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# Practical Stabilization of Passive Nonlinear Systems with Limited Control

B. Jayawardhana\* M. Z. Almuzakki\*\*\* A. Tanwani\*\*\*

\* *Engineering and Technology Institute Groningen, Faculty of Science and Engineering, University of Groningen, The Netherlands (e-mail: {b.jayawardhana; m.z.almuzakki}@rug.nl).*

\*\* *Department of Computer Science, Faculty of Science and Computer, Universitas Pertamina, Jakarta, Indonesia.*

\*\*\* *Laboratory for Analyses and Architecture of Systems (LAAS) – CNRS, Université de Toulouse, France (e-mail: aneel.tanwani@laas.fr).*

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**Abstract:** This article addresses output feedback stabilization of continuous-time nonlinear systems by choosing control actions from a finite set. Working under the assumption that the system under consideration is passive and large-time norm observable, we propose a static feedback mapping, from the output space to the finite set of control actions, which is shown to be practically stabilizing if the convex hull of certain control actions (in the chosen finite set) contains the origin in its interior. Consequently, to construct this stabilizing feedback, it suffices to have, in addition to a zero symbol, another  $m + 1$  elements in the control set which form an  $m$ -simplex in  $\mathbb{R}^m$  (the input, and output space).

*Keywords:* Nonlinear passive systems; finite control set; output feedback; binary control.

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## 1. INTRODUCTION

The design of advanced mechatronics and robotics systems often has to deal with extreme environmental conditions, such as, cryogenic temperature or ultra-high vacuum setting, and consequently several design constraints have to be taken into account. These constraints restrict the choice of the sensor systems, actuator systems, as well as, the information processing and control mechanism. For instance, in a networked robotics systems, the limited capacity of the communication network enforces the real-time information from the sensor systems and from the controller to be quantized before it can be transmitted via the network. The quantization process produces information loss that can lead to a performance degradation of the overall systems. Among many others, the seminal paper (Elia and Mitter, 2001) discusses the design of control systems under minimal information / quantization levels for linear single input systems case. The study was continued in (Kao and Venkatesh, 2002) for multiple input case of linear systems. In the networked control systems setting, the papers (Cortés, 2006; Jafarian and De Persis, 2015; De Persis and Jayawardhana, 2012) present various analysis and design methods that incorporate the quantization effect in the control design. Another example is the design of mechatronics systems with limited actuation, such as, a fixed set of constant actuator systems in Ocean Grazer (Barradas-Berglind et al., 2016; Wei et al., 2017), or a fixed configuration of constant thruster systems in the space rockets, which can only provide piecewise constant actuation with limited discrete values. In this paper, we shall address the constrained control problem, where the

focus is on designing control actions with limited information from sensors, and limited/minimal actuation levels.

To describe our control problem, let us consider linear systems described by

$$\Sigma_{\text{lin}} : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \quad (1)$$

where the state  $x(t) \in \mathbb{R}^n$ , the input and output signal  $u(t), y(t) \in \mathbb{R}^m$  and  $(A, B, C)$  are system's matrices with appropriate dimension. As we consider limited actuation / information transmission, the control input  $u$  can only take values from a finite discrete set  $\mathcal{U} := \{0, u_1, u_2, \dots, u_p\}$  with  $u_i \in \mathbb{R}^m$  for each  $i = 1, \dots, p$ . For such systems, assuming we have a stabilizing output feedback law  $y \mapsto Ky$ , two questions are particularly relevant for stabilization when the actuation set  $\mathcal{U}$  is finite: a) how to map  $Ky$  to an element in  $\mathcal{U}$  and b) how to determine the minimal number of elements in  $\mathcal{U}$ . By addressing these questions, with generic output maps and nonlinear dynamics, our aim is to design  $\phi : \mathbb{R}^m \rightarrow \mathcal{U}$ , with  $\mathcal{U}$  discrete (and minimal), such that  $u = \phi(Ky) \in \mathcal{U}$  practically stabilizes  $\Sigma_{\text{lin}}$ .

The question of finding the minimal set  $\mathcal{U}$  for feedback stabilization has received considerable attention. The first step in this regard is, what should be the minimal cardinality of the set  $\mathcal{U}$ . The primary results in this direction state that if the number of bits per sample (rate of communication) is greater than the intrinsic entropy of the system, then the system is stabilizable in some appropriate sense. Such relations are collected in the discrete-time linear setting by (Nair et al., 2007). For the continuous-time systems, the articles by (Colonius and Kawan, 2009) and (Colonius, 2012) develop similar relations for linear

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systems, but their bounds on minimal bit-rates for stabilization of nonlinear systems are rather conservative.

The question of designing the mapping  $\phi : \mathbb{R}^m \rightarrow \mathcal{U}$  has also been addressed in different forms. Note that, with  $\mathcal{U}$  being discrete, the range of  $\phi$  determines a partition of the output space containing as many regions as the number of elements in the set  $\mathcal{U}$ . When  $\mathcal{U}$  is a regular grid in a compact set in  $\mathbb{R}^m$ , for example  $\{-N, -N + 1, \dots, N - 1, N\}^m$  with  $2N + 1$  being the number of quantization level per input dimension, then  $\phi$  can be the standard (uniform) quantization operator. In this case, the existing works on quantized control systems are directly applicable. We refer the works of (Delchamps, 1990; Tatikonda, 2000; Ceragioli and De Persis, 2007; Fu and de Souza, 2009; Jafarian and De Persis, 2015; De Persis and Jayawardhana, 2012). However, in these works, the state is shown to stay in a ball around the origin whose size depends on the number of quantization levels. One has to resort to time-varying feedbacks to get desired accuracy with finitely many quantization levels (Liberzon, 2003; Tanwani et al., 2016). However, we will show in Section 3 of this paper that a static mapping that assigns  $Ky$  to the nearest element in  $\mathcal{U}$  (practically) stabilizes the system under certain structural assumptions.

