Attractors and limit cycles of discrete-time switching affine systems: nominal and uncertain cases
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A switched system consists in the association of a finite set of dynamics with a switching rule that designates at each time which is the active one among them [28]. This is a wide particular class of hybrid systems [23], that allows to model numerous applications in various fields as embedded systems, electromechanics, biology, or networked control systems and to characterize complex and not intuitive behaviors. Among them, there are discrete-time switched systems, that can be considered in their own or potentially generated by continuous-time switched systems with a switching rule that is piecewise constant according to a given sampling period [27]. It is interesting to emphasize that in the last case, linear or affine modes remain linear or affine by discretization. This kind of systems have generated a rich study concerning their stability to the origin [12,36], stabilization [21] or stabilizability [18].

Stabilizing Switched Affine Systems (SAS’s) to a desired position, that is not in the set of equilibrium points of the modes, has motivated a large collection of contributions, especially in the continuous-time framework: global quadratic stabilization via a min-switching strategy [10], use of dynamic programming to select the mode to activate and also the time to switch [31], globally stabilizing min-switching strategy taking into account a cost to minimize [15], local stabilization without requiring the existence of a Hurwitz combination of the linear parts [25], practical stabilization with minimum dwell-time guarantees [1], robust stabilization to an unknown equilibrium point [2,3]. We can emphasize also contributions in the discrete-time domain leading to the practical stabilization via different types of Lyapunov functions: quadratic forms [14], switched quadratic functions based on Lyapunov-Metzler inequalities [17] or thanks to multiple shifted quadratic functions [32].

Besides equilibrium points, dynamical systems may have as asymptotic behaviors, self-sustained oscillations, or limit cycles: that are closed and isolated trajectories [34, Section 7]. Their studies have been initiated by Poincaré and are generically related to the \( \omega \)-limit sets (set of accumulation points of the trajectories) and Poincaré-Bendixson theorem. For hybrid or switched systems, limit cycles have been mainly investigated in the continuous-time domain, motivated by switched circuits [4,24,26,30]. The Poincaré-Bendixson theorem has been extended for hybrid systems [33] and the \( \omega \)-limit sets of hybrid systems have been also investigated [11]. The main difficulty is to determine the switching times related to a limit cycle [20,22]. There are only few contributions on limit cycles for discrete-time SAS’s: one case study is provided in [29], based on prac-
tical stability [38] or clock- and state-dependent switched\nLyapunov functions [16] and also the preliminary result on\nmultiple shifted quadratic Lyapunov functions putting in\nevidence the notion of limit cycles [32].\n
This paper is focused on limit cycles for nominal and un-\ncertain discrete-time SAS’s, and their stabilizability thanks\nto state-dependent switching laws. In summary the contri-\nbutions are:

- A rigorous definition of limit cycles in the framework of\ndiscrete-time SAS’s.
- An original state-feedback min-switching control\nstrategy allows to obtain an autonomous system, which\ndiffers to the time-dependent one presented in [16].
- An exponential stabilization criteria, using a mini-\nswitching control law, expressed in terms of Linear\nMatrix Inequalities (LMI) is proven.
- A robust stabilization criteria for the uncertain case.

The paper is organized as follows. The investigated system\nis presented in Section 2. Limit cycles are rigorously defined\nin Section 3, where a necessary and sufficient condition for\ntheir existence is provided. Sections 4 and 5 deal with the\nstabilization to a limit cycle, in the nominal and uncertain\ncases. Families of optimization problems are presented in\nSection 6. Several illustrations are provided in Section 7\nbefore concluding remarks in Section 8.

Notations: Throughout the paper, \(\mathbb{N}\) denotes the set of\nnatural numbers, \(\mathbb{R}\) the real numbers, \(\mathbb{R}^n\) the \(n\)-dimensional Euclid-\n
ean space, \(\mathbb{R}^{m,n}\) the set of all real \(n \times m\) matrices and \(\mathbb{S}^n\) the\nsymmetric matrices in \(\mathbb{R}^{n,n}\). For any \(\sigma\) and \(m\) in \(\mathbb{N}\), \(I_n\) and \(\Theta_n\) denote the identity and null matrices of \(\mathbb{R}^{n,n}\), respectively.

Notation \(\mathbb{R}^{n,n}\) stands for the matrix in \(\mathbb{R}^{n,n}\) whose entries\nare all 1. For any matrix \(M\) of \(\mathbb{R}^{m,n}\), the notation \(M > 0\), \((M < 0)\) means that \(M\) is symmetric positive (negative) def-\nite. For any suitable matrices \(A = A^T, B, C = C^T\), notation\n\([A, B, C]\) stands for the symmetric matrix \([A, B, C]\). \(\|x\|\) denotes\nthe Euclidean norm of \(x\). For any \(M > 0\) in \(\mathbb{S}_n\) and any vector \(x \in \mathbb{R}^n\), we denote \(\|x\|_M = \sqrt{x^T M x}\), the weighted norm and \(\lambda_\min(M)\) and \(\lambda_\max(M)\) denote its minimal and maximal eigen-\nvalues respectively of \(M\). For any vector \(h \in \mathbb{R}^n\), we denote the\nshifted ellipsoid \(E(M, h) = \{x \in \mathbb{R}^n, (x - h)^T M (x - h) \leq 1\}\).

2 Problem formulation

Consider the discrete-time SAS given by

\[
\begin{align*}
x^+ &= A_\sigma x + B_\sigma u, \\
\sigma &\in u(x) \subset \mathbb{X},
\end{align*}
\tag{1}
\]

where \(x \in \mathbb{R}^n\) is the state vector, which adopts the following\nnotation \(\sigma = \sigma_0 + 1\) and \(x = x_0\) and \(x_0 \in \mathbb{R}^n\). Likewise,\n\(\sigma \in \mathbb{X} : = \{1, 2, ..., K\}\) characterizes the active mode. Finally,\n\(A_i \in \mathbb{R}^{n,n}\) and \(B_i \in \mathbb{R}^{n,c}\) are the matrices of mode \(i \in \mathbb{X}\).

The particularity of this class of systems relies on its control\naction, which is performed through the selection of the mode\n\(\sigma\), requiring a particular attention. The objective here is to

design a suitable set valued map \(u\) that ensures the conver-\ngence of the state trajectories to a set to be characterized ac-
accurately. Indeed, it is well-known that stabilizing system (1)\nto a single equilibrium cannot be achieved in general [38].
Its nonlinear nature, due to the affine term imposes to relax\nthe control objectives to an acceptable stability result such\nas practical stability in [14], for instance.

In this paper, the objective is to go deeper into the analysis\nand design of SAS’s and to characterize their steady-state behavior, i.e. limit cycles. Compared to [4] or [16], the fol-
lowing improvements have been performed. First, a distinc-
tion between periodic state-trajectories, as stated in [16],\nstate-limit cycles, and hybrid limit cycles is provided. Then,
a necessary and sufficient condition for their existence re-
gardless of their stability will be presented, which is not the\ncase in [16]. More interestingly, we will show that the LMI\ncondition provided here, which is the same as the one in\n[9] for periodic systems or more recently [16] for SAS’s,\nensures both the existence of a hybrid limit cycle and its\nstable. Finally, an extension to the case of uncertain sys-
tems is conducted, enhancing then the notion of robust limit\ncycles. To do so, we propose to follow our preliminary re-
results provided in [32] where a different Lyapunov function inspired from [17] is included to better understand the\nactor of the resulting control Lyapunov function. An im-
portant by-product of the following analysis is that the usual\nunderlying necessary condition consisting in the existence of\na Schur stable convex combination of matrices \(A\), is not\nrequired anymore, which, apart from [16], has been rarely\ncompleted in the literature.

