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Abstract: This paper deals with the stabilization of switched affine systems. The particularities of this class of nonlinear systems are first related to the fact that the control action is performed through the selection of the switching mode to be activated and, second, to the problem of providing an accurate characterization of the set where the solutions to the system converge to. In this paper, we propose a new method based on a control Lyapunov function, that provides a more accurate invariant set for the closed-loop systems, which is composed by the union of potentially several disjoint subsets. The main contribution is presented as a non convex optimization problem, which refers to a Lyapunov-Metzler condition. Nevertheless a gridding technique applied on some parameters allows obtaining a reasonable solution through an LMI optimization. The method is then illustrated on two numerical examples.

Keywords: Control of switched systems ; Lyapunov methods ; Stability of nonlinear systems

1. INTRODUCTION

Switched systems represent a subclass of hybrid systems (Liberzon, 2003a), (Shorten et al., 2007) encountered in many applications such event-triggered control, mobile sensor networks, damping of vibrating structures, among many others. The reader may refer to (Antunes and Heemels, 2017) to know more about these applications. While one can find a rich literature on the stabilization of switched linear systems, as for instance in (Feron, 1996; Lin and Antsaklis, 2009; Liberzon, 2003b), the problem of stabilizing switched affine systems has been less regarded even though this class of nonlinear systems is of particular interest for the analysis of sampled-data sliding mode controllers (Su et al., 2000) or for the stabilization of DC-DC converters (Deaecto et al., 2010; Beneux et al., 2019).

Stabilizing switched affine systems aims at ensuring the convergence of the state to a selected reference point that does not necessarily coincide with the equilibriums of the modes. Several control design methods have been considered in the literature as for instance in (Hetel and Fridman, 2013; Skafidas et al., 1999; Seatzu et al., 2006; Hetel and Bernuau, 2014) or in (Deaecto et al., 2010; Albea-Sanchez et al., 2015; Beneux et al., 2019), where a particular attention to the application to power converters is paid. It is worth mentioning that these design strategies are not fully satisfactory in practice because they theoretically produce, around the equilibrium, an infinite number of control updates, which is not reasonable because of implementation constraints. This problem is similar to the one arising in the implementation of sliding mode controllers, where chattering effects occur when the state enters in the sliding surface (Edwards and Spurgeon, 1998; Shtessel et al., 2014). To solve this issue, a possible solution is to introduce a minimum latency between two successive control updates, also known as a dwell time constraint (Senesky et al., 2003; Buisson et al., 2005; Theunisse et al., 2015; Albea-Sanchez et al., 2019). One can note, however, that the resulting control signals updates are sampled in an aperiodic manner and, in some occasions, due to practical constraints, one needs to impose a periodic implementation. Moreover, it is worth mentioning a robust approach with respect to aperiodic sampled-data switching controllers was investigated in (Hauroigne et al., 2011; Hetel and Fridman, 2013). Nevertheless, all the previous design solutions are based on a common quadratic Lyapunov function, which is known, in the linear case, to be conservative and/or restrictive.

Another solution provided in the literature consists in considering periodic updates of the control input and the resulting discrete-time formulation of the switched affine system. The objective here is to ensure the stabilization of the state to a neighborhood of the reference. Where this solution obviously presents a Zeno behaviour at the reference, the price to pay is that the discrete-time system cannot be stabilized to a single point but rather to a suitable region. In this situation the authors of (Deaecto and Geromel, 2017; Ventosa-Cutillas et al., 2019) provide a solution considering a common and quadratic Lyapunov function, which is conservative, leading to a practical stabilization result. This approach was latter relaxed in (Egidio and Deaecto, 2019), where the design of practically stabilizing control law was developed thanks to a switched Lyapunov function, reducing then the inherent

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conservatism of the resulting condition.

The present paper aims at providing a novel method for the design of stabilizing control laws for switched affine systems. The novelty of the method relies on the use of another switching control Lyapunov function, with a structure that notably differs from the one proposed in (Egidio and Deaecto, 2019). This method gives rise to a control law resulting from a non convex optimization problem referring to Lyapunov-Metzler conditions (see e.g. (Geromel and Colaneri, 2006; Heemels et al., 2016)). Thanks to a gridding technique, this problem can be solved iteratively by producing a convex problem formulated as a Linear Matrix Inequality (LMI). Interestingly, this new method ensures the convergence of the trajectories of the system to the interior of an invariant attractive region composed of several possibly disjoint ellipsoidal regions. A comparison with the numerical application presented in (Egidio and Deaecto, 2019) is proposed and shows the efficiency of our method, since it provides a smaller and more accurate invariant set.