With certain passivity structure in the dynamics,  $\Sigma_{\text{lin}}$  can be practically stabilized by using binary control for each input dimension which translates to  $2^m + 1$  elements in  $\mathcal{U}$ , e.g.,  $\mathcal{U} = \{0\} \cup \{-1, 1\}^m$ , see (Cortés, 2006; Jafarian and De Persis, 2015). As a relaxation of these results, and dealing with multi-input multi-output nonlinear systems, we show that practical stabilization is achievable by simply using  $m + 2$  elements in  $\mathcal{U}$  for a rather generic class of passive systems.

Passive systems are well-studied in literature on dynamical control systems, as they model physical phenomena exhibited by almost all thermo-chemo-electromechanical systems (van der Schaft et al., 2013; Ortega et al., 2013). The passivity property can be related to the dynamics of the energy variable of a system. It describes the energy dissipation process and energy exchange mechanism with the environment through the (input/output) ports. In particular, for passive systems, the rate of change of the system's "stored energy" never exceeds the power supplied by the environment through its external ports. We refer interested readers to the various expositions on passive systems in (Sepulchre et al., 2012; Ortega et al., 2013; Khalil, 2014; van der Schaft, 2016).

When quantization effect is of a particular concern, the interconnection of passive systems and quantizers has been studied for the past decade in various different contexts. The tutorial paper (Jayawardhana et al., 2011) presents the practical stability analysis of passive systems in a feedback loop with a quantizer using an adapted circle criterion for nonsmooth systems. For distributed control systems, the paper (De Persis and Jayawardhana, 2012) analyzes networked passive systems through a quantized communication channel. The contribution of this paper lies in studying control strategies for passive systems with minimal control actions.

The rest of the paper is organized as follows. In Section 2, we provide some preliminaries on set-valued dynamics re-

sulting from discontinuous controls, and formulate the control problem. Our main results are presented in Section 3, where we study practical stabilization of passive systems under the nearest neighbor control approach. Illustration via an academic example and concluding remarks are provided in Sections 4 and 5, respectively.

## 2. PRELIMINARIES AND PROBLEM FORMULATION

### 2.1 Preliminaries

Notation: For an element of a normed vector space, we denote its norm by  $\|\cdot\|$ . The set  $\mathbb{B}_\epsilon \subset \mathbb{R}^n$  is defined as,  $\mathbb{B}_\epsilon := \{\xi \in \mathbb{R}^n \mid \|\xi\| \leq \epsilon\}$ . The inner product of two vectors  $\nu, \gamma \in \mathbb{R}^m$  is denoted by  $\langle \nu, \gamma \rangle$ . For a discrete set  $\mathcal{U}$ , its cardinality is denoted by  $\text{card}(\mathcal{U})$ . The convex hull of vertices from a discrete set  $B$  is denoted by  $\text{conv}(B)$ . The interior of a set  $S \subset \mathbb{R}^n$  is denoted by  $\text{int}(S)$ . For a signal  $z : \mathbb{R}_+ \rightarrow \mathbb{R}^n$ , the essential supremum norm of  $z$  over an interval  $I \subset \mathbb{R}_+$  is denoted by  $\|z\|_I$ .

A set-valued mapping  $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  is called *upper semicontinuous* at  $x$  if for every open set  $X$  containing  $\Phi(x) \subset \mathbb{R}^n$ , there exists an open set  $\Xi$  containing  $x$  such that for all  $\xi \in \Xi$ ,  $\Phi(\xi) \subset X$ . Correspondingly,  $\Phi$  is *upper semicontinuous* if it is upper semicontinuous at every point in  $\mathbb{R}^n$ . Using the set-valued map  $\Phi$ , consider now the following differential inclusion

$$\dot{x} \in \Phi(x) \quad x(0) = x_0. \quad (2)$$

A Krasowskii solution  $x(\cdot)$  on an interval  $I \subset \mathbb{R}_+$  is an absolutely continuous function  $x : \mathbb{R}_+ \rightarrow I$  such that (2) holds almost everywhere on  $I$ . It is *maximal* if it has no right extension and it is a *global* solution if  $I = \mathbb{R}_+$ . For any upper semicontinuous set-valued map  $\Phi$  such that  $\Phi(\xi)$  is compact and convex for all  $\xi \in \mathbb{R}^n$ , the following properties have been established (see, e.g., Lemma 1 in Jayawardhana et al. (2011)): (i). the differential inclusion (2) has a solution; (ii). every solution can be extended to a maximal one; and (iii). if the maximal solution is bounded then it is global. For any discontinuous map  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , we can define an upper semicontinuous set-valued map  $\Phi$  by convexifying  $F$  as follows

$$\mathcal{K}(F(x)) := \bigcap_{\delta > 0} \overline{\text{co}}(F(x + \mathbb{B}_\delta))$$

where  $\overline{\text{co}}(S)$  is the convex closure of  $S$ .

