(t, \cdot) + a(\cdot)u_D(t) + b(\cdot)u'_D(t). \quad (10)$$

Since  $A$  is a self-adjoint operator with compact resolvent, we can consider a Hilbert basis  $\{\phi_m\}_{m \in \mathbb{N}} \subset D(A)$  of  $L^2(0, L)$  composed by eigenfunctions of  $A$ . Furthermore, the corresponding sequence of eigenvalues  $\{\sigma_m\}_{m \in \mathbb{N}} \subset \mathbb{R}$  satisfies

$$\begin{cases} \sigma_m \geq \sigma_{m+1} > -\infty \text{ for each } m \in \mathbb{N}, \\ \sigma_m \rightarrow -\infty \text{ as } m \rightarrow \infty. \end{cases} \quad (11)$$

From (Cerpa, 2010, Section 2) we know that  $\sigma_m < \lambda^2/4$  for each  $m \in \mathbb{N}$ , which tells us, in view of (11), that only a finite number of eigenvalues are unstable. Then, let  $n \in \mathbb{N}$  be the number of nonnegative eigenvalues.

Let us construct a finite-dimensional system containing all the unstable eigenvalues. Any solution  $y(t, \cdot)$  of (10) can be written in the form

$$y(t, x) = \sum_{m=1}^{\infty} y_m(t)\phi_m(x). \quad (12)$$

From (10), (12) and the fact that  $A\phi_m = \sigma_m\phi_m$  for each  $m \in \mathbb{N}$  we deduce

$$\begin{cases} y_1'(t) = \sigma_1 y_1(t) + a_1 u_D(t) + b_1 u_D'(t), \\ \vdots \\ y_n'(t) = \sigma_n y_n(t) + a_n u_D(t) + b_n u_D'(t), \end{cases} \quad (13)$$

where  $a_m = (a, \phi_m)_{L^2(0,L)}$  and  $b_m = (b, \phi_m)_{L^2(0,L)}$ . Thus, (13) is a system of  $n \in \mathbb{N}$  ordinary differential equations controlled by  $u_D(t)$  and  $u_D'(t)$ . As usual, let us set

$$\alpha_D(t) = u_D'(t), \quad (14)$$

and consider now  $u_D(t)$  as a new state and  $\alpha_D(t)$  as the control we aim to design. Finally, with the aid of the matrix notations

$$X_1 = \begin{bmatrix} u_D \\ y_1 \\ \vdots \\ y_n \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & \dots & 0 \\ a_1 & \sigma_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & 0 & \dots & \sigma_n \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ b_1 \\ \vdots \\ b_n \end{bmatrix}, \quad (15)$$

we can construct a unstable finite-dimensional system from (13) together with (14), namely

$$X_1'(t) = A_1 X_1(t) + B_1 \alpha_D(t). \quad (16)$$

Recalling that  $\alpha_D(t) = \alpha(t - D)$ , we see that the control in (16) has a input delay  $D \geq 0$ .

### 3.2 Stabilization of the finite-dimensional system

In order to deal with the delay in (16) we proceed as in Bresch-Pietri et al. (2018) and Prieur and Trélat (2018). Thus, let us consider the Artstein transform

$$Z_1(t) = X_1(t) + \int_{t-D}^t e^{(t-s-D)A_1} B_1 \alpha(s) ds. \quad (17)$$

Using (17), which has been introduced in Artstein (1982), we can transform (16) into the unstable finite-dimensional system

$$Z_1'(t) = A_1 Z_1(t) + e^{-DA_1} B_1 \alpha(t). \quad (18)$$

This time we see that the control in (18) has no input delay. Let us check that the pair of matrices  $(A_1, e^{-DA_1} B_1)$  satisfies the Kalman condition for any  $D \geq 0$ . To this end, we need the following result.

**Lemma 5.** The eigenfunctions of  $A$  satisfy  $\phi_m'''(0) \neq 0$  for each  $m \in \mathbb{N}$  if and only if  $\lambda \in (0, \infty) \setminus \mathcal{AD}$ . Furthermore, if  $\lambda \in (0, \infty) \setminus \mathcal{AD}$ , then the eigenvalues of  $A$  are simple.

