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Masayuki Sato, Dimitri Peaucelle

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M. Sato and D. Peaucelle

NOVELTY

- *Relevance to the journal aims and scope:* Our paper addresses the design problem of Linear Parameter-Varying (LPV) output feedback controllers depending on inexact scheduling parameters for LPV systems. The topic of LPV controller design is one of the hot topics in the last two decades, and the usage of inexact scheduling parameters, i.e. robustness against the uncertainties in the provided scheduling parameters, is a natural supposition from the view point of practicality. That is, our addressed problem contains uncertainties for LPV systems. Thus, our topic fits the scope of the journal. Furthermore, we designed flight controllers using our proposed method, and had several flight tests to demonstrate conservatism reduction of our method compared to one of the existing design methods. The development of design techniques for LPV systems is well illustrated. Thus, our objective also fits the aim of the journal.

- *Theoretical contributions:* We propose a new method for our addressed problem with the reverse use of Elimination lemma. Several methods on the same topic have already been proposed; however, all of them are merely sufficient conditions for the problem. This is because the adopted over-bounding methods for the scheduling parameter discrepancies are not necessary and sufficient conditions, but sufficient conditions. On this issue, our adopted approach using Elimination lemma theoretically introduces no conservatism for the over-bounding. This is a sharp contrast to existing methods.

- *Keywords:* Linear Parameter-Varying (LPV) system, flight controller, inexact scheduling parameter, Elimination lemma.

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Conservatism Reduction for LPV Control Design Facing Inexact Scheduling Parameters Illustrated on Flight Tests

Masayuki SATO*[1] | Dimitri PEAUCELLE[2,3]

[1]Aeronautical Technology Directorate, Japan Aerospace Exploration Agency, Tokyo, Japan
[2]MAC group, LAAS-CNRS, Toulouse, France
[3]Université de Toulouse, Toulouse, France

Correspondence
*Masayuki SATO. Email: sato.masayuki@jaxa.jp

Summary

This paper addresses the design problem of Linear Parameter-Varying (LPV) output feedback controllers which depend on inexact scheduling parameters for LPV systems. This problem has already been tackled and several methods have been proposed by over-bounding the discrepancies between the actual scheduling parameters and the provided ones in the derivation of controller design condition. However, all methods in literature have conservatism in the over-bounding, which is the main issue addressed in this paper. We therefore propose a new over-bounding for the discrepancies with the reverse use of Elimination lemma, which introduces *no conservatism* in theory. The new method is formulated in terms of bilinear matrix inequality which is not tractable compared to Linear Matrix Inequality (LMI), thus a practical design procedure composed of line search and iterative algorithm is shown. The effectiveness of our method is illustrated by an application to flight controller design for the lateral-directional motions of a research airplane MuPAL-α and the consequently conducted flight tests.

KEYWORDS:
Linear Parameter-Varying (LPV) system, flight controller, inexact scheduling parameter, Elimination lemma

1 | INTRODUCTION

The effectiveness of Linear Parameter-Varying (LPV) controllers, which is known as one of the realization of *gain-scheduled control* technique, for controlling systems modeled as LPV systems are widely known, and many reports have already been published[1,2,3], etc. Most of them implicitly suppose that *exact* scheduling parameters are available; however it does not always hold true in practical systems due to aging effect, finite resolutions, and other unexpected uncertainties of the measurement equipment of scheduling parameters. For example, airspeed is widely used as one of the scheduling parameters in gain-scheduled flight controller design[4], and Pitot tube is widely used to measure airspeed in aeronautical community; however, it is also well known that so-called *position error* always exists due to the effect of aircraft fuselage and thus the precise airspeed cannot be always measured, as illustrated in[5].

The research of designing LPV output feedback controllers depending on inexact scheduling parameters has been one of the hot research topics in gain-scheduled controller design, and several methods have already been proposed[6,7,8,9], etc. On the uncertainties in the inexactly provided scheduling parameters, particular uncertainties are first supposed[6,7], then the supposition

[0]**Abbreviations:** LPV, linear parameter-varying; LMI, linear matrix inequality; BMI, bilinear matrix inequality; w.r.t., with respect to; EAS, equivalent airspeed; L.H.S., left hand side

has been relaxed[10,11]. On this issue, it has been established in[5] that as long as the design matrix inequalities are given as parametrically affine, bounded uncertainties in the provided scheduling parameters can be supposed by using the convexity of design matrix inequalities w.r.t. parameters; *viz.*, it can be supposed that arbitrary uncertainties exist in the provided scheduling parameters as long as they are bounded and the design matrix inequalities are given as parametrically affine. However, in the papers above, over-bounding for the discrepancies between the actual scheduling parameters and the provided ones is used to robustify the LPV controllers against the uncertainties in the provided scheduling parameters, and all the adopted methods in the papers above are merely sufficient conditions for the over-bounding. In other words, to the best of authors' knowledge, no methods which introduce *no conservatism* in over-bounding the parameter discrepancies have yet been proposed. On the use of inexact scheduling parameters, switching LPV controllers depending on inexact scheduling parameters have been also proposed in[12] to improve control performance.

Based on the research background above, we propose a new over-bounding for the parameter discrepancies in LPV output feedback controller design. The new over-bounding theoretically introduces *no conservatism* with the reverse use of Elimination lemma[13,14,15]. Our supposed LPV systems in this paper are slightly relaxed from parametrically affine LPV systems[16,17] to parametrically multi-affine LPV systems without introducing huge numerical complexity in the design process. As a practical application of our method, we design discrete-time LPV flight controllers for the lateral-directional motions of a research airplane MuPAL-α in the same problem setup adopted in[5], and confirm that our method produces better LPV controllers than the method in[5] with respect to (w.r.t.) control performance, i.e. model-matching performance. Furthermore, the control performance is examined in practical environment, i.e. in flight tests, which has not been presented in our conference papers[16,17]. To enhance theoretical contributions of this paper, we address the design problem in both continuous-time and discrete-time cases.

Notations: $\mathbf{0}$ denotes an appropriately dimensional zero matrix, $\mathbf{I}$ denotes an appropriately dimensional identity matrix (if necessary, the dimension is denoted by the subscript), $\mathbb{R}_+$, $\mathbb{R}^n$, $\mathbb{R}^{n\times m}$ and $\mathbb{S}^n_+$ respectively denote the sets of positive real scalars, n-dimensional real vectors, $n\times m$-dimensional real matrices and $n\times n$-dimensional positive definite matrices, the symbol $*$ in a matrix represents an abbreviated asymmetric term, $\mathsf{diag}\left(X_1,\ \ldots,\ X_m\right)$ denotes a block diagonal matrix composed of $X_1,\ \ldots,\ X_m$, and $\mathsf{He}\{X\}$ for a square matrix X denotes $X+X^T$. For a time-varying vector x, $\delta[x]$ denotes $\dot{x}:=\frac{d}{dt}x$ and $x^+:=x(k+1)$ in continuous-time and discrete-time cases respectively.

The remainder of this paper is structured as follows: Section 2 shows our method for over-bounding Hermitian terms via Elimination lemma; Section 3 shows the proposed design method for discrete-time LPV output feedback controllers depending on inexactly provided scheduling parameters; Section 4 shows discrete-time LPV flight controller design using the method in Section 3 and the verification results by flight tests; and concluding remarks are finally given. The counterpart results in Section 3 for continuous-time case are all given in the appendix.

2 | A NEW OVER-BOUNDING FOR HERMITIAN TERMS

In LPV output feedback controller design addressed in the next section, the term containing the scheduling parameter discrepancies is represented as a Hermitian term, and it should be over-bounded by another tractable term to robustify LPV controllers against the discrepancies. Thus, in this section, a general problem w.r.t. over-bounding is considered, *viz.*, the verification problem of the following matrix inequalities containing a Hermitian term is considered:

$$Q_0-\mathsf{He}\left\{Q_1Q_2\right\}\succ 0, \tag{1}$$

where $Q_0\in\mathbb{S}^n_+$, $Q_1\in\mathbb{R}^{n\times l}$, and $Q_2\in\mathbb{R}^{l\times n}$ are given.

With use of Elimination lemma[13,14,15], we give the following formulation to verify the feasibility.

Lemma 1. [16] The following are both equivalent to (1).

$$\exists R\in\mathbb{R}^{l\times l}\ \text{s.t.}\ \begin{bmatrix}Q_0 & Q_1\\ * & \mathbf{0}\end{bmatrix}+\mathsf{He}\left\{\begin{bmatrix}\mathbf{0}\\ \mathbf{I}_l\end{bmatrix}R\begin{bmatrix}Q_2 & \mathbf{I}_l\end{bmatrix}\right\}\succ 0 \tag{2}$$

$$\exists \tilde{R}\in\mathbb{R}^{l\times l}\ \text{s.t.}\ \begin{bmatrix}Q_0 & *\\ Q_2 & \mathbf{0}\end{bmatrix}+\mathsf{He}\left\{\begin{bmatrix}\mathbf{0}\\ \mathbf{I}_l\end{bmatrix}\tilde{R}\begin{bmatrix}Q_1^T & \mathbf{I}_l\end{bmatrix}\right\}\succ 0 \tag{3}$$

Proof. Note that inequality (1) is equivalently represented as $\begin{bmatrix} \mathbf{I}_n & -Q_2^T \end{bmatrix} \begin{bmatrix} Q_0 & Q_1 \\ * & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{I}_n \\ -Q_2 \end{bmatrix} \succ 0$, and that $\begin{bmatrix} \mathbf{I}_n & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_l \end{bmatrix}^{T^\perp}$ and $\begin{bmatrix} \mathbf{I}_n & -Q_2^T \end{bmatrix} = \begin{bmatrix} Q_2 & \mathbf{I}_l \end{bmatrix}^{T^\perp}$ hold. Then, the existence of R satisfying (2) is equivalent to $Q_0 \in \mathbb{S}_+^n$ and (1) from Elimination lemma[13,14,15]. It is thus proved that the feasibility of (1) under $Q_0 \in \mathbb{S}_+^n$ is equivalent to the existence of R such that (2) holds. As matrix R is confirmed to be non-singular from the bottom-right block in inequality (2), the equivalence between (2) and (3) is straightforwardly confirmed by setting $\tilde{R} = R^{-1}$. □

In[7] for continuous-time case and in[5] for discrete-time case, the following formulation is used to over-bound the term related to the scheduling parameter discrepancies, i.e. $\mathsf{He}\{Q_1 Q_2\}$ in (1).

