Attractors and limit cycles of discrete-time switching affine systems : nominal and uncertain cases

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Robust stabilization to limit cycles of switching discrete-time affine systems using control Lyapunov functions ⋆

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Abstract

This paper deals with the robust stabilization of uncertain discrete-time switched affine systems using a control Lyapunov approach and a min-switching state-feedback control law. After presenting some preliminaries on cycles and limit cycles, a constructive stabilization theorem is provided and guarantees that the solutions to the nominal closed-loop system converge to a limit cycle. These conditions are expressed in terms of simple linear matrix inequalities (LMI), whose underlying necessary conditions relax the usual one in this literature. This method is extended to the case of uncertain systems, for which the notion of limit cycle needs to be adapted. The theoretical results are evaluated on academic examples and demonstrate the potential of the method over the recent literature.

Keywords: Switching affine systems, Robust Stabilization, Uncertain systems, LMI, Control Lyapunov function.

1 Introduction

A switched system consists in the association of a finite set of dynamics with a switching rule that designates at each time which is the active one among them [30]. This is a wide particular class of hybrid systems [24], that allows to model numerous applications in various fields as embedding systems, electromechanics, biology, or networked control systems and to characterize complex and not intuitive behaviors.

Among them, there are discrete-time switched systems, that can be considered in their own or potentially generated by continuous-time switched systems with a switching rule that is piecewise constant according to a given sampling period [29,27]. It is interesting to emphasize that in the last case, linear or affine modes remain linear or affine by discretization. This kind of systems have generated a rich study concerning their stability to the origin [13,39], stabilization [22] or stabilizability [19].

The question to stabilize a switched affine system to a state, that is not in the set of equilibrium points of the modes, has motivated a large collection of contributions, especially in the continuous-time framework: global quadratic stabilization via a min-switching strategy [10], use of dynamic programming to select the mode to activate and also the time to switch [34], globally stabilizing min-switching strategy taking into account a cost to minimize [15], local stabilization without requiring the existence of a Hurwitz combination of the linear parts [26], practical stabilization with dwell-time guarantees [1], robust stabilization to an unknown equilibrium point [2,3]. We can emphasize also contributions in the discrete-time domain leading to the practical stabilization via different types of Lyapunov functions: a Lyapunov function as a general quadratic form [16], or switched quadratic Lyapunov functions based on Lyapunov-Metzler inequalities [18] or thanks to multiple shifted quadratic Lyapunov functions [35].

Besides equilibrium points, dynamical systems may have as asymptotic behaviors, self-sustained oscillations, or limit cycles: that are closed and isolated trajectories [37, Section 7]. Their studies have been initiated by Poincaré and are generically related to the ω-limit sets (set of accumulation points of the trajectories) and Poincaré-Bendixson theorem. For hybrid or switched systems, limit cycles have been mainly investigated in the continuous-time domain, motivated by switched circuits [33,28,32,25,4]. The Poincaré-Bendixson theorem has been extended for hybrid systems [36] and the ω-limit sets of hybrid systems have been also investigated [11]. The main difficulty is to determine

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the switching times related to a limit cycle [23,21].
There are only a few contributions on limit cycles for discrete-time affine systems switched systems: one case study is provided in [32], based on practical stability [41] or clock-dependent switched Lyapunov functions [17] and also the preliminary result on multiple shifted quadratic Lyapunov functions putting in evidence the notion of limit cycles [35].
This paper is focused on limit cycles for discrete-time switched affine systems. We will investigate their existence, and also their stabilizability thanks to state-dependent switching laws. A particular interest of this result is the fact that these systems can be stabilized without the requirement of having a Schur combination of the linear part. We will be also interested in their extension to the framework of uncertain switched affine systems. In summary the contributions are:

- A rigorous definition of limit cycle in the framework of discrete-time switched affine systems.
- An original pure state dependent min-switching control strategy allows to obtain an autonomous system. This control law differs to the time-dependent one presented in [17], which is based on a periodic Lyapunov function.
- By considering a class of Lyapunov functions consisting in the minimum of shifted quadratic ones, an exponential stabilization is proven, first to an attractor and second with additional assumption to the (hybrid) limit cycle.
- Noting that our min-switching control strategy does not depend on the system structure (A_σ, B_σ), it is possible to propose a novel robust control law that considers parametric uncertainties.

The paper is organized as follows. The investigated system is presented in Section 2. Limit cycles are not only periodic trajectories and are rigorously defined and discussed in Section 3. In the same section, thanks to the discrete-time nature of the system, we can avoid the Poincaré-Bendixson theorem and we will provide necessary and sufficient conditions for the existence of limit cycles of a given length. Section 4 deals with the stabilization to a limit cycle. Section 5 extends these results to the framework of uncertain switched affine systems and generalizes the notion of limit cycle to the one of *robust limit cycle*. Families of optimization problems are presented in Section 6 depending on the requirements of the designer that characterize a satisfactory limit cycle or the best limit cycle among a given set. Several illustrations are provided in Section 8 allowing additional discussions before concluding remarks in Section 9.

Notations: Throughout the paper, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{R}$ the real numbers, $\mathbb{R}^n$ the n-dimensional Euclidean space, $\mathbb{R}^{n\times m}$ the set of all real $n\times m$ matrices and $\mathbb{S}^n$ the set of symmetric matrices in $\mathbb{R}^{n\times n}$. For any n and m in $\mathbb{N}$, matrices I_n and $\mathbf{0}_{n,m}$ ($\mathbf{0}_n = \mathbf{0}_{n,n}$) denote the identity matrix of $\mathbb{R}^{n\times n}$ and the null matrix of $\mathbb{R}^{n\times m}$, respectively. When no confusion is possible, the subscripts of these matrices that precise the dimension, will be omitted. For any matrix M of $\mathbb{R}^{n\times n}$, the notation $M \succ 0$, ($M \prec 0$) means that M is symmetric positive (negative) definite and $\det(M)$ represents its determinant. For any matrices $A = A^\top, B, C = C^\top$ of appropriate dimensions, matrix $\left[\begin{smallmatrix} A & B \\ * & C \end{smallmatrix}\right]$ denotes the symmetric matrix $\left[\begin{smallmatrix} A & B \\ B^\top & C \end{smallmatrix}\right]$. $\|\cdot\|$ denotes the Euclidean norm. For a symmetric positive definite matrix P and a vector x, we denote $\|x\|_P = \sqrt{x^\top P x}$, the weighted norm. For a symmetric matrix, $\lambda_m(\cdot)$ and $\lambda_M(\cdot)$ denote its minimal and maximal eigenvalues respectively. For a matrix $M \in \mathbb{S}^n$, $M \succ 0$ and a vector $h \in \mathbb{R}^n$, we denote the shifted ellipsoid $\mathcal{E}(M,h) = \{x \in \mathbb{R}^n,\ (x-h)^\top M(x-h) \leq 1\}$.

2 Problem formulation

Consider the discrete-time switched affine system given by

$$\begin{cases} x^+ = A_\sigma x + B_\sigma, \\ \sigma \in u(x) \subset \mathbb{K}, \\ x_0 \in \mathbb{R}^n, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, which adopts the following notation $x^+ = x_{k+1}$ and $x = x_k$. Likewise, $\sigma \in \mathbb{K} := \{1, 2, .., K\}$ characterizes the active mode. Finally, $A_i \in \mathbb{R}^{n\times n}$ and $B_i \in \mathbb{R}^{n\times 1}$, for any $i \in \mathbb{K}$ are the matrices of mode $i \in \mathbb{K}$. The particularity of this class of systems relies on its control action, which is only performed through the selection of the active mode σ, which requires a particular attention.
The objective here is to design a suitable set valued map u in system (1) that ensures the convergence of the state trajectories to a set to be characterized in an accurate manner. Indeed, it is well-known that asymptotic stabilization of a single equilibrium of the switched affine system (1) cannot, in general, be achieved [41]. Due to the affine term, and consequently nonlinear nature, one has to relax the control objective to derive an acceptable stability result. For instance, in [16], the authors have derived a practical stability result. More precisely, it is shown therein, that the solutions to the switched affine system converge to an invariant region characterized by a level set of a Lyapunov function centered at a desired operating point or at a slightly shifted point nearby this desired reference position.
In this paper, our objective is to go deeper into the analysis of switched affine systems and try to characterize in a finer manner their behavior to steady state solutions, that is to limit cycles. This new analysis is achieved thanks to a different class of Lyapunov functions different to the quadratic ones. Indeed, one has to use more advanced tools and Lyapunov functions arising in switched affine systems in order to derive more accurate results. A first attempt was considered in [22] for switched linear systems, where the Lyapunov function is defined using different Lyapunov matrices. It is note-worthy that the discrete-time nature of the dynamics (1) allows to consider classes of Lyapunov functions associated to possibly disconnected level sets, as pointed out in

[12]. Here, we propose to follow our preliminary results provided in [35] where a different Lyapunov function inspired from [18] is included. In this work, we would like to extend the idea presented in [35] in order to better understand the attractor of the resulting control Lyapunov function. An important by-product of the following analysis is that the usual underlying necessary condition consisting of the existence of a Schur stable convex combination of matrices A_i is not required anymore.

3 Limit cycles of switched affine systems

This section is focused on the equilibria, or more precisely time-varying steady states of switched affine systems. Indeed, this kind of systems seem to have a natural convergence to a repeated sequence, behaving as a limit cycle in discrete time as defined in [38]. In this paper, the authors mention that limit cycles represent the stationary state of sustained oscillations, which do not depend on initial conditions but depend exclusively on the parameters of the system, *i.e.* they are intrinsic properties.

3.1 Definitions of limit cycles and discussions

We first define the notion of limit cycle [37,38], adapted to discrete-time systems.

Definition 1 *For system* (1)*, a hybrid limit cycle or limit cycle in short, is a closed and isolated hybrid trajectory* $\mathbb{N} \to \mathbb{K}\times\mathbb{R}^n$, $k \mapsto (\sigma_k, x_k)$*, which is a solution of the switched dynamics* (1)*. Roughly speaking* $s : \mathbb{N} \to \mathbb{K} \times \mathbb{R}^n$, $s_k = (\sigma_k, x_k)$ *of* (1) *is a limit cycle if and only if the two following properties are satisfied:*

(i) Closed means periodic. There exists a positive definite integer K such that $s_{k+K} = s_k$, $\forall k \in \mathbb{N}$*;*
(ii) Isolated means that there exists a neighbourhood of this trajectory which does not contain an other periodic solution generated by the same switching law. There exists $\kappa > 0$*, such that, for any periodic trajectory* $\tilde{s}_k = (\sigma_k, \tilde{x}_k)$ *solution to* (1)*, with the same switching rule* σ_k *as* s_k*,*

$$\sup_{k\in\mathbb{N}} \|x_k - \tilde{x}_k\| > \kappa. \tag{2}$$

Remark 1 Definition 1 emphasizes both the state evolution and the control law of the dynamics (1), in order to ease the determination of limit cycles and to avoid Poincaré map and Poincaré-Bendixson theorem, that are not easy to generalize first in high dimensions or second in hybrid context [23],[36]. Here, since the switching law is a purely state-feedback, another approach, taking the point of view of closed-loop dynamics, can recover more classical definitions of autonomous differential equations/inclusions [38]. ⌟

Remark 2 In condition ii), it is not restrictive to consider the same switching law for the two trajectories. In fact, isolated trajectories yield that there exists $\kappa_0 > 0$, such that, $\forall \kappa \in]0, \kappa_0]$,

$$\sup_{k\in\mathbb{N}}(\|s_k - \tilde{s}_k\|) = \sup_{k\in\mathbb{N}}(|\sigma_k - \tilde{\sigma}_k| + \|x_k - \tilde{x}_k\|) \leq \kappa, \tag{3}$$

implies that $\nexists K \in \mathbb{N}$, $K \geq 1$, such that $\tilde{s}_k = \tilde{s}_{k+K}$. The switching law σ_k and $\tilde{\sigma}_k$ belonging to $\mathbb{K}$ of finite cardinal, the hybrid trajectory $\tilde{s}_k$ can be in the neighbourhood of s_k if they share the same switching law $\tilde{\sigma}_k = \sigma_k$. This isolated characteristic may be thus reduced to the statement ii), which is more tractable in practice. ⌟