### 2.2 Stabilization with limited control problem

For the rest of this paper, we consider the following nonlinear systems

$$\Sigma : \begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \quad (3)$$

where the state  $x(t) \in \mathbb{R}^n$ , the output  $y(t) \in \mathbb{R}^m$  and the input  $u(t) \in \mathcal{U} := \{0, u_1, \dots, u_p\}$  with  $u_i \in \mathbb{R}^m$  for all  $i = 1, \dots, p$ . The function  $f, g$  and  $h$  are assumed to be  $\mathcal{C}^1$  differentiable,  $f(0) = 0$ ,  $g(x)$  is full-rank for all  $x$  and  $h(0) = 0$ . We assume further that  $\Sigma$  is *passive*, i.e., for all pair of input and output signals  $u, y$  we have  $\int_0^T \langle y(t), u(t) \rangle dt > -\infty$  for all  $T > 0$  (Willems, 1972; van der Schaft, 2016; Ortega et al., 2013). By

the well-known Hill-Moylan conditions, the passivity of  $\Sigma$  implies that there exists a positive definite storage function  $H : \mathbb{R}^n \rightarrow \mathbb{R}_+^+$  such that  $\langle \nabla H(x), f(x) \rangle \leq 0$  and  $\langle \nabla H(x), g(x) \rangle = h^\top(x)$ . Without loss of generality, we assume that the storage function  $H$  is *proper*, i.e. all level sets of  $H$  are compact.

Using the passivity assumption on  $\Sigma$ , it is immediate to see that  $u \equiv 0$  implies that all level sets of  $H$  are positively invariant. More precisely, for any  $c > 0$ , if  $H(x(0)) \leq c$  then  $H(x(t)) \leq c$  for all  $t \geq 0$ . In other words, if we initialize the state of  $\Sigma$  such that  $x(0) \in \Omega_c := \{\xi | H(\xi) \leq c\}$  with  $u \equiv 0$  then  $x(t) \in \Omega_c$  for all  $t \geq 0$ . We will use this property later to establish the practical stability of our closed-loop systems in conjunction with the following observability notion from Hespanha et al. (2005). System (3) has the *large-time norm observability* property if there exist  $\tau > 0, \gamma, \chi \in \mathcal{K}_\infty$  such that the solution  $x$  of (3) satisfies

$$\|x(t)\| \leq \gamma(\|y\|_{[t_1, t_2]}) + \chi(\|u\|_{[t_1, t_2]})$$

for all  $x(0), u, t_1 \geq 0, t_2 \geq t_1 + \tau$ , and each  $t \in [t_1, t_2]$ . In particular, we will use the large-time norm observability property for the autonomous system (with  $u = 0$ ):

$$\left. \begin{array}{l} \dot{x} = f(x) \\ y = h(x). \end{array} \right\} \quad (4)$$

In this case, large-time norm observability of (4) implies

$$\exists \tau > 0, \gamma \in \mathcal{K}_\infty \text{ such that, for each } t_1 \geq 0, x(0) \in \mathbb{R}^n, \\ \|x(t)\| \leq \gamma(\|y\|_{[t_1, t_2]}), \quad \forall t_2 \geq t_1 + \tau, t \in [t_1, t_2]. \quad (5)$$

We note that in the standard passivity-based control literature, the notion of zero-state observability or zero-state detectability is typically assumed for establishing the convergence of the state to zero in the  $\Omega$ -limit set. However, these notions cannot be used to conclude the boundedness of the state trajectories given the bound on the output trajectories. Therefore, instead of using these notions, we will use the above large-time norm observability for deducing the practical stability based on the information on  $y$  in the  $\Omega$ -limit set.

**Practical output-feedback stabilization with limited control (POS-LC):** For a given system  $\Sigma$  as in (3) and for a given ball  $\mathbb{B}_\epsilon$  with  $\epsilon > 0$ , determine the (minimal) elements  $u_i, i = 1, \dots, p$  for  $\mathcal{U}$  and determine  $\phi : \mathbb{R}^m \rightarrow \mathcal{U}$  such that the closed-loop system of (3) with  $u = \phi(y)$  satisfies  $x(t) \rightarrow \mathbb{B}_\epsilon$  as  $t \rightarrow \infty$  for all initial condition  $x(0)$ .

In our problem formulation, both the determination of elements in  $\mathcal{U}$ , as well as, the design of the practical stabilization mapping  $\phi$  constitute our control problem. To solve this problem, we will work under the following basic assumption:

- (A0) The system  $\Sigma$  in (3) is passive with a proper storage function  $H$  and, the corresponding autonomous system (4) is large-time norm-observable for some  $\tau > 0$  and  $\gamma \in \mathcal{K}_\infty$ .

### 3. NEAREST NEIGHBOR CONTROL

As a solution to the aforementioned problem, we provide conditions on the set  $\mathcal{U}$  followed by the description of the static map  $\phi$  which practically stabilizes the system. For

efficiency in practical implementation, in our design, we work with minimal number of elements in the set  $\mathcal{U}$  which yield the desired performance using the static feedback only. Toward this end, the only assumption we associate with the set  $\mathcal{U}$  is the following:

- (A1) For a given set  $\mathcal{U} := \{0, u_1, u_2, \dots, u_p\}$ , there exists a minimal index set  $\mathcal{I} \subset \{1, \dots, p\}$  such that the set  $\mathcal{V} := \{u_i\}_{i \in \mathcal{I}} \subset \mathcal{U}$  defines the vertices of a convex polytope satisfying,  $0 \in \text{int}(\text{conv}(\mathcal{V}))$ .

An immediate consequence of (A1) is the following lemma, which is used in the derivation of our main result.