Lemma 2. C.f.[18] [1] If one of either conditions (4) or (5) holds then inequality (1) holds.

$$\exists \Upsilon \in \mathbb{S}_+^l \text{ s.t. } \begin{bmatrix} Q_0 & Q_1 + Q_2^T \Upsilon \\ Q_1^T + \Upsilon Q_2 & \Upsilon \end{bmatrix} \succ 0 \tag{4}$$

$$\exists \Upsilon \in \mathbb{S}_+^l \text{ s.t. } \begin{bmatrix} Q_0 & Q_1 & Q_2^T \Upsilon \\ Q_1^T & \Upsilon & \mathbf{0} \\ \Upsilon Q_2 & \mathbf{0} & \Upsilon \end{bmatrix} \succ 0 \tag{5}$$

As mentioned in[18], it has not yet been clarified which condition is less conservative between (4) and (5). We claim that both conditions are conservative compared to Lemma 1.

Lemma 3. If either (4) or (5) holds, then (2) holds.

Proof. Let us suppose that (4) holds. As $\Upsilon \succ 0$ is supposed in the bottom-right block in (4), then the following inequality holds: $\begin{bmatrix} Q_0 & Q_1 + Q_2^T \Upsilon \\ Q_1^T + \Upsilon Q_2 & \Upsilon + \Upsilon \end{bmatrix} \succeq \begin{bmatrix} Q_0 & Q_1 + Q_2^T \Upsilon \\ Q_1^T + \Upsilon Q_2 & \Upsilon \end{bmatrix}$. Thus, (2) holds with $R = \Upsilon$. Next, let us suppose that (5) holds. Pre- and post-multiplying $\begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{I} \end{bmatrix}$ and its transpose to (5) respectively lead to (2) with $R = \Upsilon$. Thus, the assertion is proved. □

In summary, with a slight abuse of mathematical expressions, the following relation holds on the feasibility of (1): Lemma 2 $\subseteq$ Lemma 1 $\Leftrightarrow$ (1).

Using Lemma 1, we propose a design method of LPV output feedback controllers depending on inexact scheduling parameters, and show the conservatism reduction compared the methods in[7,5].

Remark 1. In[9], a similar method to Lemma 1 is used for the design problem of LPV output feedback controllers depending on inexact scheduling parameters; however, the method uses $r\mathbf{I}_l$ instead of full matrices R and $\tilde{R}$. Thus, the proposed method inevitably introduces conservatism for over-bounding the discrepancies. □

3 | PROPOSED METHOD IN DISCRETE-TIME CASE

We first define our addressed problem, then show our proposed method, and finally give a practical design procedure composed of line search and iterative algorithm.

3.1 | Problem Definition

Let us consider the following LPV system (6).

$$G(\theta) : \begin{bmatrix} \delta[x] \\ z \\ y \end{bmatrix} = \begin{bmatrix} A(\theta) & B_1(\theta) & B_2 \\ C_1(\theta) & D_{11}(\theta) & D_{12} \\ C_2 & D_{21} & \mathbf{0} \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix}, \tag{6}$$

where $\theta = \begin{bmatrix} \theta_1 & \dots & \theta_q \end{bmatrix}^T$ denotes q independent measurable scheduling parameters, $x \in \mathbb{R}^n$, $w \in \mathbb{R}^{n_w}$, $u \in \mathbb{R}^{n_u}$, $z \in \mathbb{R}^{n_z}$ and $y \in \mathbb{R}^{n_y}$ respectively denote the state with its initial value as zero, the external input, the control input, the performance

[1] A slightly different version with $Q_1^T = \begin{bmatrix} \Upsilon_1 & \mathbf{0} \end{bmatrix}$ and $Q_2 = \begin{bmatrix} \mathbf{0} & \Upsilon_2 \end{bmatrix}$ is shown in[18].

output and the measurement output, and parameter-dependent matrices $A(\theta)$, $B_1(\theta)$, $C_1(\theta)$ and $D_{11}(\theta)$ are all supposed to have compatible dimensions and to be multi-affine w.r.t. scheduling parameters[19,20]. That is, they are supposed to be represented as $\sum \theta_1^{\alpha_1} \cdots \theta_q^{\alpha_q} Z_{\alpha_1 \cdots \alpha_q}$, $\alpha_i = \{0, 1\}$, where $Z_{\alpha_1 \cdots \alpha_q}$ is the coefficient matrix. This assumption on parameter-dependency is made to derive a numerically tractable multi-affine condition.

For all parameters, it is supposed that their variation ranges and their maximum deviations in a single sampling period are all known in advance. Thus, $\left(\theta_i, \delta[\theta_i]\right) \in \Lambda_i$ is supposed for $i = \{1, \ \dots, \ q\}$ with *a priori* given convex polytopes Λ_i as in[21], and the following is consequently supposed: $(\theta, \delta[\theta]) \in \Lambda_\theta := \Lambda_1 \times \cdots \times \Lambda_q$.

We would like to design a full-order LPV controller, which depends on the provided scheduling parameter vector $\hat{\theta}$, defined below.

$$K(\hat{\theta}) : \begin{bmatrix} \delta[x_K] \\ u \end{bmatrix} = \underbrace{\begin{bmatrix} A_K(\hat{\theta}) & B_K(\hat{\theta}) \\ C_K(\hat{\theta}) & D_K(\hat{\theta}) \end{bmatrix}}_{\bar{K}(\hat{\theta})} \begin{bmatrix} x_K \\ y \end{bmatrix}, \tag{7}$$

where $x_K \in \mathbb{R}^n$ is the state with its initial value as zero, and $\hat{\theta}$ denotes the provided scheduling parameter vector which is not always identical to θ. Matrix $\bar{K}(\hat{\theta})$ is supposed to be parametrically multi-affine. This supposition is made to derive a numerically tractable multi-affine condition. It is also supposed that the uncertainty in $\hat{\theta}_i$ is bounded, it is then always possible to make a convex poytope to cover the admissible region of the pair $(\theta_i, \hat{\theta}_i)$[10,5]. Thus, it is supposed that $(\theta_i, \hat{\theta}_i) \in \Xi_i$ holds with *a priori* defined convex polytope Ξ_i. Then, $(\theta, \hat{\theta}) \in \Xi_\theta := \Xi_1 \times \cdots \times \Xi_q$ holds.

The closed-loop system is straightforwardly derived as

$$G_{cl}(\theta, \hat{\theta}) : \begin{bmatrix} \delta[x_{cl}] \\ z \end{bmatrix} = \begin{bmatrix} A_{cl}(\theta, \hat{\theta}) & B_{cl}(\theta, \hat{\theta}) \\ C_{cl}(\theta, \hat{\theta}) & D_{cl}(\theta, \hat{\theta}) \end{bmatrix} \begin{bmatrix} x_{cl} \\ w \end{bmatrix}, \tag{8}$$

where $x_{cl} = \begin{bmatrix} x^T & x_K^T \end{bmatrix}^T$ denotes the state. Matrices $A_{cl}(\theta, \hat{\theta})$, etc. are straightforwardly calculated[16], and thus they are omitted here.

We address the following problem.

Problem 1. For given $\gamma \in \mathbb{R}_+$, design an LPV controller $K(\hat{\theta})$ defined in (7) such that the closed-loop system $G_{cl}(\theta, \hat{\theta})$ is stabilized and $\|z\|_2^2 < \gamma^2 \|w\|_2^2$ is satisfied for all triplets of $(\theta, \delta[\theta], \hat{\theta})$ in which $(\theta, \delta[\theta]) \in \Lambda_\theta$ and $(\theta, \hat{\theta}) \in \Xi_\theta$ hold.

3.2 | Design Method

To show our method, we define a parametrically multi-affine matrix set $\mathcal{M}(\theta, \delta[\theta], \hat{\theta})$ and a constant matrix set $\mathcal{N}$.

$$\begin{aligned} \mathcal{M}(\theta, \delta[\theta], \hat{\theta}) &= \left(\mathcal{P}(\theta), \mathcal{Q}(\theta, \delta[\theta], \hat{\theta}), \mathcal{K}(\hat{\theta})\right) \in \mathbb{S}_+^{2n} \times \mathbb{R}^{n \times n} \times \mathbb{R}^{(n+n_u) \times (n+n_y)} \\ \mathcal{N} &= (\mathcal{X}, \mathcal{Y}, \mathcal{S}) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \end{aligned} \tag{9}$$

The following is proposed for Problem 1 using $\mathcal{M}(\theta, \delta[\theta], \hat{\theta})$ and $\mathcal{N}$.