**Proof.** Let us recall that  $A$  is defined in (9). We only show the second statement of this lemma, since the first one has already been shown in (Cerpa et al., 2017, Section 2.3). We proceed as in the proof of (Cerpa, 2010, Lemma 2.1). Thus, let  $\phi_1$  and  $\phi_2$  be two eigenfunctions associated to the same eigenvalue  $\sigma$ . Then, we have that  $\Phi = \phi_1'''(0)\phi_2 - \phi_2'''(0)\phi_1$  satisfies  $A\Phi = \sigma\Phi$  with  $\Phi \in D(A)$ . The extra information of  $\Phi'''(0) = 0$  allows us to infer that  $\Phi = 0$  in  $[0, L]$ . Accordingly,  $\phi_1$  and  $\phi_2$  are linearly dependent because  $\phi_1'''(0)$  and  $\phi_2'''(0)$  are different from zero by the first statement of this lemma.  $\bullet$

**Lemma 6.** Let  $\lambda \in (0, \infty) \setminus \mathcal{AD}$ . Then, for any  $D \geq 0$  the pair of matrices  $(A_1, e^{-DA_1} B_1)$  satisfies the Kalman condition. In other words:

$$\text{rank} \begin{pmatrix} e^{-DA_1} B_1, A_1 e^{-DA_1} B_1, \\ \dots, A_1^n e^{-DA_1} B_1 \end{pmatrix} = n + 1. \quad (19)$$

**Proof.** Since the matrices  $A_1$  and  $e^{-DA_1}$  commute, and the matrix  $e^{-DA_1}$  is invertible, it follows

$$\begin{aligned} \text{rank} (e^{-DA_1} B_1, A_1 e^{-DA_1} B_1, \dots, A_1^n e^{-DA_1} B_1) \\ = \text{rank} (B_1, A_1 B_1, \dots, A_1^n B_1), \end{aligned} \quad (20)$$

which means that we have to check that the pair of matrices  $(A_1, B_1)$  satisfies the Kalman condition. Some computations lead us to

$$\begin{aligned} \det(B_1, A_1 B_1, \dots, A_1^n B_1) \\ = VdM(\sigma_1, \dots, \sigma_n) \prod_{m=1}^n (a_m + \sigma_m b_m), \end{aligned} \quad (21)$$

where

$$VdM(\sigma_1, \dots, \sigma_n) = \prod_{1 \leq i < j \leq n} (\sigma_j - \sigma_i), \quad (22)$$

is a Van der Monde determinant, which is different from zero because the eigenvalues of  $A$  are simple in virtue of Lemma 5. Some integrations by parts and the fact that  $-\phi_m''' - \lambda\phi_m'' = \sigma_m\phi_m$  yield

$$\begin{aligned} a_m + \sigma_m b_m &= (-\lambda d'' - \sigma_m d, \phi_m)_{L^2(0,L)} \\ &= (d, \phi_m''')_{L^2(0,L)} \\ &= -\phi_m'''(0). \end{aligned} \quad (23)$$

Finally, we conclude that (21) is different from zero due to (23) and Lemma 5.  $\bullet$

Thanks to Lemma 6 we have that (18) is controllable. Accordingly, we can apply the pole-shifting theorem to exponential stabilize it.

**Corollary 7.** Let  $\lambda \in (0, \infty) \setminus \mathcal{AD}$ . Then, for any  $D \geq 0$  there exists a gain matrix  $K_1(D) \in \mathbb{R}^{1 \times (n+1)}$  such that the matrix  $A_1 + e^{-DA_1} B_1 K_1(D)$  admits  $-1$  as an eigenvalue of order  $(n+1)$ . Furthermore, there exists a symmetric positive matrix  $P(D) \in \mathbb{R}^{(n+1) \times (n+1)}$  such that

$$\begin{aligned} P(D)(A_1 + e^{-DA_1} B_1 K_1(D)) \\ + (A_1 + e^{-DA_1} B_1 K_1(D))^T P(D) = -I_{n+1}. \end{aligned} \quad (24)$$

In virtue of Corollary 7 we can construct the Lyapunov function

$$V_1(t) = \frac{1}{2} Z_1(t)^T P(D) Z_1(t), \quad (25)$$

to prove that the feedback control  $\alpha(t) = K_1(D) Z_1(t)$  exponential stabilize (18). However, from (14) together with (4) we see that we actually have to select

$$\begin{cases} \alpha(t) = 0 & \text{when } t \in [-D, 0), \\ \alpha(t) = K_1(D) Z_1(t) & \text{when } t \in [0, \infty). \end{cases} \quad (26)$$

