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Stability analysis of linear coupled differential-difference systems with general distributed delays

Qian Feng and Sing Kiong Nguang and Alexandre Seuret

Abstract

We present a new approach for the stability analysis of linear coupled differential-difference systems (CDDS) with a general distributed delay. The distributed delay term in this note can contain any \mathbb{L}^2 function which is approximated via a class of elementary functions including polynomial, trigonometric and exponential functions etc. Through the application of a new proposed integral inequality, sufficient condition for the stability of the system is derived in terms of linear matrix inequalities based on the construction of a Liapunov Krasovskii functional. The methods proposed in this note can handle problems which cannot be deal with by existing approaches. Two numerical examples are presented to show the effectiveness of our proposed stability condition.

Index Terms

Distributed Delay; Integral Inequality; Coupled differential-difference systems; Approximations.

I. INTRODUCTION

Coupled differential-difference systems (CDDSs), which are mathematically related to time-delay systems [1], can characterize a broad class of models concerning delay and propagation effects [2]. CDDSs can characterize standard or neutral time-delay systems or some singular delay systems [3]. One of the advantages of using CDDS representations over standard models of time-delay systems can be demonstrated by the results of stability analysis in [4], [5], where the structure of CDDS may effectively reduce the numerical burden of the constructed stability conditions. For more information on the topic of CDDS, see [6] and the references therein.

Over the past decades, a series of results on the stability of CDDS [7], [8], [4], [5] has been proposed based on the construction of Liapunov-Krasovskii functionals. In particular, the construction of the functionals for linear CDDS with pointwise delays [6], [9] can be achieved by solving linear matrix inequalities (LMIs). To the best of our knowledge, however, no available solutions have been published in the peer-reviewed literature which may handle the stability analysis of a linear CDDS with non-trivial (non-constant) distributed delays. This is even true when frequency domain based methods are concerned. For example, the approaches in [10], [11] may not be applied to analyze the stability of a CDDS with a non-trivial distributed delay due to the mathematical structure of the corresponding system spectrum.

Generally speaking, analyzing distributed delays may require much more efforts due to the complexity of dealing with different types of distributed delay kernels. For the latest available approaches to handle distributed delays in time domain, see [12], [13], [14]. In [13], a method is proposed to cope with distributed delay terms with the kernels of \mathbb{L}^2 continuous functions based on the approximation of Legendre polynomials. However, the formulation of the stability condition in [13] is not succinct to be extended to the situation of having multiple kernels at the same time. Moreover, another drawback of Legendre polynomials is that it might perform poorly to approximate certain functions with insufficient degrees, for instance, functions with patterns of intense oscillations as pointed out in Example 1 of [14]. On the other hand, the methodology in [14] can only handle distributed delay kernels which belong to a

specific class of elementary functions. This motivates us to develop a more efficient and robust approach to handle the presence of general distributed delay terms in a system.

In this paper, we propose a new approach to handle the problem of the stability analysis of a linear CDDS with a general distributed delay term. Specifically, the distributed delay kernels can be any \mathbb{L}^2 function over an interval. Moreover, the models of many existing delay systems, such as the ones in [6], [13], [14], [15], are generalized by the model we considered. The distributed delay kernels are handled by a new approximation scheme in this paper based on a class of elementary functions which includes the Legendre polynomials approximation scheme proposed in [13]. Moreover, our approximation scenario also provides a unified matrix framework which can easily handle the situation of a CDDS having multiple distributed delay kernels. It shows that, if orthogonal functions $\{\vartheta_i(\cdot)\}_{i=1}^d$ are chosen as the approximator, then the resulting approximation-error-vector converges to zeros in a \mathbb{L}^2 sense as $d \rightarrow \infty$. Meanwhile, a new general integral inequality is derived which has an approximation-error-related term at its lower bound. By constructing a Liapunov-Krasovskii functional via the application of this inequality, sufficient conditions for the asymptotic (exponential) stability of the linear CDDS can be derived in terms of LMIs where the corresponding approximation error is expressed by a positive definite matrix. Our stability condition is further proved to have a hierarchical feasibility enhancement under the framework of orthogonal functions, which generalizes the results in [16]. Finally, two numerical examples are given to demonstrate the effectiveness and capacity of the proposed methodologies.

The novelties in this paper are summarized as follows:

- We proposed a much wider class of functions for the approximation of delay kernel functions compared to polynomials approximation. The idea behind the proposed approximation is based on the application of the Hilbert projection theorem (least square approximation in Hilbert space) [17]. Moreover, this also allows one to construct a Liapunov-Krasovskii functional with non-polynomials kernels when distributed delay terms are handled via approximations.
- A very general inequality is derived in this paper where the approximator functions belong to a weighted \mathbb{L}_2 space. This means that the proposed integral inequality has tremendously general structure and it has very good potential to be used for the situations where a general approximator is employed to deal with distributed delay kernels. Thus many future works might be done based on the proposed inequality.

Notations and rules: Empty matrices \square , which follows the rules in Matlab environment, is applied in this paper. \mathbb{N} stands for the set of natural numbers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We also define $\mathbb{R}_{\geq a} := \{x \in \mathbb{R} : x \geq a\}$ and $\mathbb{S}^n = \{X \in \mathbb{R}^{n \times n} : X = X^\top\}$. $\mathbf{M}(\mathcal{L}(\mathcal{X})/\mathcal{B}(\mathbb{R})) := \{f(\cdot) \in \mathbb{R}^{\mathcal{X}} : \forall \mathcal{Y} \in \mathcal{B}(\mathbb{R}), f^{-1}(\mathcal{Y}) \in \mathcal{L}(\mathcal{X})\}$ denotes the space of all $\mathcal{L}(\mathcal{X})/\mathcal{B}(\mathbb{R})$ measurable functions from \mathcal{X} onto \mathbb{R} , where $\mathcal{L}(\mathcal{X})$ contains all the subsets of \mathcal{X} which are Lebesgue measurable with $\mathcal{X} \in \mathcal{L}(\mathbb{R})$, and $\mathcal{B}(\mathbb{R})$ is the Borel σ -algebra on \mathbb{R} . For any $X \in \mathbb{R}^{n \times n}$, we define $\mathbf{Sy}(X) := X + X^\top$. A column vector containing a sequence of mathematical objects (scalars, matrices, functions etc) is defined as $\mathbf{Col}_{i=1}^n x_i := [\mathbf{Row}_{i=1}^n x_i^\top]^\top = [x_1^\top \cdots x_i^\top \cdots x_n^\top]^\top$. In addition, it is defined that $\mathbf{Col}_{i=1}^n = \square$ with $n < 1$, where \square is an empty matrix with appropriate column dimensions. The symbol $*$ is used to indicate $[*]YX = X^\top YX$ or $X^\top Y[*] = X^\top YX$ or $\begin{bmatrix} A & B \\ * & C \end{bmatrix} = \begin{bmatrix} A & B \\ B^\top & C \end{bmatrix}$. $\mathbf{O}_{n \times m}$ stands for a $n \times m$ zero matrix and the notation $\mathbf{O}_{n \times n}$ is abbreviated as \mathbf{O}_n , whereas $\mathbf{0}_n$ denotes a $n \times 1$ column vector. The diagonal sum of two matrices is defined as $X \oplus Y = \begin{bmatrix} X & \mathbf{0} \\ * & Y \end{bmatrix}$. \otimes denotes the Kronecker product. Finally, we assume the operator precedence of matrices to be *matrix (scalars) multiplications* $> \otimes > \oplus > +$.

II. PRELIMINARIES AND PROBLEM FORMULATION

In this paper, we consider the stability analysis of a CDDS

$$\begin{aligned} \dot{\mathbf{x}}(t) &= A_1 \mathbf{x}(t) + A_2 \mathbf{y}(t-r) + \int_{-r}^0 \tilde{A}_3(\tau) \mathbf{y}(t+\tau) d\tau \\ \mathbf{y}(t) &= A_4 \mathbf{x}(t) + A_5 \mathbf{y}(t-r) \\ \mathbf{x}(t_0) &= \boldsymbol{\xi} \in \mathbb{R}^n, \quad \mathbf{y}(t_0 + \theta) = \boldsymbol{\phi}(\theta), \quad \forall \theta \in [-r, 0] \end{aligned} \tag{1}$$

with a general distributed delay, where $t_0 \in \mathbb{R}$ and $r > 0$ is given and $\phi(\cdot) \in \widehat{\mathbf{C}}([-r, 0]; \mathbb{R}^\nu)$. $\widehat{\mathbf{C}}([-r, 0]; \mathbb{R}^n)$ here stands for the space of the functions which are bounded, right-continuous and piecewise continuous functions, endowed with the norm $\|\phi(\cdot)\|_\infty = \sup_{\tau \in \mathcal{X}} \|\phi(\tau)\|_2$. Furthermore, $\mathbf{x}(t) \in \mathbb{R}^n$; $\mathbf{y}(t) \in \mathbb{R}^\nu$ are the solution of (1) with $n, \nu \in \mathbb{N}$ which determine the size of $A_1; A_2; \tilde{A}_3(\tau)$ and $A_4; A_5$, where $\tilde{A}_3(\cdot) \in \mathbb{L}^2([-r, 0]; \mathbb{R}^{n \times \nu})$. Meanwhile, we define $\mathbf{f}(\cdot) := \mathbf{Col}_{i=1}^d f_i(\cdot) \in \mathbf{C}(\mathbb{R}; \mathbb{R}^d)$ with $d \in \mathbb{N}_0$ and we assume $\mathbf{f}(\cdot)$ satisfies the property

$$\exists M \in \mathbb{R}^{d \times d}, \quad \frac{d\mathbf{f}(\tau)}{d\tau} = M\mathbf{f}(\tau). \quad (2)$$