Theorem 1. For given $\gamma \in \mathbb{R}_+$, suppose that the following condition holds at all combinations of the vertices of Λ_θ and Ξ_θ.

$$\exists \mathcal{M}(\theta, \delta[\theta], \hat{\theta}) \text{ and } \mathcal{N} \text{ s.t. (10) or (11)}$$

Then, controller $K(\hat{\theta})$ whose state-space matrix $\bar{K}(\hat{\theta})$ is given as (14), in which matrices $U, V \in \mathbb{R}^{n \times n}$ are nonsingular matrices satisfying $VU + \mathcal{Y}\mathcal{X} = \mathcal{S}$, stabilizes the closed-loop system and $\|z\|_2^2 < \gamma^2 \|w\|_2^2$ is satisfied for all triplets of $(\theta, \delta[\theta], \hat{\theta})$ in which $(\theta, \delta[\theta]) \in \Lambda_\theta$ and $(\theta, \hat{\theta}) \in \Xi_\theta$ hold.

$$\begin{bmatrix} \Phi_0(\theta, \delta[\theta], \hat{\theta}) & * \\ \Phi_1(\theta, \hat{\theta})^T & \mathbf{0} \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix} \mathcal{Q}(\theta, \delta[\theta], \hat{\theta}) \begin{bmatrix} \Phi_2^T \\ \mathbf{I} \end{bmatrix}^T \right\} \succ 0, \tag{10}$$

where $\Phi_0(\theta, \delta[\theta], \hat{\theta})$ is defined as $\Phi_0^d(\theta, \delta[\theta], \hat{\theta})$ in (12) using (13), and

$$\begin{cases} \Phi_1(\theta, \hat{\theta}) = \begin{bmatrix} \mathbf{0} & \begin{bmatrix} \mathbf{0} & \left(A(\theta) - A(\hat{\theta})\right)^T \mathcal{Y}^T \end{bmatrix} & \mathbf{0} & \mathbf{0} \end{bmatrix}^T, \\ \Phi_2 = \begin{bmatrix} \begin{bmatrix} -\mathcal{X} & \mathbf{0} \end{bmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}. \end{cases}$$

$$\begin{bmatrix} \Phi_0(\theta, \delta[\theta], \hat{\theta}) & * \\ \Phi_2(\theta, \hat{\theta}) & \mathbf{0} \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} \Phi_1 \\ \mathbf{I} \end{bmatrix} \mathcal{Q}(\theta, \delta[\theta], \hat{\theta}) \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix}^T \right\} \succ 0, \tag{11}$$

where $\Phi_0(\theta, \delta[\theta], \hat{\theta})$ is defined as $\Phi_0^d(\theta, \delta[\theta], \hat{\theta})$ in (12) using (13), and

$$\begin{cases} \Phi_1 &= \begin{bmatrix} \mathbf{0} & \begin{bmatrix} \mathbf{0} & \mathcal{Y}^T \end{bmatrix} & \mathbf{0} & \mathbf{0} \end{bmatrix}^T, \\ \Phi_2(\theta, \hat{\theta}) &= \begin{bmatrix} \begin{bmatrix} -\left(A(\theta) - A(\hat{\theta})\right)\mathcal{X} & \mathbf{0} \end{bmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}. \end{cases}$$

$$\Phi_0^d(\theta, \delta[\theta], \hat{\theta}) = \begin{bmatrix} -\mathcal{P}(\theta) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathcal{P}(\theta^+) & \mathbf{0} & \Upsilon_B(\theta, \hat{\theta}) \\ \mathbf{0} & \mathbf{0} & \gamma^2\mathbf{I} & \Upsilon_D(\theta, \hat{\theta}) \\ \mathbf{0} & * & * & \mathbf{I} \end{bmatrix} + \mathsf{He}\left\{ \begin{bmatrix} \begin{bmatrix} \mathcal{X} & \mathbf{I} \\ \mathcal{S} & \mathcal{Y} \end{bmatrix} \\ \Upsilon_A(\theta, \hat{\theta}) \\ \Upsilon_C(\theta, \hat{\theta}) \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \right\} \tag{12}$$

$$\left[\begin{array}{c|c} \Upsilon_A(\theta, \hat{\theta}) & \Upsilon_B(\theta, \hat{\theta}) \\ \hline \Upsilon_C(\theta, \hat{\theta}) & \Upsilon_D(\theta, \hat{\theta}) \end{array}\right] = \left[\begin{array}{cc|c} A(\theta)\mathcal{X} & A(\theta) & B_1(\theta) \\ \mathbf{0} & \mathcal{Y}A(\theta) & \mathcal{Y}B_1(\theta) \\ \hline C_1(\theta)\mathcal{X} & C_1(\theta) & D_{11}(\theta) \end{array}\right] + \left[\begin{array}{cc} \mathbf{0} & B_2 \\ \mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & D_{12} \end{array}\right] \mathcal{K}(\hat{\theta}) \left[\begin{array}{cc|c} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C_2 & D_{21} \end{array}\right] \tag{13}$$

$$\bar{K}(\hat{\theta}) = \begin{bmatrix} V & \mathcal{Y}B_2 \\ \mathbf{0} & \mathbf{I} \end{bmatrix}^{-1} \left(\mathcal{K}(\hat{\theta}) - \begin{bmatrix} \mathcal{Y}A(\hat{\theta})\mathcal{X} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right) \begin{bmatrix} U & \mathbf{0} \\ C_2\mathcal{X} & \mathbf{I} \end{bmatrix}^{-1} \tag{14}$$

Proof. Since the inequalities (10) and (11) are parametrically multi-affine, if they hold at all combinations of the vertices of Λ_θ and Ξ_θ, then they also hold for all possible combinations of $(\theta, \delta[\theta], \hat{\theta})$ in which $(\theta, \delta[\theta]) \in \Lambda_\theta$ and $(\theta, \hat{\theta}) \in \Xi_\theta$ hold. Then, $\Phi_0(\theta, \delta[\theta], \hat{\theta}) - \mathsf{He}\left\{\Phi_1(\cdot)\Phi_2(\cdot)\right\} \succ 0$ holds for all possible combinations of $(\theta, \delta[\theta], \hat{\theta})$ from Lemma 1. Here $\Phi_1(\cdot)\Phi_2(\cdot)$ denotes $\Phi_1(\theta, \hat{\theta})\Phi_2$ or $\Phi_1\Phi_2(\theta, \hat{\theta})$. Note that $\Phi_1(\cdot)\Phi_2(\cdot)$ is given as follows:

$$\begin{bmatrix} \mathbf{0} \\ \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathcal{Y}\left(A(\hat{\theta}) - A(\theta)\right)\mathcal{X} & \mathbf{0} \end{bmatrix} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Then, the assertion is straightforwardly proved by following the same procedure in [8,5]. □

On conservatism of over-bounding of the parameter discrepancies, the following assertion holds.

Theorem 2. Suppose that, in Theorem 1, there exist $\mathcal{P}(\theta)$, $\mathcal{K}(\hat{\theta})$ and $\mathcal{N}$ such that $\Phi_0^d(\theta, \delta[\theta], \hat{\theta}) \succ 0$ and $\Phi_0^d(\theta, \delta[\theta], \hat{\theta}) - \mathsf{He}\left\{\Phi_1(\theta, \hat{\theta})\Phi_2\right\} \succ 0$ hold. Then, there always exists a matrix-valued function of $\mathcal{Q}(\theta, \delta[\theta], \hat{\theta})$ satisfying (10). The corresponding assertion for (11) also holds with a (possibly different) matrix-valued function of $\mathcal{Q}(\theta, \delta[\theta], \hat{\theta})$.

First, note that $\Phi_0(\theta, \delta[\theta], \hat{\theta}) \succ 0$ is a necessary condition for (10) and (11). Then, Theorem 2 is obvious from Lemma 1, i.e. Elimination lemma [13,14,15], if we accept matrix-valued function $\mathcal{Q}(\theta, \delta[\theta], \hat{\theta})$ instead of restricting multi-affine matrix $\mathcal{Q}(\theta, \delta[\theta], \hat{\theta})$. Thus, the proof is omitted.

In other words, under the condition that there exist $\mathcal{P}(\theta)$, $\mathcal{K}(\hat{\theta})$ and $\mathcal{N}$ such that $\Phi_0^d(\theta, \delta[\theta], \hat{\theta}) \succ 0$ holds, Theorem 2 does not introduce any conservatism for the over-bounding of $\mathsf{He}\left\{\Phi_1(\theta, \hat{\theta})\Phi_2\right\}$ in the formulation (10). The corresponding property for (11) holds as well. On the other hand, using multi-affine $\mathcal{Q}(\theta, \delta[\theta], \hat{\theta})$ obviously introduces conservatism; however, it leads to multi-affine matrix inequality which is attractive w.r.t. numerical complexity in solving matrix inequalities. Thus, we restrict $\mathcal{Q}(\theta, \delta[\theta], \hat{\theta})$ to be parametrically multi-affine in Theorem 1.

Remark 2. In [5], four formulations using Lemma 2 to address the parameter discrepancies are given for Problem 1, i.e. two formulations given as (4) and (5), and two factorizations, which are also used in Theorem 1, for $\mathcal{Y}\left(A(\hat{\theta}) - A(\theta)\right)\mathcal{X}$. However, they are all more conservative than or equal to Theorem 1, which is due to Lemma 3. □

3.3 | Design Procedure

The formulations in Theorem 1 are given in terms of parametrically multi-affine matrix inequalities. Though, they are Bilinear Matrix Inequalities (BMIs) which are not so tractable compared to Linear Matrix Inequalities (LMIs). Thus, similarly to [5], we show a concrete design procedure for solving them, i.e. combined method composed of line search for $\mathcal{Q}(\theta, \delta[\theta], \hat{\theta})$ and iterative algorithm.

Note that $\mathsf{He}\left\{\mathcal{Q}(\theta,\delta[\theta],\hat{\theta})\right\}$ in (10) and (11) should be positive definite from the requirement that the bottom-right block should be positive definite. Then, if we set $\mathcal{Q}(\theta,\delta[\theta],\hat{\theta})$ as $q\mathbf{I}$ with scalar q, i.e. line search for $\mathcal{Q}(\theta,\delta[\theta],\hat{\theta})$, then q should be positive. Bearing this in mind, we show the following algorithm to solve the condition in Theorem 1.

Step 0 Set the iteration number $i=0$. Matrix $\mathcal{Q}(\theta,\delta[\theta],\hat{\theta})$ is set as $q\mathbf{I}$ with positive scalar q, and conduct line search of q using the *a priori* defined variations of q for optimal γ by solving the following problem.

$$\min_{\gamma,\mathcal{P}(\theta),\mathcal{K}(\hat{\theta}),\mathcal{X},\mathcal{Y},\mathcal{S}} \gamma \text{ s.t. (10) or (11)}$$

The value q^* which produces the best γ is chosen and set $\mathcal{Q}(\theta,\delta[\theta],\hat{\theta})=q^*\mathbf{I}$. Set the best γ as γ_0. If no solutions are found, we conclude the problem to be infeasible.

Step 1 Increase the iteration number $i=i+1$. If (10) is solved in the previous step, proceed to (1). Otherwise, proceed to (2).