*Lemma 8.* Let  $\lambda \in (0, \infty) \setminus \mathcal{AD}$ . Then, for any  $D \geq 0$  we have

$$\begin{aligned} & \|\alpha_D(t)\|_{\mathbb{R}^{n+1}} \\ & \leq e^D \|K_1(D)^\top\|_{\mathbb{R}^{n+1}} \|Z_1(t)\|_{\mathbb{R}^{n+1}}, \quad t \geq 0. \end{aligned} \quad (27)$$

**Proof.** Let us recall that  $\alpha_D(t) = \alpha(t - D)$ . By (26) it follows that (27) is true for  $t \in [0, D]$ . Thus, let us prove (27) only for  $t \in [D, \infty)$ . Let us introduce the matrix  $A_2 = A_1 + e^{-DA_1} B_1 K_1(D)$ , which satisfies  $\|A_2\| = 1$  due to Corollary 7. Then, plugging (26) into (18) we find that  $Z_1(t) = \Phi(t) Z_1(0)$  for  $t \geq 0$ , where  $\Phi(t) = \exp\{A_2 t\}$ . Finally, we deduce (27) from (26), the fact that  $\Phi(t - D) = \Phi(-D)\Phi(t)$  for  $t \in [D, \infty)$ , and the application of Cauchy-Schwarz inequality. •

Let us note that if  $\sigma_{\min}(P(D)) > 0$  and  $\sigma_{\max}(P(D)) > 0$  respectively denote the lowest and greatest eigenvalue of the symmetric positive matrix  $P(D) \in \mathbb{R}^{(n+1) \times (n+1)}$ , then it holds

$$\begin{aligned} & \sigma_{\min}(P(D)) \|Z_1(t)\|_{\mathbb{R}^{n+1}}^2 \leq Z_1(t)^\top P(D) Z_1(t) \\ & \leq \sigma_{\max}(P(D)) \|Z_1(t)\|_{\mathbb{R}^{n+1}}^2. \end{aligned} \quad (28)$$

### 3.3 Stabilization of the infinite-dimensional system

Plugging (26) into (17) we get

$$Z_1(t) = X_1(t) + \int_{I(t)} e^{(t-s-D)A_1} B_1 K_1(D) Z_1(s) ds, \quad (29)$$

where  $I(t) = (t - D, t) \cap (0, \infty)$ . In the forthcoming analysis it will be necessary to keep in mind that

$$I(t) = \begin{cases} \emptyset & \text{if } t \leq 0, \\ (0, t) & \text{if } 0 < t \leq D, \\ (t - D, t) & \text{if } t \geq D. \end{cases} \quad (30)$$

So far the feedback control  $\alpha(t)$  makes  $X_1(t)$  to go exponentially to zero as  $t \rightarrow \infty$ . Indeed, this comes from (29) and the fact that  $Z_1(t)$  goes exponentially to zero as  $t \rightarrow \infty$ . Let us prove that the same feedback control  $\alpha(t)$  also exponential stabilize (10). To this end, let us introduce the Lyapunov function constructed in Prieur and Trélat (2018), which is

$$\begin{aligned} V_D(t) &= M(D) V_1(t) + M(D) \int_{I(t)} V_1(s) ds \\ &\quad - \frac{1}{2} (y(t, \cdot), Ay(t, \cdot))_{L^2(0, L)}, \end{aligned} \quad (31)$$

where  $M(D) > 0$ . With (9) and (25) in mind, we can rewrite (31) as

$$\begin{aligned} V_D(t) &= \frac{M(D)}{2} Z_1(t)^\top P(D) Z_1(t) \\ &\quad + \frac{M(D)}{2} \int_{I(t)} Z_1(s)^\top P(D) Z_1(s) ds \\ &\quad - \frac{1}{2} \sum_{m=1}^{\infty} \sigma_m y_m(t)^2. \end{aligned} \quad (32)$$

The next three lemmas give the necessary tools to prove our main result.