It is obvious that given a $\mathbf{f}(\cdot) \in \mathbf{C}(\mathbb{R}; \mathbb{R}^d)$ satisfying (2), then for any $\tilde{A}_3(\cdot) \in \mathbb{L}^2([-r, 0]; \mathbb{R}^{n \times \nu})$ in (1) there exist $A_3 \in \mathbb{R}^{n \times \rho\nu}$ with $\rho = d + \delta$, and $\varphi(\cdot) \in \mathbb{L}^2([-r, 0]; \mathbb{R}^\delta)$ with $\delta \in \mathbb{N}_0$ such that

$$\forall \tau \in [-r, 0], \quad \tilde{A}_3(\tau) = A_3 \left(\begin{bmatrix} \varphi(\tau) \\ \mathbf{f}(\tau) \end{bmatrix} \otimes I_\nu \right) \quad (3)$$

$$\int_{-r}^0 \begin{bmatrix} \varphi(\tau) \\ \mathbf{f}(\tau) \end{bmatrix}^\top [\varphi^\top(\tau) \quad \mathbf{f}^\top(\tau)] d\tau \succ 0 \quad (4)$$

Remark 1. The condition in (2) indicates that the functions in $\mathbf{f}(\cdot)$ are the solutions of linear homogeneous equations with constant coefficients, such as polynomials, trigonometric and exponential functions etc. Moreover, the matrix inequality in (4) indicates that the functions in $[\varphi^\top(\tau) \quad \mathbf{f}^\top(\tau)]^\top$ are linearly independent in a Lebesgue sense. (See Theorem 7.2.10 in [18]) Finally, we stress here that the decomposition in (3) with (4) is always achievable since $\varphi(\cdot) \in \mathbb{L}^2([-r, 0]; \mathbb{R}^\delta)$ can cover any function in $\tilde{A}_3(\cdot) \in \mathbb{L}^2([-r, 0]; \mathbb{R}^{n \times \nu})$ if such function is not covered by $\mathbf{f}(\cdot)$.

Remark 2. The distributed delay $\tilde{A}_3(\cdot)$ in (1) can be equivalently denoted by the decomposition in (3) which is employed in this paper so that well-posed stability condition can be derived. This will be illustrated later in light of the results in Lemma 3 and Theorem 1. Note that $\varphi(\cdot)$ can be a 0×1 empty vector if $\delta = 0$. Finally, (1) generalizes almost all the models of linear distributed delay (neutral) systems in the literature [12], [13], [14] without considering uncertainties or disturbances or inputs or outputs.

To tackle the function $\varphi(\cdot) \in \mathbb{L}^2([-r, 0]; \mathbb{R}^\delta)$ in (3), which may not satisfy (2), we use $\mathbf{f}(\tau)$ to approximate $\varphi(\tau)$ over $[-r, 0]$. This is denoted mathematically as

$$\forall \tau \in [-r, 0], \quad \varphi(\tau) = \Gamma_d \mathbf{f}(\tau) + \varepsilon_d(\tau) \quad (5)$$

where

$$\Gamma_d := \int_{-r}^0 \varphi(\tau) \mathbf{f}^\top(\tau) d\tau F_d, \quad F_d^{-1} := \int_{-r}^0 \mathbf{f}(\tau) \mathbf{f}^\top(\tau) d\tau \quad (6)$$

and $\varepsilon_d(\tau) := \varphi(\tau) - \Gamma_d \mathbf{f}(\tau)$ defines the error of approximations which gives

$$\mathbf{E}_d := \int_{-r}^0 \varepsilon_d(\tau) \varepsilon_d^\top(\tau) d\tau \in \mathbb{S}^\delta \quad (7)$$

to measure the error residual of (5).

Remark 3. Note that using the approximation scheme in (5) does not put extra constraints to $\mathbf{f}(\cdot)$ in (2)–(4) and $\mathbf{f}(\cdot)$ can be any function as long as (2)–(4) are satisfied. To obtain non-conservative results, we suggest that one can choose $\mathbf{f}(\cdot)$ to be in line with the functions in $\tilde{A}_3(\cdot)$ considering the structures of $\varphi(\tau)$. Namely, $\mathbf{f}(\cdot)$ should be “easy” to approximate $\varphi(\cdot)$ over $[-r, 0]$ if $\delta \neq 0$.

Remark 4. If $\mathbf{f}(\cdot)$ contains only Legendre polynomials, then (5)–(6) generalizes the polynomials approximation scenario in [13] via a matrix framework. The mathematical structure of (6) can be interpreted as a vector form of the Hilbert projection Theorem (Least square approximation) [17] via $\mathbf{f}(\cdot)$ in (3). In addition, the approximator $\mathbf{f}(\cdot)$ in (6) belongs to a class of elementary functions, not approximation by a specific type of functions such as polynomials. Thus the proposed approximation scheme in (5) is both conceptually and mathematically more general than the polynomials approximation in [13] as $\mathbf{f}(\cdot)$ in (6) can contain functions which are not orthogonal polynomials or even mutually orthogonal.

The following lemma of the properties of the Kronecker product will be used throughout the rest of our paper.

Lemma 1. For all $X \in \mathbb{R}^{n \times m}$ and $Y \in \mathbb{R}^{m \times p}$ and $Z \in \mathbb{R}^{q \times r}$,

$$(X \otimes I_q)(Y \otimes Z) = (XY) \otimes (I_q Z) = (XY) \otimes Z = (XY) \otimes (Z I_r) = (X \otimes Z)(Y \otimes I_r). \quad (8)$$

Moreover, for all $X \in \mathbb{R}^{n \times m}$, we have

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \otimes X = \begin{bmatrix} A \otimes X & B \otimes X \\ C \otimes X & D \otimes X \end{bmatrix} \quad (9)$$

for any A, B, C, D with appropriate dimensions which make the block matrix at the left hand of the equality in (9) to be compatible.

Considering (3) and (5), now (1) can be re-expressed as

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A} \boldsymbol{\vartheta}(t), \quad \mathbf{y}(t) = \Xi \boldsymbol{\vartheta}(t), \quad \Xi = [A_4 \quad A_5 \quad \mathbf{0}_{\nu \times \rho\nu}] \\ \mathbf{x}(t_0) &= \boldsymbol{\xi} \in \mathbb{R}^n, \quad \mathbf{y}(t_0 + \theta) = \boldsymbol{\phi}(\theta), \quad \forall \theta \in [-r, 0] \end{aligned} \quad (10)$$

where

$$\mathbf{A} := \begin{bmatrix} A_1 & A_2 & A_3 \left(\begin{bmatrix} \Gamma_d \\ I_d \end{bmatrix} \otimes I_\nu \right) & A_3 \left(\begin{bmatrix} \mathbf{E}_d \\ \mathbf{0}_{d \times \delta} \end{bmatrix} \otimes I_\nu \right) \end{bmatrix} \quad (11)$$

$$\boldsymbol{\vartheta}(t) := \mathbf{Col} \left(\begin{bmatrix} \mathbf{x}(t) \\ \mathbf{y}(t-r) \end{bmatrix}, \begin{bmatrix} \int_{-r}^0 F_d(\tau) \mathbf{y}(t+\tau) d\tau \\ \int_{-r}^0 E_d(\tau) \mathbf{y}(t+\tau) d\tau \end{bmatrix} \right) \in \mathbb{R}^\varkappa \quad (12)$$

with $\varkappa := n + \nu + \rho\nu$ and $E_d(\tau) := \mathbf{E}_d^{-1} \boldsymbol{\varepsilon}_d(\tau) \otimes I_\nu$ with $\boldsymbol{\varepsilon}_d(\tau)$ and \mathbf{E}_d are defined in (7). Note that \mathbf{E}_d^{-1} is well defined and we will explain it in Remark 6 in the next section. Note that also the matrix in (11) associated with the distributed delay terms are derived based on the identity

$$\left(\begin{bmatrix} \boldsymbol{\varphi}(\tau) \\ \boldsymbol{f}(\tau) \end{bmatrix} \otimes I_\nu \right) \mathbf{y}(t+\tau) = \left(\begin{bmatrix} \Gamma_d \\ I_d \end{bmatrix} \otimes I_\nu \right) F_d(\tau) \mathbf{y}(t+\tau) + \left(\begin{bmatrix} \mathbf{E}_d \\ \mathbf{0}_{d \times \delta} \end{bmatrix} \otimes I_\nu \right) E_d(\tau) \mathbf{y}(t+\tau)$$

which itself is obtained via the property in (8).

III. MAIN RESULTS ON STABILITY ANALYSIS

In this section, the main results on the stability analysis of (1) are presented in subsection III-B preceded by the presentation of some important Lemmas in Subsection III-A where we present a novel integral inequality and the Liapunov-Krasovskii stability criteria for (1).