The paper is organized as follows: the problem is stated in Section 2. Then, a switched control design is provided in Section 3. Section 4 is devoted to numerical applications of our method. The paper ends with a conclusion and several perspectives.

Notations: Throughout the paper, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{R}$ the real numbers, $\mathbb{R}^n$ the n-dimensional Euclidean space and $\mathbb{R}^{n\times m}$ the set of all real $n \times m$ matrices. For any n and m in $\mathbb{N}$, matrices I_n and $\mathbf{0}_{n,m}$ denote the identity matrix of $\mathbb{R}^{n\times n}$ and the null matrix of $\mathbb{R}^{n\times m}$, respectively. When no confusion is possible, the subscripts of these matrices that specify the dimension, will be omitted. For any matrix M of $\mathbb{R}^{n\times n}$, the notation $M \succ 0$, $(M \prec 0)$ means that M is symmetric positive (negative) definite and $\det(M)$ represents its determinant. Finally, we define Λ as the subset of $(0,1)^K$ such that an element λ in Λ has its components, λ_i in $(0,1)$ for all $i \in \mathbb{K} := \{1,\ldots,K\}$ and verifies $\sum_{i\in\mathbb{K}} \lambda_i = 1$.

2. PROBLEM FORMULATION

2.1 System data

Consider discrete-time switched affine system

$$\begin{cases} x_{k+1} = A_{\sigma_k} x_k + B_{\sigma_k}, & k \in \mathbb{N}, \\ \sigma_k = u(x_k) \in \mathbb{K}, \\ x_0 \in \mathbb{R}^n, \end{cases} \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state vector, which is assumed to be known at each time instant k in $\mathbb{N}$. The switched system is composed of K subsystems defined through the constant and known matrices $A_i \in \mathbb{R}^{n\times n}$ and $B_i \in \mathbb{R}^{n\times 1}$ for all $i \in \mathbb{K}$. The active mode is characterized by variable σ_k, which can take any value in $\mathbb{K}$. The particularity of this class of systems relies on the fact that the only control action u is performed through the selection of the active mode σ_k.

The objective of this paper is the design of a suitable control law for system (1) that ensures the convergence of the state trajectories to a set to be characterized in an accurate manner. Indeed, it is well-known that asymptotic stability of a single equilibrium of (1) cannot be achieved in general for switched affine system (1). Due to the affine, and consequently nonlinear nature, one has to relax the control objective to derive an acceptable stability result. For instance, in (Deaecto and Geromel, 2017), the authors have derived a practical stability result. More precisely, it is shown therein, that the solutions to the switched affine systems converge to an invariant region characterized by a level set of a Lyapunov function centered at a desired operating point or at a slightly shifted point nearby this operating point.

Here, our objective is to go deeper into the analysis of switched affine systems and try to characterize in a more accurate manner the region where the solutions to the system converge to. This new analysis is achieved thanks to a different class of Lyapunov functions than the quadratic ones. Indeed, one has to use more advanced tools and Lyapunov functions arising in switched affine systems in order to derive more accurate results. A first attempt was considered in (Geromel and Colaneri, 2006) for switched linear systems, where the Lyapunov function is defined using different Lyapunov matrices. It is noteworthy that the discrete-time nature of the dynamics (1) allows to consider classes of Lyapunov functions associated with possibly disconnected level sets, as pointed out in (Cavichioli Gonzaga et al., 2012). Here, we propose a different Lyapunov function inspired from (Egidio and Deaecto, 2019) and defined as follows

$$V(x) = \min_{i\in\mathbb{K}} (x - \rho_i)^\top P (x - \rho_i), \quad \forall x \in \mathbb{R}^n, \tag{2}$$

where P is a symmetric positive definite matrix of $\mathbb{R}^{n\times n}$ and where, for any $i \in \mathbb{K}$, ρ_i are vectors of $\mathbb{R}^n$. These vectors represent several possible shifted centers of the Lyapunov function and are to be characterized in a dedicated manner.

