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To cite this version:
Claire Paya, Euriell Le Corronc, Yannick Pencolé. Detection of time shift failures in (max, +)-linear systems with time intervals for assembly line. 2019 IEEE International Conference on Systems, Man and Cybernetics, Oct 2019, Bari, Italy. pp.1801-1806. hal-02376093

HAL Id: hal-02376093
https://laas.hal.science/hal-02376093
Submitted on 22 Nov 2019

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Detection of time shift failures in (max, +)-linear systems with time intervals for assembly line

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Abstract—In this paper, we address the problem of failure detection in assembly lines modeled as Timed Event Graphs (TEG). The proposed method represents TEGs as (max,+)-linear systems with time intervals and aims at detecting time shift failures in the underlying assembly lines. To do so, we propose the definition of a set of indicators relying on the residuation theory on (max,+)-linear systems that handle certain and uncertain observable outputs.

Index Terms— (max,+)-linear system, assembly line, fault detection, discrete event system, timed event graph

I. INTRODUCTION

In industrial systems such as assembly lines, fault diagnosis is usually automatized by using Discrete Event System (DES). Industry also requires that failures are rapidly identified, to avoid systems unavailability for too long. The various system failures can be loss of event information, loss of information. Among those failures, timing issues can be a problem for instance, an assembly line that slows down will put out fewer pieces. STMicroelectronics is a company that develops, produces and commercializes microchips for electronic system. The semiconductor manufacturing process is extremely complex and constantly innovating. One of the challenges is the monitoring of time drifts for the supervision of an instrumented production chain. This means detecting as soon as possible the time differences between the production plan and the so-called Work In Progress (WIP for short) in order to be able to correct the WIP quickly without having too much delay in the delivery of the product.

This problem is a subclass of the problems called failure diagnosis in timed discrete event systems. One of the first diagnostic methods is the method extracted from [SSL79] which is applied to timed automata [TRI02]. This diagnostic method refines decisions on the diagnosis by taking into account dated observations. In [GTY09] the diagnostic method uses time Petri net that models competition and/or system parallelism. Among the classes of Petri Nets, Timed Event Graph is a good candidate to represent assembly lines. TEGs are one of the subclasses of Petri nets where places are associated with a punctual duration. Recently, in the article [SLCP17] the method uses (max,+) algebra to model TEGs in a linear state representation and proposes a method to perform failure diagnosis. This method performs diagnosis on a fixed time (max,+) linear system. In this article, we extend the diagnostic method of the article [SLCP17] by dealing with time intervals in TEGs and by still using a representation with (max,+) algebra. We first propose a method for detecting time lags in system with certain outputs (i.e. we know exactly what are the output events of the system) that is secondly extended to deal with a method for detecting time lags in system with uncertain outputs (i.e. the occurrence date of the output events is within a given time interval).

The paper is organized as follows. Section II presents a motivation example inspired from the microchip industry. Section III summarizes the necessary mathematical background about (max,+)-linear systems. Section IV introduces the acceptable outputs in (max,+)-linear system with time intervals. Section V then defines detection in (max,+)-linear systems with time intervals with indicator for certain outputs and indicator for uncertain outputs.

II. MOTIVATION EXAMPLE

In the STMicroelectronics plant, several products are produced at the same time. For the production of products there is a several different production plans for the same product there are variations in production time. WIP may change during manufacture depending on equipment availability. These changes may cause delays depending on the delivery date of the product, the WIP have to be corrected. The purpose of the proposed method is to detect when the delay becomes significant and that it is absolutely necessary to make a correction of the WIP in order not to have a delay on the delivery.

Example 1. Figure 1 shows a part of such a plant. It is an assembly line composed of equipments. It is represented as a Time Event Graph where places hold time intervals. Equipment 1 and 2 do the same treatment but not will the same treatment time. The treatment on Equipment 3 need a sufficient number of wafers so you have a synchronization between equipments to do the treatment on Equipment 3. Input $u_1$ is a flow of timed events corresponding to the arrivals of the wafers on Equipment 1, for the place $p_1$ the time corresponds to the arrival time of the wafers in front of the equipment. If the equipment is ready, this corresponds to a token in the place...
the process duration is between 2 and 3 hours. Input \( u_2 \) is a flow of timed events corresponding to the arrivals of the wafers on Equipment 2. For the place \( p_2 \) the time corresponds to the arrival time of the wafers in front of the equipment. If the equipment is ready, this corresponds to a token in the place \( p_0 \), the processing of Equipment 2 is carried out on place \( p_5 \), and the process duration is between 1 and 3 hours. The processing of Equipment 3 corresponds to place \( p_{10} \) and the process lasts exactly 2 hours. This operation can only be processed if there is a sufficient number of wafers coming from Equipments 1 and 2. The time on places \( p_8 \) and \( p_7 \) corresponds to the arrival times where process lasts 3 hours for \( p_8 \) and 5 hours for \( p_7 \). Now suppose that on the input \( u_1 \) we have a wafer at \( t=0 \) then a wafer at \( t=1 \) then a wafer at \( t=2 \) and finally a wafer at \( t=3 \). Suppose now that it is the same on input \( u_2 \). Imagine now that final products are available on the output \( y \) respectively at time 12, 15, 18 and 23, is the schedule respected according the TEG of Figure 1 or is there a time drift within the assembly line.