A. Mathematical preliminaries

Lemma 2. Given $r > 0$, the trivial solution $\mathbf{x}(t) \equiv \mathbf{0}_n$, $\mathbf{y}(t) \equiv \mathbf{0}_\nu$ of (1) is globally uniformly asymptotically stable if there exist $\epsilon_1; \epsilon_2; \epsilon_3 > 0$ and a differentiable functional $v : \mathbb{R}^n \times \widehat{\mathbf{C}}([-r, 0]; \mathbb{R}^\nu) \rightarrow \mathbb{R}$ such that $v(\mathbf{0}_n, \mathbf{0}_\nu(\cdot)) = 0$ and

$$\epsilon_1 \|\boldsymbol{\xi}\|_2^2 \leq v(\boldsymbol{\xi}, \boldsymbol{\phi}(\cdot)) \leq \epsilon_2 [\max(\|\boldsymbol{\xi}\|_2, \|\boldsymbol{\phi}(\cdot)\|_\infty)]^2 \quad (13)$$

$$\dot{v}(\boldsymbol{\xi}, \boldsymbol{\phi}(\cdot)) = \left. \frac{d^+}{dt} v(\mathbf{x}(t), \mathbf{y}_t(\cdot)) \right|_{t=t_0, \mathbf{x}(t_0)=\boldsymbol{\xi}, \mathbf{y}_{t_0}(\cdot)=\boldsymbol{\phi}(\cdot)} \leq -\epsilon_3 \|\boldsymbol{\xi}\|_2^2 \quad (14)$$

for any $\boldsymbol{\xi} \in \mathbb{R}^n$ and $\boldsymbol{\phi}(\cdot) \in \widehat{\mathbf{C}}([-r, 0]; \mathbb{R}^\nu)$ in (1), where $t_0 \in \mathbb{R}$ and $\frac{d^+}{dx} f(x) = \limsup_{\eta \downarrow 0} \frac{f(x+\eta) - f(x)}{\eta}$. Furthermore, $\mathbf{y}_t(\cdot)$ in (14) is defined by $\forall t \geq t_0, \forall \theta \in [-r, 0), \mathbf{y}_t(\theta) = \mathbf{y}(t + \theta)$ where $\mathbf{x}(t)$ and $\mathbf{y}(t)$ satisfying (1).

Proof: Let $u(\cdot), v(\cdot), w(\cdot)$ in Theorem 3 of [6] be quadratic functions with the multiplier factors $\epsilon_1; \epsilon_2; \epsilon_3 > 0$. Since (1) is a particular case of the general system considered in Theorem 3 of [6], then Lemma 2 is obtained. ■

Now we derive the following new integral inequality which will be employed later in deriving our stability condition. First of all, we define the weighted Lebesgue function space

$$\mathbb{L}_{\varpi}^2(\mathcal{K}; \mathbb{R}^d) := \left\{ \phi(\cdot) \in \mathbb{M}(\mathcal{L}(\mathcal{K})/\mathcal{B}(\mathbb{R}^d)) : \|\phi(\cdot)\|_{2,\varpi} < \infty \right\} \quad (15)$$

with $d \in \mathbb{N}_0$ and $\|\phi(\cdot)\|_{2,\varpi} := \int_{\mathcal{K}} \varpi(\tau) \phi^\top(\tau) \phi(\tau) d\tau$ where $\varpi(\cdot) \in \mathbb{M}(\mathcal{L}(\mathcal{K})/\mathcal{B}(\mathbb{R}_{\geq 0}))$ and the function $\varpi(\cdot)$ has only countably infinite or finite numbers of zero values. Furthermore, $\mathcal{K} \subseteq \mathbb{R}$ and the Lebesgue measure of \mathcal{K} is non-zero.

Lemma 3. Given \mathcal{K} and $\varpi(\cdot)$ in (15) and $U \in \mathbb{S}_{>0}^n := \{X \in \mathbb{S}^n : X \succeq 0\}$ with $n \in \mathbb{N}$. Let $\mathbf{f}(\cdot) := \mathbf{Col}_{i=1}^d f_i(\cdot) \in \mathbb{L}_{\varpi}^2(\mathcal{K}; \mathbb{R}^d)$ and $\mathbf{g}(\cdot) := \mathbf{Col}_{i=1}^{\delta} g_i(\cdot) \in \mathbb{L}_{\varpi}^2(\mathcal{K}; \mathbb{R}^{\delta})$ with $d \in \mathbb{N}$ and $\delta \in \mathbb{N}_0$, in which the functions $\mathbf{f}(\cdot)$ and $\mathbf{g}(\cdot)$ satisfy

$$\int_{\mathcal{K}} \varpi(\tau) \begin{bmatrix} \mathbf{g}(\tau) \\ \mathbf{f}(\tau) \end{bmatrix} [\mathbf{g}^\top(\tau) \quad \mathbf{f}^\top(\tau)] d\tau \succ 0. \quad (16)$$

Then the inequality

$$\int_{\mathcal{K}} \varpi(\tau) \mathbf{x}^\top(\tau) U \mathbf{x}(\tau) d\tau \geq [*] (\mathcal{F}_d \otimes U) \left[\int_{\mathcal{K}} \varpi(\tau) \mathbf{F}(\tau) \mathbf{x}(\tau) d\tau \right] + [*] (\mathcal{E}_d^{-1} \otimes U) \left[\int_{\mathcal{K}} \varpi(\tau) \mathbf{E}(\tau) \mathbf{x}(\tau) d\tau \right] \quad (17)$$

holds for all $\mathbf{x}(\cdot) \in \mathbb{L}_{\varpi}^2(\mathcal{K}; \mathbb{R}^n)$, where $\mathbf{F}(\tau) = \mathbf{f}(\tau) \otimes I_n \in \mathbb{R}^{dn \times n}$ and $\mathcal{F}_d^{-1} = \int_{\mathcal{K}} \varpi(\tau) \mathbf{f}(\tau) \mathbf{f}^\top(\tau) d\tau \in \mathbb{S}_{>0}^d$. In addition, $\mathbf{E}(\tau) = \mathbf{e}(\tau) \otimes I_n \in \mathbb{R}^{\delta n \times n}$ where $\mathbf{e}(\tau) = \mathbf{g}(\tau) - \mathbf{A} \mathbf{f}(\tau) \in \mathbb{R}^{\delta}$ and $\mathbf{A} = \int_{\mathcal{K}} \varpi(\tau) \mathbf{g}(\tau) \mathbf{f}^\top(\tau) d\tau \mathcal{F}_d \in \mathbb{R}^{\delta \times d}$ and $\mathcal{E}_d := \int_{\mathcal{K}} \varpi(\tau) \mathbf{e}(\tau) \mathbf{e}^\top(\tau) d\tau \in \mathbb{S}^{\delta}$.

Proof: The proof of Lemma 3 is inspired by the proofs of Lemma 2 in [13] and the proof of Lemma 5 in [14]. Firstly, we have

$$\mathcal{E}_d := \int_{\mathcal{K}} \varpi(\tau) \mathbf{e}(\tau) \mathbf{e}^\top(\tau) d\tau = [I_{\delta} \quad -\mathbf{A}] \int_{\mathcal{K}} \varpi(\tau) \begin{bmatrix} \mathbf{g}(\tau) \\ \mathbf{f}(\tau) \end{bmatrix} [\mathbf{g}^\top(\tau) \quad \mathbf{f}^\top(\tau)] d\tau [I_{\delta} \quad -\mathbf{A}]^\top \succ 0, \quad (18)$$

where the positive definiteness in (18) can be established based on (16) and the property of congruence transformations with the fact that $\text{rank} [I_{\delta} \quad -\mathbf{A}] = \delta$. Consequently, \mathcal{E}_d^{-1} is well defined.

Let $\mathbf{y}(\tau) := \mathbf{x}(\tau) - \mathbf{F}^\top(\tau) (\mathcal{F}_d \otimes I_n) \int_{\mathcal{K}} \varpi(\theta) \mathbf{F}(\theta) \mathbf{x}(\theta) d\theta - \mathbf{E}^\top(\tau) (\mathcal{E}_d^{-1} \otimes I_n) \int_{\mathcal{K}} \varpi(\theta) \mathbf{E}(\theta) \mathbf{x}(\theta) d\theta$, where the definitions of $\mathbf{F}(\cdot)$ and $\mathbf{E}(\cdot)$ have been given in Lemma 3. By $\mathbf{A} = \int_{\mathcal{K}} \varpi(\tau) \mathbf{g}(\tau) \mathbf{f}^\top(\tau) d\tau \mathcal{F}_d$ and $\mathbf{e}(\tau) = \mathbf{g}(\tau) - \mathbf{A} \mathbf{f}(\tau) \in \mathbb{R}^{\delta}$, we have

$$\begin{aligned} \int_{\mathcal{K}} \varpi(\tau) \mathbf{e}(\tau) \mathbf{f}^\top(\tau) d\tau &= \int_{\mathcal{K}} \varpi(\tau) [\mathbf{g}(\tau) - \mathbf{A} \mathbf{f}(\tau)] \mathbf{f}^\top(\tau) d\tau = \int_{\mathcal{K}} \varpi(\tau) \mathbf{g}(\tau) \mathbf{f}^\top(\tau) d\tau - \mathbf{A} \int_{\mathcal{K}} \varpi(\tau) \mathbf{f}(\tau) \mathbf{f}^\top(\tau) d\tau \\ &= \int_{\mathcal{K}} \varpi(\tau) \mathbf{g}(\tau) \mathbf{f}^\top(\tau) d\tau - \left(\int_{\mathcal{K}} \varpi(\tau) \mathbf{g}(\tau) \mathbf{f}^\top(\tau) d\tau \right) \mathcal{F}_d \mathcal{F}_d^{-1} = \mathbf{O}_{\delta \times d}. \end{aligned} \quad (19)$$