Remark 1. In the definition of the Lyapunov function, it is assumed that the number of shifted centers is the same as the number of modes. This can be seen a restrictive assumption that could be relaxed but this goes beyond the objectives of this paper, which concerns the development of a new control-Lyapunov function for switched affine systems.

As mentioned above, ensuring asymptotic stability of an equilibrium is in general not possible for switched affine systems. However, it is still possible to derive practical stability result as a given level set of a Lyapunov function. Following the same ideas as in the literature, our objective is to guarantee the practical stability of a level set of this new Lyapunov function. This level set, also called attractor is defined as follows:

$$\mathcal{A} := \{x \in \mathbb{R}^n \quad | \quad V(x) \le 1\}, \tag{3}$$

where we recall that the Lyapunov function V is defined by (2). Depending on the selection of matrix P and on the shifted centers ρ_i, the attractor may not be a convex nor a connected set. Indeed, we will see in the example section, that the level set of the Lyapunov function may characterize several disjoint regions.

2.2 Preliminary

This preliminary is taken from (Geromel and Colaneri, 2006) and provides an equivalence between a minimum of a set of values and their convex linear combination. The following lemma taken from (Cavichioli Gonzaga, 2012) formalizes this statement.

Lemma 2. (Cavichioli Gonzaga (2012)). For any scalars v_i, where $i \in \mathbb{K}$, the following equality holds

$$\min_{i\in\mathbb{K}} v_i = \inf_{\alpha\in\Lambda} \sum_{i\in\mathbb{K}} \alpha_i v_i. \tag{4}$$

Remark 3. There is a small difference with respect to the original formulation of (Cavichioli Gonzaga, 2012). Indeed, here, Λ is an open set (i.e. the α_i's cannot be equal to 0 nor 1). Equality (4) will be a key element of the next developments. In particular, the previous lemma ensures that for any element α in Λ, the following inequality holds

$$\min_{i\in\mathbb{K}} v_i < \sum_{i\in\mathbb{K}} \alpha_i v_i. \tag{5}$$

3. STATE BASED SWITCHING CONTROL

3.1 Main result

We dedicate this section to design a novel switching control law based on the state x_k from (1) which is assumed to be known at every time instant k. This stabilization result is stated in the following theorem.

Theorem 4. Consider parameters $\mu \in (0,1)$, $\lambda^{(i)} \in \Lambda$, with $i \in \mathbb{K}$, a positive definite matrix $W \in \mathbb{R}^{n\times n} \succ 0$, and $\rho_i \in \mathbb{R}^N$ that are the solutions to the following non convex optimization problem

$$\min_{\mu,W,\rho_i,\lambda_i} \quad \mathrm{Tr}(W) \tag{6}$$

subject to the constraints

$$W \succ 0, \tag{7}$$

$$\begin{bmatrix} -(1-\mu)W & 0 & \mathbb{A}_i(\lambda^{(i)}) \\ * & -\mu & \mathbb{B}_i(\lambda^{(i)}) \\ * & * & -\mathbb{D}_i(\lambda^{(i)}) \end{bmatrix} \prec 0, \quad \forall i \in \mathbb{K}, \tag{8}$$

where

$$\mathbb{A}_i(\lambda^{(i)}) = \left[\lambda_1^{(i)} W A_i^\top \ \dots \ \lambda_K^{(i)} W A_i^\top\right],$$
$$\mathbb{B}_i(\lambda^{(i)}) = \left[\lambda_1^{(i)} (A_i\rho_i + B_i - \rho_1)^\top \ \dots \ \lambda_K^{(i)} (A_i\rho_i + B_i - \rho_K)^\top\right],$$
$$\mathbb{D}_i(\lambda^{(i)}) = \mathrm{diag}(\lambda_1^{(i)} W, \dots, \lambda_K^{(i)} W).$$

Then, the switching function control law given by

$$u(x) \in G(x) = \operatorname*{argmin}_{i\in\mathbb{K}} (x-\rho_i)^T W^{-1} (x-\rho_i), \ \forall x \in \mathbb{R}^n, \tag{9}$$

ensures that $\mathcal{A}$ is uniformly globally asymptotic stable (UGAS) for system (1).