![Fig. 1. Representation of assembly line as a Time Event Graph with time intervals](image)

Time Event Graphs, as the one presented in Figure 1, can be formally defined as \((\text{max},+)\) linear systems that are introduced in the next section. This formalization will be used to the definition of the indicators of time shift failures as the one of Example 1.

### III. Scientific Background

This section recalls the mathematical background that will be used in this paper for describing \((\text{max},+)\) linear systems with time intervals. For this section, interested reader is invited to peruse [BCOQ92], [Max91], [BHMR] and [LHCJ04].

#### A. Dioid theory

**Definition 1.** A dioid \( \mathcal{D} \) is a set composed of two internal operations \( \oplus \) and \( \circ \). The addition \( \oplus \) is associative, commutative, idempotent (i.e. \( \forall a \in \mathcal{D}, a \oplus a = a \)) and has a neutral element denoted \( \varepsilon \). The multiplication \( \circ \) is associative and distributive, neutral on the right and the left over the addition \( \oplus \) and has a neutral element denoted \( e \). When there is no ambiguity, the symbol \( \oplus \) is omitted.

**Definition 2.** A dioid is complete if it is closed for infinite sums and if the multiplication \( \circ \) is distributive over infinite sums.

**Example 2.** The dioid \( \mathbb{R}_{\text{max}} = (\mathbb{R} \cup -\infty) \) with the max operation for the law \( \oplus \) and the addition for the law \( \circ \) and with \( \varepsilon = -\infty \) and \( e = 0 \). \( \mathbb{R}_{\text{max}} \) is not complete because \( +\infty \) does not belong to the set \( \mathbb{R}_{\text{max}} \) so the infinite sum is not set to \( +\infty \). By adding \( +\infty \) to the dioid \( \mathbb{R}_{\text{max}} \), we get the complete dioid \( \mathbb{R}_{\text{max}} \).

**Theorem 1.** Let \( \mathcal{D} \) be a complete dioid, \( x = a^*b \) is a solution of the equation \( x = ax \oplus b \), where \( a^* = \bigoplus_{i \geq 0} a^i \) the Kleene star operator with \( a^0 = e \).

**Definition 3.** For a dioid \( \mathcal{D} \), \( a \preceq b \) denotes the order relation such that \( \forall a, b \in \mathcal{D}, a \preceq b \iff a \circ b = b \).

**Example 3.** The complete dioid \( \mathbb{B}[\gamma, \delta] \) is the set of formal series with two commutative variables \( \gamma \) and \( \delta \) with boolean coefficients in \( \{\varepsilon, e\} \) and exponents in \( \mathbb{Z} \). A series \( s \in \mathbb{B}[\gamma, \delta] \) is written \( s = \bigoplus_{n,t \in \mathbb{Z}} s(n, t)\gamma^n\delta^t \) where \( s(n, t) = e \) or \( \varepsilon \). \( \varepsilon = \bigoplus_{n,t \in \mathbb{Z}} \gamma^n\delta^t \) and \( e = \gamma^0\delta^0 \) are the neutral elements.

Graphically, a series of \( \mathbb{B}[\gamma, \delta] \) is described by a collection of point of coordinates \( (n, t) \) in \( \mathbb{Z}^2 \) with \( \gamma \) as horizontal axis and \( \delta \) as vertical axis. For instance, Figure 2 shows a couple of series \( p = \gamma^3\delta^0 \oplus \gamma^1\delta^1 \oplus \gamma^5\delta^2 \) and \( p = \gamma^0\delta^3 \oplus \gamma^2\delta^3 \oplus \gamma^4\delta^5 \).