Remark 3 It is noteworthy that the periods of s_k and $\tilde{s}_k$ in Definition 1 may be distinct (one is the multiple of the other), even if they share the same switching rule. In any case, the function $k \mapsto x_k - \tilde{x}_k$ is periodic. That implies that its supremum in (2) exists and is the maximum over the associated period. ⌟

Remark 4 Let denote N as the period of the limit cycle. The projected switching law trajectory $\mathbb{N} \to \mathbb{K}$, $k \mapsto \sigma_k$, is an $N-$periodic function. The projected state trajectory is also $N-$periodic function and could be called limit cycle through misuse of language. In the following, we will denote ρ_j, $j = 1, \cdots, N$, the ordered vectors consisting of this period, that is $\rho_{1+(k+\delta)\mathrm{mod}N} = x_k$ for $k \in \mathbb{N}$ and where $\delta \in \mathbb{N}$ represents a shift in cycle. ⌟

Remark 5 For linear switched systems, a non trivial periodic trajectory cannot be isolated, due to the linearity. Only switched systems with at least one nonlinear mode could exhibit (not trivial) limit cycles. The fact that a limit cycle is isolated is crucial for their determination in our context. ⌟

Remark 6 An equilibrium point of a given mode can be viewed as a particular limit cycle associated with a frozen switching law. They will not be specifically addressed here. ⌟

Remark 7 Finally, in the literature, the set $\{\rho_j\}_{j=1,2,\ldots,N}$ in $\mathbb{R}^n$ could be referred to the limit cycle or ω-limit set of the trajectory. That can be understood in the context of autonomous systems, where the trajectory can be identified with its graph in the state space (see [31, Section 3.2]). We will call it the attractor to avoid confusion. ⌟

All these comments will be of importance to characterize the limit cycles for affine switched systems. Due to the importance of periodic switching law, the following notion of cycle and associated definitions are introduced.

Definition 2 *A cycle,* ν *of a switched affine system refers to a periodic function from* $\mathbb{N}$ *to* $\mathbb{K}$*. Notations* N_ν *and* $\mathbb{D}_\nu$ *stand for the minimum period and the minimal domain of* ν*,*

respectively. More formally, they are defined as follows

$$N_\nu = \min N \in \mathbb{N}^* \text{ s.t. } \nu(\ell+N) = \nu(\ell), \forall \ell \in \mathbb{N},$$
$$\mathbb{D}_\nu = \{1, 2, \ldots, N_\nu\}.$$

Definition 3 *Denote the set of cycles from $\mathbb{N}$ to $\mathbb{K}$ by*

$$C := \{\nu : \mathbb{N} \to \mathbb{K}, \text{ s.t. } \exists N \in \mathbb{N}^*, \ \forall \ell \in \mathbb{N}, \ \nu(\ell+N) = \nu(\ell)\}.$$

Moreover, C_N denotes the set of cycles that are N-periodic: $C_N := \{\nu : \mathbb{N} \to \mathbb{K}, \text{ s.t. } \forall \ell \in \mathbb{N}, \ \nu(\ell+N) = \nu(\ell)\}$, with abuse of notation, we will explicit C_N thanks to the possible periods of ν: $\{\nu(1), \cdots, \nu(N)\}$.

Finally for readability, we introduce the following modulo notation: $\lfloor i \rfloor_\nu = ((i-1) \text{mod} N_\nu) + 1$, for any $i \in \mathbb{N}$, $i \geq 1$. That is, in particular, $\lfloor i \rfloor_\nu = i$, for any $i = 1, \cdots, N_\nu$ and $\lfloor N_\nu + 1 \rfloor_\nu = 1$.

3.2 Necessary and sufficient conditions of existence

In this section, the objective is to characterize the limit cycles of system (1). We take advantage of the associated periodic switching law and benefits of the discrete-time (linear) periodic system literature, see for instance [7,9,14,40] or the survey [8]. A limit cycle having a periodic switching law, we will determine necessary and sufficient conditions to the existence of a limit cycle for a given cycle ν. This result is provided in the following lemma, that generalizes the necessary and sufficient condition presented in [32] to the case of an arbitrary number of modes and an arbitrary period N_ν.

Lemma 1 *A cycle $\nu \in C$ generates a unique limit cycle for system (1) if and only if the spectrum of the matrix $\Phi_\nu(0)$ does not contain the eigenvalue 1, where the matrix $\Phi_\nu(\ell)$ is the monodromy matrix at time $\ell \in \mathbb{N}$, defined by*

$$\Phi_\nu(\ell) := \prod_{\iota=\ell+1}^{\ell+N_\nu} A_{\nu(\iota)} = A_{\nu(\ell+N_\nu)} A_{\nu(\ell+N_\nu-1)} \ldots A_{\nu(\ell+1)}, \quad \ell \in \mathbb{N}.$$

Moreover, if this assumption holds, the unique solution is given by

$$\rho := (I_{nN_\nu} - \mathbb{A}_\nu)^{-1} \mathbb{B}_\nu, \tag{4}$$

where

$$\mathbb{A}_\nu = \begin{bmatrix} 0 & \ldots & 0 & A_{\nu(N_\nu)} \\ A_{\nu(1)} & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & A_{\nu(N_\nu-1)} & 0 \end{bmatrix}, \ \mathbb{B}_\nu = \begin{bmatrix} B_{\nu(N_\nu)} \\ B_{\nu(1)} \\ \vdots \\ B_{\nu(N_\nu-1)} \end{bmatrix}, \ \rho = \begin{bmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_{N_\nu} \end{bmatrix}.$$

Proof. A cycle $\nu \in C$ is associated with a unique limit cycle for the switched affine system (1) if and only if there exists a unique sequence of N_ν vectors $\{\rho_i\}_{i \in \mathbb{D}_\nu}$ such that,

$$\rho_{\lfloor i+1 \rfloor_\nu} = A_{\nu(i)} \rho_i + B_{\nu(i)}, \quad \forall i \in \mathbb{D}_\nu, \tag{5}$$

which is illustrated on the schematic representation shown on Figure 1. Relations (5) can be reformulated into the following matrix affine equation, by using a cyclic augmented representation inspired by [40,20]:

$$(I_{nN_\nu} - \mathbb{A}_\nu)\rho = \mathbb{B}_\nu, \tag{6}$$

where matrices $\mathbb{A}_\nu$, $\mathbb{B}_\nu$ and ρ are defined in the Lemma.

Hence, it suffices to study the solution of (6). The cyclic augmented matrix, $\mathbb{A}_\nu \in \mathbb{R}^{nN_\nu \times nN_\nu}$, is closely related to the monodromy matrix at time $\ell \in \mathbb{N}$, denoted $\Phi_\nu(\ell)$. It is known for discrete-time periodic systems that the monodromy matrix $\Phi_\nu(\ell)$ has a spectrum that does not depend on the time ℓ (see for instance [5, Section 3.1]). These eigenvalues are called characteristic multipliers. Moreover the spectrum of the cyclic augmented matrix $\mathbb{A}_\nu$ is the set of all N_ν-roots of the n eigenvalues of the monodromy matrix $\Phi_\nu(0)$ (see the argument of [40, Proof of Theorem 4] or [7, page 322, Section 3.2]). We infer that matrix $(I_{nN_\nu} - \mathbb{A}_\nu)$ is nonsingular if and only if the spectrum of the monodromy matrix $\Phi_\nu(0)$ does not contain the eigenvalue 1. When $(I_{nN_\nu} - \mathbb{A}_\nu)$ is nonsingular, there exists a unique solution to the equation (6) and is such isolated. When the matrix $(I_{nN_\nu} - \mathbb{A}_\nu)$ is singular, two cases occur depending on whether or not $\mathbb{B}_\nu$ belong or not to the image $(I_{nN_\nu} - \mathbb{A}_\nu)$. If it does not, solution to (5) do not exist. If it does, that there exists an infinite number of solution to (5), which prevents from satisfying the isolated nature of the limit cycle. □

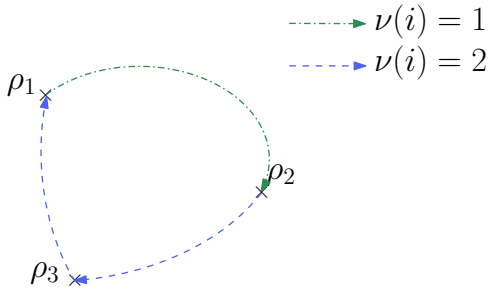


Fig. 1. Schematic representation of a cycle ν of period $N_\nu = 3$, for a system (1) with $K = 2$ modes. Here we have $\nu : \{1, 2, 2\}$. The closed state trajectory is composed of the vectors ρ_i that verify condition (5).

3.3 Invariance of limit cycles wrt. the realization

In the context of stabilization of switched affine systems, it is usual to perform a change of the coordinates in order to locate the reference position at the origin. In light of this remark, one may wonder whether a limit cycle of the system is affected by this transformation. Let us then introduce a general formulation of an affine change of coordinates given by $z = Tx + w$, where T is a nonsingular matrix and where w is a vector of $\mathbb{R}^n$. Then, the following proposition holds.

Proposition 1 *Assume that a cycle ν generates a limit cycle for system* (1)*, denoted $\{\rho_i\}_{i\in\mathbb{D}_\nu}$. Then, for any nonsingular matrix T and any vector $w \in \mathbb{R}^n$, $\{T\rho_i + w\}_{i\in\mathbb{D}_\nu}$ is the limit cycle associated to the same cycle for the same system* (1) *but expressed in the new coordinates $z = Tx + w$.*

Proof. Simple manipulations of (5) conclude the proof. □

3.4 Reduction of cycles

The search of a limit cycle of system (1) can be guided by an objective. Among the possible objectives, we can cite for instance and not exhaustively: existence of a limit cycle of shortest period; a cycle with a given maximal distance to a fixed center or optimizing robustness margins (see Section 5). Such a search can be performed by increasing step by step the period. In the situation where system (1) admits K modes, the number of possible N_ν-periodic cycles increases exponentially with the number of modes, since there are K^{N_ν}. Thus, it is important to understand if some of them are redundant and can be removed. In this section, we propose several simple rules, stated in Corollaries 1 and 2 (based on Lemma 1), to avoid redundant limit cycles. These rules lead to a sieve on the potential cycles to investigate.

Corollary 1 *For any cycle $\nu \in C$, for which 1 is not a characteristic multiplier and for any integer $M > 0$, the cycle $\bar{\nu}$ given by*

$$\bar{\nu}(\ell) = \nu(\ell + M), \quad \forall \ell \in \mathbb{N},$$

is N_ν-periodic and associated to a unique limit cycle, which is a circular permutation of the limit cycle related to ν.

Proof. The proof is straightforward by noting that $\bar{\nu}$ is a shifted version of ν, by recalling that the spectrum of the monodromy matrix $\Phi_{\bar{\nu}}(\ell) = \Phi_\nu(\ell + M)$ does not depend on ℓ and by re-ordering the vectors ρ_i's. □

Corollary 2 *For any integer $M \in \mathbb{N}$, $M > 0$ and a cycle $\nu \in C$, which does not admit an M-root of unity as characteristic multiplier, consider the MN_ν-periodic cycle $\bar{\nu}$ given by*

$$\bar{\nu}(\bar{\ell}) = \nu(\lfloor \bar{\ell} \rfloor_\nu), \quad \forall \bar{\ell} = 1, \ldots, MN_\nu.$$

Then, ν does not admit an M-root of unity as characteristic multiplier, the cycle $\bar{\nu}$ is associated to the unique limit cycle, which consists in the M-times concatenation of the limit cycle related to ν.