3 Limit cycles of switched affine systems

This section focuses on the time-varying steady-states of\nSAS’s, which seem to have a natural convergence to a re-
peated sequence, behaving as a limit cycle in discrete time.
Limit cycles represent the stationary state of sustained osc-
illations, that depend exclusively on the parameters of the\nsystem and their intrinsic properties. The notion of limit cy-
cles [34,35] is adapted here to discrete-time SAS’s.

3.1 Definitions and notations

Let us first denote as \(\mathbb{H}^n\), the set of functions from \(\mathbb{N}\) to\n\(\mathbb{R} \times \mathbb{R}^n\). The following definition of hybrid limit cycle is for-
mulated.

Definition 1 (Hybrid limit cycle) A hybrid limit cycle for\nsystem (1) or limit cycle in short, is a closed and isolated\nhybrid trajectory \(s \in \mathbb{H}^n\). More formally, this means that \(k \mapsto x_k = (\sigma_k, x_k)\) is an N-
periodic solution to (1), which verifies

\[
\exists s(\in) > 0, \text{ s.t. } \forall s \in \mathbb{H}^n(\in), \min_{k \in \mathbb{N}} \sup_{k \in \mathbb{N}} ||s_k - s_{k+1}|| > \epsilon(\in),
\]

where \(\mathbb{H}^n(\in) \subset \mathbb{H}^n\) is the set of N-periodic functions of \(\mathbb{H}^n\)
(i.e. with the same period as \(s\)), that are solution to (1) but
are not shifted version of \( s \)

\[
\mathbb{T}^n(s) = \begin{cases} 
\tilde{s} \in \mathbb{R}^m, \\
(i) \quad \tilde{x}_{k+1} = A_{\sigma_k} \tilde{x}_k + B_{\sigma_k}, \quad \forall k \in \mathbb{N}, \\
(ii) \quad \tilde{x}_{k+N} = \tilde{x}_k, \quad \forall k \in \mathbb{N}, \\
(iii) \quad \min_{\delta \in \mathbb{N}} \left( \sup_{k \in \mathbb{N}} \| \tilde{x}_k - \tilde{x}_{k+\delta} \| \right) \neq 0.
\end{cases}
\]

Studying limit cycles is usually performed using Poincaré map and Poincaré-Bendixon theorem. However, the results are not easy to generalize for high dimensions nor for hybrid systems [22,33]. Another approach, taking the point of view of closed-loop dynamics, recovers more classical definitions of autonomous differential equations/inclusions [35]. Compared to [16], the notion of hybrid limit cycles is introduced. It is composed of both the state trajectory \( x_k \) and the control input \( \sigma_k \). Here, ‘closed’ refers to periodic nature of this trajectory (avoiding heteroclinic orbits [34, Ex.6.6.1]) and ‘isolated’ means that there exists a neighborhood of this trajectory, which does not contain another periodic solution generated by the same switching law. This is highlighted in the definition of \( \mathbb{T}^n(s) \), which excludes the solutions to (1) (e.g. (ii)) that are \( N \)-periodic and that are a shifted version of \( s \) (e.g. (iii)).

For a given hybrid limit cycle \( s_k \), both the projected switching signal trajectory \( \mathbb{N} \rightarrow \mathbb{R}^n, k \mapsto \sigma_k \) and the projected state trajectory \( \mathbb{N} \rightarrow \mathbb{R}^n, k \mapsto x_k \) are periodic functions. This gives rise to the next definitions.

**Definition 2 (Cycle)** For a hybrid limit cycle \( k \mapsto \tilde{s}_k = (\sigma_k, x_k) \), the cycle refers to the \( k \mapsto \sigma_k \) function, denoted \( \nu \). A limit cycle being a closed trajectory, there exists \( N \in \mathbb{N} \setminus \{0\} \) such that \( \nu(k + N) = \nu(k), \forall k \in \mathbb{N} \).

In addition, notations \( N_\nu \) and \( \mathbb{D}_\nu \) stand for the minimum period and the minimal domain of \( \nu \), defined as follows

\[
N_\nu = \min N \in \mathbb{N} \setminus \{0\} \text{ s.t. } \nu(k + N) = \nu(k), \forall k \in \mathbb{N}, \\
\mathbb{D}_\nu = \{1, 2, \ldots, N_\nu\}.
\]

**Definition 3 (Set of cycles)** Let \( C \) be defined as

\[
C := \{ \nu : \mathbb{N} \rightarrow \mathbb{R}^n \mid \exists N \in \mathbb{N} \setminus \{0\}, \forall k \in \mathbb{N}, \nu(k + N) = \nu(k) \}
\]

being the set of cycles (periodic functions) from \( \mathbb{N} \) to \( \mathbb{R}^n \). Moreover, \( C_N \) denotes the set of cycles that are \( N \)-periodic

\[
C_N := \{ \nu : \mathbb{N} \rightarrow \mathbb{R}^n \mid \exists N \in \mathbb{N}, \forall k \in \mathbb{N}, \nu(k + N) = \nu(k) \}.
\]

To ease the readability, we introduce the following modulo notation: \( [i]_N = ((i - 1) \mod N) + 1 \), for any \( i \in \mathbb{N}, i \geq 1 \). In particular, \( [i]_N = i \), for any \( i = 1, \ldots, N_\nu \) and \( [N_\nu + 1]_N = 1 \).

**Definition 4 (State limit cycle)** For a hybrid limit cycle \( k \mapsto s_k = (\sigma_k, x_k) \), the state limit cycle refers to the \( k \mapsto x_k \).

In the following, we will denote \( \rho_i, i \in \mathbb{D}_\nu \) (\( \nu \) being the associated cycle) as the ordered vectors.

**Definition 5 (Attractor)** A compact set \( \mathcal{A} \subset \mathbb{R}^n \) is said to be an attractor of system (1) if it is attractive and invariant.

The next paragraph exhibits conditions for the existence of limit cycles for switched affine systems.

### 3.2 Necessary and sufficient conditions of existence

To characterize the limit cycles of system (1), we take advantage of the associated periodic switching law and benefits of the discrete-time (linear) periodic system literature, see for instance [7,9,13,37] or the survey [8]. A necessary and sufficient condition to the existence of a limit cycle for a given cycle \( \nu \) is provided in the following lemma, that generalizes [29] to the case of an arbitrary number of modes and an arbitrary period \( N_\nu \).