**Example 4.** The complete dioid \( \mathbb{M}^{\text{in}}_{\text{max}}[\gamma, \delta] \) is the quotient dioid \( \mathbb{B}[\gamma, \delta] \) by the modulo \( \gamma^*(\delta^{-1})^* \). The complete dioid \( \mathbb{M}^{\text{in}}_{\text{max}}[\gamma, \delta] \) with \( \forall a, b \in \mathbb{M}^{\text{in}}_{\text{max}}[\gamma, \delta] \): \( ab \leftrightarrow a\gamma^*(\delta^{-1})^* = b\gamma^*(\delta^{-1})^* \). The internal operations are the same for \( \mathbb{B}[\gamma, \delta] \) and neutral elements \( \varepsilon \) and \( e \) are identical to those of \( \mathbb{B}[\gamma, \delta] \).

**Definition 4.** Let \( s \in \mathbb{M}^{\text{in}}_{\text{max}}[\gamma, \delta] \) be a series, the dater function of \( s \) is the non-decreasing function \( D_a(n) \) from \( \mathbb{Z} \rightarrow \mathbb{Z} \) such that \( s = \bigoplus_{n \in \mathbb{Z}} \gamma^n\delta^{D_a(n)} \).

**Example 5.** Considering the example 2 for the input \( u_1 \) must start operating on a wafer at time \( t=0 \) then on a second wafer at time \( t=1 \) then on a third at time \( t=2 \) and finally on a fourth wafer at \( t=3 \). The date function of this input is written \( u_1 = \gamma^0\delta^0 \oplus \gamma^1\delta^1 \oplus \gamma^2\delta^2 \oplus \gamma^3\delta^3 \oplus \gamma^4\delta^\infty \) where \( D_{u_1}(0) = 0 \), \( D_{u_1}(1) = 1 \), \( D_{u_1}(2) = 2 \) and \( D_{u_1}(3) = 3 \).

#### B. Residuation theory

Residuation is a general notion in lattice theory which allows for the definition of “pseudo-inverse” of some isotone maps. In particular, the residuation allows to find the biggest solution of inequalities such as \( \pi(x) \preceq b \).

**Definition 5.** Let \( \pi : \mathcal{D} \rightarrow \mathcal{C} \) be an isotone mapping, where \( \mathcal{D} \) and \( \mathcal{C} \) are complete dioid. The largest solution from \( \pi(x) = b \) and noted \( \pi^w \) is called the residual of \( \pi \). Where \( \pi \) is residuated, \( \pi^w \) is the unique isotone mapping such that \( \pi \circ \pi^w \preceq I_{\text{AC}} \) and \( \pi^w \circ \pi \preceq I_{\text{ID}} \) where \( I_{\text{AC}} \) and \( I_{\text{ID}} \) are respectively the identity mapping on \( \mathcal{C} \) and \( \mathcal{D} \).
Example 6. The mapping \( L_a \mapsto a \otimes x \) and \( R_a \mapsto x \otimes a \) defined over a complete doid \( D \) are both residuated. Their residuals are denoted, respectively, by \( L_a^x (x) = a \gamma x = \frac{x}{a} \), \( R_a^x (x) = x \delta a = \frac{x}{a} \).

Thanks to the residuals defined above we will be able to define time comparison between series.

Definition 6. Let \( a, b \in M_{\text{in}}^{ax} [\gamma, \delta] \) and their respective dater function \( D_a \) and \( D_b \), the time shift function representing the time shift between \( a \) and \( b \) for each \( n \in \mathbb{Z} \) is defined by \( T_{a,b} (n) = D_a - D_b \).

**Theorem 2.** Let \( a, b \in M_{\text{in}}^{ax} [\gamma, \delta] \), the time shift function can be bounded by:

\[
\forall n \in \mathbb{Z}, \quad D_{b/a} (0) \leq T_{a,b} (n) \leq -D_{a/b} (0).
\]

This theorem defines some minimum and maximum bounds for the time shift function. The extraction of the bound is done on the series \( a \circ b \) and \( b \circ a \). The \( D_{b/a} (0) \) bound is obtained from the \( \gamma \delta^0 \delta^0_{\gamma} \) of \( b \circ a \). The \( D_{a/b} (0) \) bound is obtained from the \( \gamma \delta^0 \delta^0_{\gamma} \) of \( a \circ b \).