Proof. The monodromy matrix related to cycle $\bar{\nu}$ at time 0 satisfies $\Phi_{\bar{\nu}}(0) = (\Phi_\nu(0))^M$. Matrix $\Phi_{\bar{\nu}}(0)$ admits 1 as eigenvalue if and only if $\Phi_\nu(0)$ admits an M-root of unity as eigenvalue. Lemma 1 allows to conclude. It is easy to check that the M times concatenation of the limit cycle associated to ν is this unique limit cycle, when it exists. □

Thanks to Corollaries 1 and 2, we can build the following first sieve on the possible cycles to investigate, irrespectively on the associated characteristic multipliers. This sieve is irrespective on the associated characteristic multipliers, because if a cycle generates a stabilibizable limit cycle, the shifted one or one of multiple periods will be also stabilizable and will generate the same hybrid trajectory.

Algorithm 1 Sieve 1

Input : N
Output : C^*
$C^* \leftarrow \emptyset$
for $n \leftarrow 1$ **to** N **do**
  Consider C_n
  for $i \leftarrow 1$ **to** Card(C_n) **do**
    Consider $\nu_i \in C_n$
    For any positive $M \in \mathbb{N}$ and for any $\bar{\nu} \in C^*$
    if $\bar{\nu}(\ell) = \nu_i(\ell + M), \forall \ell \in \mathbb{N}$ **then**
      $\nu_i \leftarrow \emptyset$
    else if $\bar{\nu}(\bar{\ell}) = \nu_i(\lfloor \bar{\ell} \rfloor_{\nu_i}), \forall \bar{\ell} = 1, \ldots, Mn$ **then**
      $\nu_i \leftarrow \emptyset$
    else
      $C^* \leftarrow C^* \cup \nu_i$
    end if
  end for
end for

To illustrate the application and benefits of this sieve, let us consider the particular case where $K = 3$, i.e. $\mathbb{K} = \{1, 2, 3\}$. Then, the cycles of period at most of length 3 to be considered are $K(1 - K^3)/(1 - K) = 39$. Applying the sieve allows to reduce this number to 14 and finally leads to check only the next cycles (with some abuse of notation):

$$\begin{aligned}
&\text{Iteration 1} : C_1 \rightarrow \{1\}, \{2\}, \{3\},\\
&\text{Iteration 2} : C_2 \rightarrow \{1,2\}, \{1,3\}, \{2,3\},\\
&\text{Iteration 3} : C_3 \rightarrow \{1,1,2\}, \{1,1,3\}, \{2,2,1\}, \{2,2,3\},\\
&\qquad\qquad\qquad\quad \{3,3,1\}, \{3,3,2\}, \{1,2,3\}, \{3,2,1\}.
\end{aligned}$$

This reduction is more consequent when increasing the maximal length of period N, as emphasized in the Table 1.

Remark 8 The complexity of this algorithm may be high for large cycle by comparing two by two the cycles. Nevertheless the time to compare two cycles is less than solving feasibility problem for LMIs. This algorithm may be very useful to reduce the whole computation time in that sense. ⌟

The following Corollary will provide a useful property al-

Period at most N	1	2	3	4	5	6
Initial number	3	12	39	120	363	1092
Reduced number	3	6	14	32	80	199

Table 1
Reduction of the number of cycles to investigate by applying Sieve 1. The initial number is the number of cycles with a period at most equal to N. The reduced number is the number of cycles passing through the sieve.

lowing to ensure an equivalent class of switching laws related to the attractor.

Corollary 3 *Consider a cycle $\nu \in C$, that generates a limit cycle determined by $\{\rho_i\}_{i\in\mathbb{D}_\nu}$. If there exist $(i_0, i_1) \in \mathbb{D}_\nu^2$, $i_1 > i_0$, such that $\rho_{i_0} = \rho_{i_1}$, then $\{\rho_i\}_{i\in\mathbb{D}_\nu}$ is the union of two closed trajectories (potentially the same), associated with cycles (potentially the same too) of periods strictly less than N_ν.*

Proof. Based on the cycle ν, the proof is obtained by designing the two following cycles ν_1 and ν_2.
ν_1 is defined as a $N_{\nu_1} = (i_1 - i_0)$-periodic cycle given by $\nu_1(\ell) = \nu(i_0 - 1 + \ell)$, $\ell = 1, \cdots, (i_1 - i_0)$ and is associated with the periodic trajectory $\{\rho_{i_0}; \rho_{i_0+1}; \cdots; \rho_{i_1}\}$.
ν_2 is defined as a $N_{\nu_2} = (N - i_1 + i_0)$-periodic cycle given by $\nu_2(\ell) = \nu(i_1 - 1 + \ell)$, $\ell = 1, \cdots, (N - i_1 + i_0)$ and is associated with the periodic trajectory $\{\rho_{i_1}; \rho_{i_1+1}; \cdots \rho_{N_\nu}; \rho_1; \cdots; \rho_{i_0}\}$. □

4 Stabilization to a limit cycle

4.1 Stabilization and Control Lyapunov function

In this section, it is addressed the problem of stabilization to a limit cycle defined by a given cycle $\nu \in C$, which has been selected as discussed in the previous section. Before presenting the stabilization result, we need to introduce the following assumption, used in the second part of the Theorem allowing the convergence to the hybrid limit cycle, when the convergence to the attractor holds.

Assumption 1 *Components $\{\rho_i\}_{i\in\mathbb{D}_\nu}$ of a limit cycle associated to $\nu \in C$ are assumed to be different two by two.*

Now, we are in condition of stating the global and exponential stabilization result.

Theorem 1 *For a given cycle ν in C, associated with a limit cycle composed of vectors $\{\rho_i\}_{i\in\mathbb{D}_\nu}$ and consider there exist matrices $\{P_i\}_{i\in\mathbb{D}_\nu}$ in $\mathbb{S}^n$, such that*

$$P_i \succ 0, \quad A_{\nu(i)}^\top P_{\lfloor i+1 \rfloor_\nu} A_{\nu(i)} - P_i \prec 0, \quad \forall i \in \mathbb{D}_\nu. \tag{7}$$

Then, the following statements hold:

(i) Attractor $\mathcal{A}_\nu = \bigcup_{i\in\mathbb{D}_\nu}\{\rho_i\}$, which is related to the limit cycle associated with ν, is globally exponentially stable for system (1) *with the switching control law*

$$u(x) = \left\{ \nu(\theta),\ \theta \in \underset{i\in\mathbb{D}_\nu}{\operatorname{argmin}} (x-\rho_i)^\top P_i (x-\rho_i) \right\} \subset \mathbb{K}. \tag{8}$$

(ii) Moreover, if Assumption 1 holds, the min-switching law (8) *converges ultimately to a shifted version of cycle ν.*

Proof. For the proof, let us consider the following Lyapunov function candidate

$$V(x) = \min_{i\in\mathbb{D}_\nu} (x-\rho_i)^\top P_i (x-\rho_i), \quad \forall x \in \mathbb{R}^n, \tag{9}$$

where vector ρ_i's are the solutions to (5).

Proof of (i): Note that matrices P_i are in finite number and are positive definite, it yields the bounds:

$$0 \le c_1 d_{\mathcal{A}_\nu}^2(x) \le V(x) \le c_2 d_{\mathcal{A}_\nu}^2(x), \tag{10}$$

with $c_1 = \min_{i\in\mathbb{D}_\nu} \lambda_m(P_i) > 0$ and $c_2 = \max_{i\in\mathbb{D}_\nu} \lambda_M(P_i) > 0$ and where $d_{\mathcal{A}_\nu}(\cdot) = \min_{i\in\mathbb{D}_\nu} \|\cdot - \rho_i\|$ defines the distance to the attractor $\mathcal{A}_\nu$ over $\mathbb{R}^n$. The computation of the forward increment of the Lyapunov function along the trajectories of the system yields

$$\begin{aligned}\Delta V(x) :=& V(x^+) - V(x)\\ =& \min_{j\in\mathbb{D}_\nu} (x^+ - \rho_j)^\top P_j (x^+ - \rho_j) - \min_{i\in\mathbb{D}_\nu} (x-\rho_i)^\top P_i (x-\rho_i)\\ =& \min_{j\in\mathbb{D}_\nu} (x^+ - \rho_j)^\top P_j (x^+ - \rho_j) - (x-\rho_\theta)^\top P_\theta (x-\rho_\theta).\end{aligned}$$

The last expression has been obtained by noting that θ results from the control law in (8), and is such that it minimizes the quadratic term, by definition. The first term of ΔV, being the minimum of several values, is consequently less than or equal to any of them. In particular, this is also true by selecting the term associated to $\lfloor \theta + 1 \rfloor_\nu$, yielding

$$\Delta V(x) \le (x^+ - \rho_{\lfloor \theta+1 \rfloor_\nu})^\top P_{\lfloor \theta+1 \rfloor_\nu} (x^+ - \rho_{\lfloor \theta+1 \rfloor_\nu}) - (x-\rho_\theta)^\top P_\theta (x-\rho_\theta).$$

From the dynamics of the closed-loop switched affine system (1),(8), one has

$$\begin{aligned}x^+ - \rho_{\lfloor \theta+1 \rfloor_\nu} =& A_{\nu(\theta)} x + B_{\nu(\theta)} - \rho_{\lfloor \theta+1 \rfloor_\nu}\\ =& A_{\nu(\theta)} (x - \rho_\theta) + \underbrace{A_{\nu(\theta)} \rho_\theta + B_{\nu(\theta)} - \rho_{\lfloor \theta+1 \rfloor_\nu}}_{=0}.\end{aligned} \tag{11}$$

Re-injecting this expression into the upper bound of $\Delta V(x)$ leads to the following inequality

$$\Delta V(x) \le (x-\rho_\theta)^\top \left(A_{\nu(\theta)}^\top P_{\lfloor \theta+1 \rfloor_\nu} A_{\nu(\theta)} - P_\theta \right) (x - \rho_\theta).$$

Therefore, if matrices P_i's verify the strict inequalities in (7), there exists a small enough positive scalar $c_3 > 0$, such that $A_{\nu(i)}^\top P_{\lfloor i+1 \rfloor_\nu} A_{\nu(i)} - P_i \prec -c_3 I_n$, for all $i \in \mathbb{D}_\nu$, yielding

$$\Delta V(x) \le -c_3 \|x - \rho_\theta\|^2 \le -c_3 d_{\mathcal{A}_\nu}^2(x) \le -\frac{c_3}{c_2} V(x), \quad \forall x \in \mathbb{R}^n,$$

due to $\|x - \rho_\theta\| \ge d_{\mathcal{A}_\nu}(x)$ and inequality (10). $V(x)$ is a control Lyapunov function for closed-loop system (1),(8). The map $k \to V(x_k)$ converges globally exponentially to zero. That proves the global exponential convergence of the closed-loop trajectory to attractor $\mathcal{A}_\nu$ and ends the proof.