Substituting the expression of $\mathbf{y}(\cdot)$ into $\int_{\mathcal{K}} \varpi(\tau) \mathbf{y}^\top(\tau) U \mathbf{y}(\tau) d\tau$ and considering (19) yields

$$\begin{aligned} \int_{\mathcal{K}} \varpi(\tau) \mathbf{y}^\top(\tau) U \mathbf{y}(\tau) d\tau &= \int_{\mathcal{K}} \varpi(\tau) \mathbf{x}^\top(\tau) U \mathbf{x}(\tau) d\tau - 2 \int_{\mathcal{K}} \varpi(\tau) \mathbf{x}^\top(\tau) U \mathbf{F}^\top(\tau) d\tau (\mathcal{F}_d \otimes I_n) \boldsymbol{\zeta} \\ &+ \boldsymbol{\zeta}^\top \int_{\mathcal{K}} \varpi(\tau) (\mathcal{F}_d \otimes I_n)^\top \mathbf{F}(\tau) U \mathbf{F}^\top(\tau) (\mathcal{F}_d \otimes I_n) d\tau \boldsymbol{\zeta} - 2 \int_{\mathcal{K}} \varpi(\tau) \mathbf{x}^\top(\tau) U \mathbf{E}^\top(\tau) d\tau (\mathcal{E}_d^{-1} \otimes I_n) \boldsymbol{\omega} \\ &+ \boldsymbol{\omega}^\top \int_{\mathcal{K}} \varpi(\tau) (\mathcal{E}_d^{-1} \otimes I_n)^\top \mathbf{E}(\tau) U \mathbf{E}^\top(\tau) (\mathcal{E}_d^{-1} \otimes I_n) d\tau \boldsymbol{\omega} \end{aligned} \quad (20)$$

where $\boldsymbol{\zeta} := \int_{\mathcal{K}} \varpi(\theta) \mathbf{F}(\theta) \mathbf{x}(\theta) d\theta$ and $\boldsymbol{\omega} := \int_{\mathcal{K}} \varpi(\theta) \mathbf{E}(\theta) \mathbf{x}(\theta) d\theta$. Now apply (8) to the term $U \mathbf{F}^\top(\tau)$ and $U \mathbf{E}^\top(\tau)$ and consider $\mathbf{F}(\tau) = \mathbf{f}(\tau) \otimes I_n$ and $\mathbf{E}(\tau) = \mathbf{e}(\tau) \otimes I_n$. Then we have

$$U \mathbf{F}^\top(\tau) = \mathbf{F}^\top(\tau) (I_d \otimes U), \quad U \mathbf{E}^\top(\tau) = \mathbf{E}^\top(\tau) (I_{\delta} \otimes U) \quad (21)$$

given $(X \otimes Y)^\top = X^\top \otimes Y^\top$. Now applying (21) with (8) to the integral terms in (20) related to (21) yields

$$\begin{aligned} \int_{\mathcal{K}} \varpi(\tau) \mathbf{x}^\top(\tau) U \mathbf{F}^\top(\tau) d\tau (\mathcal{F}_d \otimes I_n) \boldsymbol{\zeta} &= \boldsymbol{\zeta}^\top (\mathcal{F}_d \otimes U) \boldsymbol{\zeta} \\ \int_{\mathcal{K}} \varpi(\tau) \mathbf{x}^\top(\tau) U \mathbf{E}^\top(\tau) d\tau (\mathcal{E}_d^{-1} \otimes I_n) \boldsymbol{\omega} &= \boldsymbol{\omega}^\top (\mathcal{E}_d^{-1} \otimes U) \boldsymbol{\omega}, \end{aligned} \quad (22)$$

$$\int_{\mathcal{K}} (\mathcal{F}_d \otimes I_n)^\top \varpi(\tau) \mathbf{F}(\tau) U \mathbf{F}^\top(\tau) (\mathcal{F}_d \otimes I_n) d\tau = (\mathcal{F}_d \otimes I_n) \int_{\mathcal{K}} \varpi(\tau) \mathbf{F}(\tau) \mathbf{F}^\top(\tau) d\tau (\mathcal{F}_d \otimes U) \quad (23)$$

$$\int_{\mathcal{K}} (\mathcal{E}_d^{-1} \otimes I_n)^\top \varpi(\tau) \mathbf{E}(\tau) U \mathbf{E}^\top(\tau) (\mathcal{E}_d^{-1} \otimes I_n) d\tau = (\mathcal{E}_d^{-1} \otimes I_n) \int_{\mathcal{K}} \varpi(\tau) \mathbf{E}(\tau) \mathbf{E}^\top(\tau) d\tau (\mathcal{E}_d^{-1} \otimes U). \quad (24)$$

Meanwhile, since $\mathbf{F}(\tau) = \mathbf{f}(\tau) \otimes I_n$ and $\mathbf{E}(\tau) = \mathbf{e}(\tau) \otimes I_n$, we have

$$\int_{\mathcal{K}} \varpi(\tau) \mathbf{F}(\tau) \mathbf{F}^\top(\tau) d\tau = \left(\int_{\mathcal{K}} \varpi(\tau) \mathbf{f}(\tau) \mathbf{f}^\top(\tau) d\tau \right) \otimes I_n = \mathcal{F}_d^{-1} \otimes I_n \quad (25)$$

$$\int_{\mathcal{K}} \varpi(\tau) \mathbf{E}(\tau) \mathbf{E}^\top(\tau) d\tau = \left(\int_{\mathcal{K}} \varpi(\tau) \mathbf{e}(\tau) \mathbf{e}^\top(\tau) d\tau \right) \otimes I_n = \mathcal{E}_d \otimes I_n. \quad (26)$$

Moreover, (23)–(24) can be reformulated into

$$\begin{aligned} \int_{\mathcal{K}} (\mathcal{F}_d \otimes I_n)^\top \varpi(\tau) \mathbf{F}(\tau) U \mathbf{F}^\top(\tau) (\mathcal{F}_d \otimes I_n) d\tau &= (\mathcal{F}_d \otimes I_n) \int_{\mathcal{K}} \varpi(\tau) \mathbf{F}(\tau) \mathbf{F}^\top(\tau) d\tau (\mathcal{F}_d \otimes U) = \mathcal{F}_d \otimes U, \\ \int_{\mathcal{K}} (\mathcal{E}_d^{-1} \otimes I_n)^\top \varpi(\tau) \mathbf{E}(\tau) U \mathbf{E}^\top(\tau) (\mathcal{E}_d^{-1} \otimes I_n) d\tau &= \\ (\mathcal{E}_d^{-1} \otimes I_n) \int_{\mathcal{K}} \varpi(\tau) \mathbf{E}(\tau) \mathbf{E}^\top(\tau) d\tau (\mathcal{E}_d^{-1} \otimes U) &= \mathcal{E}_d^{-1} \otimes U \end{aligned} \quad (27)$$

by (25) and (26). Now by (27) and (22), (20) can be simplified into

$$\begin{aligned} \int_{\mathcal{K}} \varpi(\tau) \mathbf{y}^\top(\tau) U \mathbf{y}(\tau) d\tau &= \int_{\mathcal{K}} \varpi(\tau) \mathbf{x}^\top(\tau) U \mathbf{x}(\tau) d\tau - [*] (\mathcal{F}_d \otimes U) \left[\int_{\mathcal{K}} \varpi(\tau) \mathbf{F}(\tau) \mathbf{x}(\tau) d\tau \right] \\ &\quad - [*] (\mathcal{E}_d^{-1} \otimes U) \left[\int_{\mathcal{K}} \varpi(\tau) \mathbf{E}(\tau) \mathbf{x}(\tau) d\tau \right]. \end{aligned} \quad (28)$$

Given $U \succeq 0$, one can obtain (17) via (28). ■

Remark 5. By Theorem 7.2.10 in [18], we know (16) indicates that the functions in $\mathbf{f}(\cdot)$ and $\mathbf{g}(\cdot)$ are linearly independent in a Lebesgue sense. Since $\mathbf{f}(\cdot) \in \mathbb{L}_{\varpi}^2(\mathcal{K}; \mathbb{R}^d)$ and $\mathbf{g}(\cdot) \in \mathbb{L}_{\varpi}^2(\mathcal{K}; \mathbb{R}^\delta)$ with plenty options for $\varpi(\cdot) \in \mathbb{L}_f(\mathcal{K}; \mathbb{R}_{\geq 0})$, then it is certain that the structure of the inequality (17) is tremendously general. If $\mathbf{f}(\cdot)$ contains only Legendre polynomials, then the result of Lemma 1 in [13] is covered by (17). With $\delta = 0$, then one can conclude that the inequalities in [19], [20], [21], [22], [14] are generalized by (17) with appropriate $\varpi(\cdot)$, $\mathbf{f}(\cdot)$ and $\mathbf{x}(\cdot)$. Note that we assume the inverse of a 0×0 matrix is still a 0×0 matrix which implies that \mathcal{E}_d^{-1} becomes a 0×0 matrix with $\delta = 0$.

Remark 6. In (17), $\mathbf{f}(\cdot)$ can be interpreted as to approximate $\mathbf{g}(\cdot)$. By letting $\mathbf{f}(\tau) = \mathbf{f}(\tau)$ and $\mathbf{g}(\tau) = \varphi(\tau)$ with $\varpi(\tau) = 1$ in Lemma 3, then we have $\mathcal{E}_d = \mathbf{E}_d$ where \mathbf{E}_d is given in (7). This also indicates that \mathbf{E}_d is invertible. Furthermore, if $\mathbf{f}(\cdot)$ contains only orthogonal functions, then the behavior of \mathcal{E}_d can be quantitatively characterized by the following corollary. Finally, since $\mathbf{f}(\cdot)$ and $\mathbf{g}(\cdot)$ in (17) belong to very general function space and $\varpi(\cdot)$ can be adjusted based on specific requirements, thus the results in Lemma 3 is much more superior both conceptually and mathematically than the result of Lemma 1 in [13].