Definition 7. Let \( a, b \in M_{\text{in}}^{ax} [\gamma, \delta] \), the time shift between series \( a \) and \( b \) is

\[
\Delta (a, b) = |D_{b/a} (0); -D_{a/b} (0)|,
\]

where \( \gamma \delta^0 \delta^0_{\gamma} \) and \( \gamma \delta^0 \delta^0_{\gamma} \) are both residuated. Their residuals \( \overset{\circ}{\gamma} \) are such that

\[
\text{for each } \mathbb{R}^{n \times 1} = [p, q] = [\gamma^3 \delta^0 + \gamma^4 \delta^1 + \gamma^5 \delta^2, \gamma^0 \delta^1 + \gamma^2 \delta^3 + \gamma^4 \delta^5].
\]

D. Models of (max,+) linear systems with time intervals

The objective is to define a Time Event Graph with time intervals (see Figure 1) as a (max,+) linear system with time intervals. The elements of the TEG will be represented by equations in \( \mathbb{I}(M_{\text{in}}^{ax} [\gamma, \delta]) \). The equations can be grouped into a set of matrices \( A, B \) and \( C \) called the state representation of the system that defines the relations between any set of input event flows \( u \) and the state \( x \) and the relations between the state \( x \) and the output event flows \( y \). Let \( \mathbb{I} \in (M_{\text{in}}^{ax} [\gamma, \delta])^{n \times 1} \) be the input vector of size \( p \). \( \mathbb{I} \in (M_{\text{in}}^{ax} [\gamma, \delta])^{n \times 1} \) be the state vector of size \( n \) and \( \gamma \in (M_{\text{in}}^{ax} [\gamma, \delta])^{n \times 1} \) be the output vector of size \( q \). The state representation is:

\[
\begin{align*}
\mathbb{I} &= Ax \oplus Bu, \\
\gamma &= Cx.
\end{align*}
\]

where \( A \in (M_{\text{in}}^{ax} [\gamma, \delta])^{n \times n} \), \( B \in (M_{\text{in}}^{ax} [\gamma, \delta])^{n \times p} \) and \( C \in (M_{\text{in}}^{ax} [\gamma, \delta])^{q \times n} \).

\[
\mathbb{I} = Ax \oplus Bu
\]

where \( A \) and \( B \) is the transfer function of the TEG.

Example 8. For the system of Figure 2 the matrices \( A \in (M_{\text{in}}^{ax} [\gamma, \delta])^{6 \times 6} \), \( B \in (M_{\text{in}}^{ax} [\gamma, \delta])^{6 \times 2} \) and \( C \in (M_{\text{in}}^{ax} [\gamma, \delta])^{1 \times 6} \) of the state representation are:

\[
\begin{align*}
\mathbb{I} &= Ax \oplus Bu, \\
\gamma &= Cx.
\end{align*}
\]

The order relation \( \leq \) in \( \mathbb{I}(D) \) induced by the additive law \( \oplus \) is such that

\[
x \oplus y = y \Leftrightarrow x \leq y \in D \quad \frac{\mathbb{I}}{x} \leq \frac{\mathbb{I}}{y} \in D
\]

Let \( I(M_{\text{in}}^{ax} [\gamma, \delta]) \) denote the set of intervals of the doid \( M_{\text{in}}^{ax} [\gamma, \delta] \). An interval defined over \( I(M_{\text{in}}^{ax} [\gamma, \delta]) \) corresponds to all the series between the series of the minimum bound and the series of the maximum bound. For this purpose, the representation of the series in \( M_{\text{in}}^{ax} [\gamma, \delta] \) is used for the plotting of the series of the lower and upper bounds.

Example 9. Back to Figure 2 the couple of polynomials represents the interval

\[
p = [p, q] = [\gamma^3 \delta^0 + \gamma^4 \delta^1 + \gamma^5 \delta^2, \gamma^0 \delta^1 + \gamma^2 \delta^3 + \gamma^4 \delta^5].
\]

Fig. 2. Interval representation of \( p \in I(M_{\text{in}}^{ax} [\gamma, \delta]) \).
A(1; 2) = [γ^10δ^0, γ^1δ^0]; A(2; 1) = [γ^0δ^2, γ^0δ^5];
A(3; 4) = [γ^10δ^0, γ^1δ^0]; A(4; 3) = [γ^0δ^1, γ^0δ^3];
A(5; 2) = [γ^0δ^3, γ^0δ^5]; A(5; 4) = [γ^0δ^5, γ^0δ^2];
A(5; 6) = [γ^1δ^0, γ^0δ^3]; A(6; 5) = [γ^0δ^2, γ^0δ^2].
B(1; 1) = [γ^0δ^0, γ^0δ^0]; B(3; 2) = [γ^0δ^0, γ^0δ^0];
C(1; 6) = [γ^0δ^0, γ^0δ^0];

Any other elements in the matrices correspond to ε.