(1) Solve the following problem with the matrix $\mathcal{X}$ being fixed as obtained in the previous step.

$$\min_{\gamma,\mathcal{P}(\theta),\mathcal{K}(\hat{\theta}),\mathcal{Y},\mathcal{S},\mathcal{Q}(\theta,\delta[\theta],\hat{\theta})} \gamma \text{ s.t. (10)}$$

(2) Solve the following problem with the matrix $\mathcal{Y}$ being fixed as obtained in the previous step.

$$\min_{\gamma,\mathcal{P}(\theta),\mathcal{K}(\hat{\theta}),\mathcal{X},\mathcal{S},\mathcal{Q}(\theta,\delta[\theta],\hat{\theta})} \gamma \text{ s.t. (11)}$$

Step 2 Solve the following problem with the matrix $\mathcal{Q}(\theta,\delta[\theta],\hat{\theta})$ being fixed as obtained in the previous step, and set the optimal γ as γ_i.

$$\min_{\gamma,\mathcal{P}(\theta),\mathcal{K}(\hat{\theta}),\mathcal{X},\mathcal{Y},\mathcal{S}} \gamma \text{ s.t. (10) or (11)}$$

If $\gamma_{i-1}-\gamma_i<\eta$ holds with a predefined threshold $\eta\in\mathbb{R}_+$, then stop the iteration and the finally obtained solution is the design result. Otherwise, go back to **Step 1**.

Remark 3. In the algorithm shown above, all conditions are given in terms of multi-affine LMIs, thus they are easily solved by using SeDuMi[22], SDPT3[23], etc. In our controller design shown in the next section, we use SDPT3. □

4 | LPV FLIGHT CONTROLLER USING INEXACT AIRSPEED

We show LPV flight controller design for the lateral-directional motions of research airplane MuPAL-α[24]. In our controller design, constant scaling matrix is introduced to Theorem 1 for conservatism reduction due to multiple uncertainty blocks. The corresponding modifications to the design procedure shown in the previous section are necessary and they are given after the definition of our flight controller design problem.

The problem setup is the same as in conference paper[17] and our previous report[5], thus we give a simplified description on it.

4.1 | LPV Flight Controller Design Setup

MuPAL-α has been developed as an in-flight simulator[25], which has an ability to mimic other aircraft dynamics in flight, and thus, model-matching and disturbance suppression are imposed as controller design specifications in[17,5]. The same design specifications are adopted in this paper. To this end, the block diagram shown in Fig. 1 is set for designing an LPV flight controller. The definitions of blocks and signals in the figure are given below.

Blocks: $G_{L/D}(V_{true})$, $G_{act}(V_{true})$, G_{dex} and $K(V_{prov})$ respectively denote the lateral-directional motions of MuPAL-α scheduled by true EAS (Equivalent Air Speed) which is denoted by V_{true}, the actuator dynamics scheduled by V_{true}, one step delay model and the to-be-designed LPV controller scheduled by provided EAS which is denoted by $V_{prov}(=V_{true}+\delta_{EAS})$ with the uncertainty in the provided EAS data δ_{EAS}; W, W_g and W_M respectively denote the weighting functions for uncertainties related to the onboard actuators, gust suppression performance and model-matching performance; $\Delta=\mathsf{diag}\left(\Delta_{per},\Delta_{gust},\Delta_a,\Delta_r\right)$ denotes the structured uncertainty block composed of 2×2-dimensional model-matching performance block (Δ_{per}), 2×2-dimensional gust suppression performance block (Δ_{gust}), and two scalar uncertainty blocks to represent the uncertainties related to the onboard

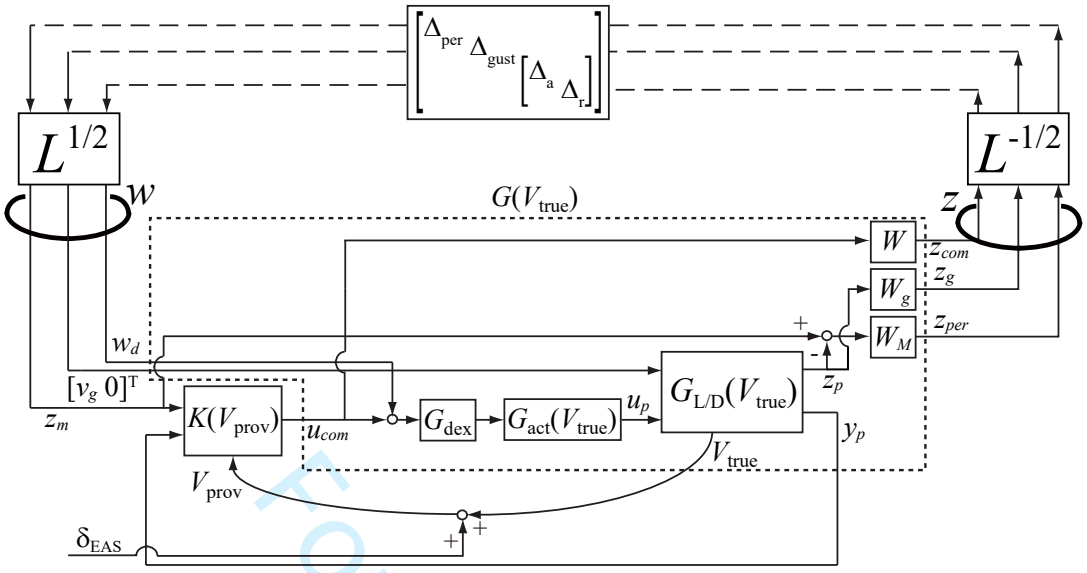

FIGURE 1 Block diagram for simultaneous realization of model-matching and gust suppression

aileron and rudder actuators (Δ_a and Δ_r); and $L^{1/2}$ and its inverse denote the constant scaling matrix to reduce conservatism due to the structured uncertainty block Δ. It is supposed that $\|\Delta\| \leq 1$ holds.

Signals: u_{com}, u_p, z_p, y_p and δ_{EAS} respectively denote the control command produced by $K(V_{prov})$, i.e. $u_{com} = [\delta_{a_c}\ \delta_{r_c}]^T$ (δ_{a_c} and δ_{r_c} respectively denote aileron and rudder deflection commands), actual control input to MuPAL-α, i.e. $u_p = [\delta_a\ \delta_r]^T$ (δ_a and δ_r respectively denote aileron and rudder deflections), plant output to be controlled, i.e. $z_p = [v_a\ \phi]^T$ (v_a and ϕ respectively denote lateral airspeed and roll angle), measurement output, i.e. $y_p = [v_a\ p\ \phi\ r]^T$ (p and r respectively denote roll and yaw rates) and uncertainties in the provided EAS; the pair of w_d and z_{com} denotes the signals to evaluate the uncertainties related to the onboard actuators, *viz.*, they respectively denote fictitious external input and weighted u_{com} to compensate the uncertainties of the onboard actuators; the pair of $[v_g\ 0]^T$ and z_g denotes the signals to evaluate gust suppression performance, *viz.*, v_g and z_g respectively denote sideway gust and performance output z_p multiplied by the weighting function W_g; and the pair of z_m and z_{per} denotes the signals to evaluate model-matching performance, *viz.*, they respectively denote the model output which is to be reproduced by MuPAL-α and performance output z_p multiplied by the weighting function W_M. Note that v_g is augmented as $[v_g\ 0]^T$ to comply with the size of z_g.

As the design specifications and the onboard actuator uncertainties (uncertain delay ranging in $[0.06, 0.10]$ [s] in both aileron and rudder channels) are the same as in[5], the following weighting functions which are the same as in[5] are used after the discretization via Euler method with the onboard computer's sampling period $\Delta T = 0.02$ [s]: $W_M = \frac{20}{T^s s^2 + T\sqrt{2}s + 1}\mathbf{I}_2$, $W_g = \frac{0.5}{0.1s+1}\mathbf{I}_2$, $W = \frac{2.5s}{s+30}\mathbf{I}_2$, where T in W_M is a tuning design parameter. Small values of T widen the frequency range in which model-matching is realized, thus small T is preferable.

We address the following problem with help of Fig. 1.

Problem 2. Design a constant scaling matrix $L \in \mathbb{S}^6_+$ compatible with the structured uncertainty block Δ, i.e. $\Delta L = L\Delta$, and a stabilizing LPV controller $K(V_{prov})$, which depends on the provided scheduling parameter V_{prov} corrupted from V_{true} with the bounded uncertainty δ_{EAS} given by (15), satisfying $\|L^{-1/2}z\|_2^2 < \gamma^2\|L^{-1/2}w\|_2^2$ with $\mathbb{R}_+ \ni \gamma \leq 1$ for all possible triplets $(V_{true}, V_{true}^+, V_{prov})$ belonging to a convex polytope whose vertex set is given in (16).

$$\delta_{EAS} \in [0.5 - 0.04 \times V_{true},\ 1.875 - 0.035 \times V_{true}] \tag{15}$$

$$\left\{\begin{array}{l}(54.012, 54.012, 52.352), (54.012, 54.012, 53.997),\\ (54.012, 54.112, 52.352), (54.012, 54.112, 53.997),\\ (54.112, 54.012, 52.448), (54.112, 54.012, 54.093),\\ (99.179, 99.279, 95.712), (99.179, 99.279, 97.583),\\ (99.279, 99.179, 95.808), (99.279, 99.179, 97.679),\\ (99.279, 99.279, 95.808), (99.279, 99.279, 97.679)\end{array}\right\} \tag{16}$$

The vertex set of (16) means that the supposed EAS range is $V_{true} \in [54.012, 99.279]$ [m/s] and the supposed maximum and minimum deviations of EAS for a single sampling period ΔT are ± 0.1 [m/s/sampling].

The problem needs to design an LPV output feedback controller $K(V_{prov})$ and a scaling matrix L as well, thus solving the problem is not a simple application of Theorem 1; however, the difference is only the introduction of the constant matrix L. Thus, the combined method of line search and iterative algorithm, which has already shown in the previous section, can be applied with the following modifications with $\mathcal{K}(\hat{\theta})$ being partitioned as $\begin{bmatrix}\mathcal{K}_A(\hat{\theta}) & \mathcal{K}_B(\hat{\theta})\\ \mathcal{K}_C(\hat{\theta}) & \mathcal{K}_D(\hat{\theta})\end{bmatrix}$ and L being set as $\mathcal{L}$.