Corollary 1. *Given all the parameters defined in Lemma 3 and assume that $\mathbf{f}(\cdot) = \mathbf{Col}_{i=1}^d f_i(\cdot)$ where $\{f_i(\cdot)\}_{i=1}^\infty$ contains only orthogonal functions. Then we have that for all $d \in \mathbb{N}_0$*

$$0 \prec \mathcal{E}_{d+1} = \mathcal{E}_d - \left(\int_{\mathcal{K}} \varpi(\tau) f_{d+1}^2(\tau) d\tau \right) \mathbf{a}_{d+1} \mathbf{a}_{d+1}^\top \preceq \mathcal{E}_d \quad (29)$$

where $\mathbf{a}_{d+1} := \left(\int_{\mathcal{K}} \varpi(\tau) \mathbf{g}(\tau) f_{d+1}(\tau) d\tau \right) \left(\int_{\mathcal{K}} \varpi(\tau) f_{d+1}^2(\tau) d\tau \right)^{-1} \in \mathbb{R}^\delta$ and $f_{d+1}(\cdot) \in \mathbb{L}_{\varpi}^2(\mathcal{K}; \mathbb{R})$.

Proof: Note that only the dimension of $\mathbf{f}(\cdot)$ is related to d , whereas δ as the dimension of $\mathbf{g}(\cdot)$ is independent from d . Since $\mathbf{f}(\cdot)$ contains only orthogonal functions, it is obvious that $\mathcal{F}_{d+1} = \mathcal{F}_d \oplus \left(\int_{\mathcal{K}} \varpi(\tau) f_{d+1}^2(\tau) d\tau \right)^{-1}$ (See the Definition 1 in [23]). By using this property considering the definition of $\mathbf{e}(\tau)$ in Lemma 3, it follows that

$$\begin{aligned} \mathbf{e}_{d+1}(\tau) = \mathbf{g}(\tau) - \left(\int_{\mathcal{K}} \varpi(\tau) \mathbf{g}(\tau) [\mathbf{f}^\top(\tau) \quad \mathbf{f}_{d+1}(\tau)] d\tau \right) \left[\mathcal{F}_d \oplus \left(\int_{\mathcal{K}} \varpi(\tau) \mathbf{f}_{d+1}^2(\tau) d\tau \right)^{-1} \right] \begin{bmatrix} \mathbf{f}(\tau) \\ \mathbf{f}_{d+1}(\tau) \end{bmatrix} = \mathbf{g}(\tau) \\ - [\mathbf{A}_d \quad \mathbf{a}_{d+1}] \begin{bmatrix} \mathbf{f}(\tau) \\ \mathbf{f}_{d+1}(\tau) \end{bmatrix} = \mathbf{e}_d(\tau) - \mathbf{f}_{d+1}(\tau) \mathbf{a}_{d+1} \end{aligned} \quad (30)$$

for all $d \in \mathbb{N}_0$, where \mathbf{a}_{d+1} has been defined in (29) and $\mathbf{e}_d(\tau) = \mathbf{g}(\tau) - \mathbf{A}_d \mathbf{f}(\tau)$. Note that the index d is added to the symbols \mathbf{A} and $\mathbf{e}(\tau)$ in Lemma 3 without causing ambiguity. Considering (30) and (18), we have

$$\begin{aligned} 0 \prec \mathcal{E}_{d+1} = \int_{\mathcal{K}} \varpi(\tau) \mathbf{e}_{d+1}(\tau) \mathbf{e}_{d+1}^\top(\tau) d\tau = \mathcal{E}_d - \mathbf{S} \mathbf{y} \left(\mathbf{a}_{d+1} \int_{\mathcal{K}} \varpi(\tau) \mathbf{f}_{d+1}(\tau) \mathbf{e}_d^\top(\tau) d\tau \right) \\ + \left(\int_{\mathcal{K}} \varpi(\tau) \mathbf{f}_{d+1}^2(\tau) d\tau \right) \mathbf{a}_{d+1} \mathbf{a}_{d+1}^\top. \end{aligned} \quad (31)$$

By (19) and the fact that $\int_{\mathcal{K}} \varpi(\tau) \mathbf{f}_{d+1}(\tau) \mathbf{f}(\tau) d\tau = \mathbf{0}_d$ due to the orthogonality among $\{f_i(\cdot)\}_{i=1}^\infty$, we have

$$\begin{aligned} \mathbf{O}_{\delta \times (d+1)} &= \int_{\mathcal{K}} \varpi(\tau) \mathbf{e}_{d+1}(\tau) [\mathbf{f}^\top(\tau) \quad \mathbf{f}_{d+1}(\tau)] d\tau = \int_{\mathcal{K}} \varpi(\tau) (\mathbf{e}_d(\tau) - \mathbf{a}_{d+1} \mathbf{f}_{d+1}(\tau)) [\mathbf{f}^\top(\tau) \quad \mathbf{f}_{d+1}(\tau)] d\tau \\ &= \int_{\mathcal{K}} \varpi(\tau) [\mathbf{e}_d(\tau) \mathbf{f}^\top(\tau) \quad \mathbf{f}_{d+1}(\tau) \mathbf{e}_d(\tau)] d\tau - \mathbf{a}_{d+1} \int_{\mathcal{K}} \varpi(\tau) [\mathbf{f}_{d+1}(\tau) \mathbf{f}^\top(\tau) \quad \mathbf{f}_{d+1}^2(\tau)] d\tau \\ &= [\mathbf{O}_{\delta \times d} \quad \int_{\mathcal{K}} \varpi(\tau) \mathbf{f}_{d+1}(\tau) \mathbf{e}_d(\tau) d\tau] - [\mathbf{O}_{\delta \times d} \quad \int_{\mathcal{K}} \varpi(\tau) \mathbf{f}_{d+1}^2(\tau) d\tau \mathbf{a}_{d+1}] = \mathbf{O}_{\delta \times (d+1)}. \end{aligned} \quad (32)$$

which leads to the conclusion that $\int_{\mathcal{K}} \varpi(\tau) \mathbf{f}_{d+1}(\tau) \mathbf{e}_d(\tau) d\tau = \int_{\mathcal{K}} \varpi(\tau) \mathbf{f}_{d+1}^2(\tau) d\tau \mathbf{a}_{d+1}$. Substituting this equality into (31) yields (29) given $\int_{\mathcal{K}} \varpi(\tau) \mathbf{f}_{d+1}^2(\tau) d\tau > 0$ and $\mathbf{a}_{d+1} \mathbf{a}_{d+1}^\top \succeq 0$. ■

Remark 7. The conclusion of Lemma 1 in [13] is generalized by Corollary 1 as $\mathbf{f}(\cdot)$ can have Legendre polynomials.

B. Stability Analysis

The main result on the stability analysis of (1) is presented in the following theorem.

Theorem 1. *Given (1) with $r > 0$ and $\mathbf{f}(\cdot) \in \mathbf{C}(\mathbb{R}; \mathbb{R}^d)$, $\mathbf{A}_3 \in \mathbb{R}^{n \times \nu}$, $\varphi(\cdot) \in \mathbb{L}^2([-r, 0]; \mathbb{R}^\delta)$ in (3) with $d \in \mathbb{N}$ and $\delta \in \mathbb{N}_0$ and the matrices Γ_d and \mathbf{E}_d in (5)–(7), then the trivial solution of (1) is globally asymptotically stable if there exist $P \in \mathbb{S}^{n+d\nu}$ and $Q; R \in \mathbb{S}^\nu$ such that*

$$\mathbf{P} := P + (\mathbf{O}_n \oplus [\mathbf{F}_d \otimes Q]) \succ 0 \quad (33)$$

$$Q \succ 0, \quad R \succ 0, \quad \Phi \prec 0 \quad (34)$$

where $\mathbf{F}_d^{-1} = \int_{-r}^0 \mathbf{f}(\tau) \mathbf{f}^\top(\tau) d\tau$ and

$$\Phi := \mathbf{S} \mathbf{y} (\Theta_2^\top P \Theta_1) - (\mathbf{O}_n \oplus Q \oplus [\mathbf{F}_d \otimes R] \oplus [\mathbf{E}_d \otimes R]) + \Xi^\top (Q + rR) \Xi \quad (35)$$

$$\Theta_1 := \mathbf{Col} [\mathbf{A}, (H \otimes I_\nu) \Pi], \quad \Theta_2 = [\Upsilon \quad \mathbf{O}_{(n+d\nu) \times \delta\nu}] \quad (36)$$

$$H = [\mathbf{f}(0) \quad -\mathbf{f}(-r) \quad -M \quad \mathbf{O}_{d \times \delta}] \quad (37)$$

$$\Upsilon := \begin{bmatrix} I_n & \mathbf{O}_{n \times \nu} & \mathbf{O}_{n \times d\nu} \\ \mathbf{O}_{d\nu \times n} & \mathbf{O}_{d\nu \times \nu} & I_{d\nu} \end{bmatrix} \quad (38)$$

$$\Pi = \mathbf{Col} [\Xi, [\mathbf{O}_{(\nu+d\nu+\delta\nu) \times n} \quad I_{\nu+d\nu+\delta\nu}]] \quad (39)$$

with \mathbf{A} in (11) and Ξ in (10) and M in (2).