In any element of the matrices, the exponent n of γ represents the backward event shift between transitions (the n + 1st firing of x1 depend on the nth firing of x2) and the exponent of δ represents the backward time shift between transition (the firing date of x2 depend on the firing date of x1 and time between 5 and 7).

IV. ACCEPTABLE OUTPUT IN (MAX,+)-LINEAR SYSTEMS WITH TIME INTERVALS

This section presents the tool using to define acceptable output of this system. This section introduces the acceptable outputs of a (max,+)-linear system with time intervals. Acceptable outputs correspond to the observable outputs that are included in the interval of the predicted output. The hypothesis for the different definitions that we will define inputs are observable and outputs are observable.

A. Time comparison between output in (max,+)-linear systems with time intervals

For a (max,+)-linear system with time interval with 
H ∈ I(M_{max}^{γ, δ}[γ, δ]^{q×p}) as a transfer function and u ∈ I(M_{max}^{γ, δ}[γ, δ]^{q×p}) as an observed input one can obtain the predicted outputs of the system y_p = Hu = [y_p, y_p]. The predicted outputs are the observable trajectories obtained with the knowledge of the system. With the time shift bounds given in Definition 9 we define the time shifts of the (max,+)-linear systems as time interval: that is the difference between y_p and y_p.

**Definition 10.** Let y_p = Hu = [y_p, y_p], with u ∈ I(M_{max}^{γ, δ}[γ, δ]^{q×p}), H ∈ I(M_{max}^{γ, δ}[γ, δ]^{q×p}), with u a degenerated input and H a transfer function. The system gap with y_p in reference is:

\[ \Delta_{y_p} = [D_{y_p}/y_p(0); -D_{y_p}/y_p(0)] = [\Delta y_p, \Delta y_p] \]

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The system gap system gap with y_p in reference corresponds to the gap of y_p from y_p. Since y_p is an interval we have y_p ≥ y_p so \( D_{y_p}/y_p(0) ≥ 0 \) and \( -D_{y_p}/y_p(0) ≥ 0 \).

**Example 10.** Given this TEG of Figure 4 suppose that Equipment I (resp.) results in the following observable inputs \( u_1 = [u_1, u_1] = [γ^0δ^0 ⊕ γ^0δ^1 ⊕ γ^0δ^2 ⊕ γ^0δ^0 ⊕ γ^0δ^1 ⊕ γ^0δ^2 + γ^0δ^0, γ^0δ^1 ⊕ γ^0δ^2 ⊕ γ^0δ^2 + γ^0δ^0] \). When \( u_1 = \pi_1 \) and \( u_2 = [u_2, u_2] = [γ^0δ^0 ⊕ γ^0δ^1 ⊕ γ^0δ^2 + γ^0δ^0, γ^0δ^1 ⊕ γ^0δ^2 + γ^0δ^0, γ^0δ^0 ⊕ γ^0δ^1 ⊕ γ^0δ^2 + γ^0δ^0] \). With this observable inputs we get the predictive output \( y_p = [y_p, y_p] = [γ^0δ^0 ⪰ γ^0δ^1 ⊕ γ^0δ^2 + γ^0δ^1 ⊕ γ^0δ^2 + γ^0δ^0, γ^0δ^0 ⊕ γ^0δ^1 ⊕ γ^0δ^2 + γ^0δ^0] \). Any other elements in the matrices correspond to ε.

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The system gap with y_p, in reference corresponds to the gap form y_p to y_p. Since y_p is an interval y_p ≤ y_p so \( D_{y_p}/y_p(0) ≤ 0 \) and \( -D_{y_p}/y_p(0) ≤ 0 \).

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**B. Acceptable outputs**

**Definition 11.** Let H ∈ I(M_{max}^{γ, δ}[γ, δ]^{q×p}) transfer function, u ∈ I(M_{max}^{γ, δ}[γ, δ]^{q×p}) a observable interval input, Hu = [y_p, y_p] the observable predicted output, a observable output y_{acc} is a output acceptable if y_{acc} must respect the following condition y_{acc} ≤ y_{acc} ≤ y_p. The set of acceptable outputs denote y_{acc} ∈ M_{max}^{γ, δ}[γ, δ].