- In matrix inequalities (10) and (11), $\Phi_0^d(\theta, \delta[\theta], \hat{\theta})$ defined in (12) is revised as (17) with a constant scaling matrix $\mathcal{L} \in \mathbb{S}_+^6$ which is also a decision matrix.

$$\Phi_0^d(\theta, \delta[\theta], \hat{\theta}) = \begin{bmatrix}-\mathcal{P}(\theta) & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathcal{P}(\theta^+) & \mathbf{0} & \Upsilon_B(\theta, \hat{\theta})\\ \mathbf{0} & \mathbf{0} & \gamma^2\mathcal{L} & \Upsilon_D(\theta, \hat{\theta})\\ \mathbf{0} & * & * & \mathcal{L}^{-1}\end{bmatrix} + \mathrm{He}\left\{\begin{bmatrix}\begin{bmatrix}\mathcal{X} & \mathbf{I}\\ \mathcal{S} & \mathcal{Y}\end{bmatrix}\\ \Upsilon_A(\theta, \hat{\theta})\\ \Upsilon_C(\theta, \hat{\theta})\\ \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix}\right\} \tag{17}$$

- In **Step 0**, decision matrix $\mathcal{L}$ is set as $\mathbf{I}$.
- In (1) in **Step 1**, the problem is revised as follows:

$$\min_{\gamma, \mathcal{P}(\theta), \mathcal{K}_A(\hat{\theta}), \mathcal{K}_B(\hat{\theta}), \mathcal{Y}, \mathcal{S}, \mathcal{Q}(\theta, \hat{\theta}), \mathcal{L}^{-1}} \gamma \text{ s.t. } \mathsf{diag}\left(\mathbf{I}, \mathbf{I}, \mathcal{L}^{-1}, \mathbf{I}, \mathbf{I}\right) \times \text{L.H.S. of (10)} \times \mathsf{diag}\left(\mathbf{I}, \mathbf{I}, \mathcal{L}^{-1}, \mathbf{I}, \mathbf{I}\right) \succ 0,$$

  where $\mathcal{K}_c(\hat{\theta})$, $\mathcal{K}_D(\hat{\theta})$ and $\mathcal{X}$ are fixed as ones obtained in the previous step.

- In (2) in **Step 1**, the problem is revised as follows:

$$\min_{\gamma, \mathcal{P}(\theta), \mathcal{K}_A(\hat{\theta}), \mathcal{K}_C(\hat{\theta}), \mathcal{X}, \mathcal{S}, \mathcal{Q}(\theta, \hat{\theta}), \mathcal{L}} \gamma \text{ s.t. } \mathsf{diag}\left(\mathbf{I}, \mathbf{I}, \mathbf{I}, \mathcal{L}, \mathbf{I}\right) \times \text{L.H.S. of (11)} \times \mathsf{diag}\left(\mathbf{I}, \mathbf{I}, \mathbf{I}, \mathcal{L}, \mathbf{I}\right) \succ 0,$$

  where $\mathcal{K}_B(\hat{\theta})$, $\mathcal{K}_D(\hat{\theta})$ and $\mathcal{Y}$ are fixed as ones obtained in the previous step.

- In **Step 2**, decision matrix $\mathcal{L}$ is fixed as obtained in the previous step.

With the revisions above and ten gridding points over a logarithmic scale in the interval $[10^{-5}, 10^5]$ for the line search of q, we design LPV flight controllers depending on V_{prov} for various values of T in W_M. In our design, the decision matrix $\mathcal{Q}(\theta, \delta[\theta], \hat{\theta})$ is set as parametrically affine w.r.t. V_{true} and V_{prov}, i.e. $\mathcal{Q}(\theta, \hat{\theta})$, because controller performance dependence on V_{true}^+ in $\mathcal{Q}(\theta, \delta[\theta], \hat{\theta})$ is confirmed to be small in a primitive design and this setup reduces numerical complexity in solving LMIs.

4.2 | Controller Design and *A Posteriori* Analysis

We design LPV controllers using Theorem 1 combined with constant scaling matrix L. For comparison, we design LPV controllers using the method in[5], i.e. Theorem 1 in the paper. After designing LPV controllers, we also conduct *a posteriori* analysis to calculate γ satisfying $\|L^{-1/2}z\|_2^2 < \gamma^2\|L^{-1/2}w\|_2^2$ for all possible triplets $(V_{true}, V_{true}^+, V_{prov})$ belonging to a convex polytope whose vertex set is given in (16) using Theorem 2 in[5]. This is because the use of parametrically affine $\mathcal{Q}(\theta, \hat{\theta})$ in Theorem 1 and Theorem 1 in[5] inevitably introduces conservatism (see Theorem 2 and the paragraph beneath in the previous section) and therefore *a posteriori* analysis is necessary for precise performance evaluation.

The minimized γ's with two methods are shown in Tables 1 and 2. In the same tables, the guaranteed γ's which are obtained in *a posteriori* analysis are also shown.

TABLE 1 Minimized γ's via Theorem 1 combined with constant scaling matrix and *a posteriori* analysis

T	7.0	7.5	8.0	8.5	9.0
Design	1.14	1.09	1.06	1.02	1.00
Analysis	1.09	1.03	1.00	0.97	0.95

TABLE 2 Minimized γ's via method in[5] and *a posteriori* analysis

T	8.5	9.0	9.5	10.0
Design	1.07 (d)	1.03 (b)	1.00 (b)	0.96 (b)
Analysis	1.02	0.99	0.96	0.93

Note that Theorem 1 has two formulations which come from two possible factorizations for $\mathcal{Y}\left(A(\hat{\theta}) - A(\theta)\right)\mathcal{X}$, and both are solved to obtain the best controllers. In our design, the best controllers for all T in Table 1 are obtained by solving the formulation (11). Similarly, the method in[5] has four formulations (a) $\sim$ (d), and all formulations are solved to obtain the best controllers. The choice for the best controllers is indicated in the parentheses in Table 2.

The results in Tables 1 and 2 clearly indicate that the controllers designed by Theorem 1 which is based on Lemma 1 to compensate δ_{EAS} in V_{prov} have better performance than the controllers designed by the design method in[5] which is based on Lemma 2 to compensate δ_{EAS} in V_{prov}. Thus, conservatism reduction using Lemma 1 compared to using Lemma 2 in LPV controller design is well illustrated, which is reasonable from Lemma 3.

In *a posteriori* analysis, it is confirmed that the LPV controller with $T = 8.0$ in Table 1 and the LPV controller with $T = 9.0$ in Table 2 satisfy $\gamma \leq 1$. Therefore, these two LPV controllers are implemented to the onboard computer of MuPAL-α to examine practical performance.

Figs. 2 and 3 respectively show the Bode plots of the closed-loop system using the LPV controller with $T = 8.0$ in Table 1 and the LPV controller with $T = 9.0$ in Table 2 for $\mathsf{diag}\left(\delta_a, \delta_r\right) = \pm\mathbf{I}_2$ and frozen V_{true} and δ_{EAS}. For comparison, the corresponding Bode plots for open-loop systems are also depicted. Both figures indicate that the designed LPV controllers have good model-matching and good disturbance suppression for $\mathsf{diag}\left(\delta_a, \delta_r\right) = \pm\mathbf{I}_2$ and frozen V_{true} and δ_{EAS}, and it is not so easy to find the difference between the two controllers w.r.t. control performance. However, the former has better model-matching performance than the latter, *viz.*, control performance is improved. In particular, at 0.6 [rad/s], phase delay from $z_m(1)$ to v_a is decreased from about 35 [deg] in Fig. 3 to less than 30 [deg] in Fig. 2, and phase delay from $z_m(2)$ to ϕ is similarly decreased from over 30 [deg] in Fig. 3 to about 25 [deg] in Fig. 2.

These figures indicate that the model-matching in gain characteristics is realized up to around 1.5 [rad/s] using the designed two LPV controllers; however, phase delay becomes so large as the frequency increases. Thus, roughly speaking, the motions whose frequencies are less than 0.6 [rad/s] are expected to be reproduced as MuPAL-α's motions in v_a and ϕ.

4.3 | Practical Performance in Flight

We conducted flight tests to compare control performance of the implemented two LPV controllers. The flight data were obtained in two days (Feb. 9th and 19th in 2018) for unbiased results. Furthermore, the experiments were conducted one after another with very short breaks in both two flights to equalize the flight conditions as much as possible.

We show one pair of disturbance suppression check results (Figs. 4 and 5), and two pairs of model-matching performance check results under disturbance (Figs. 6 and 7, and Figs. 8 and 9). Note that flight tests inevitably suffer wind gust, thus we cannot have flight tests focusing purely on model-matching performance check. Thus, flight tests in Figs. 6 $\sim$ 9 are the results for checking model-matching performance under disturbance. The command signals $z_m(1)$ and $z_m(2)$ in Figs. 6 $\sim$ 9 respectively have the peak gains at 0.57 and 0.44 [rad/s]. These frequencies are less than the estimated upper limit to achieve model-matching, but not so far from the limit. Thus, the commands in flight tests are suitable to compare control performance.