**Lemma 1** A cycle \( \nu \in C \) generates a unique limit cycle for system (1) if and only if \( \nu \) is not an eigenvalue of matrix \( (\Phi(0))_M \) for all \( M \in \mathbb{N} \setminus \{0\} \), where \( \Phi(0) \) is the monodromy matrix at time \( \ell \in \mathbb{N} \), defined by

\[
\Phi(0) := \prod_{\ell=1}^{N} A_{\nu(\ell+1)} A_{\nu(\ell+2)} \cdots A_{\nu(\ell+N-1)} A_{\nu(\ell-1)}, \quad \forall \ell \in \mathbb{N}.
\]

Moreover, the state-limit cycle is given by \( \rho := (I_{N_\nu} - A_0)^{-1} B_\nu \), where

\[
A_\nu = \begin{bmatrix}
0 & \cdots & 0 & A_{\nu(N)} \\
A_{\nu(1)} & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & A_{\nu(N-1)} & 0
\end{bmatrix}, \\
B_\nu = \begin{bmatrix}
B_{\nu(1)} \\
B_{\nu(2)} \\
\vdots \\
B_{\nu(N)}
\end{bmatrix}, \\
\rho = \begin{bmatrix}
\rho_1 \\
\rho_2 \\
\vdots \\
\rho_{N_\nu}
\end{bmatrix}.
\]

**Proof.** A cycle \( \nu \in C \) is associated with a unique limit cycle for the switched affine system (1) if and only if there exists a unique sequence of \( N_\nu \) vectors \( \{\rho_i\}_{i \in \mathbb{D}_\nu} \) such that

\[
\rho_{[i]_N} = A_{\nu(i)}(\rho_i + B_\nu(i)), \quad \forall i \in \mathbb{D}_\nu,
\]

which is illustrated on the schematic representation shown on Fig. 1. In this figure, the attractor refers to \( \mathcal{A}_\nu \in \mathbb{N}^{nN_\nu \times nN_\nu} \), the state limit cycle \( k \mapsto x_k = \rho_{k[N_\nu]} \), with \( \delta \in \{1, 2, 3\} \). Finally, the hybrid limit cycle is \( k \mapsto s_k = (\sigma_k, x_k) = (\nu(k + \delta), \rho_{k[N_\nu]}(\delta)) \), with \( \delta \in \{1, 2, 3\} \).

Relations (2) can be reformulated into the following equation, by using a cyclic augmented representation inspired by [19,37]:

\[
(I_{N_\nu} - A_\nu)\rho = B_\nu,
\]

where \( A_\nu \), \( B_\nu \), and \( \rho \) are defined in the lemma. Let us first recall that matrix \( A_\nu \in \mathbb{R}^{nN_\nu \times nN_\nu} \) is closely related to the monodromy matrix at time \( \ell \in \mathbb{N} \), \( \Phi(0) \). For discrete-time periodic systems, the spectrum of the monodromy matrix does not depend on the time \( \ell \) (see [5, Section 3.1]). These eigenvalues are called characteristic multipliers. Moreover the spectrum of \( A_\nu \) is the set of all \( N_\nu \)-roots of the \( n \) eigenvalues of \( \Phi(0) \) (see [7, page 322, Section 3.2] or [37, Th.4]).

**Sufficiency:** Assume that 1 is not an eigenvalue of matrix \( (\Phi(0))_M = \Phi(0)^M \), where \( M \) is any strictly positive integer. In particular 1 is not an eigenvalue of \( \Phi(0) \) and we infer that matrix \( (I_{N_\nu} - A_\nu) \) is nonsingular. It follows that \( \rho \) defined in Lemma 1 is a periodic solution of (2). This ensures that a periodic solution in the neighborhood of this solution shares the same cycle \( \nu \). The period of the associated hybrid solution is necessarily a multiple of \( N_\nu \). Then, it is \( MN_\nu \)-periodic, with
Consider the Lyapunov function candidate defined by $V(\rho) = \rho_1^{\nu_1} \cdots \rho_N^{\nu_N}$, where $\nu_i$ is the integer satisfying $\nu_i \geq \nu_1$. If $\rho_1 = \rho_2$, then $V(\rho)$ is the union of two closed trajectories associated with cycles of periods strictly less than $N_r$.

**Proposition 1** Assume that a cycle $\nu$ generates a limit cycle for system (1), with the components of the state limit cycle denoted $[\rho_i]_{i=1}^N$. Then, for any nonsingular matrix $T$ and any vector $w \in \mathbb{R}^n$, $[T \rho_1 + w]_{i=1}^N$ are the components of the state limit cycle associated to the same cycle for the same system (1) but expressed in the new coordinates $z = T x + w$.

**Proposition 2** Consider a cycle $\nu \in C$ and its limit cycle $[\rho_i]_{i=1}^N$. If there exist $(i_0, i_1) \in \mathbb{Z}^2$, $i_1 > i_0$, such that $\rho_{i_0} = \rho_{i_1}$, then $[\rho_i]_{i=1}^N$ is the union of two closed trajectories associated with cycles of periods strictly less than $N_r$.

**Proof.** Based on the cycle $\nu$, the proof is obtained by denoting the two following cycles $\nu_1$ and $\nu_2$. $\nu_1$ is defined as a $N_{\nu_1} = (i_1 - i_0)$-periodic cycle given by $\nu_1(\ell) = \nu(i_0 - 1 + \ell), \ell = 1, \ldots, (i_1 - i_0)$ and is associated with the periodic trajectory $[\rho_{i_0}, \rho_{i_0 + 1}, \ldots, \rho_{i_1}]$. $\nu_2$ is defined as a $N_{\nu_2} = (N - i_1 + i_0)$-periodic cycle given by $\nu_2(\ell) = \nu(i_1 - 1 + \ell), \ell = 1, \ldots, (N - i_1 + i_0)$ and is associated with the periodic trajectory $[\rho_{i_1}, \rho_{i_1 + 1}, \ldots, \rho_{N_r}, \rho_{1}, \ldots, \rho_{i_0}]$. □

4 Stabilization to a limit cycle

4.1 Stabilization and Control Lyapunov function

This section presents the first stabilization theorem to a limit cycle defined by a given cycle $\nu \in C$.

**Theorem 1** For a given cycle $\nu \in C$, assume that there exist matrices $[P_i]_{i=1}^N$ in $\mathbb{R}^n$, such that

$$P_i > 0, \quad A_i^T P_{i+1} A_i - P_i < 0, \quad \forall i \in \mathbb{D}_\nu. \quad (4)$$

Then, the following statements hold:

(i) Cycle $\nu$ generates a unique limit cycle for system (1).
(ii) $\mathcal{A}_\nu := \bigcup_{i \in \mathbb{D}_\nu} [\rho_i]$, with $\rho_i$ solution to (2) is globally exponentially stable for system (1) with the switching control law

$$u(x) = \begin{cases} \nu(x), & \theta \in \arg\min_{i \in \mathbb{D}_\nu} (x - \rho_i)^T P_i (x - \rho_i) \in \mathbb{R}. \end{cases} \quad (5)$$

(iii) If $\rho_i \neq \rho_j$ for any $i \neq j$ in $\mathbb{D}_\nu$, then the switching signal $\sigma$ resulting from the closed-loop system (1),(5) converges ultimately to a shifted version of $\nu$.

**Proof.** The proof is split into three parts. Proof of (i): Introduce $\hat{P} = \text{diag}([P_i]_{i=1}^N)$ solution to (4).

Simple calculations show that

$$A_i^T \hat{P} A_i - \hat{P} = \text{diag} \left( \{ A_i^T P_{i+1} A_i - P_i \}_{i=1}^N \right) < 0.$$

Hence, matrix $A_i$ is Schur stable, which, according to Corollary 1, implies that (2) thus admits a unique solution.

Proof of (ii): Consider the Lyapunov function candidate defined as follows

$$V(x) = \min_{i \in \mathbb{D}_\nu} ||x - \rho_i||_{P_i}^2 \min_{i \in \mathbb{D}_\nu} (x - \rho_i)^T P_i (x - \rho_i), \quad \forall x \in \mathbb{R}^n. \quad (6)$$

Another useful property to ensure an equivalent class of switching laws related to the attractor is provided here.