**Proposition 1.** y_{acc} ∈ y_{acc} is acceptable output if and only if the following 4 conditions hold:

\[ D_{y_{acc}}/y_p(0) \in [\Delta y_p, 0], \]
\[ -D_{y_{acc}}/y_p(0) \in [\Delta y_p, 0], \]
\[ D_{y_{acc}}/y_p(0) \in [0, \Delta y_p], \]
\[ -D_{y_{acc}}/y_p(0) \in [0, \Delta y_p]. \]

**Proof.** For the superior bound case \( y_{acc} = y_p \), the time shift between \( y_p = [y_p, y_p] \) and the observable output y_{acc} is:

\[ D_{y_{acc}}/y_p(0) = 0 \]
\[ -D_{y_{acc}}/y_p(0) = 0 \]

Thus, the set of acceptable outputs denote y_{acc} ∈ M_{max}^{γ, δ}[γ, δ].

As \( y_{acc} \leq y_{acc} \leq y_p \), the time shift is:

\[ D_{y_{acc}}/y_p(0) = 0 \]
\[ -D_{y_{acc}}/y_p(0) = 0 \]

Thus, the set of acceptable outputs denote y_{acc} ∈ M_{max}^{γ, δ}[γ, δ].

As \( y_{acc} \leq y_{acc} \leq y_p \), the time shift is:

\[ D_{y_{acc}}/y_p(0) = 0 \]
\[ -D_{y_{acc}}/y_p(0) = 0 \]

Thus, the set of acceptable outputs denote y_{acc} ∈ M_{max}^{γ, δ}[γ, δ].
and $-D_{y_p,y_{acc}}(0)$. So if $y_{acc} \in Y_{acc}$ then conditions \(2\), \(3\), \(4\) and \(5\) hold.

If $D_{y_{acc} \notin Y_{acc}}(0) \in [\Delta \theta_p, 0]$, it involve the worst case $D_{y_{acc} \notin Y_{acc}}(0) = \Delta \theta_p$ that gives us $y_{acc} = y_p$ or $D_{y_{acc} \notin Y_{acc}}(0) = 0$ that gives us $y_{acc} = \bar{y}_p$ so $y_p \leq y_{acc} \leq \bar{y}_p$.

If $-D_{y_p,y_{acc}}(0) \in [\Delta \theta_p, 0]$, it involve the worst case $-D_{y_p,y_{acc}}(0) = \Delta \theta_p$ that gives use $y_{acc} = y_p$ where $-D_{y_p,y_{acc}}(0) = 0$ that gives us $y_{acc} = \bar{y}_p$ so $y_p \leq y_{acc} \leq \bar{y}_p$. If $D_{y_{acc} \notin Y_{acc}}(0) \in [0, \Delta \theta_p]$ it involve the worst case $D_{y_{acc} \notin Y_{acc}}(0) = 0$ that gives us $y_{acc} = y_p$ where $D_{y_{acc} \notin Y_{acc}}(0) = \Delta \theta_p$ that gives us $y_{acc} = \bar{y}_p$ so $y_p \leq y_{acc} \leq \bar{y}_p$.

If $-D_{y_p,y_{acc}}(0) \in [0, \Delta \theta_p]$, it involve the worst case $-D_{y_p,y_{acc}}(0) = 0$ that gives us $y_{acc} = y_p$ where $-D_{y_p,y_{acc}}(0) = \Delta \theta_p$ that gives us $y_{acc} = \bar{y}_p$.

Example 11. $y_p = \{y_{p1}, y_{p2}\} = \{\gamma^0 \delta^7 + \gamma^1 \delta^9 + \gamma^2 \delta^{11} + \gamma^3 \delta^{13} + \gamma^4 \delta^{15}, \gamma^0 \delta^{10} + \gamma^1 \delta^{15} + \gamma^2 \delta^{20} + \gamma^3 \delta^{25} + \gamma^4 \delta^{+\infty}\}$ is represented by the series in plain line in figure 3 with the system gap $\Delta \theta_p = [-12, -3]$ and $\Delta \gamma = [3, 12]$.

The observable output $y_1 = \gamma^0 \delta^8 + \gamma^1 \delta^{11} + \gamma^2 \delta^{14} + \gamma^3 \delta^{17} + \gamma^4 \delta^{+\infty}$ is represented by the series in dashed line and the observable output $y_2 = \gamma^0 \delta^8 + \gamma^2 \delta^8 + \gamma^3 \delta^{13} + \gamma^4 \delta^{+\infty}$ is represented by the series in dotted line.

![Graphical representation of the acceptable output](image)

The output $y_1$ belongs to the set of acceptable outputs, it indicates $-8 \in [-12, 0]$ and $-2 \in [-3, 0]$ and $1 \in [0, 3]$ and $4 \in [0, 12]$. The output $y_2$ does not belongs to the set of acceptable outputs, it indicates $-12 \in [-12, 0]$ and $-3 \in [-3, 0]$ and $-3 \notin [0, 3]$ and $0 \in [0, 12]$.