It is not so easy to see the difference of the two LPV flight controllers w.r.t. control performance in those figures. In order to see the difference, Tables 3, 4 and 5 summarize control performance of the implemented two LPV controllers in exposed flight tests. In those tables, Δv_a and $\Delta\phi$ respectively denote $z_m(1) - v_a$ and $z_m(2) - \phi$, e.g. the discrepancies between commands and

TABLE 3 Performance comparison of controllers in Figs. 4 and 5

	Controller via Theorem 1 with L	Controller via method in[5]
duration time (T_{eval} [s])	205.28	199.24
$\frac{\int_0^{T_{eval}} \lvert\Delta v_a\rvert^2 dt}{T_{eval}}$ [(m/s)^2]	0.106	0.203
$\frac{\int_0^{T_{eval}} \lvert\Delta \phi\rvert^2 dt}{T_{eval}}$ [(deg)^2]	0.057	0.081

TABLE 4 Performance comparison of controllers in Figs. 6 and 7

	Controller via Theorem 1 with L	Controller via method in[5]
duration time (T_{eval} [s])	173.25	175.03
$\frac{\int_0^{T_{eval}} \lvert\Delta v_a\rvert^2 dt}{T_{eval}}$ [(m/s)^2]	0.740	0.885
$\frac{\int_0^{T_{eval}} \lvert\Delta \phi\rvert^2 dt}{T_{eval}}$ [(deg)^2]	7.571	10.755

TABLE 5 Performance comparison of controllers in Figs. 8 and 9

	Controller via Theorem 1 with L	Controller via method in[5]
duration time (T_{eval} [s])	182.67	185.93
$\frac{\int_0^{T_{eval}} \lvert\Delta v_a\rvert^2 dt}{T_{eval}}$ [(m/s)^2]	0.523	0.660
$\frac{\int_0^{T_{eval}} \lvert\Delta \phi\rvert^2 dt}{T_{eval}}$ [(deg)^2]	4.160	6.164

the controlled outputs of MuPAL-α. The tables clearly illustrate that the LPV controller designed using Theorem 1 has better performance than the LPV controller designed using the method in[5] w.r.t. disturbance suppression as well as model-matching under disturbance.

5 | CONCLUSIONS

This paper addresses the design problem of Linear Parameter-Varying (LPV) output feedback controllers depending on inexactly provided scheduling parameters for LPV systems. For this problem, under some mild assumptions for the state-space matrices of LPV systems, we derive a design condition in terms of multi-affine matrix inequality in which the discrepancies between the actual and the provided scheduling parameters are over-bounded by using Elimination lemma. The derived condition theoretically introduces no conservatism in addressing the parameter discrepancies, which is a sharp contrast to existing methods and consequently means that our proposed method is no more conservative than existing methods; however, the design condition is given as multi-affine Bilinear Matrix Inequality (BMI) which is not tractable compared to multi-affine Linear Matrix Inequality (LMI). We thus show a design procedure composed of line search and iterative algorithm, in which only multi-affine LMIs should be solved.

As a practical application of our method, we address LPV flight controller design for the lateral-directional motions of a research airplane MuPAL-α which has been developed as an in-flight simulator, and design LPV flight controllers using our proposed method and a previously proposed method. In controller design and *a posteriori* analysis, it is confirmed that the LPV controllers designed by our method have better control performance than the LPV controllers designed by the previously proposed method. Furthermore, in experiments under practical environment, i.e. in flight tests, control performance improvement of the LPV controller designed by our method compared to the LPV controller designed by the previously proposed method is confirmed. We thus illustrate the effectiveness of our proposed method in theory as well as in practical performance.

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APPENDIX

A PROPOSED METHOD IN CONTINUOUS-TIME CASE

In the appendix, the counterpart results in Section 3 for continuous-time case are shown.

Similarly to the discrete-time case, we consider LPV controller (7) for LPV system (6). For all parameters, it is supposed that their variation ranges and their derivative ranges are all known in advance, and thus $(\theta_i, \delta[\theta_i]) \in \Lambda_i$ is supposed for $i = \{1, \dots, q\}$ with *a priori* given convex polytopes Λ_i. Then, $(\theta, \delta[\theta]) \in \Lambda_\theta := \Lambda_1 \times \cdots \times \Lambda_q$ holds. Other suppositions are all the same as in discrete-time case.

Then, the following is proposed for Problem 1 using $\mathcal{M}(\theta, \delta[\theta], \hat{\theta})$ and $\mathcal{N}$ defined in (9).

Theorem 3. For given $\gamma \in \mathbb{R}_+$, suppose that the following condition holds at all combinations of the vertices of Λ_θ and Ξ_θ.

$$\exists \alpha \in \mathbb{R}_+, \mathcal{M}(\theta, \hat{\theta}) \text{ and } \mathcal{N} \text{ s.t. (10) or (11)}$$

In (10), $\Phi_0(\theta, \delta[\theta], \hat{\theta})$ is defined as $\Phi_0^c(\theta, \delta[\theta], \hat{\theta})$ in (A1) using (13) and

$$\begin{cases} \Phi_1(\theta, \hat{\theta}) = \begin{bmatrix} \mathbf{0} \begin{bmatrix} \mathbf{0} & \left(A(\theta) - A(\hat{\theta})\right)^T \mathcal{Y}^T \end{bmatrix} & \mathbf{0} & \mathbf{0} \end{bmatrix}^T, \\ \Phi_2 = \begin{bmatrix} \begin{bmatrix} \alpha\mathcal{X} & \mathbf{0} \end{bmatrix} & \begin{bmatrix} \mathcal{X} & \mathbf{0} \end{bmatrix} & \mathbf{0} & \mathbf{0} \end{bmatrix}. \end{cases}$$

In (11), $\Phi_0(\theta, \delta[\theta], \hat{\theta})$ is defined as $\Phi_0^c(\theta, \delta[\theta], \hat{\theta})$ in (A1) using (13) and

$$\begin{cases} \Phi_1 = \begin{bmatrix} \mathbf{0} \begin{bmatrix} \mathbf{0} & \mathcal{Y}^T \end{bmatrix} & \mathbf{0} & \mathbf{0} \end{bmatrix}^T, \\ \Phi_2(\theta, \hat{\theta}) = \begin{bmatrix} \begin{bmatrix} \alpha \left(A(\theta) - A(\hat{\theta})\right)\mathcal{X} & \mathbf{0} \end{bmatrix} & \begin{bmatrix} \left(A(\theta) - A(\hat{\theta})\right)\mathcal{X} & \mathbf{0} \end{bmatrix} & \mathbf{0} & \mathbf{0} \end{bmatrix}. \end{cases}$$

Then, controller $K(\hat{\theta})$ whose state-space matrix $\bar{K}(\hat{\theta})$ is given as (14), in which matrices $U, V \in \mathbb{R}^{n\times n}$ are nonsingular matrices satisfying $VU + \mathcal{Y}\mathcal{X} = \mathcal{S}$, stabilizes the closed-loop system and $\|z\|_2^2 < \gamma^2\|w\|_2^2$ is satisfied for all triplets of $(\theta, \delta[\theta], \hat{\theta})$ in which $(\theta, \delta[\theta]) \in \Lambda_\theta$ and $(\theta, \hat{\theta}) \in \Xi_\theta$ hold.

$$\Phi_0^c(\theta, \dot{\theta}, \hat{\theta}) = -\begin{bmatrix} \mathbf{0} & \mathcal{P}(\theta) & \mathbf{0} & \mathbf{0} \\ * & -\dot{\mathcal{P}}(\theta) & \mathbf{0} & \Upsilon_B(\theta, \hat{\theta}) \\ \mathbf{0} & \mathbf{0} & -\gamma^2\mathbf{I} & \Upsilon_D(\theta, \hat{\theta}) \\ \mathbf{0} & * & * & -\mathbf{I} \end{bmatrix} - \mathrm{He}\left\{ \begin{bmatrix} -\begin{bmatrix} \mathcal{X} & \mathbf{I} \\ \mathcal{S} & \mathcal{Y} \end{bmatrix} \\ \Upsilon_A(\theta, \hat{\theta}) \\ \Upsilon_C(\theta, \hat{\theta}) \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \alpha\mathbf{I} & \mathbf{I} & \mathbf{0} & \mathbf{0} \end{bmatrix} \right\} \tag{A1}$$

A brief proof is given below

Proof. Note that $\Phi_1(\cdot)\Phi_2(\cdot)$ is given as follows:

$$\begin{bmatrix} \mathbf{0} \\ \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathcal{Y}\left(A(\theta) - A(\hat{\theta})\right)\mathcal{X} & \mathbf{0} \end{bmatrix} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \alpha\mathbf{I} & \mathbf{I} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Then, by following the same procedure as in [26], the following inequality with $P(\theta) = T^{-T}\mathcal{P}(\theta)T^{-1}$ and $H = T^{-T}\begin{bmatrix} \mathcal{X} & \mathbf{I} \\ \mathcal{S} & \mathcal{Y} \end{bmatrix} T^{-1}$ with $T = \begin{bmatrix} \mathbf{I} & \mathcal{Y}^T \\ \mathbf{0} & V^T \end{bmatrix}$ is derived from $\Phi_0(\theta, \delta[\theta], \hat{\theta}) - \mathsf{He}\{\Phi_1(\cdot)\Phi_2(\cdot)\} \succ 0$:

$$\begin{bmatrix} \mathbf{0} & P(\theta) & \mathbf{0} & \mathbf{0} \\ * & -\dot{P}(\theta) & \mathbf{0} & B_{cl}(\theta, \hat{\theta}) \\ \mathbf{0} & \mathbf{0} & -\gamma^2\mathbf{I} & D_{cl}(\theta, \hat{\theta}) \\ \mathbf{0} & * & * & -\mathbf{I} \end{bmatrix} + \mathsf{He}\left\{ \begin{bmatrix} -\mathbf{I} & A_{cl}(\theta, \hat{\theta})^T & C_{cl}(\theta, \hat{\theta})^T & \mathbf{0} \end{bmatrix}^T H \begin{bmatrix} \alpha\mathbf{I} & \mathbf{I} & \mathbf{0} & \mathbf{0} \end{bmatrix} \right\} \prec 0.$$

Then, the assertion is proved by following the same argument in dilated LMI technique summarized in [27]. □

Remark 4. In contrast to Theorem 1, the formulation in Theorem 3 has a single line search parameter α. Thus, if we apply the design procedure in Section 3, the line search for α should be conducted at each step. Thus, the numerical complexity in designing continuous-time LPV controllers is so huge, similarly to [9]. Furthermore, if the implemented computers are digital computers, the online discretization of the state-space matrices of continuous-time LPV controllers should be also conducted by using Euler method, trapezoidal approximation [28], etc. Due to these reasons, in our previous report [5] and this paper, we directly design discrete-time LPV controllers for the discretized lateral-directional motion model of MuPAL-α. □

On conservatism for over-bounding the parameter discrepancies, a similar assertion as in Theorem 2 also holds for continuous-time case.