Proof: To analyze the stability of (1), we consider the Krasovskii functional

$$v(\xi, \phi(\cdot)) = \boldsymbol{\eta}^\top P \boldsymbol{\eta} + \int_{-r}^0 \phi^\top(\tau) [Q + (\tau + r)R] \phi(\tau) d\tau \quad (40)$$

to be constructed, where $P \in \mathbb{S}^{n+d\nu}$ and $Q, R \in \mathbb{S}^\nu$ and

$$\boldsymbol{\eta} := \mathbf{Col} \left[\boldsymbol{\xi}, \int_{-r}^0 F_d(\tau) \boldsymbol{\phi}(\tau) d\tau \right] \quad (41)$$

with $F_d(\tau) = \mathbf{f}(\tau) \otimes I_\nu \in \mathbb{R}^{d\nu \times \nu}$. Since (1) can be equivalently denoted by (10) via $\mathbf{f}(\cdot)$, A_3 , $\boldsymbol{\varphi}(\cdot)$, differentiate $v(\mathbf{x}(t), \mathbf{y}_t(\cdot))$ with $\mathbf{x}(t)$, $\mathbf{y}_t(\cdot)$ in Lemma 2 at $t = t_0$ in light of (10), it produces

$$\begin{aligned} \left. \frac{d^+}{dt} v(\mathbf{x}(t), \mathbf{y}_t(\cdot)) \right|_{t=t_0, \mathbf{x}(t_0)=\boldsymbol{\xi}, \mathbf{y}_{t_0}(\cdot)=\boldsymbol{\phi}(\cdot)} &= \boldsymbol{\vartheta}^\top(t_0) \mathbf{S} \mathbf{y} (\Theta_2^\top P \Theta_1) \boldsymbol{\vartheta}(t_0) \\ &\quad - \boldsymbol{\phi}^\top(-r) Q \boldsymbol{\phi}(-r) + \boldsymbol{\phi}^\top(0) (Q + rR) \boldsymbol{\phi}(0) - \int_{-r}^0 \boldsymbol{\phi}^\top(\tau) R \boldsymbol{\phi}(\tau) d\tau \end{aligned} \quad (42)$$

where $\boldsymbol{\vartheta}(t)$ and Θ_1, Θ_2 have been defined in (12) and (36), respectively, and

$$\boldsymbol{\vartheta}(t_0) := \mathbf{Col} \left(\begin{bmatrix} \boldsymbol{\xi} \\ \boldsymbol{\phi}(-r) \end{bmatrix}, \begin{bmatrix} \int_{-r}^0 F_d(\tau) \boldsymbol{\phi}(\tau) d\tau \\ \int_{-r}^0 E_d(\tau) \boldsymbol{\phi}(\tau) d\tau \end{bmatrix} \right) \in \mathbb{R}^z \quad (43)$$

given the initial conditions in (10). Note that the relation

$$\begin{aligned} \int_{-r}^0 F_d(\tau) \dot{\mathbf{y}}(t+\tau) d\tau \Big|_{t=t_0} &= F_d(0) \boldsymbol{\phi}(0) - F_d(-r) \boldsymbol{\phi}(-r) \\ &\quad - [M \otimes I_\nu] \int_{-r}^0 F_d(\tau) \boldsymbol{\phi}(\tau) d\tau = (H \otimes I_\nu) \Pi \boldsymbol{\vartheta}(t_0), \end{aligned} \quad (44)$$

has been applied to derive H in (37), where (44) itself can be derived via (1)–(2) with (8)–(9). To obtain an upper bound for (42), let $R \succ 0$, $\varpi(\tau) = 1$, $\mathbf{f}(\tau) = \mathbf{f}(\cdot)$ and $\mathbf{g}(\tau) = \boldsymbol{\varphi}(\tau)$, then one can derive

$$\int_{-r}^0 \boldsymbol{\phi}^\top(\tau) R \boldsymbol{\phi}(\tau) d\tau \geq [*] (F_d \otimes R) \int_{-r}^0 F_d(\tau) \boldsymbol{\phi}(\tau) d\tau + [*] (E_d \otimes R) \int_{-r}^0 E_d(\tau) \boldsymbol{\phi}(\tau) d\tau \quad (45)$$

via (17) given (4) and the forms in (5)–(7), where $F_d(\tau)$ and $E_d(\tau)$ are given in (12). Now applying (45) with $R \succ 0$ to (42) yields

$$\left. \frac{d^+}{dt} v(\mathbf{x}(t), \mathbf{y}_t(\cdot)) \right|_{t=t_0, \mathbf{x}(t_0)=\boldsymbol{\xi}, \mathbf{y}_{t_0}(\cdot)=\boldsymbol{\phi}(\cdot)} \leq \boldsymbol{\vartheta}^\top(t_0) \boldsymbol{\Phi} \boldsymbol{\vartheta}(t_0) \quad (46)$$

for all $t_0 \in \mathbb{R}$ where $\boldsymbol{\Phi}$ is given in (35). It is obvious that there exists $\epsilon_3 > 0$ such that

$$\left. \frac{d^+}{dt} v(\mathbf{x}(t), \mathbf{y}_t(\cdot)) \right|_{t=t_0, \mathbf{x}(t_0)=\boldsymbol{\xi}, \mathbf{y}_{t_0}(\cdot)=\boldsymbol{\phi}(\cdot)} \leq -\epsilon_3 \|\boldsymbol{\xi}\|_2 \quad (47)$$

if $\boldsymbol{\Phi} \prec 0$. Thus it follows that if $\boldsymbol{\Phi} \prec 0$ and $R \succ 0$ are satisfied then (40) satisfies (14). Hence the feasible solutions of (34) infer the existence of (40) satisfying (14).

Now we start to prove that (33) with $Q \succ 0$ and $R \succ 0$ in (34) infers that (40) satisfies (13). First of all, given the structure of (40), it follows that $\exists \lambda; \eta > 0$:

$$\begin{aligned} v(\boldsymbol{\xi}, \boldsymbol{\phi}(\cdot)) &\leq \left[\int_{-r}^0 F(\tau) \boldsymbol{\phi}(\tau) d\tau \right]^\top \lambda \left[\int_{-r}^0 F(\tau) \boldsymbol{\phi}(\tau) d\tau \right] + \int_{-r}^0 \boldsymbol{\phi}^\top(\tau) \lambda \boldsymbol{\phi}(\tau) d\tau \\ &\leq \lambda \|\boldsymbol{\xi}\|_2^2 + \int_{-r}^0 \boldsymbol{\phi}^\top(\tau) F^\top(\tau) d\tau \lambda \int_{-r}^0 F(\tau) \boldsymbol{\phi}(\tau) d\tau + \lambda r \|\boldsymbol{\phi}(\cdot)\|_\infty^2 \\ &\leq \lambda \|\boldsymbol{\xi}\|_2^2 + \lambda r \|\boldsymbol{\phi}(\cdot)\|_\infty^2 + \left[\int_{-r}^0 F(\tau) \boldsymbol{\phi}(\tau) d\tau \right]^\top (\eta F \otimes I_n) \left[\int_{-r}^0 F(\tau) \boldsymbol{\phi}(\tau) d\tau \right] \\ &\leq \lambda \|\boldsymbol{\xi}\|_2^2 + \lambda r \|\boldsymbol{\phi}(\cdot)\|_\infty^2 + \int_{-r}^0 \boldsymbol{\phi}^\top(\tau) \eta \boldsymbol{\phi}(\tau) d\tau \leq \lambda \|\boldsymbol{\xi}\|_2^2 + \\ &\quad (\lambda r + \eta r) \|\boldsymbol{\phi}(\cdot)\|_\infty^2 \leq (\lambda + \lambda r + \eta r) (\|\boldsymbol{\xi}\|_2^2 + \|\boldsymbol{\phi}(\cdot)\|_\infty^2) \end{aligned}$$

$$\leq 2(\lambda + \lambda r + \eta r) [\max(\|\boldsymbol{\xi}\|_2, \|\boldsymbol{\phi}(\cdot)\|_\infty)]^2 \quad (48)$$

which is derived via the property of quadratic forms: $\forall X \in \mathbb{S}^n, \exists \lambda > 0 : \forall \mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}, \mathbf{x}^\top (\lambda I_n - X) \mathbf{x} > 0$ with (17) with $\mathbf{f}(\tau) = \mathbf{f}(\tau)$ and $\delta = 0$. This shows that (40) satisfies the upper bound property in (13).

Now to construct a lower bound for $v(\boldsymbol{\xi}, \boldsymbol{\phi}(\cdot))$, apply (17) to (40) with $\varpi(\tau) = 1, \delta = 0$ and $\mathbf{f}(\tau) = \mathbf{f}(\tau)$ matching the term $F_d(\tau)$ in (40), which gives

$$\int_{-r}^0 \boldsymbol{\phi}^\top(\tau) Q \boldsymbol{\phi}(\tau) d\tau \geq [*] (F_d \otimes Q) \int_{-r}^0 F_d(\tau) \boldsymbol{\phi}(\tau) d\tau. \quad (49)$$

Considering the conclusion in (48) and by further applying (49) to (40) with $Q \succ 0$ and $R \succ 0$ in (34), it yields that (13) is satisfied if (33) and $Q \succ 0$ and $R \succ 0$ hold. This shows that feasible solutions of (33)–(34) infers the existence of (40) satisfying all the conditions in (13) and (14). ■

Remark 8. If one wants to increase the dimension of $\mathbf{f}(\cdot)$ in (40), then extra zeros need to be added to A_3 in (3) to ensure compatibility. Note that there is no upper bound imposed on the dimension of $\mathbf{f}(\cdot)$. Furthermore, if (1) is a retarded system, then the functional in (40) generalizes the Krasovskii functional in [13] where $\mathbf{f}(\cdot)$ contains only Legendre polynomials. Finally, the use of $\mathbf{f}(\cdot)$ in (2) to approximate $\varphi(\cdot)$ allows one to construct functional with more sophisticated kernels in (40) other than only the option of polynomials for (40).