*Lemma 9.* Let  $\lambda \in (0, \infty) \setminus \mathcal{AD}$ . Let us set the positive constants

$$\begin{aligned} C_1(D) &= \max \left\{ 2, 2e^{4D\|A_1\|} \|B_1 K_1(D)\| \right\}, \\ C_2(D) &= \min \left\{ \frac{\sigma_1}{2}, \frac{1}{2} \right\}, \\ C_3(D) &= \max \left\{ \lambda^2, 2 - \frac{\lambda^2}{\sigma_{n+1}} \right\}, \\ C_4(D) &= \frac{2\sigma_1 C_1(D)}{\sigma_{\min}(P(D))}. \end{aligned} \quad (33)$$

Also, let us assume that  $M(D) \geq C_4(D)$ . Then, for any  $D \geq 0$  we have

$$\begin{aligned} & \frac{C_2(D)}{2} \|X_1(t)\|_{\mathbb{R}^{n+1}}^2 \\ & + \frac{C_2(D)}{2C_3(D)} \|y(t, \cdot)\|_{H_0^2(0, L)}^2 \leq V_D(t), \quad t \geq 0. \end{aligned} \quad (34)$$

**Proof.** Firstly, by (29) we have

$$\begin{aligned} \|X_1(t)\|_{\mathbb{R}^{n+1}}^2 &= X_1(t)^\top Z_1(t) \\ &\quad - X_1(t)^\top \int_{I(t)} e^{(t-s-D)A_1} B_1 K_1(D) Z_1(s) ds, \end{aligned} \quad (35)$$

from which we get, by applying the Cauchy-Schwarz and Cauchy inequalities,

$$\begin{aligned} \|X_1(t)\|_{\mathbb{R}^{n+1}}^2 &\leq C_1(D) \|Z_1(t)\|_{\mathbb{R}^{n+1}}^2 \\ &\quad + C_1(D) \int_{I(t)} \|Z_1(s)\|_{\mathbb{R}^{n+1}}^2 ds. \end{aligned} \quad (36)$$

Then, combining (32) with (28) and (36) we find

$$\begin{aligned} V_D(t) &\geq M(D) \frac{\sigma_{\min}(P(D))}{2C_1(D)} \|X_1(t)\|_{\mathbb{R}^{n+1}}^2 \\ &\quad - \frac{1}{2} \sum_{m=1}^{\infty} \sigma_m y_m(t)^2. \end{aligned} \quad (37)$$

Taking into account that  $\sigma_1 > \sigma_m \geq 0$  for  $m \in \{2, \dots, n\}$  and  $0 > \sigma_{n+1} > \sigma_m$  for  $m \in \{n+2, \dots\}$ , in (37) we use

$$\begin{aligned} & -\frac{1}{2} \sum_{m=1}^{\infty} \sigma_m y_m(t)^2 \\ & \geq -\frac{\sigma_1}{2} \|X_1(t)\|_{\mathbb{R}^{n+1}}^2 - \frac{1}{2} \sum_{m=n+1}^{\infty} \sigma_m y_m(t)^2, \end{aligned} \quad (38)$$

and then consider that  $M(D) \geq C_4(D)$  to obtain

$$\begin{aligned} V_D(t) &\geq C_2(D) \|X_1(t)\|_{\mathbb{R}^{n+1}}^2 \\ &\quad - C_2(D) \sum_{m=n+1}^{\infty} \sigma_m y_m(t)^2. \end{aligned} \quad (39)$$

Secondly, some integrations by parts, (12), (9), the fact that  $-\phi_i'''' - \lambda\phi_i'' = \sigma_i\phi_i$  and Cauchy inequality yield

$$\begin{aligned}
\|y(t, \cdot)\|_{H_0^2(0,L)}^2 &= \sum_{(i,j) \in \mathbb{N}^2} y_i(t)y_j(t) (\phi_i'', \phi_j'')_{L^2(0,L)} \\
&= \sum_{(i,j) \in \mathbb{N}^2} y_i(t)y_j(t) (\phi_i''''', \phi_j)_{L^2(0,L)} \\
&= -\lambda (y'', y)_{L^2(0,L)} - \sum_{m=1}^{\infty} \sigma_m y_m(t)^2 \\
&\leq \lambda^2 \sum_{m=1}^{\infty} y_m(t)^2 - 2 \sum_{m=n+1}^{\infty} \sigma_m y_m(t)^2 \\
&\leq \lambda^2 \|X_1(t)\|_{\mathbb{R}^{n+1}}^2 + \sum_{m=n+1}^{\infty} (\lambda^2 - 2\sigma_m) y_m(t)^2.
\end{aligned} \tag{40}$$