Theorem 4. Suppose that, in Theorem 3, there exist α, $\mathcal{P}(\theta)$, $\mathcal{K}(\hat{\theta})$ and $\mathcal{N}$ such that $\Phi_0^c(\theta, \delta[\theta], \hat{\theta}) \succ 0$ and $\Phi_0^c(\theta, \delta[\theta], \hat{\theta}) - \mathsf{He}\left\{\Phi_1(\theta, \hat{\theta})\Phi_2\right\} \succ 0$ hold. Then, there always exists a matrix-valued function of $\mathcal{Q}(\theta, \delta[\theta], \hat{\theta})$ satisfying (10). The corresponding assertion for (11) also holds with a (possibly different) matrix-valued function of $\mathcal{Q}(\theta, \delta[\theta], \hat{\theta})$.

Due to the same reason of Theorem 2, the proof is omitted.

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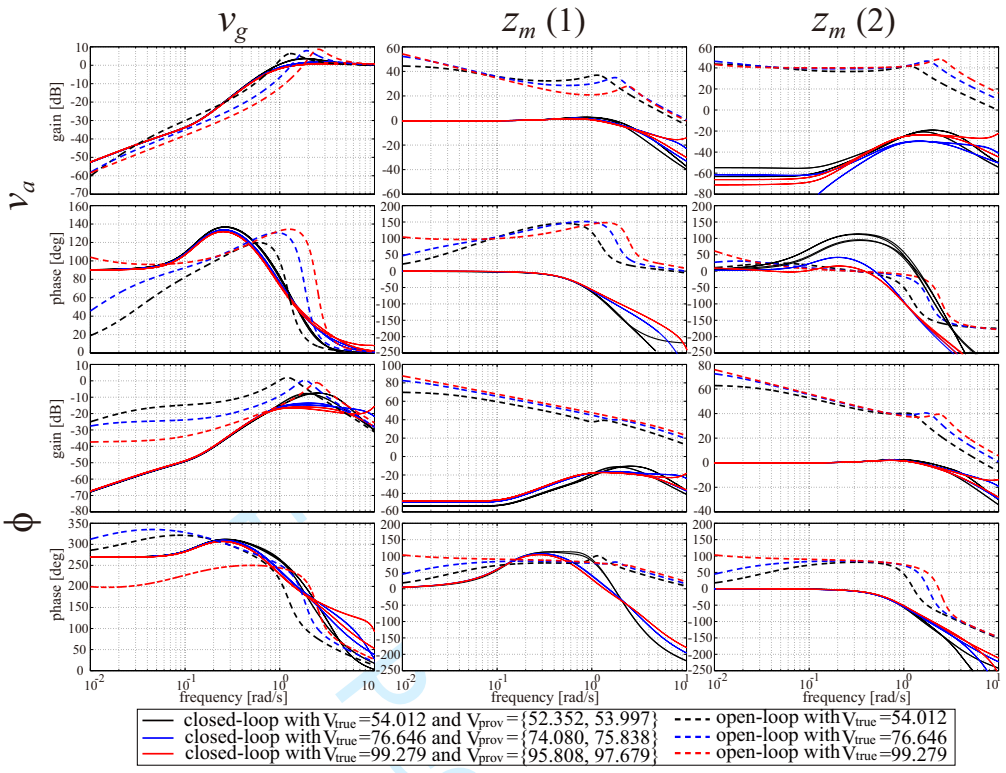

FIGURE 2 Bode plots from $[v_g \; z_m^T]^T$ to z_p using LPV controller with $T = 8.0$ in Table 1

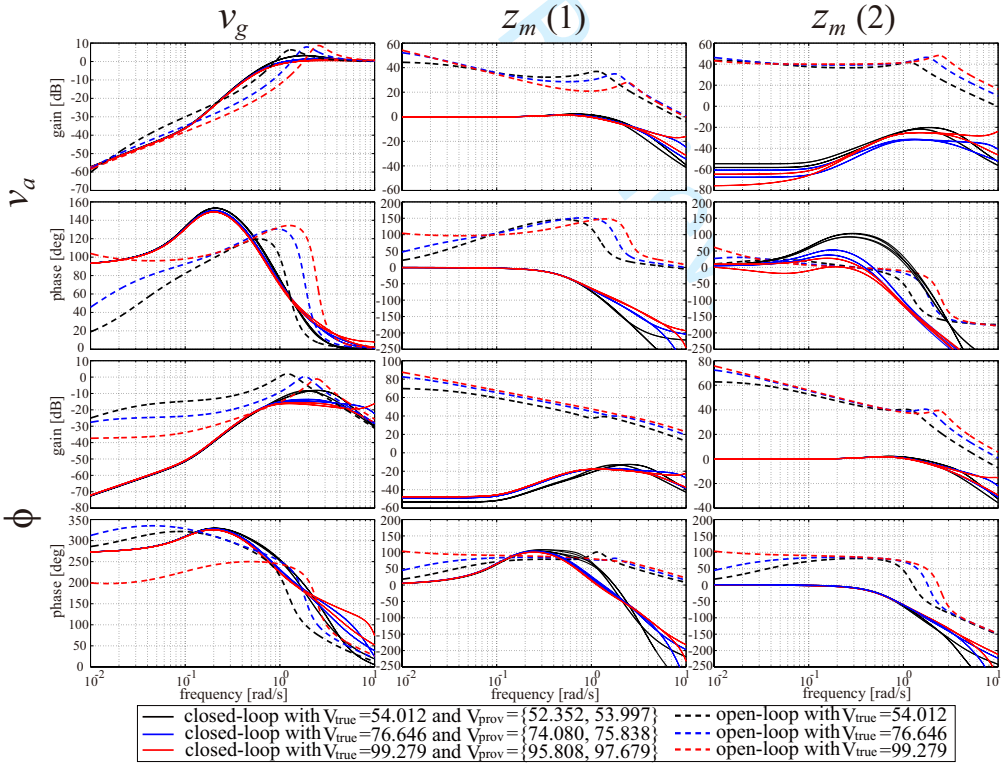

FIGURE 3 Bode plots from $[v_g \; z_m^T]^T$ to z_p using LPV controller with $T = 9.0$ in Table 2

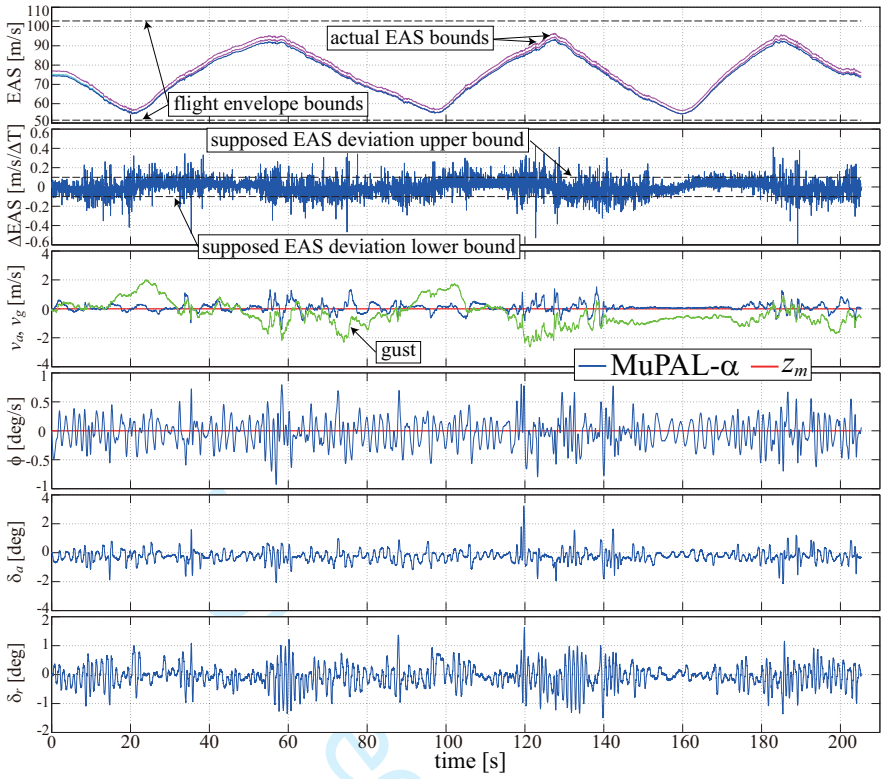

FIGURE 4 Flight test result with LPV controller via Theorem 1 with $T = 8.0$ for disturbance suppression check

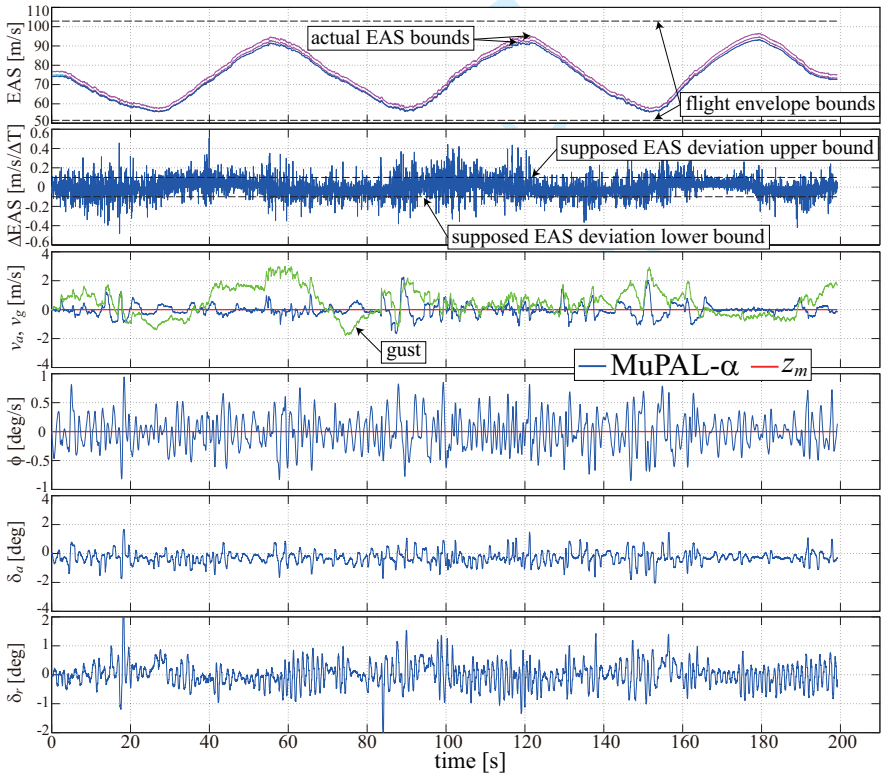

FIGURE 5 Flight test result with LPV controller via Theorem 1 in[5] with $T = 9.0$ for disturbance suppression check