*Lemma 1.* Consider a discrete set  $\mathcal{V} := \{v_1, \dots, v_q\} \subset \mathbb{R}^m$  such that  $0 \in \text{int}(\text{conv}(\mathcal{V}))$ . Then, there exists  $C_{\mathcal{V}} > 0$  such that the following implication holds for each  $y \in \mathbb{R}^m$

$$\|y\| > C_{\mathcal{V}} \Rightarrow \exists v_i \in \mathcal{V} \text{ s.t. } \|v_i + y\| < \|y\|. \quad (6)$$

**Proof.** First we observe that any point  $\tilde{v} \in \mathbb{R}^m$  that is closer to the origin than to any point  $v_i \in \mathcal{V}$  is contained in the solution set of the linear inequality

$$[v_1 \ \dots \ v_q]^\top \tilde{v} \leq \frac{1}{2} [\|v_1\|^2 \ \dots \ \|v_q\|^2]^\top \quad (7)$$

where  $v_1, \dots, v_q \in \mathcal{V}$ . Let us denote the solution set of (7) by  $\tilde{\mathcal{V}}$ . In other words, the set  $\tilde{\mathcal{V}}$  is defined by the intersection of  $q$  half-spaces described by each row of the inequality (7), so that  $\tilde{\mathcal{V}}$  is a closed convex polyhedron. It remains to show that  $\tilde{\mathcal{V}}$  is bounded. Indeed, boundedness implies that we can choose  $C_{\mathcal{V}} = \max_{\tilde{v} \in \tilde{\mathcal{V}}} (\|\tilde{v}\|)$ , such that  $\mathbb{B}_{C_{\mathcal{V}}}$

is the smallest ball containing the set  $\tilde{\mathcal{V}}$ . By construction of  $\tilde{\mathcal{V}}$ , for any point  $y \in \mathbb{R}^m \setminus \mathbb{B}_{C_{\mathcal{V}}}$ , there is an element  $v_i \in \mathcal{V}$  such that  $\|v_i + y\| < \|y\|$ .

To show that  $\tilde{\mathcal{V}}$  is bounded, we observe that, under (A1), there exists  $\delta > 0$  such that  $\mathbb{B}_\delta \subset \text{conv}(\mathcal{V})$ . Thus, for every  $\tilde{v} \in \tilde{\mathcal{V}}$ ,  $\delta \frac{\tilde{v}}{\|\tilde{v}\|} \in \text{conv}(\mathcal{V})$ . Hence, there exist  $\lambda_i \geq 0$  such that  $\sum_{i=1}^q \lambda_i = 1$  and  $\delta \frac{\tilde{v}}{\|\tilde{v}\|} = \sum_{i=1}^q \lambda_i v_i$ . Consequently, from (7), it follows that

$$\delta \frac{\tilde{v}^\top \tilde{v}}{\|\tilde{v}\|} = \sum_{i=1}^q \lambda_i v_i^\top \tilde{v} \leq \frac{1}{2} \sum_{i=1}^q \lambda_i \|v_i\|^2$$

and hence  $\|\tilde{v}\| \leq \frac{1}{2\delta} \sum_{i=1}^q \lambda_i \|v_i\|^2$ .  $\square$

Using the result of Lemma 1 and the assumptions introduced thus far, we can define the function  $\phi$  which maps the measured outputs to the discrete set  $\mathcal{U}$  to achieve practical stabilization.

*Proposition 2.* Consider a nonlinear system  $\Sigma$  as in (3) satisfying (A0), along with a discrete set  $\mathcal{U} \supset \mathcal{V}$  satisfying (A1) and a scalar  $C_{\mathcal{V}}$  satisfying (6). For a given  $\epsilon > 0$  assume that

$$\gamma(C_{\mathcal{V}}) \leq \epsilon.$$

Then by defining

$$\phi(y) := \arg \min_{v \in \mathcal{U}} \{\|v + y\|\} \quad (8)$$

the control law  $u = \phi(y)$  globally practically stabilizes  $\Sigma$  with respect to  $\mathbb{B}_\epsilon$ .

**Proof.** Suppose that  $\phi(y) = \{u_i\}_{i \in J}$  for some  $J \subset \{1, \dots, p\}$  and  $0 \notin \{u_i\}_{i \in J}$ . It follows from (8) that  $\{u_i\}_{i \in J}$

are the closest points to  $-y$  and it implies that

$$\begin{aligned} \langle \phi(y), -y \rangle &\in \{ \|u_i\| \|y\| \cos(\theta_{i,y}) \mid i \in J \} \\ \Leftrightarrow \langle \phi(y), y \rangle &\in \{ -\kappa_{i,y} \|u_i\| \|y\| \mid i \in J \} \end{aligned}$$

where  $\theta_{i,y}$  is the angle between the vectors  $u_i$  and  $-y$  which lies strictly between  $-\pi/2$  to  $\pi/2$ , and  $\kappa_{i,y} := \cos(\theta_{i,y}) > 0$ , accordingly. Otherwise, 0 is (also one of) the nearest element(s) of  $\mathcal{U}$  to  $-y$ , in which case,

$$\langle \phi(y), y \rangle \in \{0\} \cup \{ -\kappa_{i,y} \|u_i\| \|y\| \mid i \in J \}$$

or  $\langle \phi(y), y \rangle = 0$  if  $\phi(y)$  is a singleton and is given by 0.

Based on this property of  $\langle \phi(y), y \rangle$ , we can now analyze the behaviour of the closed system which is given by

$$\begin{aligned} \dot{x} &= f(x) + g(x)\phi(y) \\ y &= h(x). \end{aligned} \quad (9)$$

As  $\phi(y)$  is a non-smooth operator, we consider instead the following differential inclusion

$$\begin{aligned} \dot{x} &\in \mathcal{K}(f(x) + g(x)\phi(y)) = f(x) + g(x)\mathcal{K}(\phi(y)) \\ y &= h(x). \end{aligned} \quad (10)$$

We note that the solution of (9) is also a solution of (10). Therefore, the asymptotic behaviour of all solutions of (10) determines the asymptotic behavior of the solution of (9).