Proof. The proof aims at demonstrating that $\mathcal{A}$ defined in (3) is UGAS provided that the conditions of Theorem 4 are verified. To do so, the two following items have to be considered

- V given in (2) is a Lyapunov function for the system (1), (9), with $P = W^{-1}$.
- $\mathcal{A}$ is invariant for the system (1),(9).

In order to prove the first item, let us compute the increment of the Lyapunov function. This leads to

$$\begin{aligned} \Delta V(x_k) = & \min_{j\in\mathbb{K}} (x_{k+1} - \rho_j)^\top P (x_{k+1} - \rho_j) \\ & - \min_{i\in\mathbb{K}} (x_k - \rho_i)^\top P (x_k - \rho_i). \end{aligned}$$

According to the switching control law (9), the active mode σ_k corresponds to the mode that minimizes the Lyapunov function at time k, which allows us to write

$$\begin{aligned} \Delta V(x_k) = & \min_{j\in\mathbb{K}} (x_{k+1} - \rho_j)^\top P (x_{k+1} - \rho_j) \\ & - (x_k - \rho_{\sigma_k})^\top P (x_k - \rho_{\sigma_k}). \end{aligned}$$

Thanks to inequality (5), the following inequality holds for any element $\lambda^{(\sigma_k)}$ in Λ.

$$\begin{aligned} \Delta V(x_k) < & \sum_{j\in\mathbb{K}} \lambda_j^{(\sigma_k)} (x_{k+1} - \rho_j)^\top P (x_{k+1} - \rho_j) \\ & - (x_k - \rho_{\sigma_k})^\top P (x_k - \rho_{\sigma_k}), \end{aligned}$$

where $\lambda_j^{(\sigma_k)}$ is the j^{th} component of $\lambda^{(\sigma_k)}$.

Let us now focus on the first positive terms of the previous expression. Replacing x_{k+1} by its expression from (1), we note that $x_{k+1} - \rho_j = A_{\sigma_k} x_k + B_{\sigma_k} - \rho_j$. Our objective is to rewrite the previous expression using $x_k - \rho_{\sigma_k}$, in order to take the full benefits of the negative terms of the Lyapunov increment. Simple manipulations yield

$$x_{k+1} - \rho_j = A_{\sigma_k}(x_k - \rho_{\sigma_k}) + A_{\sigma_k}\rho_{\sigma_k} + B_{\sigma_k} - \rho_j.$$

Let us now introduce a new vector, χ_k, given by

$$\chi_k^\top = \left[(x_k - \rho_{\sigma_k})^\top P \quad 1\right]^\top, \tag{10}$$

and matrix $W = P^{-1} \succ 0$ and then, we obtain the following expression

$$x_{k+1} - \rho_j = \left[A_{\sigma_k} W \quad A_{\sigma_k}\rho_{\sigma_k} + B_{\sigma_k} - \rho_j\right] \chi_k.$$

Hence, gathering all the terms in the sum and using the notations introduced in Theorem 4, we have

$$\Delta V(x_k) = \chi_k^\top \Phi(\sigma_k, \lambda^{(\sigma_k)}) \chi_k, \tag{11}$$

where matrix $\Phi(\sigma_k, \lambda^{(\sigma_k)})$ is given by

$$\begin{aligned} \Phi(\sigma_k, \lambda^{(\sigma_k)}) = & \begin{bmatrix} \mathbb{A}_{\sigma_k}(\lambda^{(\sigma_k)}) \\ \mathbb{B}_{\sigma_k}(\lambda^{(\sigma_k)}) \end{bmatrix} \mathbb{D}_{\sigma_k}^{-1}(\lambda^{(\sigma_k)}) \begin{bmatrix} \mathbb{A}_{\sigma_k}(\lambda^{(\sigma_k)}) \\ \mathbb{B}_{\sigma_k}(\lambda^{(\sigma_k)}) \end{bmatrix}^\top \\ & - \begin{bmatrix} W & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