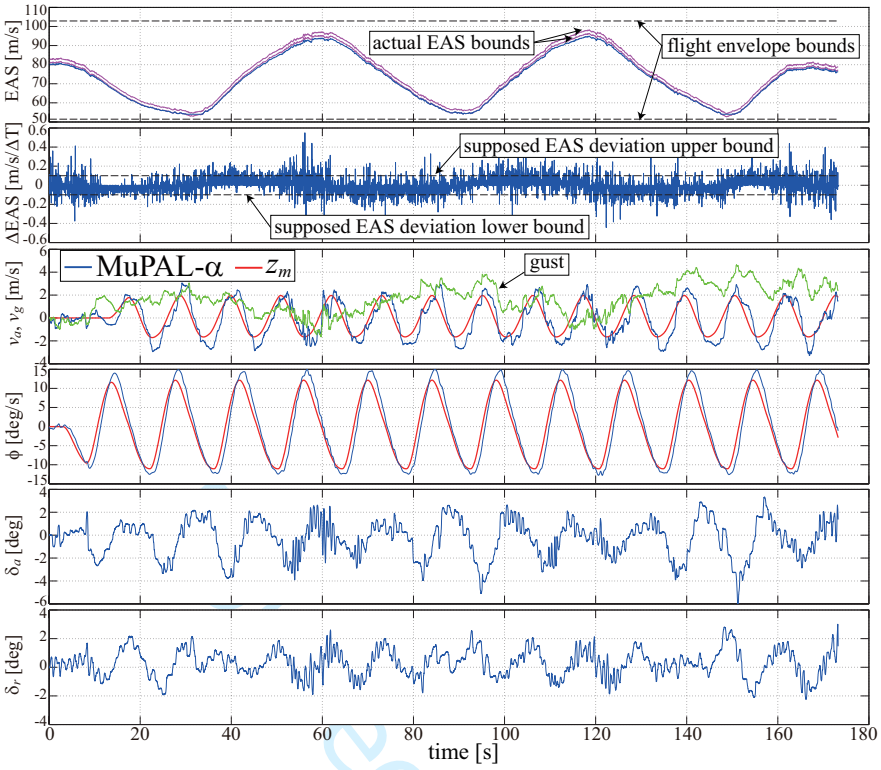

FIGURE 6 Flight test result with LPV controller via Theorem 1 with $T = 8.0$ for model-matching under disturbance

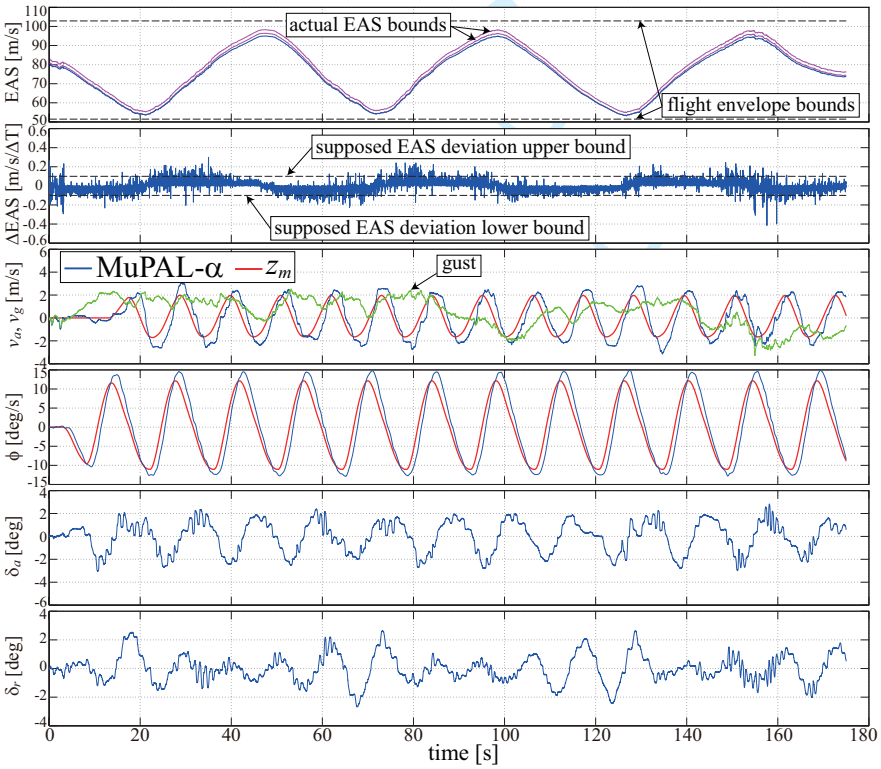

FIGURE 7 Flight test result with LPV controller via Theorem 1 in[5] with $T = 9.0$ for model-matching under disturbance

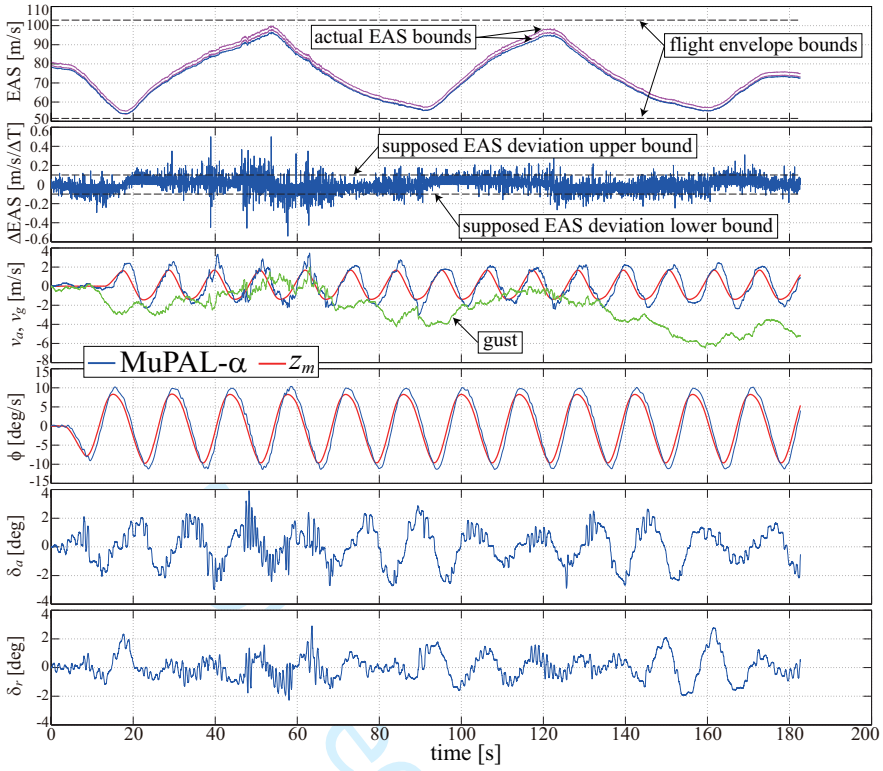

FIGURE 8 Flight test result with LPV controller via Theorem 1 with $T = 8.0$ for model-matching under disturbance

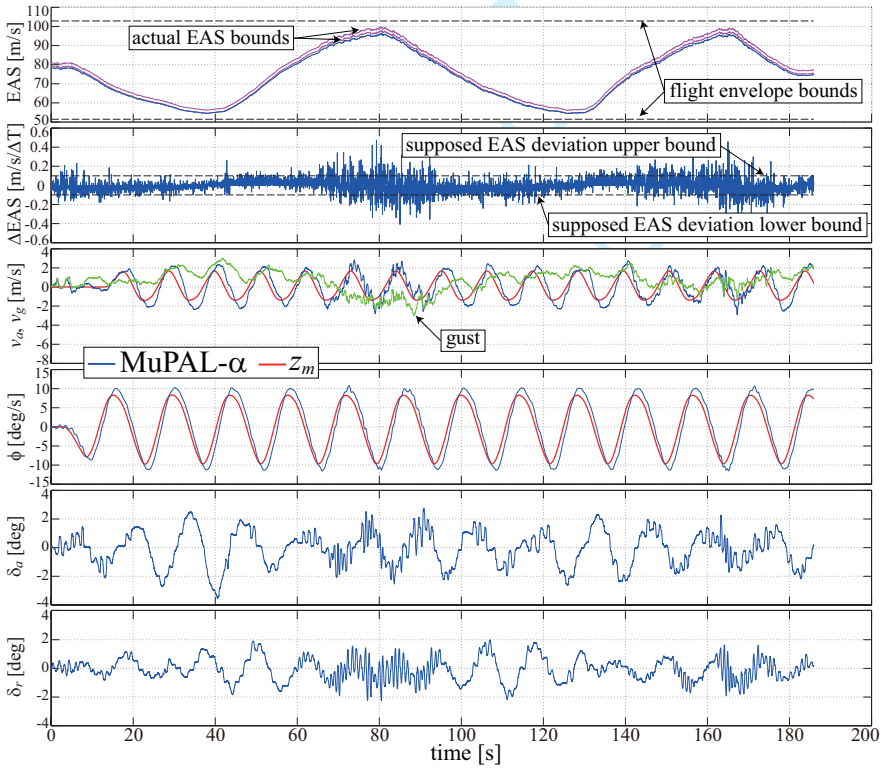

FIGURE 9 Flight test result with LPV controller via Theorem 1 in[5] with $T = 9.0$ for model-matching under disturbance