Proof of (ii): The idea is to prove that there exist $k_0 \in \mathbb{N}$ and $\delta \in \mathbb{K}$ such that

$$u(x_k) = \{\nu(k - \delta)\}, \quad \sigma_k = \nu(k - \delta), \quad \forall k \ge k_0. \tag{12}$$

The proof is obtained by showing that there exists a sufficiently small scalar $\epsilon > 0$ (to be determined in this proof), such that we have the implication: $x \in \mathcal{S}_\epsilon = \{x \in \mathbb{R}^n,\ V(x) \leq \epsilon^2\}$ and $\theta \in \arg\min_{i\in\mathbb{D}_\nu}(x-\rho_i)^\top P_i(x-\rho_i)$ implies

$$(x-\rho_{\lfloor\theta+1\rfloor_\nu})^\top P_{\lfloor\theta+1\rfloor_\nu}(x-\rho_{\lfloor\theta+1\rfloor_\nu}) < (x-\rho_j)^\top P_j(x-\rho_j), \quad (13)$$

for all $j \in \mathbb{D}_\nu$, with $j \neq \lfloor\theta+1\rfloor_\nu$, that is the solution of the next optimization problem (8) is $\lfloor\theta+1\rfloor_\nu$. To sum up, k_0 is related to the time to reach the level set $\mathcal{S}_\epsilon$, which is always possible to reach thanks to the convergence of the Lyapunov function to zero. The shift δ is determined thanks to the solution θ of the optimization problem at time k_0, that is on the initial condition x_0 and the selection of the previous switchings. First, notice that thanks to the equivalence of weighted norms, there exist constants $c_{i,j} > 0$, $\forall (i,j) \in \mathbb{D}_\nu$, such that

$$\|x\|_{P_i} \leq c_{i,j}\|x\|_{P_j}, \quad (14)$$

(for instance, select $c_{i,j} \geq \sqrt{\lambda_M(P_i)/\lambda_m(P_j)}$). Thanks to LMIs (7) and $x \in \mathcal{S}_\epsilon$, we have, with the notation $x^+ = A_{\nu(\theta)}x + B_{\nu(\theta)}$,

$$\|x^+ - \rho_{\lfloor\theta+1\rfloor_\nu}\|_{P_{\lfloor\theta+1\rfloor_\nu}} \leq \|x-\rho_\theta\|_{P_\theta} \leq \epsilon. \quad (15)$$

Thanks to (11) and having x in $\mathcal{S}_\epsilon$, the inequalities

$$\begin{aligned}\|x^+ - \rho_{\lfloor\theta+1\rfloor_\nu}\|_{P_\theta} = \|A_{\nu(\theta)}(x-\rho_\theta)\|_{P_\theta} &\leq \|A_{\nu(\theta)}\|_{P_\theta}\|x-\rho_\theta\|_{P_\theta}, \\ &\leq \|A_{\nu(\theta)}\|_{P_\theta}\epsilon \quad (16)\end{aligned}$$

hold, where $\|A_{\nu(\theta)}\|_{P_\theta}$ denotes the matrix norm induced by the weighted norm $\|\cdot\|_{P_\theta}$. That yields, due to the triangular inequality and relations (14),

$$\begin{aligned}\|\rho_{\lfloor\theta+1\rfloor_\nu}-\rho_j\|_{P_\theta} - \|A_{\nu(\theta)}\|_{P_\theta}\epsilon &\leq \|\rho_{\lfloor\theta+1\rfloor_\nu}-\rho_j\|_{P_\theta} - \|A_{\nu(\theta)}(x-\rho_\theta)\|_{P_\theta}, \\ &\leq \|\rho_{\lfloor\theta+1\rfloor_\nu} - \rho_j + A_{\nu(\theta)}(x-\rho_\theta)\|_{P_\theta}, \\ &\leq \|x^+ - \rho_j\|_{P_\theta}, \\ &\leq c_{\theta,j}\|x^+ - \rho_j\|_{P_j}, \quad \forall j \in \mathbb{D}_\nu. \quad (17)\end{aligned}$$

Thanks to Assumption 1, it is always possible to find a positive scalar ϵ such that the strict inequalities $0 < c_{\theta,j}\epsilon < \|\rho_{\lfloor\theta+1\rfloor_\nu}-\rho_j\|_{P_\theta} - \|A_{\nu(\theta)}\|_{P_\theta}\epsilon$ hold for any $j \in \mathbb{D}_\nu$, $j \neq \lfloor\theta+1\rfloor_\nu$. Combining the two latter inequalities leads to

$$\epsilon < \|x^+ - \rho_j\|_{P_j}, \quad \forall j \in \mathbb{D}_\nu \backslash \{\lfloor\theta+1\rfloor_\nu\}. \quad (18)$$

Comparing inequalities (15) and (18) concludes

$$\|x^+ - \rho_{\lfloor\theta+1\rfloor_\nu}\|_{P_{\lfloor\theta+1\rfloor_\nu}} \leq \epsilon < \|x^+ - \rho_j\|_{P_j}, \quad \forall j \in \mathbb{D}_\nu \backslash \{\lfloor\theta+1\rfloor_\nu\},$$

which ends the proof. □

In the remainder of the section, Theorem 1 and results in Section 3.2 are commented and several of important consequences are emphasized. More particularly, the properties of the feasibility of conditions (7), the nature of the Lyapunov function (9) are investigated.

4.2 Feasibility of the sufficient conditions

Theorem 1 is based on the feasibility of the Linear Matrix Inequalities (LMIs) (7). Such inequalities have been already encountered in the framework of discrete-time linear periodic systems. By the periodic Lyapunov lemma (see [6]), we have the following result:

Lemma 2 ([9]) *For a given cycle ν, there exist positive definite matrices $\{P_i\}_{i\in\mathbb{D}_\nu}$ satisfying LMIs* (7) *if and only if the monodromy matrix $\Phi_\nu(0)$ is Schur.*

One of the main advantage of Lemma 2 is that the condition dealing with the monodromy matrix can be moved closer to the condition in Lemma 1: for a given cycle $\nu \in C$, if the monodromy matrix $\Phi_\nu(0)$ is Schur, then there exists a unique limit cycle (thanks to Lemma 1) , which is stabilizable (thanks to Theorem 1). The sieve can be adapted to look for Schur-stable monodromy matrices without requiring to check the feasibility of LMIs, from the practical point of view.
The importance of having Schur monodromy matrices being revealed, the question is now to understand whether there exists a cycle $\nu \in C$ for a given system (1), which is associated to a stable monodromy matrix. The literature about the (periodic)-stabililizability of switched linear system provides useful conditions as for instance, the following lemma, which provides sufficient conditions based on discrete-time Lyapunov-Metzler inequalities (see [22]):

Lemma 3 (Theorems 6 and 22 in [19]) *If there exist K symmetric positive definite matrices $\{\tilde{P}_i\}_{i\in\mathbb{K}}$ and a matrix $\pi \in \mathbb{R}^{K\times K}$ such that $\pi_{j,i} \geq 0$, $\forall (i,j) \in \mathbb{K}^2$, $\sum_{j\in\mathbb{K}}\pi_{j,i} = 1$, $\forall i \in \mathbb{K}$ and $A_i^\top\left(\sum_{j\in\mathbb{K}}\pi_{j,i}P_j\right)A_i - P_i < 0_n$, for all $i \in \mathbb{K}$, then there exists a cycle $\nu \in C$ such that the monodromy matrix $\Phi_\nu(0)$ is Schur.*

Moreover we have the following equivalence, that allows to guarantee that the existence of a stable limit cycle:

Lemma 4 (Theorem 22 in [19]) *There exist $N \in \mathbb{N}$, and a cycle $\nu \in C_N$ such that $\Phi_\nu(0)$ is Schur if and only if there exist $M \in \mathbb{N}$, and scalars $\pi_\nu \geq 0$, for any $\nu \in \tilde{C}_M = \cup_{j=1,\cdots,M}C_j$, such that $\sum_{\nu\in\tilde{C}_M}\pi_\nu = 1$ and $\sum_{\nu\in\tilde{C}_M}\pi_\nu\Phi_\nu^\top(0)\Phi_\nu(0) < I_n$.*

Here, we are interesting in periodic-stabilizable linear switched systems. It should be recalled that stabilizable linear switched systems is not necessarily periodic-stabilizable, as shown in [19, Proposition 21 and Counter-example 17].

Remark 9 Lemma 1 demonstrates that there is no need to require the existence of weighting parameters $\lambda_i \geq 0$ and $\sum_{i\in\mathbb{K}}\lambda_i = 1$, such that matrix $\sum_{i\in\mathbb{K}}\lambda_i A_i$ is Schur stable unlike [16], for instance. This represents a major relaxation with respect to this class of stabilization results. ⌟

4.3 Structure of the Lyapunov function

The (control)-Lyapunov function $V : \mathbb{R}^n \mapsto \mathbb{R}^+$, $x \to V(x)$ defined by (9), is build only on the knowledge of the couples $\{\rho_i, P_i\}_{i\in\mathbb{D}_\nu}$ and does not depend on their order, that is roughly speaking on the cycle ν. This characteristic differs from the periodic Lyapunov function considered in [17], $\tilde{V} : (x, k) \in \mathbb{R}^n \times \mathbb{N} \to \tilde{V}(x, \lfloor k \rfloor_\nu) \in \mathbb{R}^+$, where $\lfloor k \rfloor_\nu$ can be interpreted as a counter/index in the period of ν. One major benefit of this time-independency is that the min-switching argument in (8) is a pure state-feedback. In practice, this state-feedback is designed by a state-space partition, which is *a priori* given and only dependent on the couples $\{\rho_i, P_i\}_{i\in\mathbb{D}_\nu}$. The associated state-space partition is bounded by arcs of the solutions of $x^\top(P_i - P_j)x - 2(\rho_i^\top P_i - \rho_j^\top P_j)x + \rho_i^\top P_i \rho_i - \rho_j^\top P_j \rho_j = 0$, $(i, j) \in \mathbb{D}_\nu)^2$, $i \neq j$. Geometrically these curves are conic ones (for instance, a line when $P_i = P_j$, an ellipsoid when $P_i - P_j$ is definite and an hyperboloid when $P_i - P_j$ is not definite without 0 as eigenvalue). Finally, the Lyapunov function being defined as the minimum of a set of shifted quadratic forms, its level sets are the union of given ellipsoids (with weighted matrix P_i) centered in ρ_i.

4.4 Comparison with [17]

This section aims at comparing Theorem 1 with respect to [17, Theorem 2]. While the LMI conditions are exactly the same for a given cycle ν, the contributions are different. Indeed, the control law given in [17, Theorem 2] is

$$u(x, \nu, k) = \underset{j\in\mathbb{K}}{\operatorname{argmin}} \begin{bmatrix} x - \rho([k]_\nu) \\ 1 \end{bmatrix}^\top \mathcal{L}_{k,j} \begin{bmatrix} x - \rho([k]_\nu) \\ 1 \end{bmatrix} \subset \mathbb{K}, \tag{19}$$

where

$$\mathcal{L}_{i,j} = \begin{bmatrix} A_j^\top P_{\lfloor i+1 \rfloor_\nu} A_j - P_i & A_j^\top P_{\lfloor i+1 \rfloor_\nu} b_{i,j} \\ * & b_{i,j}^\top P_{\lfloor i+1 \rfloor_\nu} b_{i,j} \end{bmatrix}$$

with $b_{i,j} = A_j \rho_i + B_j - \rho_{\lfloor i+1 \rfloor_\nu}$. Moreover, the Lyapunov function used in [17] is

$$V(x, \nu, k) = (x - \rho([k]_\nu))^\top P_{([k]_\nu)} (x - \rho([k]_\nu)), \quad \forall x \in \mathbb{R}^n. \tag{20}$$

The authors in [17] provide stabilization sufficient conditions to the state trajectory (and not to the hybrid trajectory including the switching law), if (7) are satisfied. Note that this control law depends on a time counter $\nu(k)$ such that in each instant time k, the control input selects the point in the cycle ν_K from an *argmin* function that goes down the K-functioning modes.