**Corollary 2** Consider a cycle $\nu \in C$ and its limit cycle $[\rho_i]_{i=1}^N$. If there exist $(i_0, i_1) \in \mathbb{Z}^2$, $i_1 > i_0$, such that $\rho_{i_0} = \rho_{i_1}$, then $[\rho_i]_{i=1}^N$ is the union of two closed trajectories associated with cycles of periods strictly less than $N_r$. □
The idea is to prove that there exist small enough positive scalars where vectors $\rho_i$'s are solution to (2). Since $\mathbb{D}_i$ is bounded and $P > 0$, inequality

$$0 \leq c_1 d_{\mathcal{A}}(x) \leq c_2 d_{\mathcal{A}}^2(x),$$

(7)

holds with $c_1 = \min_{\nu \in \mathbb{D}_i} \lambda_{\max}(P_i) > 0$ and $c_2 = \max_{\nu \in \mathbb{D}_i} \lambda_{\min}(P_i) > 0$ where $d_{\mathcal{A}}(x) = \min_{\nu \in \mathbb{D}_i} \|x - \rho_i\|$ defines the distance of a vector $x$ in $\mathbb{R}^n$ to $\mathcal{A}$. Moreover, $\Delta V$ writes

$$\Delta V(x) := V(x^+) - V(x) = \min_{j \in \mathbb{D}_i} \|x^+ - \rho_j\|_{P_j} - \min_{j \in \mathbb{D}_i} \|x^+ - \rho_j\|_{P_j} = \min_{j \in \mathbb{D}_i} \|x^+ - \rho_j\|^2_{P_j} - \|x^+ - \rho_j\|^2_{P_j}.$$  

The last expression has been obtained by noting that the $\theta$ results from the control law in (5), and minimizes the quadratic term, by definition. The first term of $\Delta V$, being the minimum of several values, is consequently less than or equal to any of them, and, in particular, to $[\theta + 1]$. This yields $\Delta V(x) \leq \|x^+ - \rho_{\theta+1}\|^2_{P_{\theta+1}} - \|x^+ - \rho\|_{P_i}$. The closed-loop system (1),(5) imposes that $x^+ - \rho_{\theta+1} = A_{\theta+1}(x + B_{\theta+1} - \rho_{\theta+1}) = A_{\theta+1}(x - \rho)$, since $A_{\theta+1}\rho = B_{\theta+1} - \rho_{\theta+1} = 0$ from (2). Re-injecting this expression into the upper bound of $\Delta V(x)$ yields

$$\Delta V(x) \leq (x - \rho_0)^T \left( A_{\theta+1}^T \rho_{\theta+1} - \rho_0 \right)^T (x - \rho_0).$$

Therefore, if matrices $P_i$'s verify the strict inequalities (4), then there exists a small enough positive scalar $c_1 > 0$, such that $\lambda_{\min}(A_{\theta+1}) = P_i < -c_1 I_n$, for all $i \in \mathbb{D}_i$, so that

$$\Delta V(x) \leq -c_1 \|x - \rho\|^2 \leq -c_3 d_{\mathcal{A}}^2(x) \leq -c_3/c_2 V(x), \quad \forall x \in \mathbb{R}^n,$$

due to $\|x - \rho\| \geq d_{\mathcal{A}}(x)$ and inequality (7), which proves the global exponential convergence to attractor $\mathcal{A}$.

Proof of (iii): The idea is to prove that there exist $k_0 \in \mathbb{N}$ and $\delta \in \mathbb{K}$ such that $u(\lambda_k) = [\nu(k - \delta)]$, $c_1 = \nu(k - \delta)$, for all $k \geq k_0$. To do so, one needs to show that there exists a sufficiently small scalar $\epsilon > 0$ (to be determined), such that having $x \in \mathcal{S}_\delta = \{x \in \mathbb{R}^n, \quad V(x) \leq \epsilon \}$ and $\theta \in \arg \min_{\nu \in \mathbb{D}_i} \|x^+ - \rho\|_{P_j}$ implies $\|x^+ - \rho_{\theta+1}\|^2_{P_{\theta+1}} < \|x^+ - \rho\|^2_{P_j}$, for all $j \in \mathbb{D}_i$, with $j \neq \theta + 1$, that is the solution to (5) is $[\theta + 1]_\nu$.

To sum up, $k_0$ is related to the time to reach the level set $\mathcal{S}_\delta$, which is always possible to reach thanks to the convergence of the Lyapunov function to zero. The shift $\delta$ only depends on the initial condition $x_0$. To prove this item, the equivalence of weighted norms ensures that

$$\exists c_{ij} > 0, \quad \forall i, j \in \mathbb{D}_i, \text{ s.t. } \forall y \in \mathbb{R}^n, \quad \|y\|_{P_i} \leq c_{ij} \|y\|_{P_j},$$

(8)

(for instance, select $c_{ij} \geq \sqrt{\lambda_{\min}(P_i)/\lambda_{\max}(P_j)}$). Thanks to (4) and $x \in \mathcal{S}_\delta$, we have, with the notation $x^+ = A_{\theta+1}x + B_{\theta+1}$,

$$\|x^+ - \rho_{\theta+1}\|^2_{P_{\theta+1}} \leq \|x - \rho\|_{P_i}.$$

(9)

As $x^+ - \rho_{\theta+1} = A_{\theta+1}(x - \rho)$ and $x \in \mathcal{S}_\epsilon$, inequalities

$$\|x^+ - \rho_{\theta+1}\|_{P_{\theta+1}} = \|A_{\theta+1}(x - \rho)\|_{P_{\theta+1}} \leq \|A_{\theta+1}\|_{P_{\theta+1}} \|x - \rho\|_{P_i} \leq \|A_{\theta+1}\|_{P_{\theta+1}} \|x - \rho\|_{P_i} \epsilon,$$

(10)

hold, where $\|A_{\theta+1}\|_{P_i}$ denotes the matrix norm induced by the weighted norm $\|x\|_{P_i} = \sqrt{x^TP_i x}$, for any vector $x$ in $\mathbb{R}^n$. The triangular inequality and relations (8) yield, for all $j \in \mathbb{D}_i$,

$$\|\rho_{\theta+1} - \rho_j\|_{P_j} - \|A_{\theta+1}\|_{P_{\theta+1}} \|x - \rho\|_{P_i} \leq \|\rho_{\theta+1} - \rho_j\|_{P_j}, \quad \forall j \in \mathbb{D}_i \setminus \{\theta + 1\}.$$

Since having $\rho_i \neq \rho_j$ for any $i \neq j$ in $\mathbb{D}_i$, it is always possible to find $\epsilon$ such that $c_{ij} \epsilon < \|\rho_{\theta+1} - \rho_j\|_{P_j} - \|A_{\theta+1}\|_{P_{\theta+1}} \|x - \rho\|_{P_i}$ holds for any $j \in \mathbb{D}_i \setminus \{\theta + 1\}$. Combining both inequalities leads to $\epsilon < \|x^+ - \rho\|_{P_i}$, for all $j \in \mathbb{D}_i \setminus \{\theta + 1\}$. Comparing this with (9) yields $\|x^+ - \rho_{\theta+1}\|_{P_{\theta+1} \epsilon} \leq \epsilon < \|x^+ - \rho\|_{P_i}$, for all $j \in \mathbb{D}_i \setminus \{\theta + 1\}$, which concludes the proof.