Using the storage function  $H$  of the original system, for all solutions of (10) we have that either

(i):  $\phi(y) = \{u_i\}_{i \in J} =: \mathcal{W}$  for some  $J \subset \{1, \dots, p\}$ ,  $0 \notin \mathcal{W}$ , so that

$$\begin{aligned} \dot{H}(x) &= \langle \nabla H(x), \dot{x} \rangle \in \langle \nabla H(x), f(x) + g(x)\mathcal{K}(\phi(y)) \rangle \\ &= \langle \nabla H(x), f(x) \rangle + \langle y, \text{conv}(\mathcal{W}) \rangle. \end{aligned}$$

Based on the computation of  $\langle \phi(y), y \rangle$  for non-zero  $\phi(y)$  as before, it follows that

$$\begin{aligned} \langle y, \text{conv}(\mathcal{W}) \rangle &\in [ -\bar{\kappa}_{y,\max} \|u_{i,\max}\| \|y\|, -\bar{\kappa}_{y,\min} \|u_{i,\min}\| \|y\| ], \end{aligned}$$

where  $\|u_{i,\max}\| := \max_{w \in \mathcal{W}} \|w\|$ ,  $\|u_{i,\min}\| := \min_{w \in \mathcal{W}} \|w\|$ ,  $\bar{\kappa}_{y,\min} = \cos(\theta_{y,\max})$  with  $\theta_{y,\max}$  be the largest angle between  $-y$  and  $u_i \in \mathcal{W}$  and correspondingly,  $\bar{\kappa}_{y,\max} = \cos(\theta_{y,\min})$  with  $\theta_{y,\min}$  be the smallest angle between  $-y$  and  $u_i \in \mathcal{W}$ . Therefore,

$$\dot{H}(x) \leq -\bar{\kappa}_{y,\min} \|u_{i,\min}\| \|y\|;$$

or

(ii):  $\phi(y) = \{0\} \cup \{u_i\}_{i \in J} =: \mathcal{O}$  for some  $J \subset \{1, \dots, p\}$ , in which case, following the same arguments as in case (i)

$$\dot{H}(x) \in \langle H(x), f(x) \rangle + \langle y, \text{conv}(\mathcal{O}) \rangle.$$

Since  $\{0\}$  is an element of  $\mathcal{O}$ ,

$$\langle y, \text{conv}(\mathcal{W}) \rangle \in [ -\bar{\kappa}_{y,\max} \|u_{i,\max}\| \|y\|, 0 ],$$

where  $\|u_{i,\max}\| := \max_{w \in \mathcal{O}} \|w\|$ . This implies that

$$\dot{H}(x) \leq 0$$

As  $H(x(t))$  is non-increasing in both cases of (i) and (ii) and since  $H$  is proper, all solutions  $x(t)$  are bounded. By the LaSalle invariance principle, all of such compact trajectories converge to the largest invariant set  $M \subset \mathbb{R}^n$  where  $h(M) \subset Z$  where  $Z := \{y \in \mathbb{R}^m \mid 0 = \phi(y)\}$ .

In the invariant set  $M$ ,  $\phi(y(t)) = 0$  for all  $t$  and correspondingly,  $\|y(t)\| \leq C_{\mathcal{V}}$  for each  $t \geq 0$ . By the property of large-time norm-observability of (4), it holds that in the invariant set  $M$

$$\|x(t)\| \leq \gamma(C_{\mathcal{V}}) \leq \epsilon \quad \forall t \geq 0,$$

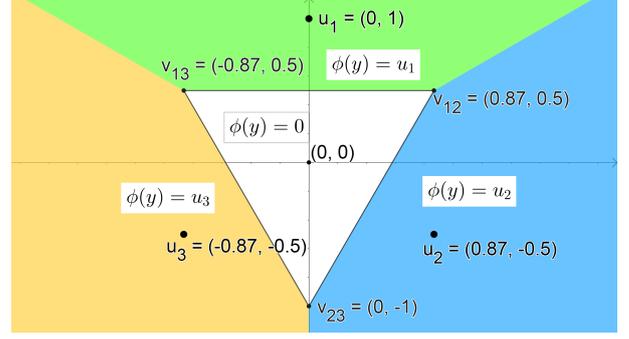


Fig. 1. Illustration of the region of the solution of  $\phi(y)$  for the discrete set  $\mathcal{U}_{\text{ex}}$  as in (11) with  $\theta_{\text{ex}} = 0$  and  $\alpha = 1$ .

where the last inequality is due to the hypotheses of the proposition. Thus, by the LaSalle invariance principle

$$x(t) \rightarrow M \subset \mathbb{B}_{\epsilon} \quad \text{as } t \rightarrow \infty.$$

Due to the properness of  $H$ , the above arguments hold for all initial conditions  $x(0) \in \mathbb{R}^n$ .  $\square$

*Example 1.* A simple example of  $\mathcal{U}$  in  $\mathbb{R}^2$ , satisfying (A1) is as follows:

$$\begin{aligned} \mathcal{U}_{\text{ex}} &:= \left\{ 0, \alpha \begin{bmatrix} \sin(\theta_{\text{ex}}) \\ \cos(\theta_{\text{ex}}) \end{bmatrix}, \alpha \begin{bmatrix} \sin(\theta_{\text{ex}} + \frac{2\pi}{3}) \\ \cos(\theta_{\text{ex}} + \frac{2\pi}{3}) \end{bmatrix}, \alpha \begin{bmatrix} \sin(\theta_{\text{ex}} + \frac{4\pi}{3}) \\ \cos(\theta_{\text{ex}} + \frac{4\pi}{3}) \end{bmatrix} \right\} \\ &= \{0, u_{\text{ex},1}, u_{\text{ex},2}, u_{\text{ex},3}\} \end{aligned} \quad (11)$$

with some  $\theta_{\text{ex}} \in \mathbb{R}$  and  $\alpha \in (0, \infty)$ . For this example, (A1) holds by taking  $\mathcal{V} := \mathcal{U} \setminus \{0\}$ . Following the proof of Lemma 1, we have  $\{y \mid \phi(y) = 0\} = \tilde{\mathcal{V}} := \text{conv}(\tilde{\mathcal{V}}_p)$  where