Our first result concerns the convergence to the attractor, while [17, Theorem 2] provides the convergence to the state trajectory of the limit cycle. However if Assumption 1 is satisfied, our second result in Theorem 1 provides a periodic solution for the hybrid trajectory with the switching law converging to $\nu(k - \delta)$. Notice that this is not the case for [17, Theorem 2], even if Assumption 1 holds (see Example in Section 8.2). This value of shift, δ, depends on the initial state x_0 and possibly on the choice of the switching law in the inclusion (8). Notice also that, for $k \geq k_0$, the set $u(x_k)$ reduces to a singleton and there is a unique selection of the mode to activate. A similar result may be obtained in [17, Theorem 2], when a unique mode allows to steer a ρ_i to its successor in the state trajectory, that is there exists a unique j_0 such that $b_{i,j_0} = 0$. Item (ii) of Theorem 1 emphasizes that even if the Lyapunov function does not depend on the cycle ν, the min-switching strategy recovers a shifted version of ν, as an element of hybrid trajectory to the equivalent relation of attractors.

It is worth noting that for a given cycle ν our proposed control law (8) aims at selecting the best mode that minimizes the quadratic term in V, looking for the best position in the cycle. Alternatively, control law (19) selects the mode that minimizes (20). Hence, the computational complexity of both control laws are different, depending on the length of the cycle and on the number of modes. For instance, depending on whether $N_\nu > K$ or $N_\nu < K$, control (8) or control (19) can reduce the computational cost and reduces the transient time, respectively.

To sum up, the two contributions are different and their use depends on the context. It is hard to decide whether one is better than the other.

An important property of the min-switching algorithm (8) is that this control law does not depend on the system parameters different to (19), enabling to develop a robust control law that takes into account parametric uncertainties. Hence, next section is devoted to present a robust control law that ensures the states to converge to a *robust limit cycle*, i.e., an invariant set of shifted ellipsoids associated to a cycle ν.

5 Robust stabilization of uncertain switched affine systems

5.1 Motivations

In many occasions, the system may suffer from parameter uncertainties or variations. From now matrices A_σ and B_σ will be assumed to be unknown and/or time-varying, considering that they belong to a polytopic set given by

$$[A_\sigma, B_\sigma] \in \operatorname{Co}\left(\left[A_\sigma^\ell, B_\sigma^\ell\right]\right)_{\ell\in\mathbb{L}}, \ \forall \sigma \in \mathbb{K}, \tag{21}$$

where $\mathbb{L}$ is a bounded subset of $\mathbb{N}$ and where matrices A_σ^ℓ and B_σ^ℓ are known and constant for any $\sigma \in \mathbb{K}$ and any $\ell \in \mathbb{L}$. Note that set $\mathbb{L}$ may be dependent on the mode σ but is avoided here without lack of generality.

It is easy to see, that the results of the previous section fail, about stabilization and stabilizability. Indeed, the main problem appears in the selection of the limit cycle solving equations (5) in the situation of uncertain and/or time-varying

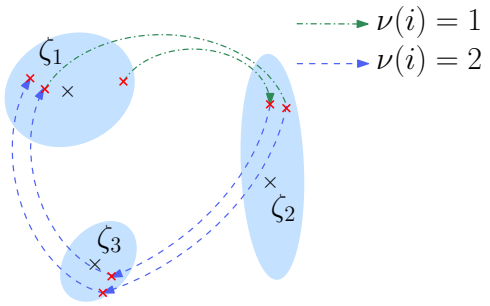


Fig. 2. Schematic representation of a robust limit cycle associated to a cycle ν of period $N_\nu = 3$, for a system (1) with $K = 2$ modes.

system matrices. Therefore, it is important to investigate in this direction, and to provide an alternative solution dedicated to this relevant situation from a practical point of view.

5.2 Robust stabilization

A solution to this robust stabilization problem will be provided in this section, taking into account that the asymptotic convergence to the union of the limit cycle cannot be guaranteed anymore, but an asymptotic convergence to a neighborhood of each one of these limit cycles, which are characterized as a level set of the Lyapunov function. Before providing our result, we consider the following assumption.

Assumption 2 *Consider matrices $W_i \in \mathbb{S}^n$ and vectors $\zeta_i \in \mathbb{R}^n$ such that the ellipsoids*

$$\mathcal{E}(W_i^{-1}, \zeta_i) = \left\{ x \in \mathbb{R}^n, i \in \mathbb{D}_\nu \middle| (x-\zeta_i)^\top W_i^{-1} (x-\zeta_i) \leq 1 \right\},$$

associated to $\nu \in C$ are disjoint and different two by two.

Now, we can formalize the robust stabilization in the following theorem.

Theorem 2 *For a given cycle ν in C and for a parameter $\mu \in (0,1)$, assume that there exist $\{(W_i, \zeta_i)\}_{i\in\mathbb{D}_\nu}$ in $\mathbb{S}^n \times \mathbb{R}^n$ and that are solutions to the following matrix inequalities*

$$W_i \succ 0, \quad \Psi_i(A^\ell_{\nu(i)}, B^\ell_{\nu(i)}) \succ 0, \quad \forall (i,\ell) \in \mathbb{D}_\nu \times \mathbb{L}, \tag{22}$$

where Ψ_i depend on the system matrices and on the decision variables [1] *$\{(W_i, \zeta_i)\}_{i\in\mathbb{D}_\nu}$ in $\mathbb{S}^n \times \mathbb{R}^n$ and are given by*

$$\Psi_i\left(A^\ell_{\nu(i)}, B^\ell_{\nu(i)}\right) = \begin{bmatrix} (1-\mu)W_i & 0 & W_i(A^\ell_{\nu(i)})^\top \\ * & \mu & (A^\ell_{\nu(i)}\zeta_i + B^\ell_{\nu(i)} - \zeta_{\lfloor i+1 \rfloor_\nu})^\top \\ * & * & W_{\lfloor i+1 \rfloor_\nu} \end{bmatrix}. \tag{23}$$

Then, the following statements hold:

[1] For the sake of simplicity, variables $\{(W_i, \zeta_i)\}_{i\in\mathbb{D}_\nu}$ are omitted in the arguments of Ψ_i.

(i) Attractor

$$\mathcal{S}_\nu := \bigcup_{i\in\mathbb{D}_\nu} \mathcal{E}(W_i^{-1}, \zeta_i) \tag{24}$$

is robustly globally exponentially stable for system (1) with the switching control law

$$u(x) = \left\{ \nu(\theta), \ \theta \in \operatorname*{argmin}_{i\in\mathbb{D}_\nu} (x-\zeta_i)^\top W_i^{-1} (x-\zeta_i) \right\} \subset \mathbb{K}. \tag{25}$$

(ii) Moreover, if Assumption 2 holds, the min-switching law (25) converges ultimately to a shifted version of ν.

Proof. Consider the same Lyapunov function given in (9) but with the Lyapunov matrices P_i replaced by W_i^{-1} and with vectors ρ_i's replaced by ζ_i's, solution to the matrix inequalities (22).

Proof of (i): Following the proof of Theorem 1, one has

$$\begin{aligned} \Delta V(x) &= \min_{j\in\mathbb{D}_\nu} (x^+ - \zeta_j)^\top W_j^{-1} (x^+ - \zeta_j) - \min_{i\in\mathbb{D}_\nu} (x-\zeta_i)^\top W_i^{-1} (x-\zeta_i) \\ &\leq (x^+ - \zeta_{\lfloor\theta+1\rfloor_\nu})^\top W^{-1}_{\lfloor\theta+1\rfloor_\nu} (x^+ - \zeta_{\lfloor\theta+1\rfloor_\nu}) - (x-\zeta_\theta)^\top W_\theta^{-1} (x-\zeta_\theta). \end{aligned}$$

The objective is to express this last expression as a quadratic term. As for Theorem 1, we have that

$$\begin{aligned} x^+ - \zeta_{\lfloor\theta+1\rfloor_\nu} &= A_{\nu(\theta)} x + B_{\nu(\theta)} - \zeta_{\lfloor\theta+1\rfloor_\nu} \\ &= A_{\nu(\theta)}(x-\zeta_\theta) + \underbrace{A_{\nu(\theta)}\zeta_\theta + B_{\nu(\theta)} - \zeta_{\lfloor\theta+1\rfloor_\nu}}_{:=\mathcal{B}_{\nu(\theta)}}. \end{aligned} \tag{26}$$

However, compared to the proof of Theorem 1, vectors $A_{\nu(\theta)}\zeta_\theta + B_{\nu(\theta)} - \zeta_{\lfloor\theta+1\rfloor_\nu}$ are not necessarily zero, since vectors ζ_i's are decision variables in inequalities (22), while in Theorem 1, vectors ρ_i were computed *a priori*. In order to find an alternative solution, we introduce

$$\chi_\theta^\top = \left[\left(W_\theta^{-1}(x-\zeta_\theta)\right)^\top \ 1 \right]^\top,$$

so that $x^+ - \zeta_{\lfloor\theta+1\rfloor_\nu} = \left[A_{\nu(\theta)} W_\theta \ \mathcal{B}_{\nu(\theta)}\right] \chi_\theta$. Hence, $\Delta V(x)$ can be rewritten in a more compact form

$$\Delta V(x) \leq -\chi_\theta^\top \Phi_\theta\left(A_{\nu(\theta)}, B_{\nu(\theta)}\right) \chi_\theta, \tag{27}$$

with

$$\Phi_\theta\left(A_{\nu(\theta)}, B_{\nu(\theta)}\right) = \begin{bmatrix} W_\theta & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} W_\theta A_{\nu(\theta)}^\top \\ \mathcal{B}_{\nu(\theta)}^\top \end{bmatrix} W^{-1}_{\lfloor\theta+1\rfloor_\nu} \begin{bmatrix} W_\theta A_{\nu(\theta)}^\top \\ \mathcal{B}_{\nu(\theta)}^\top \end{bmatrix}^\top.$$

Note that ΔV is not required to be negative in the whole state space, but only outside of set $\mathcal{S}_\nu$. From the definition of the Lyapunov function, the control law given in (25) ensures that $V(x) = (x-\zeta_\theta)^\top W_\theta^{-1} (x-\zeta_\theta)$. Therefore, if x is assumed to be

outside of $\mathcal{S}_\nu$, inequality $(x-\zeta_\theta)^\top W_\theta^{-1}(x-\zeta_\theta) > 1$ holds and has to be rewritten using the augmented vector χ_θ as follows

$$\begin{bmatrix} x-\zeta_\theta \\ 1 \end{bmatrix}^\top \begin{bmatrix} W_\theta^{-1} & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x-\zeta_\theta \\ 1 \end{bmatrix} = \chi_\theta^\top \begin{bmatrix} W_\theta & 0 \\ 0 & -1 \end{bmatrix} \chi_\theta > 0. \quad (28)$$

Then, the problem can be summarized as the satisfaction of $\chi_\theta^\top \Phi_\theta(A_{\nu(\theta)}, B_{\nu(\theta)})\chi_\theta > 0$ for all x such that condition (28) holds. An S-procedure ensures that, if there exists a scalar $\mu \in (0,1)$ such that

$$\begin{bmatrix} (1-\mu)W_\theta & 0 \\ 0 & \mu \end{bmatrix} - \begin{bmatrix} W_\theta A_{\nu(\theta)}^\top \\ \mathcal{B}_{\nu(\theta)}^\top \end{bmatrix} W_{\lfloor\theta+1\rfloor_\nu}^{-1} \begin{bmatrix} W_\theta A_{\nu(\theta)}^\top \\ \mathcal{B}_{\nu(\theta)}^\top \end{bmatrix}^\top \succ 0, \quad (29)$$

then, $\Delta V(x) < 0$ for all $x \notin \mathcal{S}_\nu$. Finally, a Schur's complement yields $\Psi_\theta(A_{\nu(\theta)}, B_{\nu(\theta)}) \succ 0$, for a fixed parameter μ, where matrix Ψ_θ is defined in (23). Since matrices $A_{\nu(\theta)}$ and $B_{\nu(\theta)}$ are uncertain, it is not yet possible to evaluate numerically these LMIs for all possible values of θ. However, since they belong to the polytopic set (21), one can define those matrices as convex combinations, with possibly time-varying weights

$$A_{\nu(\theta)} = \sum_{\ell\in\mathbb{L}} \lambda_\ell A_{\nu(\theta)}^\ell \quad \text{and} \quad B_{\nu(\theta)} = \sum_{\ell\in\mathbb{L}} \lambda_\ell B_{\nu(\theta)}^\ell, \quad (30)$$

where parameters $\lambda_\ell \in [0,1]$ and hold $\sum_{\ell\in\mathbb{L}} \lambda_\ell = 1$. By noting that Ψ_θ are affine with respect to $A_{\nu(\theta)}$ and $B_{\nu(\theta)}$, it follows

$$\Psi_\theta(A_{\nu(\theta)}, B_{\nu(\theta)}) = \sum_{\ell\in\mathbb{L}} \lambda_\ell \Psi_\theta\left(A_{\nu(\theta)}^\ell, B_{\nu(\theta)}^\ell\right) \succ 0,$$

which is guaranteed by conditions (22). This guarantees that $\Delta V(x)$ is negative definite outside of $\mathcal{S}_\nu$. Exponential stability is obtained thanks to the strict inequalities (22).