Hereafter, Theorem 1 and results in Section 3.2 are commented and several important consequences are emphasized.

4.2 Feasibility of the sufficient conditions

Theorem 1 is based on the feasibility of LMIs (4), which have been already encountered in the framework of discrete-time linear periodic systems. Indeed the periodic Lyapunov lemma (see [6]) states the following result:

**Lemma 2** ([9]) For a given cycle $\nu$, there exist positive definite matrices $[P_i]_{\nu \in \mathbb{D}_i}$ satisfying LMIs (4) if and only if the monodromy matrix $\Phi(0)$ is Schur.

One of the main advantage of Lemma 2 is that the condition dealing with the monodromy matrix can be moved closer to the condition in Lemma 1: for a given cycle $\nu \in \mathcal{C}$, if the monodromy matrix $\Phi(0)$ is Schur, then there exists a unique limit cycle (thanks to Lemma 1). The importance of having Schur monodromy matrices being revealed, the question is now to understand whether there exists a cycle $\nu \in \mathcal{C}$ for a given system (1), which is associated to a stable monodromy matrix. The literature about the (periodic)-stabilizability of switched linear system provides useful conditions as for instance, [21] or more recently [18, Th. 6 and 22]).

4.3 Comparison with [16]

This section aims at comparing Theorem 1 with respect to [16, Th. 2]. While the LMI conditions are exactly the same for a given cycle $\nu$, the contributions are notably different. Indeed the control law given in [16, Th. 2] is

$$u(x, v, k) = \arg \min_{j \in \mathcal{K}} \begin{bmatrix} x - \rho_{[k]} \end{bmatrix}^T L_{ij} \begin{bmatrix} x - \rho_{[k]} \end{bmatrix} \subset \mathbb{K},$$

(11)

where $L_{ij} = \begin{bmatrix} A_i^T P_{[k]} A_j + B_j^T P_{[k]} B_i \end{bmatrix}$ and with $b_{ij} = A_i \rho_i + B_j - \rho_{[k+1]}$ and with the clock-dependent Lyapunov function $V(x, k) = (x - \rho_{[k+1]})^T P_{[k]} (x - \rho_{[k+1]}), \quad \forall x \in \mathbb{R}^n.$

(12)
The authors in [16] provide sufficient conditions for stabilization to the state trajectory (and not to the hybrid trajectory including the switching law), if (4) are satisfied. Note that this control law depends on a time counter \( v(k) \) such that at each instant time \( k \), the control input selects the point in the cycle \( v_k \) from an \( \text{argmin} \) function over all \( K \) modes. Our first result concerns the convergence to the attractor, while [16, Th. 2] provides the convergence to the state trajectory of the limit cycle. However if the assumption of item (iii) is satisfied, Theorem 1 provides a periodic solution for the hybrid trajectory with the switching law converging to \( v(k - \delta) \). Notice that this is not the case for [16, Th. 2], even if the additional assumption on the state limit cycle holds. This value of shift, \( \delta \), depends on the initial state \( x_0 \) and possibly on the choice of the switching law in the inclusion (5). Notice also that, for \( k \geq k_0 \), the set \( u(x_k) \) reduces to a singleton and there is a unique selection of the mode to activate. A similar result may be obtained in [16, Th. 2], when a unique mode allows to steer a \( \rho_i \) to its successor in the state trajectory, that is there exists a unique \( j_0 \) such that \( b_{i,j_0} = 0 \). Item (iii) of Theorem 1 emphasizes that even if the Lyapunov function does not depend on \( v \), the min-switching strategy recovers a shifted version of \( \gamma \) as an element of hybrid trajectory to the equivalent relation of attractors.

For a given cycle \( \gamma \), control law (5) aims at selecting the best mode that minimizes the quadratic term in \( V \), looking for the best position in the cycle. Alternatively, control law (11) selects the mode that minimizes (12), evaluated a \( k + 1 \). Hence, the computational complexity of both control laws are different, depending on the length of the cycle and on the number of modes. For instance, depending on whether \( N \geq K \) or \( N < K \), control (5) or (11) can reduce the computational cost and the transient respectively.

To sum up, both contributions are different and their use depends on the context. An important property of the min-switching algorithm (5) is that this control law does not depend on the system parameters, enabling to develop a robust control law that takes into account parametric uncertainties. Hence, the following section presents a robust control law that ensures the states to converge to a robust limit cycle.

### 5 Robust stabilization of uncertain systems

Here matrices \( A_{\sigma} \) and \( B_{\sigma} \) are assumed to be known and/or time-varying, with a polytopic representation given by

\[
[A_{\sigma}, B_{\sigma}] \in \text{Co} \left( [A_{\sigma}^i, B_{\sigma}^i] \right)_{i \in \mathbb{L}}, \quad \forall \sigma \in \mathbb{R},
\]

where \( \mathbb{L} \) is a bounded subset of \( \mathbb{N} \) and \( A_{\sigma}^i \) and \( B_{\sigma}^i \) are known and constant for any \( \sigma \in \mathbb{R} \) and any \( i \in \mathbb{L} \). Note that set \( \mathbb{L} \) may depend on the mode \( \sigma \) but is avoided here without lack of generality. The results of the previous section fail, about stabilization and stabilizability. Indeed, the main problem appears in the selection of the limit cycle solving equations (2) in the situation of uncertain and/or time-varying system matrices. Therefore, it is important to provide an alternative solution dedicated to this relevant situation from a practical point of view. The robust stabilization is formalized here.

![Fig. 2. Schematic representation of a robust limit cycle associated to a cycle \( \nu \) of period \( N_r = 3 \), for a system (1) with \( K = 2 \) modes.](image)

**Theorem 2** For a given cycle \( \nu \) in \( C \) and for a parameter \( \mu \in (0, 1) \), assume that there exist \( \{(W_i, \zeta_i)\}_{i \in \mathbb{D}_\nu} \) in \( \mathbb{S}^n \times \mathbb{R}^n \) and that are solutions to the following matrix inequalities

\[
W_i > 0, \quad \Psi_i(A^{(\ell)}_i, B^{(\ell)}_i) > 0, \quad \forall (i, \ell) \in \mathbb{D}_\nu \times \mathbb{L},
\]

where \( \Psi_i \) depend on the system matrices and on the decision variables \( \{(W_i, \zeta_i)\}_{i \in \mathbb{D}_\nu} \) in \( \mathbb{S}^n \times \mathbb{R}^n \) and are given by

\[
\begin{align*}
\Psi_i(A^i, B^i) = & \begin{pmatrix}
(1 - \mu)W_i & 0 \\
W_i & A^i \zeta_i + B^i \zeta_{i+1}
\end{pmatrix} \in \mathbb{S}^n \times \mathbb{R}^n \quad \text{and}
\end{align*}
\]

Then, the following statements hold:

(i) \( \mathcal{S}_r := \bigcup_{i \in \mathbb{D}_\nu} \mathcal{E}(W_i^{-1}, \zeta_i) \) is robustly globally exponentially stable for system (1) with the control law

\[
u(x) = \left\{ \begin{array}{ll}
\nu(\theta), & \theta \in \text{argmin}(x - \zeta_0)^T W_0^{-1}(x - \zeta_0) \in \mathbb{R}, \\
\end{array} \right.
\]

(ii) Moreover, if \( \mathcal{E}(W_i^{-1}, \zeta_i) \cap \mathcal{E}(W_j^{-1}, \zeta_j) = \emptyset \) for all \( i \neq j \in \mathbb{D}_\nu \), then the switching signal \( \nu \) of closed-loop system (1), (16) converges ultimately to a shifted version of \( \nu \).