Remark 9. To apply (17) at (45) with $\varpi(\tau) = 1$ and $\mathbf{f}(\tau) = \mathbf{f}(\tau)$ and $\mathbf{g}(\tau) = \varphi(\tau)$, the linear independent condition (4) must be satisfied which guarantees an invertible E_d . Moreover, the absence of $\varphi(\cdot)$ in (40) enables us to avoid dealing with $\dot{\varphi}(\cdot)$. These are the major motivations for using the structure of the decomposition in (3) to handle distributed delay terms in this paper.

Remark 10. If $\mathbf{f}(\cdot)$ in (3) and (40) contains only Legendre polynomials, then Theorem 1 with the approximation scheme in (6) cover the stability results in Theorem 4 of [13]. Note that one can apply congruence transformations to $\boldsymbol{\Phi}$ to avoid potential numerical problems caused by the position of the error matrix E_d in (34) if some of its eigenvalues are too small. Namely, the matrix inequality

$$\widehat{\boldsymbol{\Phi}} = \left[I_{n+\nu+\varrho} \oplus \left(\eta E_d^{-\frac{1}{2}} \otimes I_\nu \right) \right]^\top \boldsymbol{\Phi} \left[I_{n+\nu+\varrho} \oplus \left(\eta E_d^{-\frac{1}{2}} \otimes I_\nu \right) \right] \prec 0 \quad (50)$$

is equivalent to (34), where $\eta \in \mathbb{R}$ and the block-diagonals of $\widehat{\boldsymbol{\Phi}}$ are no longer related to E_d .

The following theorem shows that the LMI conditions in Theorem 1 exhibit a hierarchy with respect to d when $\mathbf{f}(\cdot)$ contains only orthogonal functions.

Theorem 2. Let $\mathbf{f}(\cdot) := \mathbf{Col}_{i=1}^d f_i(\cdot) \in \mathbf{C}(\mathbb{R}; \mathbb{R}^d)$ in (2) where $\{f_i(\cdot)\}_{i=1}^\infty$ contains only functions which are mutually orthogonal over $[-r, 0]$. (See the Definition in [23]). Given Γ_d, E_d (5)–(7), we have

$$\forall d \in \mathbb{N}_0, \quad \mathcal{F}_d \subseteq \mathcal{F}_{d+1} \quad (51)$$

where $\mathcal{F}_d := \{r > 0 \mid (33)–(34) \text{ hold } \& \mathcal{P} \in \mathbb{S}^{n+d\nu}, Q; R \in \mathbb{S}^\nu\}$.

Proof: The proof here is inspired by the proof of Theorem 8 in [13]. Note that $\mathbf{P} \succ 0$ in (33) and $\boldsymbol{\Phi} \succ 0$ in (34) are indexed by the value of d . Hence d might be automatically attached to the related variables throughout the entire proof. Let $r \in \mathcal{F}_d$ with $\mathcal{F}_d \neq \emptyset$ which infers that there exist feasible solutions for (33) and (34) at d . Moreover, let $P_d \in \mathbb{S}^{n+d\nu}$ and $Q; R \in \mathbb{S}^\nu$ to be a feasible solution of (33)–(34). We will show that the feasible solutions of (33)–(34) exist at $d+1$ if the feasible solutions of (33) and (34) at d exist.

Assume that

$$P_{d+1} := P_d \oplus O_\nu, \quad \boldsymbol{\eta}_{d+1}(t) := \left[\int_{-r}^0 f_{d+1}(\tau) \mathbf{y}(t + \tau) d\tau \right]. \quad (52)$$

with reference to the structures in (40). Since $\{f_i(\cdot)\}_{i=1}^\infty$ contains functions which are orthogonal over $[-r, 0]$, we have $F_{d+1} = F_d \oplus \widehat{f}_{d+1}$ with $\widehat{f}_{d+1}^{-1} = \int_{-r}^0 f_{d+1}^2(\tau) d\tau$. Now by (52) and (9) with $F_{d+1} = F_d \oplus \widehat{f}_{d+1}$, it follows that $\mathbf{P}_{d+1} = \mathbf{P}_d \oplus \widehat{f}_{d+1} Q$ with $Q \succ 0$ and $\widehat{f}_{d+1} > 0$. Thus the feasible solutions of $\mathbf{P}_d \succ 0$ infer the existence of the feasible solutions of $\mathbf{P}_{d+1} \succ 0$.

Now we start to prove the hierarchical results for $\Phi_d \prec 0$ in (34) with respect to d . Note that $\Phi_d \prec 0$ can be written as

$$\Phi_d = \begin{bmatrix} \dot{\Phi}_d & T_d(\mathbf{E}_d \otimes I_\nu) \\ * & -\mathbf{E}_d \otimes R \end{bmatrix} \prec 0 \quad (53)$$

where $\dot{\Phi}_d$ can be easily obtained based on the structure of $\Phi_{d'}$ and

$$T_d = \Upsilon_d^\top P_d \begin{bmatrix} I_n \\ \mathbf{O}_{(\nu+d\nu) \times n} \end{bmatrix} A_3 \left(\begin{bmatrix} I_\delta \\ \mathbf{O}_{d \times \delta} \end{bmatrix} \otimes I_\nu \right) \quad (54)$$

with $\Upsilon_d = \Upsilon$ given in (38) at d . Now apply Schur complement to (53) considering the fact that $\mathbf{E}_d \succ 0$ and $R \succ 0$. Then it yields

$$\begin{aligned} \dot{\Phi}_d + T_d \left[\left(\mathbf{E}_{d+1} + \widehat{f}_{d+1}^{-1} \gamma_{d+1} \gamma_{d+1}^\top \right) \otimes R^{-1} \right] T_d^\top = \\ \dot{\Phi}_d + T_d[*] \begin{bmatrix} \widehat{f}_{d+1}^{-1} R^{-1} & \mathbf{O}_{\nu \times \nu \delta} \\ * & \mathbf{E}_{d+1}^{-1} \otimes R^{-1} \end{bmatrix} \begin{bmatrix} \gamma_{d+1}^\top \otimes I_\nu \\ \mathbf{E}_{d+1} \otimes I_\nu \end{bmatrix} T_d^\top \prec 0 \end{aligned} \quad (55)$$

based on (29) and (8), where $\gamma_{d+1} = \int_{-r}^0 \varphi(\tau) f_{d+1}(\tau) d\tau \widehat{f}_{d+1}$. Apply Schur complement again to (55) yields that given $R \succ 0$, (55) holds if and only if

$$\Theta = \begin{bmatrix} \dot{\Phi}_d & T_d(\gamma_{d+1} \otimes I_\nu) & T_d(\mathbf{E}_{d+1} \otimes I_\nu) \\ * & -\widehat{f}_{d+1} R & \mathbf{O}_{\nu \times \nu \delta} \\ * & * & -\mathbf{E}_{d+1} \otimes R \end{bmatrix} \prec 0. \quad (56)$$

Now the fact is

$$\Phi_{d+1} = \Theta = \begin{bmatrix} \dot{\Phi}_{d+1} & T_{d+1}(\mathbf{E}_{d+1} \otimes I_\nu) \\ * & -\mathbf{E}_{d+1} \otimes R \end{bmatrix} \prec 0 \quad (57)$$

given the structure of Φ_d at d in (53) and the relations

$$\begin{aligned} [A_3 \quad \mathbf{O}_{n \times \nu}] \left(\begin{bmatrix} \Gamma_{d+1} \\ I_{d+1} \end{bmatrix} \otimes I_\nu \right) &= [A_3 \quad \mathbf{O}_{n \times \nu}] \left(\begin{bmatrix} \Gamma_{d+1} \otimes I_\nu & \gamma_{d+1} \otimes I_\nu \\ I_{d\nu} & \mathbf{O}_{d\nu \times \nu} \\ \mathbf{O}_{\nu \times d\nu} & I_\nu \end{bmatrix} \right) \\ &= \left(A_3 \begin{bmatrix} \Gamma_d \otimes I_\nu \\ I_{d\nu} \end{bmatrix} \quad A_3 \begin{bmatrix} \gamma_{d+1} \otimes I_\nu \\ \mathbf{O}_{d\nu \times \nu} \end{bmatrix} \right) = \left(A_3 \begin{bmatrix} \Gamma_d \otimes I_\nu \\ I_{d\nu} \end{bmatrix} \quad A_3 \begin{bmatrix} I_{\delta\nu} \\ \mathbf{O}_{d\nu \times \delta\nu} \end{bmatrix} \right) (\gamma_{d+1} \otimes I_\nu), \end{aligned} \quad (58)$$

$$T_{d+1} = (\Upsilon_d \oplus I_\nu)^\top (P_d \oplus \mathbf{O}_\nu) \begin{bmatrix} I_n \\ \mathbf{O}_{(2\nu+d\nu) \times n} \end{bmatrix} [A_3 \quad \mathbf{O}_{n \times \nu}] \left(\begin{bmatrix} I_\delta \\ \mathbf{O}_{(d+1) \times \delta} \end{bmatrix} \otimes I_\nu \right) = \begin{bmatrix} T_d \\ \mathbf{O}_{\nu \times \delta\nu} \end{bmatrix} \quad (59)$$

in view of (54), and the coefficient $[A_5 \quad \mathbf{O}_{n \times \nu}]$ is obtained based on the relation

$$\widetilde{A}_3(\tau) = [A_3 \quad \mathbf{O}_{n \times \nu}] \left(\begin{bmatrix} \varphi^\top(\tau) & \mathbf{f}^\top(\tau) & f_{d+1}(\tau) \end{bmatrix} \otimes I_\nu \right). \quad (60)$$

Hence we have proved that given $Q \succ 0$ and $R \succ 0$ in (34), the feasible solutions of $\Phi_d \prec 0$ infers the existence of the feasible solutions of $\Phi_{d+1} \prec 0$. \blacksquare

Remark 11. Theorem 2 generalizes the result of Theorem 8 in [13] as the latter one only considers $\mathbf{f}(\cdot)$ to contain Legendre polynomials.