$$\tilde{\mathcal{V}}_p := \alpha \left\{ \begin{bmatrix} \sin(\theta_{\text{ex}} + \frac{\pi}{3}) \\ \cos(\theta_{\text{ex}} + \frac{\pi}{3}) \end{bmatrix}, \begin{bmatrix} \sin(\theta_{\text{ex}} + \frac{3\pi}{3}) \\ \cos(\theta_{\text{ex}} + \frac{3\pi}{3}) \end{bmatrix}, \begin{bmatrix} \sin(\theta_{\text{ex}} + \frac{5\pi}{3}) \\ \cos(\theta_{\text{ex}} + \frac{5\pi}{3}) \end{bmatrix} \right\}.$$

Here,  $\tilde{\mathcal{V}}_p$  contains all vertices of the convex polytope  $\tilde{\mathcal{V}}$ . By choosing  $C_{\mathcal{V}} = \alpha$ , one can check that  $\tilde{\mathcal{V}} \subset \mathbb{B}_{\alpha}$ . In the proof of Proposition 2, it is established that the trajectories converge to an invariant set  $M$ , such that  $h(M) \subseteq \tilde{\mathcal{V}}$ .

*Remark 3.* If the dynamics in system (4) are linear, that is,  $\dot{x} = Ax$ ,  $y = Cx$ , and the pair  $(A, C)$  is observable, then one can quantify  $\gamma$  in (5) using observability Gramian. In particular, if for  $\tau > 0$

$$W_{\tau} = \int_0^{\tau} e^{A^{\top}s} C^{\top} C e^{As} ds$$

then  $x(t) = W_{\tau}^{-1} \int_t^{t+\tau} e^{A^{\top}s} C^{\top} y(s) ds$ , for each  $t \geq 0$ , and  $\tau > 0$ , which in particular yields

$$|x(t)| \leq \|W_{\tau}^{-1}\| \int_0^{\tau} \|e^{A^{\top}s} C^{\top}\| ds \sup_{s \in [t, t+\tau]} |y(s)|$$

for each  $t \geq 0$ , and any  $\tau > 0$ .

*Corollary 4.* Consider the system  $\Sigma$  as in Proposition 2 with  $\mathcal{U} = \lambda\{-N, -N+1, \dots, N-1, N\}^m$ ,  $\lambda > 0$  being the step size and  $N$  a positive integer. Then the control law  $u = \phi(y)$ , where  $\phi$  is as in (8), globally practically stabilizes  $\Sigma$  with respect to  $\mathbb{B}_{\epsilon}$  where  $\epsilon > 0$  satisfies  $\gamma(\lambda\sqrt{m}) \leq \epsilon$ .

**Proof.** The proof follows *mutatis mutandis* the proof of Proposition 2. The set  $\mathcal{U}$  satisfies (A1) by taking  $\mathcal{V} = \lambda\{-1, 0, 1\}^m \setminus \{0\}$ . It is also seen that  $C_{\mathcal{V}} = \lambda\sqrt{m}$ , and by requiring  $\gamma(\lambda\sqrt{m}) \leq \epsilon$ , all the hypotheses of Proposition 2 hold.  $\square$

In contrast to the previous example where we used (11) to construct the discrete set  $\mathcal{U}$  in  $\mathbb{R}^2$ , the constant  $C_{\mathcal{V}}$  in Corollary 4 is less than  $\max_{v \in \tilde{\mathcal{V}}} \|v\|$ . This is due to the choice of the set  $\mathcal{V}$  in the proof of Corollary 4 that is dense enough such that  $\{y \mid \phi(y) = 0\} \subset \text{conv}(\mathcal{V})$ . From this corollary, one can conclude that two-level quantization with  $N = 1$  suffices to get a global practical stabilization property. This binary control law restricts however the convergence rate of the closed-loop system. It converges to the desired compact ball in a linear fashion and may not be desirable when the initial condition is very far from the origin. The use of higher quantization level (e.g.,  $N \gg 1$ ) can provide a better convergence rate when it is initialized within the quantization range.

Before we present our next result, let us recall again the result in Proposition 2. In this proposition, when  $\mathcal{U}$  is the continuum space of  $\mathbb{R}^m$ , the resulting control law is simply given by  $u = -y$ , i.e., it is a unity output feedback law. Using standard result in passive systems theory, the closed-loop system will satisfy  $\dot{H} \leq -\|y\|^2$  and the application of La-Salle invariance principle with zero-state detectability allows us to conclude that  $x(t) \rightarrow 0$  asymptotically. As the underlying system is passive, we can in fact stabilize it with any sector-bounded nonlinearity  $u = -F(y)$  where  $F$  satisfies  $k_1\|y\|^2 \leq \langle F(y), y \rangle \leq k_2\|y\|^2$  with  $0 < k_1 \leq k_2$ . There are a number of reasons for considering such feedback laws rather than the unity output feedback law. For instance, we can attain a prescribed  $L_2$ -gain disturbance attenuation level or we can shape the transient behaviour by adjusting the gains on different domain of  $y$ . In the following proposition, we will consider such sector-bounded output feedback law  $F(y)$  which is mapped to our limited control input set  $\mathcal{U}$ .