It is worth noting that the components of $\lambda^{(i)}$ are assumed to be strictly positive. Now, that the difference Lyapunov function has been properly expressed, the next step consists in ensuring its negative definiteness only outside the attractor defined in (3). To do so, we use notations χ_k and W introduced above and we note that any vector x_k outside of the attractor verifies

$$\begin{bmatrix} x_k - \rho_{\sigma_k} \\ 1 \end{bmatrix}^\top \begin{bmatrix} P & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_k - \rho_{\sigma_k} \\ 1 \end{bmatrix} = \chi_k^\top \begin{bmatrix} W & 0 \\ 0 & -1 \end{bmatrix} \chi_k > 0. \tag{12}$$

The previous problem can be rewritten as

$$\chi_k^\top \Phi(\sigma_k, \lambda^{(\sigma_k)}) \chi_k < 0$$

for all vector χ_k that verifies (12). Using an S-procedure, this problem is equivalent to the existence of a positive scalar μ, such that,

$$\Phi(\sigma, \lambda^{(\sigma_k)}) + \mu \begin{bmatrix} W & 0 \\ 0 & -1 \end{bmatrix} \prec 0.$$

Then, the proof of the first item is concluded by application of Schur complement to the first term of $\Phi(\sigma_k, \lambda^{(\sigma_k)})$, leading to inequality (8).

To conclude the proof, it remains to prove that $\mathcal{A}$ is invariant, corresponding to the second item. Assume that

x_k is in the attractor at a given instant k, i.e. $V(x_k) < 1$. Together with (8), we know that the following inequality

$$\begin{aligned} V(x_{k+1}) &= V(x_k) + \Delta V(x_k) \\ &= V(x_k) - \mu(V(x_k) - 1) \\ &\quad + \chi_k^\top \left(\Phi(\sigma_k, \lambda^{(\sigma_k)}) + \mu \left[\begin{smallmatrix} W & 0 \\ 0 & -1 \end{smallmatrix}\right]\right) \chi_k \\ &\leq (1-\mu)V(x_k) + \mu \end{aligned}$$

holds where the last inequality has been obtained from the negative definiteness of inequality (8). The assumptions that x_k is in the attractor and that $\mu \in (0,1)$ yield

$$V(x_{k+1}) \leq (1-\mu) + \mu = 1,$$

which guarantees that x_{k+1} also belongs to $\mathcal{A}$.

3.2 Comment on the centers ρ_i and on translated models

Usually for this class of switched affine systems, it is first required to define a translated system where the origin becomes located at a desired operating point. The reader may refer to (Deaecto and Geromel, 2017), for instance. It is then important to stress whether the attractor is affected by this translation. To better understand this issue, let us define the translated variable $z = x - \delta$, where δ is any vector in $\mathbb{R}^n$. The new dynamics are given by

$$\begin{cases} z_{k+1} = A_{\sigma_k} z_k + \tilde{B}_{\sigma_k}, & k \in \mathbb{N}, \\ \sigma_k \in \mathbb{K}, \\ z_0 \in \mathbb{R}^n, \end{cases} \tag{13}$$

where $\tilde{B}_{\sigma_k} = (A_{\sigma_k} - I)\delta + B_{\sigma_k}$. Then, the following proposition holds.

Proposition 5. Assume that $(\mu, W, \{\rho_i, \lambda^{(i)}\}_{i\in\mathbb{K}})$ is a solution to the optimization problem of Theorem 4 for the original systems $(A_i, B_i)_{i\in\mathbb{K}}$. Then, $(\mu, W, \{\rho_i - \delta, \lambda^{(i)}\}_{i\in\mathbb{K}})$ is a solution to the same optimization problem but for the translated system $(A_i, \tilde{B}_i)_{i\in\mathbb{K}}$.

Proof. The proof simply consists in noting that the only difference between the original and the translated system appears in the definition of the affine terms that are gathered in matrices $\mathbb{B}_i(\lambda^{(i)})$.

Proposition 5 stresses that the shifted centers ρ_i's are intrinsically the same, whatever the translation of coordinates. This is an important remark, since it proves that there is no need to apply any change of coordinates before applying Theorem 4.