Here we consider

$$\begin{aligned}
\lambda^2 - 2\sigma_m &= -\sigma_m \left(2 - \frac{\lambda^2}{\sigma_m}\right) \\
&\leq -\sigma_m \left(2 - \frac{\lambda^2}{\sigma_{n+1}}\right), \quad m \in \{n+1, \dots\},
\end{aligned} \tag{41}$$

to obtain

$$\begin{aligned}
\|y(t, \cdot)\|_{H_0^2(0,L)}^2 &\leq C_3(D) \|X_1(t)\|_{\mathbb{R}^{n+1}}^2 \\
&\quad - C_3(D) \sum_{m=n+1}^{\infty} \sigma_m y_m(t)^2.
\end{aligned} \tag{42}$$

Finally, from the combination of (39) and (42) we arrive at (34). •

*Lemma 10.* Let  $\lambda \in (0, \infty) \setminus \mathcal{AD}$ . Let us set the positive constants

$$\begin{aligned}
C_5(D) &= 4\|a\|_{L^2(0,L)}^2 C_1(D) \\
&\quad + 4e^{2D} \|b\|_{L^2(0,L)}^2 \|K_1(D)^\top\|_{\mathbb{R}^{n+1}}^2, \\
C_6(D) &= \max \left\{ -\frac{1}{\sigma_{n+1}}, C_5(D), 2\sigma_{\max}(P(D)) \right\}.
\end{aligned} \tag{43}$$

Also, let us assume that  $M(D) \geq C_6(D)$ . Then, for any  $D \geq 0$  we have

$$V_D(t) \leq \exp \left\{ -\frac{1}{M(D)} t \right\} V_D(0), \quad t \geq 0. \tag{44}$$

**Proof.** Let us note that (44) is true for  $t = 0$ . Thus, let us prove (44) only for  $t > 0$  by computing  $V_D'(t)$ . Firstly, by (25) and the use of (24) we get

$$\begin{aligned}
\frac{d}{dt} V_1(t) + \frac{d}{dt} \int_{I(t)} V_1(s) ds \\
= -\frac{1}{2} \|Z_1(t)\|_{\mathbb{R}^{n+1}}^2 - \frac{1}{2} \int_{I(t)} \|Z_1(s)\|_{\mathbb{R}^{n+1}}^2 ds.
\end{aligned} \tag{45}$$

Secondly, taking into account that  $A$  is a self-adjoint operator it follows

$$\begin{aligned}
\frac{d}{dt} (y(t, \cdot), Ay(t, \cdot))_{L^2(0,L)} \\
= 2 (Ay(t, \cdot), y_t(t, \cdot))_{L^2(0,L)}.
\end{aligned} \tag{46}$$

Here we plug (10), then apply Cauchy inequality and finally consider (15) together with (27) to obtain

$$\begin{aligned}
\frac{d}{dt} (y(t, \cdot), Ay(t, \cdot))_{L^2(0,L)} \\
\leq -\|Ay(t, \cdot)\|_{L^2(0,L)}^2 + 2\|a\|_{L^2(0,L)}^2 \|X_1(t)\|_{\mathbb{R}^{n+1}}^2 \\
+ 2e^{2D} \|b\|_{L^2(0,L)}^2 \|K_1(D)^\top\|_{\mathbb{R}^{n+1}}^2 \|Z_1(t)\|_{\mathbb{R}^{n+1}}^2.
\end{aligned} \tag{47}$$

Thirdly, since  $0 > \sigma_{n+1} > \sigma_m$  for  $m \in \{n+2, \dots\}$  and  $M(D) \geq -1/\sigma_{n+1}$  we infer

$$-\frac{1}{2} \|Ay(t, \cdot)\|_{L^2(0,L)}^2 \leq \frac{1}{2M(D)} \sum_{m=1}^{\infty} \sigma_m y_m(t)^2. \tag{48}$$