*Proposition 5.* Consider the system  $\Sigma$  satisfying (A0), along with a discrete set  $\mathcal{U} \supset \mathcal{V}$  satisfying (A1) so that there is a scalar  $C_{\mathcal{V}}$  such that (6) holds. Let  $\phi$  be given as in (8); and let  $F : \mathbb{R}^m \rightarrow \mathbb{R}^m$  satisfy the sector bound  $k_1\|y\|^2 \leq \langle F(y), y \rangle \leq k_2\|y\|^2$ ,  $0 < k_1 < k_2$  and  $\|F(y)\| \leq k_3\|y\|$  with  $k_3 > k_1$  for all  $y$ . Suppose that there exists  $\theta_{1,\max} > 0$  such that for all  $z \in \mathbb{R}^m$

$$\phi(z) \neq 0 \Rightarrow \langle \phi(z), -z \rangle \geq \|\phi(z)\| \|z\| \cos(\theta_{1,\max}). \quad (12)$$

Assume that  $\gamma(C_{\mathcal{V}}/k_1) \leq \epsilon$ , for a given  $\epsilon > 0$ . In addition, if  $\arccos(k_1/k_3) + \theta_{1,\max} < \pi/2$ , then the control law  $u = \phi(F(y))$  globally practically stabilizes  $\Sigma$  with respect to  $\mathbb{B}_{\epsilon}$ .

**Proof.** We prove the theorem by showing that for any given  $y \in \mathbb{R}^m$ , we have either

$$\langle \phi(F(y)), y \rangle \in \{-\kappa_{i,y} \|u_i\| \|y\| \mid i \in J\}$$

for some  $J \subset \{1, \dots, p\}$  with  $\kappa_{i,y} > 0$  and  $\{u_i\}_{i \in J}$  be the nearest element(s) of  $\mathcal{U}$  to  $-F(y)$ ; or when  $0 \in \phi(F(y))$

$$\langle \phi(F(y)), y \rangle \in \{0\} \cup \{-\kappa_{i,y} \|u_i\| \|y\| \mid i \in J\}$$

for some  $J \subset \{1, \dots, p\}$ , or  $\langle \phi(F(y)), y \rangle = 0$  if  $\phi(F(y))$  is a singleton and is given by 0. The rest of the proof follows similarly to that of Proposition 2.

Similar to the arguments used in the proof of that proposition, suppose that  $\phi(F(y)) = \{u_i\}_{i \in J}$  for some  $J$  and  $0 \notin \phi(F(y))$ . It follows from (8) that  $\{u_i\}_{i \in J}$  are the closest points to  $-F(y)$  which implies that

$$\langle \phi(F(y)), -F(y) \rangle \in \{\|u_i\| \|F(y)\| \cos(\theta_{i,1}) \mid i \in J\},$$

where  $\theta_{i,1}$  is the angle between the vectors  $u_i$  and  $-F(y)$ . By the hypothesis of the proposition,  $|\theta_{i,1}| \leq \theta_{1,\max}$  for any  $i$ . On the other hand,

$$\langle -F(y), -y \rangle = \|F(y)\| \|y\| \cos(\theta_2),$$

where  $\theta_2$  is the angle between the vectors  $F(y)$  and  $y$ . Since  $k_1\|y\|^2 \leq \langle F(y), y \rangle$  and  $\|F(y)\| \leq k_3\|y\|$ , the maximum angle  $\theta_2$  is given by  $\theta_{2,\max} = \arccos(k_1/k_3)$ .

Hence the inner product between the vectors  $\phi(F(y))$  and  $-y$  satisfies

$$\langle \phi(F(y)), -y \rangle \in \{\|u_i\| \|y\| \cos(\theta_{i,3}) \mid i \in J\},$$

where  $|\theta_{i,3}| \leq |\theta_{i,1}| + |\theta_2| \leq \theta_{1,\max} + \theta_{2,\max} < \pi/2$ . Therefore

$$\langle \phi(F(y)), y \rangle \leq \{-\kappa_y \|u_i\| \|y\| \mid i \in J\}$$

where  $\kappa_y = \cos(\theta_{1,\max} + \arccos(k_1/k_3)) > 0$ .

Following the same line of arguments as in the proof of Proposition 2, the solutions  $x(t)$  of the differential inclusion (10) converge to the invariant set  $M$ , in which  $\phi(F(y(t))) = 0$  for all  $t$ . Hence in  $M$ ,  $\|F(y(t))\| \leq C_{\mathcal{V}}$  for all  $t$ . Since  $k_1\|v\|^2 \leq \langle F(v), v \rangle \leq \|F(v)\| \|v\|$  holds for all  $v \in \mathbb{R}^m$ , it follows that  $\|y(t)\| \leq \frac{C_{\mathcal{V}}}{k_1}$  for all  $t$  in  $M$ . By the property of large-time norm-observability of (4), it holds that in the invariant set  $M$

$$\|x(t)\| \leq \gamma(C_{\mathcal{V}}/k_1) \leq \epsilon \quad \forall t,$$

where the last inequality is due to the hypotheses of the proposition. Thus, by the LaSalle invariance principle

$$x(t) \rightarrow M \subset \mathbb{B}_{\epsilon} \quad \text{as } t \rightarrow \infty.$$

#### 4. EXAMPLE AND SIMULATION RESULTS

In this section, we will apply our main results to an academic example and illustrate the behaviour of the closed-loop system through a numerical simulation.

*Example 2.* Consider the following nonlinear system

$$\Sigma_{\text{ex}} : \begin{cases} \dot{x} = \begin{bmatrix} -x_2 + x_3^2 \\ x_1 + x_3^2 \\ -x_1 x_3 - x_2 x_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u \\ y = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} \end{cases} \quad (13)$$

where  $x := [x_1 \ x_2 \ x_3]^T \in \mathbb{R}^3$  and  $y := [y_1 \ y_2]^T, u := [u_1 \ u_2]^T \in \mathbb{R}^2$ . It can be checked that by using the proper storage function  $H(x) = \frac{1}{2}x^T x$ , the system  $\Sigma_{\text{ex}}$  is passive.