3.3 Comments on the resolution of the non convex problem

As indicated in its statement, the optimization problem of Theorem 4 is non convex due to the multiplication of decision variables, such as for instance $\lambda_j^{(i)} W$ in the definition of matrices $\mathbb{D}_i$. However, this problem can be made convex by fixing $\mu \in (0,1)$ and $\lambda^{(i)} \in (0,1)^K$, with $i \in \mathbb{K}$. Of course, this is not realistic for large values of K, but for $K = 2$, the number of parameters to fix is only 3, which is reasonable. This is formulated in the following proposition.

Proposition 6. For given parameters $\mu, \gamma_1, \gamma_2 \in (0,1)$, the solution including the symmetric positive definite matrix $W \in \mathbb{R}^{n\times n} \succ 0$ and the vectors $\rho_i \in \mathbb{R}^N$ to the convex optimization problem

$$\min_{W,\rho_i} \quad \mathrm{Tr}(W) \tag{14}$$

subject to the constraints

$$W \succ 0, \tag{15}$$

$$\begin{bmatrix} -(1-\mu)W & 0 & \bar{\mathbb{A}}_i(\gamma_i) \\ * & -\mu & \bar{\mathbb{B}}_i(\gamma_i) \\ * & * & -\bar{\mathbb{D}}_i(\gamma_i) \end{bmatrix} \prec 0, \quad \forall i = 1, 2, \tag{16}$$

where

$$\begin{aligned} \bar{\mathbb{A}}_i(\gamma_i) &= \left[\gamma_i W A_i^\top \;\; (1-\gamma_i) W A_i^\top\right], \\ \bar{\mathbb{B}}_i(\gamma_i) &= \left[\gamma_i (A_i\rho_i + B_i - \rho_1)^\top \;\; (1-\gamma_i)(A_i\rho_i + B_i - \rho_2)^\top\right], \\ \bar{\mathbb{D}}_i(\gamma_i) &= \mathrm{diag}(\gamma_i W, (1-\gamma_i)W). \end{aligned}$$

Then, the switching function control law $\sigma_k \in G_2(x_k)$ given by

$$G_2(x) = \underset{i=1,2}{\mathrm{argmin}} (x - \rho_i)^\top W^{-1} (x - \rho_i), \; \forall x \in \mathbb{R}^n, \tag{17}$$

ensures that $\mathcal{A}$ is UGAS for system (1).

Proof. The proof is obtained by the introduction of parameters γ_i, such that, for $K = 2$, we have $\lambda_1^{(i)} = \gamma_i$ and $\lambda_2^{(i)} = 1 - \gamma_i$.

4. SIMULATION RESULTS

Through this section, we aim at illustrating our contributions through two examples that have been already treated in the literature.

Example 7. Consider the discrete-time switched affine system borrowed from (Deaecto and Geromel, 2017), as modeled in (1), with two modes ($K = 2$) and the following matrices

$$A_i = e^{F_i T}, \quad B_i = \int_0^T e^{F_i \tau} d\tau g_i, \quad \forall i \in \{1, 2\}, \tag{18}$$

where T, referring to a sampling period is taken equal to 1 and where matrices F_i and g_i for $i \in \mathbb{K}$ are given by

$$F_1 = \left[\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{smallmatrix}\right], \; F_2 = \left[\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{smallmatrix}\right], \; g_1 = \left[\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right], \; g_2 = \left[\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right].$$

In (Deaecto and Geromel, 2017), the authors considered the convergence of the state trajectories to an invariant set around a desired equilibrium. Therefore, they solved the problem by introducing an auxiliary variable and defining the translated system with that variable. In Section 3.2, we comment and prove that the solution found for system (1) is a solution for system (13).

As it has been commented in Section 3.3, the optimization problem is non convex. Using a gridding procedure to fix the parameters μ and γ_i, the resulting optimization problem becomes convex and is solvable using SDP software as the CVX solver in Matlab (see (Grant and Boyd, 2014)). The following numerical results are obtained:

$$\mu = 0.7929, \quad \gamma_1 = 10^{-5}, \quad \gamma_2 = 1 - 10^{-5},$$

$$W = \left[\begin{smallmatrix} 5.6 & -4 & 2 \\ -4 & 6.2 & -4.2 \\ 2 & -4.2 & 7.4 \end{smallmatrix}\right] .10^{-10}, \rho_1 = \left[\begin{smallmatrix} 0.1 \\ 0.4 \\ 0.37 \end{smallmatrix}\right], \rho_2 = \left[\begin{smallmatrix} 0.69 \\ -0.2 \\ -0.6 \end{smallmatrix}\right].$$