V. FAULT DETECTION IN (MAX,+) LINEAR SYSTEMS WITH TIME INTERVAL

In this section we will present indicators to detect faults in (max,+) linear systems with time interval. Thanks to what we learned in the section IV we can tell if any observable output belongs to the set of acceptable outputs or not so that we can make an indicator to tell us if a detectable failure is detected.

A. Indicator for certain output

For the certain output indicator, the outputs and inputs are observable. To be able to make the diagnosis it is necessary to detect the failures for that it is necessary to define the detectable failures.

Definition 12. A system failure is detectable if the observations of the system it generates do not correspond to the expected observations.

Definition 13. Let $(max, +)$-linear system with $H \in \mathcal{I}(\mathcal{M}^{in}_{\mathbb{R}}[\gamma, \delta]^{q \times p})$ the transfer function, $u \in \mathcal{I}(\mathcal{M}^{in}_{\mathbb{R}}[\gamma, \delta]^{p \times 1})$ the observed inputs, $y_o \in \mathcal{M}^{in}_{\mathbb{R}}[\gamma, \delta]^{q \times 1}$ the observed output. The indicator $I_f(u, y_o)$ of this system is a function

$$I_f(u, y_o) = \begin{cases} \text{true} & \text{if } y_o = Hu, y_o \notin \bar{y}_p, \Delta \theta_p(0) + \Delta \gamma(0) \\ \text{false} & \text{otherwise} \end{cases}$$

Proposition 2. The indicator returns true if and only if the system receive a detectable failure.

Proof. Let a system sudden a detectable failure, this implies that the observed output is not in the set of acceptable outputs of the system and is therefore one of the conditions \(2\), \(3\), \(4\) and \(5\) is true. Yet if \(2\) is true with \(3\) is true and if \(5\) is true with \(3\) is true. Suppose the indicator returns false then \(4\) and \(5\) are false: the indicator cannot therefore be false at the same time as the system is with detectable failure.

Let be a system to which the indicator refers true. The indicator that returns true implies that \(4\) and \(5\) are false. Suppose that the system not sudden a detectable failure then the output is in the set of acceptable outputs of the system so the conditions \(2\), \(3\), \(4\) and \(5\) are true. So a system without detectable failure this indicator don’t cannot be true.

Example 12. Back to Figure 4 consider the scenario that is defined by the inputs of Example 10 and suppose that in reality there was an incident on Equipment 1 the operation lasts longer will a processing time of 6 hours in place P1. The TEG has thus the following system gap: $\Delta \theta_p = [-11, -2]$ and $\Delta \gamma = [2, 11]$. The incident on Equipment 1 gives us the following observable output $y_o = \gamma^0 \delta^{12} + \gamma^1 \delta^{10} + \gamma^2 \delta^{21} + \gamma^3 \delta^{26} + \gamma^4 \delta^{+\infty}$. Computations of $y_o$, $y_p$ and $\bar{y}_p$ for give for $y_o$: $y_o \notin \bar{y}_p$ or $y_o \notin \bar{y}_p$. The residuals gives us the following results

$$D_{\bar{y}_p}(0) = 3 \notin [0, \Delta \theta_p] = [0, 2]$$

and $-D_{\bar{y}_p}(0) = -4 \notin \bar{y}_p$.

This indicator $I_f(u, y_o)$ indicates false because $D_{\bar{y}_p}(0) = 3 \notin [0, \Delta \theta_p] = [0, 2]$ is false.
B. Indicator for uncertain output

This section presents an indicator with uncertain output. For the uncertain output indicator, the inputs and outputs are observable. We will have to define the output sets acceptable intervals.

Definition 14. Let \((\max,+)\)-linear system with \(H \in I(\mathcal{M}_{\text{in}}^{\alpha} [\gamma, \delta]^{m \times p})\) the transfer function, \(u \in I(\mathcal{M}_{\text{in}}^{\alpha} [\gamma, \delta]^{p \times 1})\) the observed inputs, \(y_o \in I(\mathcal{M}_{\text{in}}^{\alpha} [\gamma, \delta])\) the observed outputs. The indicator \(I_f(u, y_o)\) is a function:

\[
I_f(u, y_o) = \begin{cases} 
\text{false} & \text{if for } y_p = Hu, \quad -D_{\eta_p, \eta_o}(0) \in [\Delta_{\eta_p}, 0] \\
\text{true otherwise.} & \text{and } D_{\eta_p, \eta_o}(0) \in [0, \Delta_{\eta_p}] 
\end{cases}
\]

\[\text{(7)}\]

The indicator returns true if and only if the system receive a detectable failure.