Indeed, a straightforward computation gives us  $\dot{H} = \langle y, u \rangle$ .

We will now show that  $\Sigma_{\text{ex}}$  satisfies the large-time norm observability condition. As the bound on  $x_1$  and  $x_3$  for the large-time norm observability can directly be obtained from the output  $y$ , we need to compute the bound on  $x_2$ . If we consider the sub-system of  $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$  with  $x_1$  as its output (and is equal to  $y_1$ ), it is a linear system with  $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ ,  $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ,  $C = [1 \ 0]$  and its input is  $\begin{bmatrix} x_3^2 & x_3^2 \end{bmatrix}^T$ . Thus, as  $(A, C)$  is observable, we can choose the following observability Gramian

$$W_{\pi} = \int_0^{\pi} e^{A^T s} C^T C e^{As} ds = \frac{\pi}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

whose inverse is simply given by  $W_{\pi}^{-1} = \frac{2}{\pi} I_2$  and  $\|W_{\pi}^{-1}\| = \frac{2}{\pi}$ . Then for any  $t > 0$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = W_{\pi}^{-1} \int_t^{t+\pi} e^{A^T s} C^T \left( x_1(s) - \left( L * \begin{bmatrix} x_3^2 \\ x_3^2 \end{bmatrix} \right)(s) \right) ds,$$

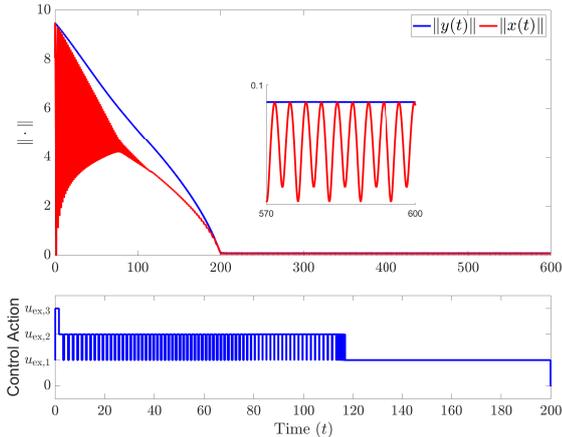


Fig. 2. Simulation results of  $\Sigma_{\text{ex}}$  using nearest neighbor control approach with discrete input set  $\mathcal{U}_{\text{ex}}$  as in (11) and fixed parameters  $\theta_{\text{ex}} = 0$  and  $\alpha = 0.1$ . The evolution of control actions is plotted with shorter time frame to show the switch before the state  $x$  and the output  $y$  finally enter the desired forward invariant set.

where  $*$  denotes the convolution operation and  $L$  is the convolution matrix kernel given by  $L(t) = Ce^{At}$ . Since  $\|e^{At}\| = 1$  for all  $t$ , it follows then that

$$\begin{aligned} & \left\| \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \right\| \\ & \leq \|W_{\pi}^{-1}\| \left\| \int_t^{t+\pi} e^{A^{\top}s} C^{\top} \left\| x_1(s) - \left( L * \begin{bmatrix} x_3^2 \\ x_3 \end{bmatrix} \right) (s) \right\| ds \right\| \\ & \leq \frac{2}{\pi} \pi \left( \|y_1\|_{[t,t+\pi]} + \sqrt{2} \|y_2\|_{[t,t+\pi]}^2 \right) \\ & \leq 2\|y\|_{[t,t+\pi]} + 4\|y\|_{[t,t+\pi]}^2. \end{aligned}$$

Since by the definition of  $y$ ,  $\|x_1\|_{[t,t+\pi]} = \|y_1\|_{[t,t+\pi]} \leq \|y\|_{[t,t+\pi]}$  and  $\|x_3\|_{[t,t+\pi]} = \|y_2\|_{[t,t+\pi]} \leq \|y\|_{[t,t+\pi]}$ , it follows from the inequality above that

$$\|x\|_{[t,t+\pi]} \leq 4(\|y\|_{[t,t+\pi]} + \|y\|_{[t,t+\pi]}^2).$$

In other words, the function  $\gamma$  in (5) is given by  $\gamma(s) = 4(s + s^2)$ . Note that when the output of  $\Sigma_{\text{ex}}$  is only  $x_1$  or  $x_3$ , it is not large-time norm observable and it is even not zero-state observable/detectable.

We will now use the result in Proposition 2 for practical stabilization of  $\Sigma_{\text{ex}}$ . We choose the control set to be  $\mathcal{U}_{\text{ex}}$  given in (11), and the desired stability margin to be  $\epsilon = 0.5$ . Then, based on the function  $\gamma$  computed for the system  $\Sigma_{\text{ex}}$ , we get  $\gamma(C_{\mathcal{V}}) < 0.5$  if  $C_{\mathcal{V}} \in \left( 0, -\frac{1}{2} + \frac{1}{4}\sqrt{6} \right]$ . By letting  $\theta_{\text{ex}} = 0$  and  $\alpha = 0.1$  in (11), it is seen that  $\Sigma_{\text{ex}}$  is globally practically stable with respect to  $\mathbb{B}_{\epsilon}$ , with  $\epsilon = 0.5$ , as confirmed by simulations reported in Figure 2.

## 5. CONCLUSIONS AND FURTHER RESEARCH

We have considered stabilization of continuous-time passive nonlinear systems, under appropriate observability assumption, using output feedback where the control input can switch between finitely many values. Our results provide a lower bound on the number of control elements, and conditions on their configuration in input space, which guarantee practical stability. We are currently investigating an algorithmic procedure for designing the finite control actions with minimal cardinality such that

the value of the constant  $C_{\mathcal{V}}$  is minimized. Questions related to improving the convergence rate with more (than necessary) control elements also require further research.

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