**Remark 1** Note that condition (14) becomes an LMI once parameter \( \mu \in (0, 1) \) is fixed.

**Proof.** Consider the same Lyapunov function given in (6) but with the Lyapunov matrices \( P_i \) replaced by \( W_i^{-1} \) and with vectors \( \rho_i \)’s replaced by \( \zeta_i \)’s, solution to (14).

**Proof of (i):** As for Theorem 1, we can show that \( \Delta V(x) \leq \frac{\|x^T - \zeta_{(l+1)}}{W_{l+1}} - \|x - \zeta_0\|^2_{W_0^{-1}} \) and we write

\[
x^T - \zeta_{(l+1)}, = A_{\nu}(x - \zeta_0) + B_{\nu}(\theta).
\]

where \( B_{\nu}(\theta) = A_{\nu}(0) \zeta_0 + B_{\nu}(0) - \zeta_{(l+1)} \), which are not necessarily zero, compared to the proof of Theorem 1 because vectors \( \zeta_i \)’s are now decision variables in inequalities (14).

In order to find an alternative solution, we introduce \( \chi_\theta = \left( (W_\theta^{-1}(x - \zeta_0)) \right) \right), \) so that \( x^T - \zeta_{(l+1)} = A_{\nu}(0) W_\theta B_{\nu}(\theta) \chi_\theta \). Hence, \( \Delta V(x) \) can be rewritten in a more compact form

\[
\Delta V(x) \leq -\chi_\theta^T \Phi_\theta(A_{\nu}(\theta), B_{\nu}(\theta)) \chi_\theta.
\]

---

1. For simplicity, variables \( \{(W_i, \zeta_i)\}_{i \in \mathbb{D}_\nu} \) are omitted in \( \Psi_i \).
with \( \Phi_\theta(A_{i(r)}, B_{i(r)}) = \begin{bmatrix} W_0 & 0 & 0 & 0 & W_0A_{i(r)}^T & W_0^{-1} & W_0A_{i(r)}^T \end{bmatrix}^T \).

Note that \( \Delta V \) is not required to be negative in the whole state space, but only outside of set \( S_r \). From the definition of the Lyapunov function, the control law given in (16) ensures that \( V(x) = ||x - \xi_0||^2 \). Therefore, \( x \) being outside of \( S_r \), inequality \( V(x) \leq 1 \) writes

\[
\begin{bmatrix} x - \xi_0^T \\ 1 \end{bmatrix} \begin{bmatrix} W_0^{-1} & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x - \xi_0 \\ 1 \end{bmatrix} = \chi_0^T W_0 \begin{bmatrix} 0 \\ -1 \end{bmatrix} \chi_0 > 0. \tag{19}
\]

Then, the problem can be summarized as the satisfaction of \( \chi_0^T \Phi_\theta(A_{i(r)}, B_{i(r)}) \chi_0 > 0 \) for all \( x \) such that (19) holds. Using an S-procedure, if there exists \( \mu \in (0, 1) \) where

\[
\begin{bmatrix} (1 - \mu)W_0 & 0 \\ 0 & \mu \end{bmatrix} - \begin{bmatrix} W_0A_{i(r)}^T & W_0^{-1} & W_0A_{i(r)}^T \end{bmatrix}^T \geq 0, \tag{20}
\]

then, \( \Delta V(x) < 0 \) for all \( x \not\in S_r \). Finally, a Schur’s complement yields \( \Psi_\theta(A_{i(r)}, B_{i(r)}) > 0 \), for a fixed parameter \( \mu \), where matrix \( \Psi_\theta \) is defined in (15). Since matrices \( A_{i(r)} \) and \( B_{i(r)} \) are uncertain, it is not yet possible to evaluate numerically these LMs for all possible values of \( \theta \). However, since they belong to the polytopic set (13), one can define those matrices as convex combinations, with possibly time-varying weights \( A_{i(r)} = \sum_{i \in \mathbb{L}} \lambda_i A_{i(r)}^i \) and \( B_{i(r)} = \sum_{i \in \mathbb{L}} \lambda_i B_{i(r)}^i \), where parameters \( \lambda_i \in [0, 1] \) and hold \( \sum_{i \in \mathbb{L}} \lambda_i = 1 \). Since \( \Psi_\theta \) are affine with respect to \( A_{i(r)} \) and \( B_{i(r)} \), it follows

\[
\Psi_\theta(A_{i(r)}, B_{i(r)}) = \sum_{i \in \mathbb{L}} \lambda_i \Psi_\theta(A_{i(r)}^i, B_{i(r)}^i) > 0,
\]

which is guaranteed by conditions (14). This guarantees that \( \Delta V(x) \) is negative definite outside of \( S_r \). Exponential stability is obtained thanks to the strict inequalities (14).

To conclude the proof, it remains to prove that the attractive set \( S_r \) is invariant. By noting that item (i) ensures that \( \Delta V(x) + \mu(V(x) - 1) < 0 \), for all \( x \in \mathbb{R}^n \). This establishes

\[
V(x) = V(x) - \mu(V(x) - 1) + \Delta V(x) + \mu(V(x) - 1) \leq (1 - \mu)V(x) + \mu.
\]

Since \( x \) belongs to \( S_r \) and \( \mu \in (0, 1) \), \( V(x^+) \leq (1 - \mu) + \mu = 1 \) holds true, guaranteeing that \( x^+ \) also belongs to \( S_r \).\( \square \)

Proof of (ii): The proof of this result is omitted because is similar to the proof of item (iii) in Theorem 1.\( \square \)

A relevant byproduct of this theorem is an extension of the definition of limit cycles in equation (2) to the case of uncertain switched affine systems, which can be now expressed in terms of series of inclusions. More specifically, Theorem 2 states that the invariance of the attractor \( S_r \) ensures that

\[
A^i_{i(t)} E(W^{-1}_{i(t)}, \xi_i(t)) + B^i_{i(t)} E(W^{-1}_{i(t)}, \xi_i(t)) \in \mathbb{D}_r, \quad \forall (i, \ell) \in \mathbb{D}_r \times \mathbb{L}, \tag{21}
\]

where the left-hand-side of the inclusion means, with a slight abuse of notations, that, for any \( i \in \mathbb{D}_r \) and for all \( x \in \mathcal{E}(W_{i(t)}, \xi_i(t)) \), vector \( A^i_{i(t)} + B^i_{i(t)} \) belongs to \( \mathcal{E}(W_{i(t)}, \xi_i(t)) \) for all \( \ell \in \mathbb{L} \). This inclusion can be seen as the natural extension of (2) to uncertain systems. The union of \( \mathcal{E}(W_{i(t)}, \xi_i) \),\( i \in \mathbb{D}_r \), can be viewed as a stable robust limit cycle. It is also relevant to understand how conservative the previous theorem is with respect to the nominal case, presented in Theorem 1. The following proposition is stated.

**Proposition 2** For a given cycle \( v \) in \( C \), consider \( (P_i, \rho_i) \) in \( \mathbb{D}_r \times \mathbb{R}^n, (4) \) for a nominal system. Then, \( (W_i, \xi_i) = (B^i P_i - 1, \rho_i) \) is solution to (14), for any arbitrary scalar \( \beta > 0 \), with a sufficiently small value of \( \mu \in (0, 1) \). Moreover, \( \lim_{\beta \to 0} S_r \to \mathcal{A}_v \).