To conclude the proof, it remains to prove that the attractive set $\mathcal{S}_\nu$ is invariant. To do so, note that for any x is in $\mathcal{S}_\nu$, i.e. $V(x) < 1$, we have

$$\begin{aligned} V(x^+) =& \; V(x) + \Delta V(x) \\ =& \; V(x) - \mu(V(x)-1) + (\Delta V(x) + \mu(V(x)-1)) \\ \leq& \; V(x) - \mu(V(x)-1) \\ & -\chi_\theta^\top \left(\begin{bmatrix} (1-\mu)W_\theta & 0 \\ 0 & \mu \end{bmatrix} - \begin{bmatrix} W_\theta A_{\nu(\theta)}^\top \\ \mathcal{B}_{\nu(\theta)}^\top \end{bmatrix} W_{\lfloor\theta+1\rfloor_\nu}^{-1} \begin{bmatrix} W_\theta A_{\nu(\theta)}^\top \\ \mathcal{B}_{\nu(\theta)}^\top \end{bmatrix}^\top \right) \chi_\theta \\ \leq& \; (1-\mu)V(x) + \mu, \end{aligned}$$

which is guaranteed by (22). Then, since x is assumed to be in $\mathcal{S}_\nu$ and μ in $(0,1)$, it holds $V(x^+) \leq (1-\mu)+\mu = 1$, guaranteeing that x^+ also belongs to $\mathcal{S}_\nu$.

Proof of (ii): The proof of this result is omitted because is similar to the proof of item (ii) in Theorem 1. □

The previous theorem allows to design a control law that stabilizes uncertain system (1)–(21), to the attractor defined by set $\mathcal{S}_\nu$ given in (24), which is a union of shifted ellipsoids $\mathcal{S}_\nu = \cup_{i\in\mathbb{D}_\nu} \mathcal{E}(W_i^{-1}, \zeta_i)$. Several comments on this robust stabilization result are provided in the next subsection.

5.3 Remarks on Theorem 2

A relevant byproduct of this theorem is an extension of the definition of limit cycles in equations (5) to the case uncertain switched affine systems, which can be now expressed in terms of series of inclusions. More specifically, Theorem 2 states that, under the satisfaction of the conditions, the invariance of the attractor $\mathcal{S}_\nu$ ensures that the following inclusions hold

$$A_{\nu(i)}^\ell \mathcal{E}(W_i^{-1}, \zeta_i) + B_{\nu(i)}^\ell \subset \mathcal{E}(W_{\lfloor i+1\rfloor_\nu}^{-1}, \zeta_{\lfloor i+1\rfloor_\nu}), \quad \forall (i,\ell) \in \mathbb{D}_\nu \times \mathbb{L}, \quad (31)$$

where the left-hand-side of the inclusion means, with a light abuse of notations, that, for any $i \in \mathbb{D}_\nu$ and for all $x \in \mathcal{E}(W_i^{-1}, \zeta_i)$, vector $A_{\nu(i)}^\ell x + B_{\nu(i)}^\ell$ belongs to $\mathcal{E}(W_{\lfloor i+1\rfloor_\nu}^{-1}, \zeta_{\lfloor i+1\rfloor_\nu})$ for all $\ell \in \mathbb{L}$. This inclusion can be seen as the natural extension of (5) to uncertain systems. The set of ellipsoids $\mathcal{E}(W_i^{-1}, \zeta_i)$, $i \in \mathbb{D}_\nu$, can be viewed as a robustly stable *robust limit cycle*. It is also relevant to understand how conservative the previous theorem is with respect to the nominal case, presented in Theorem 1. The following proposition is stated.

Proposition 2 *For a cycle ν in C, such that there exist (P_i, ρ_i) in $\mathbb{S}^n \times \mathbb{R}^n$ for $i \in \mathbb{D}_\nu$, solution to* (4) *and* (7)*. Then, for any positive scalar $\beta > 0$, $(W_i, \zeta_i) = (\beta P_i^{-1}, \rho_i)$ is solution to* (22) *with a sufficiently small value of μ. Moreover,* $\lim_{\beta\to 0^+} \mathcal{S}_\nu = \mathcal{A}_\nu$.

Proof. Let us consider matrices P_i's solution to inequalities (7) of Theorem 1 and vectors ρ_i solution to Lemma 1. For any positive scalar $\beta > 0$, let us set $(W_i, \zeta_i) = (\beta P_i^{-1}, \rho_i)$. Then $\mathcal{B}_{\nu(i)} = A_{\nu(i)}\rho_i + B_{\nu(i)} - \rho_{\lfloor i+1\rfloor_\nu} = 0$, as defined in (26). For any $\mu \in (0,1)$, $\Psi_i\left(A_{\nu(i)}^\ell, B_{\nu(i)}^\ell\right) \succ 0$ in (22) is equivalent to

$$\begin{bmatrix} (1-\mu)P_i^{-1} & P_i^{-1}A_{\nu(i)}^\top \\ * & P_{\lfloor i+1\rfloor_\nu}^{-1} \end{bmatrix} \succ 0,$$

or, with a standard manipulation, equivalent to $A_{\nu(i)}^\top P_{\lfloor i+1\rfloor_\nu} A_{\nu(i)} - P_i \prec -\mu P_i$. The latter inequality being true for a small enough value $\mu > 0$, thanks to the strict inequality (7). Moreover, by noting that attractor $\mathcal{S}_\nu$ is composed by the ellipsoids given by $\mathcal{E}(P_i/\beta, \rho_i)$, for all $i \in \mathbb{D}_\nu$ reduces to the union of singletons $\{\rho_i\}_{i\in\mathbb{D}_\nu}$ as β tends to zero, which concludes the proof. □

Remark 10 In light of the proof, since ellipsoids $\mathcal{E}(P_i/\beta, \rho_i)$ shrink to $\{\rho_i\}$, for all $i \in \mathbb{D}_\nu$, inclusions (31) become exactly conditions (5) for the existence of a limit cycle in the nominal case. This demonstrates the consistency of the method. ⌟

The previous proposition states that there is no conservatism induced by Theorem 2 with respect to Theorem 1. Therefore, in the next section dealing with the introduction of optimization problems, only the LMI constraints presented in Theorem 2 will be considered for the sake of simplicity, knowing that the same optimization problem could also be presented using the LMI constraints of Theorem 1. Similarly, in the sequel, we will refer to the attractor only as $\mathcal{S}_\nu$ knowing that in the nominal case, $\mathcal{S}_\nu = \mathcal{A}_\nu$.

6 Optimization algorithms

6.1 Motivations and preliminaries

The objective of this section is to include to the previous developments some additional constraints to conditions (22) aiming at selecting the decisions variables $\{W_i, \zeta_i\}_{i\in\mathbb{D}_\nu}$ that optimize a given cost function. This cost function has to be defined for each cycle and will not depend on the switched law. In addition, in practical situations and, in particular, in the context of power converters, the objective is to drive the solutions to the system as close as possible to a desired reference position, $x_d \in \mathbb{R}^n$.
Hence, it appears highly relevant that the cost function reflects not only the "distance" of the reference position to the attractor, but also the "size" of the attractor in order to limit the amplitude of the trajectories within the attractor. This will be considered in the remainder of this section. If the attractor is reduced to a single point of $\mathbb{R}^n$, the notion of distance is easy to be formalized. However, since this situation corresponds to a very particular case for the class of switched affine systems, we have to provide a sensible metric that also defines the distance of a point to the attractor. To go further in this direction, let us introduce the ellipsoid $\mathcal{E}(Q_\nu^{-1}, h_\nu)$ defined for some positive definite matrix Q_ν in $\mathbb{S}^n$ and some shifting vector h_ν in $\mathbb{R}^n$ to be optimized for a given cycle so that $\mathcal{E}(Q_\nu^{-1}, h_\nu)$ is the "smallest" ellipsoid verifying

$$(\{x_d\} \cup \mathcal{S}_\nu) \subset \mathcal{E}(Q_\nu^{-1}, h_\nu).$$

The next lemma helps expressing this inclusion as an LMI.

Lemma 5 *For a given matrix Q in $\mathbb{S}^n$ and some shifting vector h in $\mathbb{R}^n$, let us define*

$$\mathcal{K}_{Q,h}(W,\zeta,\eta) = \begin{bmatrix} \eta W & 0 & W \\ * & 1-\eta & \zeta^\top - h^\top \\ * & * & Q \end{bmatrix}, \tag{32}$$

for some matrix $W \in \mathbb{S}^n_+$, a shifting vector ζ and a positive scalar η. Then, the following statements hold

(i) ζ belongs to $\mathcal{E}(Q^{-1}, h)$ if and only if $\mathcal{K}_{Q,h}(0,\zeta,0) \succeq 0$.
(ii) $\mathcal{E}(W^{-1},\zeta)$ is included in $\mathcal{E}(Q_\nu^{-1}, h_\nu)$ if and only if there exists a strictly positive scalar η such that $\mathcal{K}_{Q,h}(W,\zeta,\eta) \succeq 0$.

Proof. The proof of (i) is a direct application of the Schur complement. The proof of (ii) deserves a detailed proof. Therefore the problem is to ensure that $\mathcal{E}(W^{-1},\zeta) \subset \mathcal{E}(Q^{-1}, h)$, meaning that inequality $(x-h)^\top Q^{-1}(x-h) \leq 1$ holds for all $x \in \mathbb{R}^n$ such that $(x-\zeta)^\top W^{-1}(x-\zeta) \leq 1$. The inclusion can be rewritten as follows

$$\begin{bmatrix}\tilde{x}\\1\end{bmatrix}^\top \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} W \\ \zeta^\top - h^\top \end{bmatrix} Q^{-1} \begin{bmatrix} W \\ \zeta^\top - h^\top \end{bmatrix}^\top \right) \begin{bmatrix}\tilde{x}\\1\end{bmatrix} \geq 0,$$
$$\forall x \in \mathbb{R}^n \text{ s.t. } \begin{bmatrix}\tilde{x}\\1\end{bmatrix}^\top \begin{bmatrix} -W & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix}\tilde{x}\\1\end{bmatrix} \geq 0,$$

where notation $\tilde{x} = W^{-1}(x-\zeta)$ simplifies the notation. The application of an S-procedure ensures that this problem is equivalent to the existence of a parameter $\eta > 0$ such that

$$\begin{bmatrix} \eta W & 0 \\ 0 & 1-\eta \end{bmatrix} - \begin{bmatrix} W \\ \zeta^\top - h^\top \end{bmatrix} Q^{-1} \begin{bmatrix} W \\ \zeta^\top - h^\top \end{bmatrix}^\top \succeq 0.$$

A Schur complement yields the result. □

In light of the previous lemma, we can state that inclusion $(\{x_d\} \cup \mathcal{S}_\nu) \subset \mathcal{E}(Q_\nu^{-1}, h_\nu)$ is equivalent to

$$Q_\nu \succ 0, \quad \mathcal{K}_{Q_\nu,h_\nu}(0, x_d, 0) \succeq 0, \quad \mathcal{K}_{Q_\nu,h_\nu}(W_i,\zeta_i,\eta) \succeq 0, \quad i \in \mathbb{D}_\nu. \tag{33}$$

In the nominal case, the last inequality of the previous equation can be reduced to $\mathcal{K}_{Q_\nu,h_\nu}(0,\rho_i,0) \succeq 0$, for all $i \in \mathbb{D}_\nu$.