Finally, with (31) in mind, we combine (45), (47), (36), and (48) to obtain

$$\begin{aligned}
V_D'(t) &\leq -\left(\frac{M(D)}{2} - \|a\|_{L^2(0,L)}^2 C_1(D)\right. \\
&\quad \left.- e^{2D} \|b\|_{L^2(0,L)}^2 \|K_1(D)^\top\|_{\mathbb{R}^{n+1}}^2\right) \|Z_1(t)\|_{\mathbb{R}^{n+1}}^2 \\
&\quad - \left(\frac{M(D)}{2} - \|a\|_{L^2(0,L)}^2 C_1(D)\right) \int_{I(t)} \|Z_1(s)\|_{\mathbb{R}^{n+1}}^2 ds \\
&\quad + \frac{1}{2M(D)} \sum_{m=1}^{\infty} \sigma_m y_m(t)^2,
\end{aligned} \tag{49}$$

from which we deduce  $V_D'(t) \leq M(D)^{-1} V_D(t)$  for  $t > 0$ , and hence (44), thanks to the choice  $M(D) \geq C_5(D)$ , (28), the choice  $M(D) \geq 2\sigma_{\max}(P(D))$ , and (32). •

*Lemma 11.* Let  $\lambda \in (0, \infty) \setminus \mathcal{AD}$ . Then, for any  $D \geq 0$  there exists  $C_7(D) > 0$  such that

$$V_D(0) \leq C_7(D) \|y(0, \cdot)\|_{H_0^2(0,L)}^2. \tag{50}$$

**Proof.** In view of  $I(0) = \emptyset$  it follows from (29) that  $Z_1(0) = X_1(0)$ . Accordingly, by (25) together with (31) we find

$$\begin{aligned}
V_D(0) &= \frac{M(D)}{2} X_1(0)^T P(D) X_1(0) \\
&\quad - \frac{1}{2} (y(0, \cdot), Ay(0, \cdot))_{L^2(0,L)}.
\end{aligned} \tag{51}$$

On the one hand, from (15), (4), and (28) we get

$$X_1(0)^\top P(D) X_1(0) \leq \sigma_{\max}(P(D)) \|y(0, \cdot)\|_{L^2(0,L)}^2. \tag{52}$$

On the other hand, with (9) in mind we perform some integrations by parts and then apply the Cauchy and Poincaré inequalities to obtain

$$-\frac{1}{2} (y(0, \cdot), Ay(0, \cdot))_{L^2(0,L)} \leq C \|y(0, \cdot)\|_{H_0^2(0,L)}^2. \tag{53}$$

Finally, we arrive at (50) from the combination of (51)-(53) and the application of Poincaré inequality. •

Let us prove (6). Let  $\lambda \in (0, \infty) \setminus \mathcal{AD}$  and let us set  $M(D) = C_6(D)$ . Then, by (34), (44), and (50) it follows

$$\begin{aligned} & \frac{C_2(D)}{2} \|X_1(t)\|_{\mathbb{R}^{n+1}}^2 + \frac{C_2(D)}{2C_3(D)} \|y(t, \cdot)\|_{H_0^2(0,L)}^2 \\ & \leq C_7(D) \exp \left\{ -\frac{1}{M(D)} t \right\} \|y(0, \cdot)\|_{H_0^2(0,L)}^2, \quad t \geq 0, \end{aligned} \quad (54)$$

from which we deduce (6) thanks to (7), (4) and (15). We have finished the proof of Theorem 1.

### 3.4 Inversion of the Artstein transform

We would like to express  $\alpha(t)$  in terms of  $X_1(t)$ . To this end, we present the inversion of the Artstein transform realized in Bresch-Pietri et al. (2018). Thus, taking into account that  $(t - D, t) = (t - D, \max\{t - D, 0\}) \cup J(t)$ , where  $J(t) = (\max\{t - D, 0\}, t)$ , we plug (17) into (26) and then use the fact that  $\alpha(t) = 0$  when  $t \in [-D, 0]$  to get

$$\begin{aligned} \alpha(t) &= K_1(D)X_1(t) \\ &+ K_1(D) \int_{J(t)} e^{(t-s-D)A_1} B_1 \alpha(s) ds, \quad t \geq 0. \end{aligned} \quad (55)$$

Then, for a function  $F$  let us define

$$\begin{aligned} (T_D F)(t) &= K_1(D) \int_{J(t)} e^{(t-s-D)A_1} B_1 F(s) ds, \quad t \geq 0. \end{aligned} \quad (56)$$

Let us consider  $t \geq 0$ . In virtue of (56) we can rewrite (55) as  $\alpha(t) = K_1(D)X_1(t) + (T_D \alpha)(t)$ , which is equivalent to  $(I - T_D)\alpha(t) = K_1(D)X_1(t)$ . The convergence of the Neumann series associated to  $T_D$  has been shown in (Bresch-Pietri et al., 2018, Section 4.1), which implies the existence of  $(I - T_D)^{-1}$ . Accordingly, we have

$$\alpha(t) = \sum_{m=0}^{\infty} (T_D^m K_1(D)X_1)(t), \quad t \in [0, \infty). \quad (57)$$

*Remark 12.* The series in (57) converges for any  $D \geq 0$ . Let us note that the value of  $\alpha(t)$  depends on  $X_1(s)$  over  $0 < s < t$ . Further details may be found in Bresch-Pietri et al. (2018).