Proof. For matrices \( P_i \) and vectors \( \rho_i \) solution to inequalities (4) and to Lemma 1, respectively, let us define \( (W_i, \xi_i) = (B^i P_i - 1, \rho_i) \), for any scalar \( \beta > 0 \). Then \( 2 \xi_i = A_{i(r)}(\rho_i - B_{i(r)} - \rho_i)_i = 0 \), as defined in (17). For any \( \mu \in (0, 1), \Psi_i(A_{i(r)}(\beta^i P_i - 1), B_{i(r)}(\beta^i P_i - 1)) = 0 \) is equivalent to \( A^i_{i(t)} P_{i(t)} A_{i(t)} - B^i_{i(t)} B_{i(t)} - 1 \beta P_{i(t)} \) by application of the Schur complement. The latter inequality being true for a small enough value \( \mu > 0 \), thanks to the strict inequality (4). Moreover, by noting that the attractor \( S_r \) is composed by the ellipsoids given by \( \mathcal{E}(P_i, \beta P_i) \), for all \( i \in \mathbb{D}_r \), it reduces to \( \mathcal{A}_v \), as \( \beta \) tends to zero.\( \square \)

The previous proposition states that there is no conservatism induced by Theorem 2 with respect to Theorem 1. Therefore, in the following section dealing with the introduction of optimization problems, only the LMI constraints presented in Theorem 2 (for a fixed parameter \( \mu \)) will be considered for the sake of simplicity, knowing that the same optimization problem could also be presented using the LMI constraints of Theorem 1. Then, in the sequel, we will refer to the attractor only as \( S_r \) knowing that in the nominal case, \( S_r = \mathcal{A}_v \).

### 6 Optimization algorithms

The objective of this section is to include to the previous developments some additional constraints to conditions (14) aiming at selecting the decision variables \( (W_i, \xi_i)_{i \in \mathbb{L}} \), that optimize a given cost function, evaluated for each cycle under consideration. In practical situations such as in power converters, an additional objective to stabilization could be to drive the solutions to the system as close as possible to a desired reference position, \( x^r \in \mathbb{R}^n \), referring to a desired voltage. Hence, it appears highly relevant that the cost function does not only reflect the “size” of the attractor in order to limit the amplitude of the trajectories within the attractor but also the “distance” between the reference position to the attractor.

To go further in this direction, let us introduce the ellipsoid \( \mathcal{E}(Q_r, h_r) \) defined for some positive definite matrix \( Q_r \) in \( \mathbb{R}^n \) and some shifting vector \( h_r \) in \( \mathbb{R}^n \) to be optimized for
a given cycle so that $E(Q^{-1}, h_i)$ is the "smallest" ellipsoid verifying $(\{x_d| S_i \}) \subset E(Q^{-1}, h_i)$. The following lemma helps expressing this inclusion as a matrix inequality.

**Lemma 3** For some $(Q, h)$ in $\mathbb{S}^n \times \mathbb{R}^n$, define

$$K_Q(h, \zeta, \eta) = \begin{bmatrix} \eta W & 0 & W \\ \ast & 1 - \eta & \zeta^T - h^T \end{bmatrix},$$

for some matrix $0 < W \in \mathbb{S}^n$, a shifting vector $\zeta$ and a positive scalar $\eta$. Then, the following statements hold:

(i) $\zeta$ belongs to $E(Q^{-1}, h)$ if and only if $K_Q(h, \zeta, 0) \succeq 0$.

(ii) $E(W^{-1}, \zeta)$ is included in $E(Q^{-1}, h)$ if and only if there exists $\eta > 0$ such that $K_Q(h, \zeta, \eta) \succeq 0$.

**Proof.** The proof relies on the Schur complement (i) and an S-procedure (ii) and is omitted due to space limitations. □

Inclusion $(\{x_d| S_i \}) \subset E(Q^{-1}, h_i)$ is equivalent to

$$Q > 0,$$  \hspace{1cm}  $K_{Q,h_i}(0, x_d, 0) \succeq 0,$  \hspace{1cm}  $K_{Q,h_i}(W, \zeta, \eta) \succeq 0,$  \hspace{1cm}  $i \in \mathbb{D}_v.$

(23)

In the nominal case, inequalities $K_{Q,h_i}(W, \zeta, \eta) \succeq 0$, $i \in \mathbb{D}_v$ can be reduced to $K_{Q,h_i}(0, \rho_i, 0) \succeq 0$, for all $i \in \mathbb{D}_v$. A possible way to formalize the notion of a cost related to a "distance" and/or a "size", can be formulated as follows

$$J^*(v, x_d) := \min \left\{ J(v, x_d, \{W, \zeta\}|_{\mathbb{D}_v}) \right\}$$

s.t. (14) and potential additional inequalities.

where $J$ is the cost function to be optimized and is defined as a barycenter of several families of not exhaust costs

$$J(v, x_d, \{W, \zeta\}|_{\mathbb{D}_v}) = \sum_{m=1}^{4} a_m J_m(v, x_d, \{W, \zeta\}|_{\mathbb{D}_v}),$$

(25)

where $a_m \geq 0$ and $\sum_{m=1}^{4} a_m = 1$ and where $J_m$'s are given by

(i) $J_1(v, x_d, \{W, \zeta\}|_{\mathbb{D}_v}) = \text{Tr} (Q)$, with the additional inequalities given in (23), which aim at optimizing the attractor, or more precisely at evaluating the "chattering" effect when the solution reaches the attractor.

(ii) $J_2(v, x_d, \{W, \zeta\}|_{\mathbb{D}_v}) = \sum_{i=1}^{4} \text{Tr} (W)$, which aims at minimizing $S_i$, in the uncertain case.

(iii) $J_3(v, x_d, \{W, \zeta\}|_{\mathbb{D}_v}) = X \omega$ with $\left[ \begin{array}{c} \omega \end{array} \right] \succ 0$, $\forall(i,j) \in \mathbb{D}_v$, $i \neq j$, which aims at enforcing the shifts $\zeta_i$'s to be the same value, that is to have a single shift for the ellipsoids.

(iv) $J_4([\zeta]|_{\mathbb{D}_v}, x_d) = \omega_4$ with either

$$\omega_4 \geq \omega$$

and vectors $\omega_4$ with either

$$\omega_4 \geq \omega$$

(26)

which minimizes the distance between the average value of the limit cycle and the desired reference. In [16], another but similar cost function was presented with a projection matrix $F$. Here, $J_3$ will allow to select a limit cycle closed to a desired position that is predefined by the designer.

The optimization problem can now be properly stated.

<table>
<thead>
<tr>
<th>Cycles (C)</th>
<th>Associated limit cycles</th>
<th>$J^*(v, 0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_1$</td>
<td>$[0.2, 0.4, 0.3]$</td>
<td>$2.37$</td>
</tr>
<tr>
<td>$v_2$</td>
<td>$[0.3, 0.5, 0.8]$</td>
<td>$8.88$</td>
</tr>
<tr>
<td>$v_3$</td>
<td>$[0.4, 0.1, 0.6]$</td>
<td>$5.17$</td>
</tr>
</tbody>
</table>

Table 1 Limit cycles generated by a selection of cycles, which verify the assumption of Lemma 1 for system (1) with (26). The cost function (25) was computed with $\alpha = [0.5 0.0 0.5]$ and $x_d = [0 0 0]^T$.