6.2 Definition of cost functions

Following the previous developments, a possible way to formalize the notion of a cost related to a "distance" and/or a "size", can be formulated as follows

$$J^*(\nu, x_d) := \min_{\{W_i,\zeta_i\}_{i\in\mathbb{D}_\nu}} J(\nu, x_d, \{W_i,\zeta_i\}_{i\in\mathbb{D}_\nu}) \tag{34}$$
$$\text{s.t. (22) and potential additional inequalities.}$$

where J is the cost function to be optimized and is defined as a barycenter of several families of costs (of course, not exhaustively), for instance

$$J(\nu, x_d, \{W_i,\zeta_i\}_{i\in\mathbb{D}_\nu}) = \sum_{m=1}^{4} \alpha_m J_m(\nu, x_d, \{W_i,\zeta_i\}_{i\in\mathbb{D}_\nu}), \tag{35}$$

where $\alpha_m \geq 0$ and $\sum_{m=1}^4 \alpha_m = 1$ and where J_i's are given by

- $J_1(\nu, x_d, \{W_i,\zeta_i\}_{i\in\mathbb{D}_\nu}) = \mathrm{Tr}\,(Q_\nu)$ with the additional inequalities given in (33). Hence, cost J_1 aims at optimizing the attractor. Indeed, this optimization problem

is relevant to evaluate the "chattering" effect when the solution reaches the attractor.

- $J_2(\nu, x_d, \{W_i, \zeta_i\}_{i\in\mathbb{D}_\nu}) = \sum_{i=1}^{N} \mathrm{Tr}\,(W_i)$. In the uncertain case, it might be also pertinent to add a term under the form $\sum_{i=1}^{N} \mathrm{Tr}\,(W_i)$, which consists in minimizing $\mathcal{S}_\nu$.

- $J_3(\nu, x_d, \{W_i, \zeta_i\}_{i\in\mathbb{D}_\nu}) = \omega_3$ with the additional inequalities

$$\begin{bmatrix} \omega_3 & (\zeta_i - \zeta_j)^\top \\ * & I \end{bmatrix} \succ 0, \quad \forall (i,j) \in \mathbb{D}_\nu^2,\ i \neq j.$$

  Cost J_3 aims at enforcing the shifts ζ_i's to be the same value, that is to have a single shift for the ellipsoids.
- $J_4(\{\zeta_i\}_{i\in\mathbb{D}_\nu}, x_d) = \omega_4$ with either

$$\begin{bmatrix} \omega_4 & x_d^\top - \frac{1}{N_\nu}\sum_{i\in\mathbb{D}_\nu} \zeta_i^\top \\ * & I_n \end{bmatrix} \succ 0,$$

  which minimizes the distance between the average value of the limit cycle and the desired reference, or

$$\begin{bmatrix} \omega_4 & (x_d - \zeta_1)^\top\Gamma & \cdots & (x_d - \zeta_{N_\nu})^\top\Gamma \\ * & I_n & & \\ * & * & \ddots & \\ * & * & * & I_n \end{bmatrix} \succ 0,$$

  where Γ is a projection matrix (see for instance [17]), depending on an adequate notion of distance to a desired position denoted as $x_d \in \mathbb{R}^n$. Here, J_4 will allow to select a limit cycle closed in a certain sense to a desired position that is predefined by the designer.

Remark 11 This optimization problem can be stated using the formulation of Theorem 2 dealing with robust stabilization. However, as commented in Remark 10, the same problem can be formulated for the nominal case by simply replacing ζ_i by ρ_i in (34). ⌟

7 Optimal selection of cycles and precision on the sieve

In light of the previous section, we are now able to precise the sieve and formulate the two following optimization problems.

Problem 1 *For a given bounded subset $\Omega \subset C$ and a given desired reference x_d, the optimal control law is associated to $\nu^* = \mathrm{argmin}_{\nu\in\Omega} J^*(\nu, x_d)$.*

This problem consists in spanning the bounded subset Ω to find the cycle, among a finite number, that minimizes the cost function. As soon as there exists a solution of the stabilization condition for at least one cycle in Ω, then the previous optimization problem has a solution. However, there is no guarantee about the level of the cost function for this cycle. Therefore, one may consider the second optimization problem that precises the sieve on the allowable cycles.

Problem 2 *For a given desired reference x_d, the optimal control law is the solution of the following sieve*

Algorithm 2 Sieve 2

```
Input : C*, x_d                                  ▷ C* from Sieve 1
Output : ν*
J_0 = +Inf
ν* ← ∅
for i ← 1 to Card(C*) do
    Compute J*(ν_i, x_d)                         ▷ ν_i in C*
    if J*(ν_i, x_d) ≤ J_0 then
        ν* ← ν_i
        J_0 ← J*(ν_i, x_d)
    end if
end for
```

8 Numerical applications

8.1 Example 1

Consider the discrete-time system (1), borrowed from [16], where the matrices A_i and B_i are defined as follows

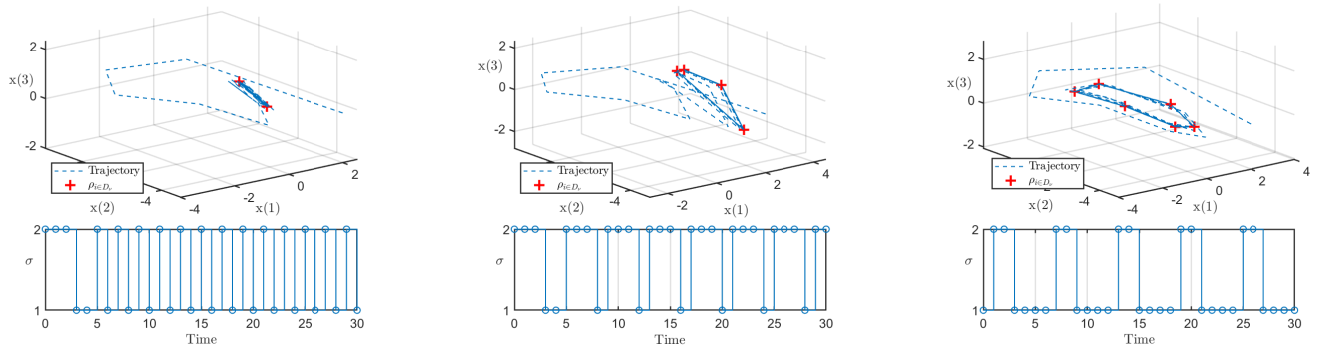
$$A_i = e^{F_i T},\ B_i = \int_0^T e^{F_i \tau} \mathrm{d}\tau g_i,\ \forall i \in \{1,2\}, \tag{36}$$

where $T = 1$ refers to the sampling period. Matrices F_i and g_i for $i = 1, 2$ are given by

$$F_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix}, F_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}, g_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, g_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

It is worth noting that, for this example, there exists a linear combination of matrices A_i which is Schur stable as shown in [16]. At a first stage, the Figure 3 shows on different graphs the limit cycles $\{\rho_i\}_{i\in\mathbb{D}_\nu}$, represented by the red crosses, obtained thanks to Lemma 1 for the different cycles $\nu_1 = \{1,2\}$, $\nu_2 = \{1,2,2,2\}$. and $\nu_3 = \{1,1,1,1,2,2\}$. The figure also shows the trajectories of the closed-loop system, started at the initial state $x_0 = [2, -5, 0]^\top$, with the control law presented in (8) in Theorem 1. It can be seen that each trajectory converges to different limit cycles. The control signal is represented at the bottom of each phase portrait on Figure 3. One can see that this switching signal tends to the presumed cycle after a small transient time as pointed out in item (ii) of Theorem 1.

It is worth mentioning that the results obtained here are very similar to the ones presented in [18] and [35]. However, the method provided in [35] is limited by the constraints that the



(a) $\nu_1 = \{1, 2\}$ (b) $\nu_2 = \{1, 2, 2, 2\}$ (c) $\nu_3 = \{1, 1, 1, 1, 2, 2\}$

Fig. 3. Evolution of the state variables (blue dotted line) for three different cycles ν_1, ν_2 and ν_3 with their associated limit cycles (red crosses). The figures at the bottom show the switching control law in each case.

Cycles (C)	Associated limit cycles	$J^*(\nu_i, 0)$
ν_1	$\begin{bmatrix}1.66\\-0.2\\-0.04\end{bmatrix}, \begin{bmatrix}2.24\\-0.81\\-1.01\end{bmatrix}$	2.37
ν_2	$\begin{bmatrix}3.2\\0.99\\-0.3\end{bmatrix}, \begin{bmatrix}4.49\\-0.81\\-2.8\end{bmatrix}, \begin{bmatrix}3.3\\-1.15\\-0.26\end{bmatrix}, \begin{bmatrix}2.67\\-0.03\\0.24\end{bmatrix}$	8.88
ν_3	$\begin{bmatrix}-0.65\\1.06\\0.08\end{bmatrix}, \begin{bmatrix}1.32\\0.71\\-0.89\end{bmatrix}, \begin{bmatrix}2.42\\-0.64\\-1.63\end{bmatrix}, \begin{bmatrix}2\\-2.1\\-1.08\end{bmatrix}, \begin{bmatrix}0.59\\-2.48\\0.39\end{bmatrix}, \begin{bmatrix}-0.98\\-0.55\\1.03\end{bmatrix}$	5.17

Table 2
Limit cycles generated by a selection of cycles, which verify the assumption of Lemma 1 for system (1) with (36) (Example 1). The last column depicts the optimal cost function (35) with $\alpha = [0.5\ 0\ 0\ 0.5]$ and $x_d = [0\ 0\ 0]^\top$.

length of the cycle needs to be equal to the number of mode. i.e. restricted to ν_1. In addition, the stabilization condition of [35] have a higher complexity since the size of each LMI increases with the number of modes. Finally compared to [18], our control law does not depend on time even though it converges to a periodic signal.
Table 2 provides a numerical point of view by gathering on the first two columns the cycles and their associated limit cycle as in Figure 3. In addition, as commented in Section 6 and resumed in Section 7, we can use a cost function like (35) for the nominal case. Only, the cost J_3 presented does make sense since vectors ζ_i's are given by the ρ_i's in (2). Therefore, with $\alpha = \begin{bmatrix}\alpha_1, \alpha_2, \alpha_3, \alpha_4\end{bmatrix} = \begin{bmatrix}0.5, 0, 0, 0.5\end{bmatrix}$ in (35), the cycle for which the cost function is minimized is, in this case, cycle ν_1 as indicated in Table 2.