## 4. CONCLUSION

In this paper we have exponentially stabilized in  $H_0^2(0, L)$  the linear Kuramoto-Sivashinsky equation by means of a feedback control designed from a finite-dimensional system with input delay  $D \geq 0$ . As in the null controllability, our main assumption was  $\lambda \in (0, \infty) \setminus \mathcal{AD}$ . To prove our main result we have followed the approach developed in Prieur and Trélat (2018), which relies on the Artstein transform, a careful spectral analysis and the pole-shifting theorem.

## REFERENCES

- Artstein, Z. (1982). Linear systems with delayed controls: A reduction. *IEEE Transactions on Automatic Control*, 27(4), 869–879.
- Baudouin, L., Crépeau, E., and Valein, J. (2018). Two approaches for the stabilization of the nonlinear KdV equation with boundary time-delay feedback. *IEEE Transactions on Automatic Control*. DOI:10.1109/TAC.2018.2849564 (To Appear).
- Bresch-Pietri, D., Prieur, C., and Trélat, E. (2018). New formulation of predictors for finite-dimensional linear control systems with input delay. *Systems and Control Letters*, 113, 9–16.
- Cerpa, E. (2010). Null controllability and stabilization of the linear Kuramoto-Sivashinsky equation. *Communications on Pure and Applied Analysis*, 9(1), 91–102.
- Cerpa, E., Guzmán, P., and Mercado, A. (2017). On the control of the linear Kuramoto-Sivashinsky equation. *ESAIM: Control, Optimisation and Calculus of Variations*, 23(1), 165–194.
- Coron, J.M. and Lü, Q. (2015). Fredholm transform and local rapid stabilization for a Kuramoto-Sivashinsky equation. *Journal of Differential Equations*, 259(8), 3683–3729.
- Datko, R. (1988). Not all feedback stabilized hyperbolic systems are robust with respect to small time delays in their feedbacks. *SIAM Journal on Control and Optimization*, 26(3), 697–713.
- Datko, R., Lagnese, J., and Polis, M. (1986). An example on the effect of time delays in boundary feedback stabilization of wave equations. *SIAM Journal on Control and Optimization*, 24(1), 152–156.
- Fridman, E., Nicaise, S., and Valein, J. (2010). Stabilization of second order evolution equations with unbounded feedback with time-dependent delay. *SIAM Journal on Control and Optimization*, 48(8), 5028–5052.
- Kuramoto, Y. and Tsuzuki, T. (1975). On the formation of dissipative structures in reaction-diffusion systems. *Progress of Theoretical Physics*, 54(3), 687–699.
- Liu, W. and Krstić, M. (2001). Stability enhancement by boundary control in the Kuramoto-Sivashinsky equation. *Nonlinear Analysis*, 43(4), 485–507.
- Nicaise, S. and Pignotti, C. (2006). Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks. *SIAM Journal on Control and Optimization*, 45(5), 1561–1585.
- Nicaise, S. and Rebiai, S. (2011). Stabilization of the Schrödinger equation with a delay term in boundary feedback or internal feedback. *Portugaliae Mathematica*, 68(1), 19–39.
- Prieur, C. and Trélat, E. (2018). Feedback stabilization of a 1D linear reaction-diffusion equation with delay boundary control. *IEEE Transactions on Automatic Control*. DOI:10.1109/TAC.2018.2849560 (To Appear).
- Sivashinsky, G. (1977). Nonlinear analysis of hydrodynamic instability in laminar flames-I. Derivation of basic equations. *Acta Astronautica*, 4(11-12), 1177–1206.
- Xu, G. and Wang, H. (2013). Stabilisation of Timoshenko beam system with delay in the boundary control. *International Journal of Control*, 86(6), 1165–1178.