Figure 1 shows the trajectories of the system. The centers are indicated by the two red crosses. With the full view of the temporal evolution, we cannot see the ellipsoids drawing the attractor. However, they appear after performing a zoom of them in the two windows. These views allow us to see the convergence of the trajectories toward the interior

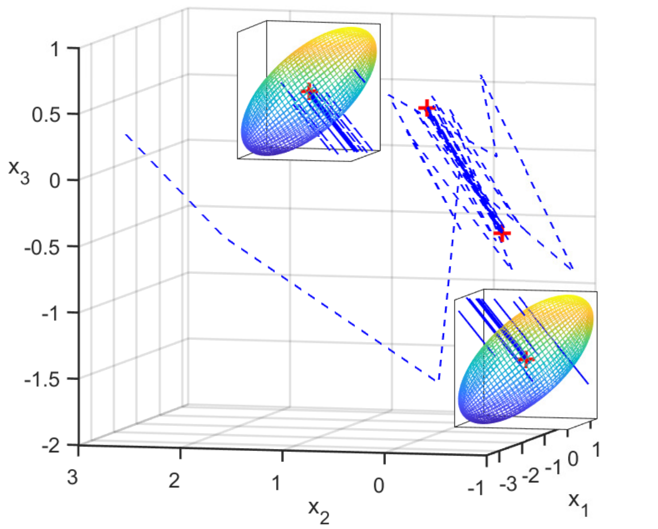

Fig. 1. Trajectories of system (1) for Example 1 in the state space. Two windowsshow the attractor located around the shifted centers.

of the two ellipsoids, which differ only by their center.

An alternative interpretation of the previous figure is shown on Figures 2, where the evolution of $x_k - \rho_{\sigma_k}$ with respect to k is plotted. One can see from this figure that the trajectories are indeed converging to the centers ρ_i. It is also of interest to point out that the switching law tends to a periodic behaviour and that the state converges to a cycle $k \mapsto \rho_{\sigma_k}$.

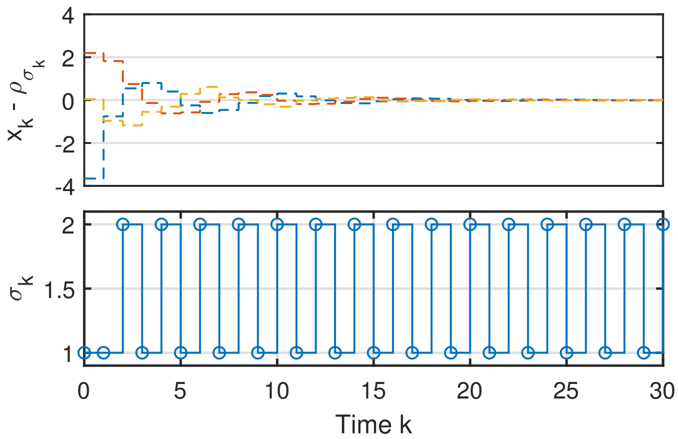

Fig. 2. Evolution of $x_k - \rho_{\sigma_k}$ with the switching function $\sigma \in \{1, 2\}$ computed at instant k for Example 1.

Example 8. Now, we take the example 1 given in (Egidio and Deaecto, 2019). The considered system is a discrete-time switched affine system discretized using (18) with $T = 0.5$ which provides the following matrices :

$$F_1 = \begin{bmatrix} -5.8 & -5.9 \\ -4.1 & -4 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0.1 & -0.5 \\ -0.3 & -5 \end{bmatrix},$$

$$g_1 = \begin{bmatrix} 0 & -2 \end{bmatrix}^\top, \qquad g_2 = \begin{bmatrix} -2 & 2 \end{bmatrix}^\top.$$

Considering the gridding procedure used in Example 1 to find the parameters μ and γ_i $\forall i \in \{1, 2\}$, we have

$$\mu = 0.997, \quad \gamma_1 = 10^{-5}, \quad \gamma_2 = 1 - 10^{-5},$$

$$W = \begin{bmatrix} 1.3 & 0.5 \\ 0.5 & 1.8 \end{bmatrix} .10^{-10}, \rho_1 = \begin{bmatrix} -1.7 \\ 0.47 \end{bmatrix}, x\rho_2 = \begin{bmatrix} -0.54 \\ 0.33 \end{bmatrix}.$$