**Proposition 3** For a given bounded subset $\Omega \subset C$ and a given desired reference $x_d$, the optimal control law with respect to $J$ is the one associated to $v^* = \arg\min_{v \in C} J^*(v, x_d)$.

### 7 Numerical Applications

**Example 1:** Consider system (1), borrowed from [14], where the matrices $A_i$ and $B_i$ are defined as follows

$$A_i = e^{F_i T}, \hspace{1cm} B_i = \int_0^T e^{F_i T} \rho_i \text{d} t, \hspace{1cm} \forall i \in \{1, 2\},$$

where $T = 1$ refers to the sampling period and with $F_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $F_2 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $g_1 = [1 0 0]^T$, $g_2 = [0 1 0]^T$.

For this example, there exists a linear combination of matrices $A_i$, which is Schur as shown in [14]. Fig. 3 shows on different graphs the limit cycles $\rho_i(1)\in \mathbb{C}$, represented by the red crosses, obtained thanks to Lemma 1 for three different cycles. The figure also shows the trajectories of the closed-loop system, initialized at $x_0 = [2, -5, 0]^T$, with control law (5) from Theorem 1. Each trajectory converges to different limit cycles. The control signal is represented at the bottom of Fig. 3 and tends to the presumed cycle after a small transient time as pointed out in item (iii) of Theorem 1. The results are interestingly very similar to the ones in [16] and [32]. However, the method provided in [32] is limited by the length of the cycle that needs to be equal to the number of modes, i.e. restricted to $v_1$. In addition, the stabilization condition of [32] has a higher complexity since the size of each LMI increases with the number of modes. Finally compared to [16], our control law does not depend on time even though it converges to a periodic signal.

Table 1 provides informations on the cycles and their associated limit cycle as in Fig. 3. A cost function (25) has been also considered according to Section 6 with $\alpha = [a_1, a_2, a_3, a_4] = [0.5, 0.0, 0.5]$ in (25). The cycle, for which the cost function, is minimized is $v_1$ as in Table 1.

**Example 2:** Consider system (1) with $A_1 = \begin{bmatrix} 0.3 & 0.1 \\ 0.9 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0.3 & 0.1 \\ 0.9 & 0 \end{bmatrix}$, $A_3 = \begin{bmatrix} 0.4 & 0 \\ 0 & 0 \end{bmatrix}$, and $A_4 = A_1 + 0.2 \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ to illustrate the comparison made in Section 4.3. The affine terms are built so that $(\nu, \rho)$ are $[1, 2, 3]$ and vectors $\rho = [0, 1]^T$, $[1, 1]^T$, $[-1, 0]^T$. Hence, we have $B_1 = B_2 = -A_1 \rho_1 + \rho_2$, $B_2 = -A_1 \rho_1 + \rho_3$, and $B_3 = -A_3 \rho_1 + \rho_1$. Fig. 4 shows the state trajectory $(x, \sigma)$ in a phase plane. The trajectories of system (1) with the switching control law (5)
seems to have a better convergence rate. However, one must notice the differences on the switching signals. Whereas the given limit cycle should only involve three of the system functioning modes, the mode associated to the matrices $A_4$ and $B_4$ is selected periodically instead of the mode 1 with the control law from [16, Th.2]. Hence, this example exposes the remark made in Section 4.3 concerning the possible cases where at least two modes can steer one vector $\rho_i$ to its successor. Control (5) provides for this example a better result than control (11), as illustrated in Fig. 4c.

Example 3: Consider system (1) with

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad A_2 = 4\sqrt{2} \begin{bmatrix} \cos \left( \frac{\nu}{4} \right) \\ \sin \left( \frac{\nu}{4} \right) \cos \left( \frac{\nu}{4} \right) \end{bmatrix}$$

and with $B_1 = [0.01, 0]^T$ and $B_2 = [0, 0]^T$. This example is adapted from [18], in which affine terms have been included. As noticed in [18], the monodromy matrix with cycle $\nu = (1^{11}, 2^2)$ is Schur, guaranteeing then, from Lemma 2, that a solution to Theorem 1 exists. However, there does not exist a linear combination of $A_1, A_2$ that is Schur stable. Fig. 5 depicts the evolution of the state variables converging to the limit cycle associated to $\nu$, as well as the state-space partition where the dark areas are the regions where mode 2 is activated.

To illustrate the uncertain case, consider now

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \pm \kappa_1 \delta_{\text{max}} \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.01 \\ \pm \kappa_2 \delta_{\text{max}} \end{bmatrix}$$

(28)

where $(\kappa_1, \kappa_2) = 10^{-3}(10, 5)$ and where $\delta_{\text{max}}$ is a parameter. Matrices $A_2$ and $B_2$ remain the same. Fig. 6 shows the different attractors $S_\nu$ obtained for various values of $\delta_{\text{max}}$ after performing a gridding procedure on parameter $\mu \in (0, 1)$ to optimally solve conditions (2) minimizing the cost function $J$ with $\alpha = [0, 1, 0, 0]$. As expected, the size of the attractor grows with $\delta_{\text{max}}$. Vectors $\zeta_i$’s, solution to Theorem 2, are very close to the limit cycle $[\rho_i]_{\nu \in \mathcal{E}_i}$ of the nominal case (for example, with $\delta_{\text{max}} = 0.01, \| \rho_i - \zeta_i \|_\infty \leq 10^{-3}$), which illustrates Proposition 2. The attractor obtained for the two smallest values of $\delta_{\text{max}}$ is the union of disjoint ellipsoids, so that the control law converges ultimately to a periodic law, verifying item (ii) of Theorem 2. Lastly, a trajectory of the closed-loop system affected by a perturbation of amplitude $\delta_{\text{max}} = 0.01$ is plotted on Fig. 7 (in a semi-log scale). The figure shows the convergence of the state into the attractor and to the limit cycle.

8 Conclusion

A new solution to the problem of stabilizing SAS’s has been derived based on a control Lyapunov function. For an a priori selected sequence of modes, LMI conditions, related to existing results on periodic systems, ensure the convergence to a limit cycle characterized by system’s matrices and this sequence. Interestingly, this solution is suitable to be extended to the uncertain case, where the notion of limit cycles was necessarily modified. Then, keys to embed optimization problems have been provided, especially to select the optimal cycle or limit cycle that minimizes a given cost function. Finally, several examples have been presented, emphasizing the potential of the contributions.

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References


Fig. 4. Comparison of the solutions to Example 2 with the control law obtained with Theorem 1 (left) and with [16, Th.2] and the distance to the same attractor $\mathcal{A}_\nu$. The solution of the closed-loop systems for Example 2 (blue dashed line) has been obtained for $\nu = \{1, 2, 3\}$ with their associated state vector $\rho$ (red crosses) of the limit cycle and the figures at the bottom show the switching control law in each case.

Fig. 5. Example 3: State trajectory and the switching control law.

Fig. 6. Example 3: Attractors obtained for several values of $\delta_{\max}$ for system (27) subject to uncertainties (28) in a logarithmic scale.

Fig. 7. Example 3: Trajectories of (27)-(28) obtained for a perturbation $\delta_{\max} = 0.01$. The grey areas correspond to the regions of the state space, where the control law imposes to select mode 2.