Parameters	Example 1	Example 2
A_1	0.315	$\begin{bmatrix} -7 & 0 \\ -0.1 & 0.1 \end{bmatrix}$
A_2	0	O_2
$\tilde{A}_3(\tau)$	$5 \sin(\cos(12\tau))$	$-\begin{bmatrix} 4 \sin(\sin(12\tau)) & 0.02e^{\sin(12\tau)} \\ 0.01e^{\cos(12\tau)} & 10 \sin(\cos(12\tau)) \end{bmatrix}$
A_4	1	I_2
A_5	0.1	$\begin{bmatrix} -0.1 & -0.1 \\ 0.12 & 0.11 \end{bmatrix}$

TABLE I
PARAMETERS OF NUMERICAL EXAMPLES FOR (1)

IV. NUMERICAL SIMULATION

In this section, two numerical examples are presented to show the effectiveness of our proposed method. Numerical calculations are conducted in Matlab environment with Yalmip [24] and SDPT3 [25].

Consider the parameters in the following table for the CDDS system in (1). To the best of our knowledge, no existing methods may analyze the stability of (1) with the parameters in Example 1 and 2. This is even true for the frequency domain approaches in [10], [11] due to the CDDS structures and non-trivial distributed delay terms in Table I.

In order to demonstrate the impact of having different $f(\cdot)$ for Theorem 1, we consider the following functions: $\ell_d(\tau) = \mathbf{Col}_{i=0}^d \ell_i(\tau) \in \mathbb{R}^{d+1}$ and $\mathbf{h}_d(\tau) = \mathbf{Col} \left[1, \mathbf{Col}_{i=1}^{d/2} \sin 12i\tau, \mathbf{Col}_{i=1}^{d/2} \cos 12i\tau \right] \in \mathbb{R}^{d+1}$, where $\ell_d(\tau)$ contains Legendre polynomials $\ell_i(\tau)$

$$\ell_d(\tau) := \sum_{k=0}^d \binom{d}{k} \binom{d+k}{k} \left(\frac{\tau}{r}\right)^k \quad (61)$$

which corresponds to $F_d = r^{-1} \bigoplus_{i=0}^d 2i+1$ and the corresponding M in (2) for $\mathbf{f}(\tau) = \ell_d(\tau)$ can be easily determined. Note that $\mathbf{h}_d(\tau)$ corresponds to $M = 0 \oplus \begin{bmatrix} O_{d/2} & \bigoplus_{i=1}^{d/2} 12i \\ -\bigoplus_{i=1}^{d/2} 12i & O_{d/2} \end{bmatrix}$ in (2) for $\mathbf{f}(\tau) = \mathbf{h}_d(\tau)$, and d must be a positive even number for $\mathbf{h}_d(\tau)$. Note that also the functions in $\mathbf{h}_d(\tau)$ are not orthogonal over $[-r, 0]$.

Now apply Theorem 1 with (50) and $\eta = 1$ and a testing vector $r = (500 : 2500)/1000$ to (1) with the parameters in Table I and

$$A_3 = [-5 \quad \mathbf{0}_{d+1}^\top] \quad (62)$$

$$A_3 = \begin{bmatrix} 0 & 0.02 & 0 & 0 & -4 & 0 & 0 & 0 & O_{2 \times (2d+2)} \\ 0 & 0 & 0.01 & 0 & 0 & 0 & 0 & -10 & \end{bmatrix}, \quad (63)$$

respectively, where (62) corresponds to both $\mathbf{f}(\tau) = \mathbf{h}_d(\tau)$ and $\mathbf{f}(\tau) = \ell_d(\tau)$ for Example 1 with $\varphi(\tau) = \sin(\cos(12\tau))$, and (63) corresponds to both $\mathbf{f}(\tau) = \mathbf{h}_d(\tau)$ and $\mathbf{f}(\tau) = \ell_d(\tau)$ with

$$\varphi(\tau) = [e^{\sin(12\tau)} \quad e^{\cos(12\tau)} \quad \sin(\sin(12\tau)) \quad \sin(\cos(12\tau))]^\top$$

for Example 2, respectively. The computing results of detectable delay margins are summarized in Table II–V where NoV denotes the number of decision variables required by each optimization program for a pointwise value of r .

$f(\tau)$	First interval	Second interval	NoV
$h_d(\tau), d = 6$	[0.599, 0.71]	[1.123, 1.233]	38
$\bar{h}_d(\tau), d = 10$	[0.599, 0.71]	[1.122, 1.234]	80
$\ell_d(\tau), d = 10$	[0.599, 0.71]	[1.126, 1.237]	80
$\bar{\ell}_d(\tau), d = 23$	[0.599, 0.71]	[1.122, 1.234]	327

TABLE II
DETECTABLE STABLE DELAY MARGINS OF EXAMPLE 1

$f(\tau)$	Third Interval	Fourth Interval	NoV
$h_d(\tau), d = 6$	[1.647, 1.757]	[2.171, 2.28]	38
$\bar{h}_d(\tau), d = 10$	[1.646, 1.758]	[2.168, 2.281]	80
$\ell_d(\tau), d = 10$	not detectable	not detectable	80
$\bar{\ell}_d(\tau), d = 23$	[1.661, 1.739]	not detectable	327

TABLE III
DETECTABLE STABLE DELAY MARGINS OF EXAMPLE 1

$f(\tau)$	First interval	Second interval	NoV
$h_d(\tau), d = 6$	[0.535, 0.774]	[1.059, 1.297]	142
$\bar{h}_d(\tau), d = 10$	[0.535, 0.774]	[1.059, 1.297]	306
$\ell_d(\tau), d = 10$	[0.535, 0.773]	[1.062, 1.287]	306
$\bar{\ell}_d(\tau), d = 24$	[0.535, 0.774]	[1.058, 1.298]	1384

TABLE IV
DETECTABLE STABLE DELAY MARGINS OF EXAMPLE 2

$f(\tau)$	Third Interval	Fourth Interval	Fifth Interval	NoV
$h_d(\tau), d = 6$	[1.583, 1.82]	[2.107, 2.168]	[2.283, 2.344]	142
$\bar{h}_d(\tau), d = 10$	[1.582, 1.821]	[2.106, 2.168]	[2.283, 2.344]	306
$\ell_d(\tau), d = 10$	not detectable	not detectable	not detectable	306
$\ell_d(\tau), d = 24$	$[1.59, 1.69] \cup [1.717, 1.809]$	not detectable	not detectable	1384

TABLE V
DETECTABLE STABLE DELAY MARGINS OF EXAMPLE 2

Remark 12. Note that when one applies Theorem 1, the numerical results produced by $f(\tau) = h_d(\tau)$ and $f(\tau) = \ell_d(\tau)$ are not mutually exclusive. Namely, users can choose the best results considering both side (and potentially even more if one wants to use more options for $f(\tau)$) as the final conclusion produced by Theorem 1. This clearly shows one of the advantages of the proposed method in this paper.

For Example 1 and 2 in Table I, we did not run our program with higher $d > 24$ for $f(\tau) = \ell_d(\tau)$ via the testing vector $r = (500 : 2500)/1000$. This is because the expected computing time becomes too long due to the calculation of Γ_d, E_d in (5)–(7) via the function `vpaintegral` with variable precision in Matlab. On the other hand, the numerical integration function `integral` in Matlab is not an alternative option in this case to calculate Γ_d and E_d because of its limited numerical accuracy. The results in Tables II–V can be explained by the fact that the functions in $\varphi(\tau)$ over $[-r, 0]$ is not “easy” to be approximated by polynomials if r becomes relatively large. Thus the numerical results in this section give examples demonstrating the advantage of our proposed approximation scheme over the existing polynomials only approximation approach [13].

V. CONCLUSION

In this paper, a new method for the stability analysis of a linear CDDS (1) with a general distributed delay has been proposed. The kernel of $\tilde{A}_3(\cdot)$ can be any L^2 function over the delay interval, which

has been handled by the decomposition (3) with a general form of approximation (5) using a class of elementary functions. The structure of the proposed approximation in (5)–(7) is based on the idea of least square approximation in Hilbert Space. Thanks to the newly proposed inequality, the approximation error has been included in the proposed stability condition. It is also proved that the stability condition possesses a hierarchy when (6) is utilized with a $f(\cdot)$ containing functions which are mutually orthogonal. Two numerical examples have been presented which have shown the strength and effectiveness of our proposed methodology.

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