8.2 *Example 2*

Let us illustrate the comparison made in Section 4.4 by an academical example. Consider the matrices A_i's as follows

$$A_1 = \begin{bmatrix}0.3 & 0.1\\0.1 & 0.7\end{bmatrix},\ A_2 = \begin{bmatrix}0.9 & 0.5\\0 & -0.8\end{bmatrix},\ A_3 = \begin{bmatrix}0.4 & 0\\1 & -0.1\end{bmatrix},$$

and a matrix slightly different from A_1 associated to a fourth mode $A_4 = A_1 + 0.2\begin{bmatrix}1 & 0\\-1 & 0\end{bmatrix}$. The affine terms are constructed

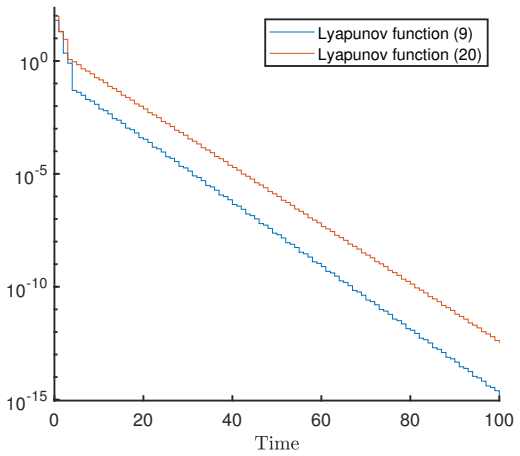
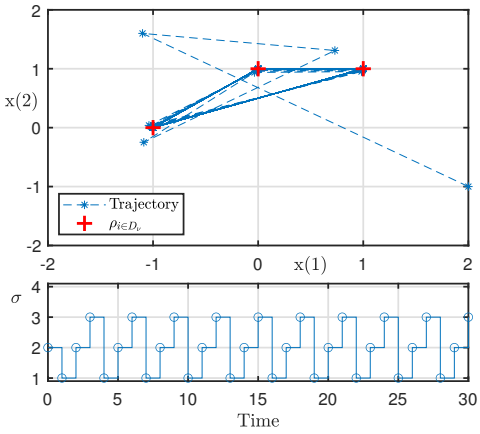


Fig. 4. Evolution of Lyapunov functions with respect to time.

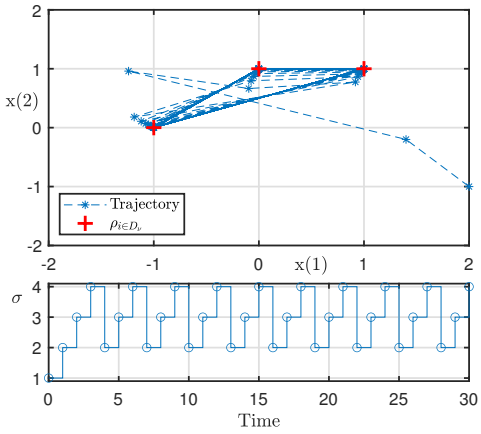
with the limit cycle (ν, ρ) chosen with the cycle $\nu = \{1, 2, 3\}$ and vectors $\rho = \{[0, 1]^\top, [1, 1]^\top, [-1, 0]^\top\}$. Hence, we have

$$B_1 = B_4 = -A_1\rho_1 + \rho_2,\ B_2 = -A_2\rho_2 + \rho_3,\ B_3 = -A_3\rho_3 + \rho_1.$$

On Figure 5 is represented the state trajectory in a phase plane in addition to the state periodic sequence associated to the limit cycle (ν, ρ). As it appears, the evolution of the state variables of system (1) with the switching control law (8) seems to have a better convergence rate. However, one must notice the differences on the switching signals. Whereas the given limit cycle should only involve three of the system functioning modes, the mode associated to the matrices A_4 and B_4 is selected periodically instead of the mode 1. Hence, this example exposes the remark made in Section 4.4 concerning the possible cases where at least two modes can steer one vector ρ_i to its successor. Control (8) provides for this example a better result than control (19). Indeed, we see as system with control (8) converges faster to the limit cycle. One can make that extra comparison through Figure 4, where the evolution of the two Lyapunov functions (10),(20)



(a) Application of the switching control law (8).



(b) Application of the switching control law (22) from [17].

Fig. 5. The first subfigure is an illustration of the application of Theorem 1; the second, the application of Theorem 2 from [17]. In each subfigure is represented the evolution of the state variables for Example 2 (blue dotted line) for a given cycle $\nu = \{1, 2, 3\}$ with their associated state vector ρ (red crosses) of the limit cycle and the figures at the bottom show the switching control law in each case.

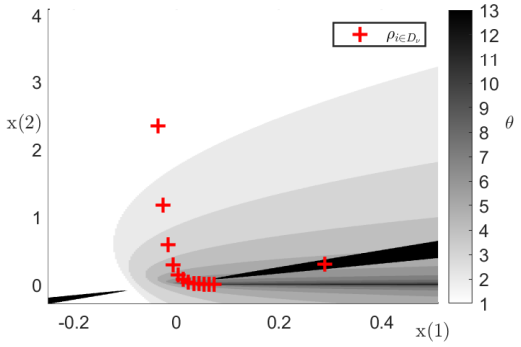


Fig. 6. State-space partition $\theta(x(1), x(2))$ for Example 2 in the nominal case. The figure also depicts the components of the limit cycle (ρ_i's) by the red crosses.

is depicted on the same diagram.

8.3 *Example 3*

Consider the discrete-time switched system described by the following matrices A_i

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}, \; A_2 = 4\sqrt{2} \begin{bmatrix} \cos\left(\frac{\Pi}{4}\right) & -\sin\left(\frac{\Pi}{4}\right) \\ \sin\left(\frac{\Pi}{4}\right) & \cos\left(\frac{\Pi}{4}\right) \end{bmatrix} \tag{37}$$

and with matrices $B_1 = [0.01, 0]^\top$ and $B_2 = [0, 0]^\top$.
This example has been adapted from [19], which does not consider affine terms. As it is notified in [19], the monodromy matrix with cycle $\nu = \{1^{11}, 2^2\}$ is Schur. Hence, Lemma 2 ensures that there exists a solution to Theorem 1. However, it is important to point out that there does not exist a linear combination of matrices A_1 and A_2 that is Schur stable, originating a fail of the stabilization theorem given

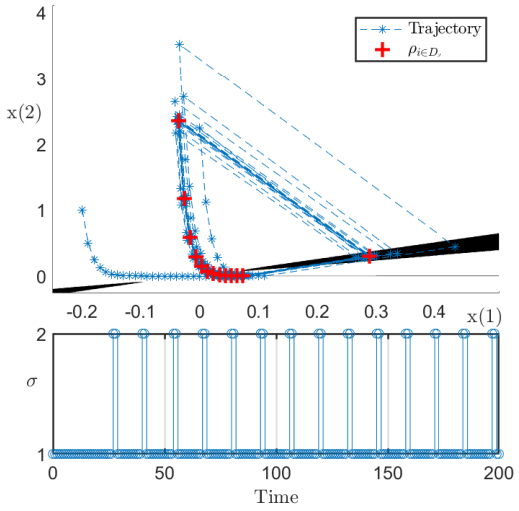


Fig. 7. From top to bottom : State trajectory with the state-space partition $\sigma(x) = \nu(\theta(x))$ and the switching control law. The dark areas are the regions of the state space where mode 2 is activated.

in [16]. Subsection 4.3 demonstrated the feature "pure state-feedback" of the min-switching argument in (8). Then, considering the tuples $\{\rho_i, P_i\}_{i\in\mathbb{D}_\nu}$, we are able to partition the state-space, where each color on Figure 6 designates the areas $\theta(x)$. Figure 7 performs not only the evolution of the state variables converging to the limit cycle associated to ν, but also the state-space partition. Indeed, compared to Figure 6, the black and white areas indicate the regions of the state space where $\nu(\theta(x))$ is equal to 1 or 2. Such state space representation of the switching control law emphasizes the simplicity of the control law and appears to be very sensitive to understand.

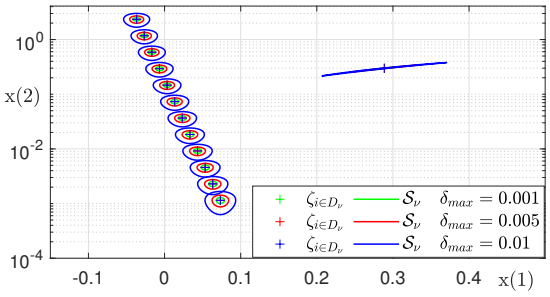


Fig. 8. Illustration of the growth of the attractor's size is the state space obtained for several values of δ_{max} for system (37) subject to uncertainties (38). The ellipsoids are deformed due to the logarithmic scale.

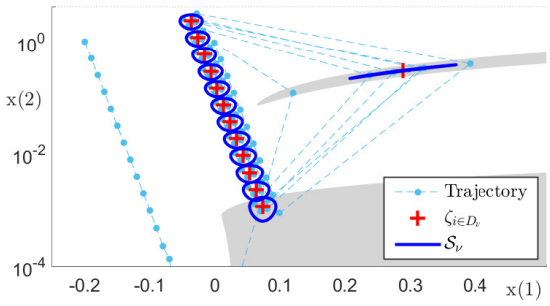


Fig. 9. Trajectories of uncertain system (37)-(38) obtained for a perturbation $\delta_{max} = 0.01$ and converging to the attractor represented by the union of the blue ellipsoids in a semi-logarithmic space. The grey areas correspond to the regions of the state space, where the control law imposes to select mode 2, the complementary set being associated to mode 1.

Let us assume from now on that the system's matrices suffer from parameter uncertainties, described by

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \pm \alpha_1 \delta_{max} \end{bmatrix}, B_1 = \begin{bmatrix} 0.01 \\ \pm \alpha_2 \delta_{max} \end{bmatrix}. \quad (38)$$

where $\alpha_1 = 0.01$ and $\alpha_2 = 0.005$ and where δ_{max} is a parameter. Matrices A_2 and B_2 remain the same. Figure 8 shows the different attractors $\mathcal{S}_\nu$ obtained for various values of δ_{max} after performing a gridding procedure on parameter $\mu \in (0, 1)$ to the optimal solution to conditions to solve conditions (2) minimizing the cost function J with $\alpha = [0, 1, 0, 0]$. As expected, the size of the attractor grows with δ_{max}. Vectors ζ_i's, solution to Theorem 2, are very close to the limit cycle $\{\rho_i\}_{i\in\mathbb{D}_\nu}$ of the nominal case (for example, with $\delta_{max} = 0.01$, $\| \rho_i - \zeta_i \|_\infty \leq 10^{-3}$), which illustrates Proposition 2. One can see that the attractor obtained for the two smallest values of δ_{max} is the union of disjoint ellipsoid, so that the control law converges ultimately to a periodic law, verifying item 2 of Theorem (2). Lastly, the evolution of the state variable with its dynamic affected by a perturbation $\delta_{max} = 0.01$ is plotted on Figure 9. The ordinate-axis is presented in a logarithmic scale to ease the differentiation between the ellipses. The figure shows the convergence of the state into the attractor. One can also see from this figure that the trajectory converges to the limit cycle

9 Conclusion

A new solution to the problem of stabilizing switched affine systems has been considered. The novelty with respect to the literature relies on the construction of a control Lyapunov function. Thanks to an a priori selected sequence of modes, i.e. cycle, simple LMI conditions, related to existing results on periodic systems, have been provided to ensure that the trajectories of the nominal closed-loop system converge to a limit cycle characterized by system's matrices and this sequence of modes. Several properties of the control law have been provided, such as the convergence to a periodic control law. Unlike other approaches from the literature, this solution is suitable to be extending to the uncertain case, where the notion of limit cycles was lightly modified. In this case, the attractor is designed by the solutions of the decision variables of an LMI problem. Then, keys to embed optimization problems have been provided, especially to select the optimal cycle or limit cycle that minimizes a given cost function. Finally, several examples have been presented, emphasizing the potential of the method for both nominal and uncertain cases.

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