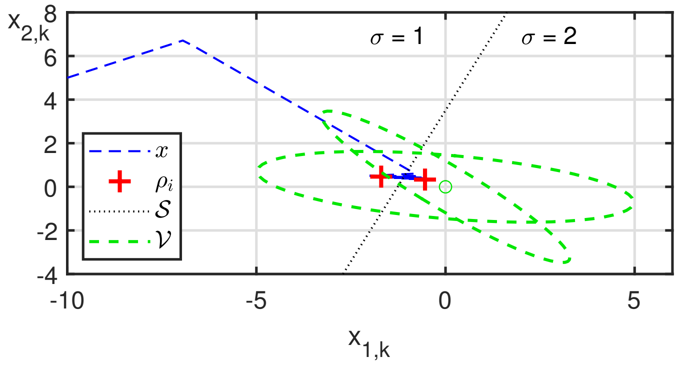

Fig. 3. Trajectories of system (1) for Example 2 in the state space. The switching surface (dotted line), the centers ρ_1 and ρ_2 (red crosses) and the invariant set (dashed line) from (Egidio and Deaecto, 2019).

Figure 3 shows the trajectory of the state x_k from the initial state toward the two shifted ellipses. Note that the ellipses depicted on this figure present such a reduced size that they are represented by a cross in each center, ρ_i. In addition, the dotted line $\mathcal{S}$, defined by $(x - \rho_1)^\top W^{-1} (x - \rho_1) = (x - \rho_2)^\top W^{-1} (x - \rho_2)$ portrays the switched surface, which separates the space into two regions. One can see which mode is active depending on which side the state x_k is.

Moreover, in the concerned figure, we compare the result given in (Egidio and Deaecto, 2019) with our main result. The readers are able to see the different set sizes. The dashed ellipsoids represent the invariant set obtained in (Egidio and Deaecto, 2019) while the invariant set obtained from Proposition 6 is illustrated by the crosses as commented above. Note that our approach provides a attractor at least 10^9 smaller than the one provided in (Egidio and Deaecto, 2019).

Finally, one can note the decreasing value of the Lyapunov function on Figure 4. It is highlighted the invariant character of the attractor. Once $V(x_k)$ goes under 1, it remains under this value.

5. CONCLUSIONS

In this paper, the problem of designing a stabilizing switched control law for switched affine systems has been addressed. Thanks to a new control Lyapunov function,

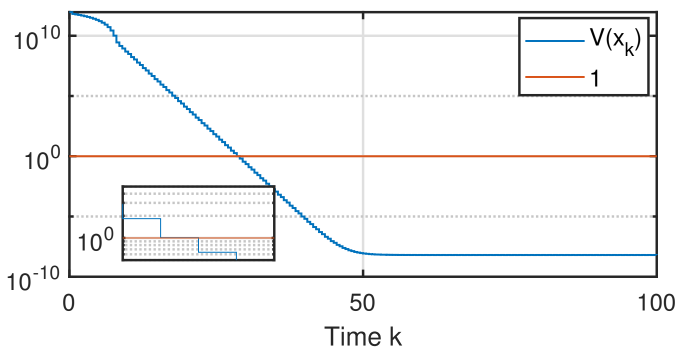

Fig. 4. Evolution of the Lyapunov function $V(x_k)$ for Example 2.

arising from the stability analysis of switched systems, an accurate characterisation of the attractor is formulated. The parameter of the control law are obtained through the solution of a non convex optimization problem, that can be efficiently solved using a gridding procedure in the situation of 2-modes switched affine systems.

This contribution opens many directions for future investigations. First, the numerical results exposed in this paper lead to the reasonable idea of considering attractor that are defined by the union of several points. A formal proof of this needs to be investigated. A second direction is related to the fact that the main contributions of this paper present a non convex optimization problem that may be difficult to solve in case of large number of modes. This problem requires a particular attention and producing a method that can overcome this issue would be highly relevant.

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