Proof. Let a system sudden a detectable failure, this implies that the observed output is not in the set of acceptable outputs of the system so either \(y_o < y_p\), either \(\eta_p < \eta_o\) or \(y_o < y_p \leq \eta_p < \eta_o\). When \(y_o < y_p\) \& \(\eta_p < \eta_o\) with the acceptable outputs of the proposition it gives us as a result \(\text{false}\). When \(\eta_p < \eta_o\) compare with the acceptable outputs of the proposition it gives us as a result \(\text{false}\). Suppose the indicator returns false then \(-D_{\eta_p, \eta_o}(0) \in [\Delta_{\eta_p}, 0]\) and \(D_{\eta_p, \eta_o}(0) \in [0, \Delta_{\eta_p}]\) is true: the indicator cannot therefore be false at the same time as the system is with detectable failure. Let be a system to which the indicator refer true. The indicator that returns true implies that \(-D_{\eta_p, \eta_o}(0) \in [\Delta_{\eta_p}, 0]\) and \(D_{\eta_p, \eta_o}(0) \in [0, \Delta_{\eta_p}]\) are false. Suppose that the system not sudden a detectable failure then the output is in the set of acceptable outputs of the system so \(y_o \leq y_p \leq \eta_o \leq \eta_p\) which gives us \(-D_{\eta_p, \eta_o}(0) \in [\Delta_{\eta_p}, 0]\) and \(D_{\eta_p, \eta_o}(0) \in [0, \Delta_{\eta_p}]\) true. So a system with detectable failure this indicator don’t cannot be true.

Example 13. Let a TEG in example \(7\) with the system gap following \(\Delta_{\eta_p} = [-11, -2]\) and \(\Delta_{\eta_p} = [2, 11]\) for the same inputs of example \(7\). For our TEG we will observe outputs \(y_o\) (i.e delivery of final products at time between 9 and 12, 11 and 16, 13 and 21, and 15 and 21). \(y_p = [1^0 \delta^0 + \gamma^1 \delta^1 + \gamma^2 \delta^2 + \gamma^3 \delta^3 + \gamma^4 \delta^4 + \gamma^5 \delta^5 + \gamma^6 \delta^6 + \gamma^7 \delta^7 + \gamma^8 \delta^8 + \gamma^9 \delta^9 + \gamma^{10} \delta^{10}]\). Computation of \(y_o \mathcal{D}_{\eta_p, \eta_o}(0)\) give for \(y_o\) : \(y_o \mathcal{D}_{\eta_p, \eta_o}(0) = \gamma^0 \delta^0 + \gamma^1 \delta^1 + \gamma^2 \delta^2 + \gamma^3 \delta^3 + \gamma^4 \delta^4 + \gamma^5 \delta^5 + \gamma^6 \delta^6 + \gamma^7 \delta^7 + \gamma^8 \delta^8 + \gamma^9 \delta^9 + \gamma^{10} \delta^{10}\). The residuals gives us the following results \(D_{\eta_p, \eta_o}(0) = 3 \notin [0, \Delta_{\eta_p}] = [0, 2]\) and \(-D_{\eta_p, \eta_o}(0) = -2 \in [\Delta_{\eta_p}, 0] = [-2, 0]\). This indicator \(I_f(u, y_o)\) indicate false because \(D_{\eta_p, \eta_o}(0) = 3 \notin [0, \Delta_{\eta_p}] = [0, 2]\) is false.

VI. CONCLUSION

In this article, we have extended the problem of time failure diagnosis in TEG by using time intervals. Using \((\max,+)\) algebraic techniques we have proposed indicators that detect time shift with certain or uncertain outputs. The indicators are based on the doid \(I(\mathcal{M}_{\text{in}}^{\alpha} [\gamma, \delta])\) to define our system but the indicator calculations are finally reduced to comparing the max and min bounds of the outputs. This study is motivated by the development of algorithms for time shift failure detection in assembly lines.

We have several perspectives for the use of indicators on a real production line such as that of STMicroelectronics. We will set up the C++ coding of this indicator in the \textit{MaxPlus-Diag} library of the article \[LCSSP18\]. We plan to extend this method by adding failure localisation and isolation modules to refine the diagnosis. The distances measured between the expected and observed outputs can use helpful information for locating a fault in a system. Another perspective is to deal with event uncertainty (interval of tokens in a Place of the TEG).

REFERENCES


[Max91] \textit{MaxPlus}. Second order theory of min-linear systems and its application to discrete event systems.

