Attractors and limit cycles of discrete-time switching affine systems : nominal and uncertain cases

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Attractors and limit cycles of discrete-time switching affine systems : nominal and uncertain cases ⋆

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Abstract

This paper deals with the robust stabilization of uncertain discrete-time switched affine systems using a control Lyapunov approach and a min-switching state-feedback control law. After presenting some preliminaries on limit cycles, a constructive stabilization theorem, expressed as linear matrix inequalities, guarantees that the solutions to the nominal closed-loop system converge to a limit cycle. This method is extended to the case of uncertain systems, for which the notion of limit cycle needs to be adapted. The theoretical results are evaluated on academic examples and demonstrate the potential of the method over the recent literature.

Keywords: Switching affine systems, robust stabilization, uncertain systems, linear matrix inequalities, control Lyapunov function.

1 Introduction

A switched system consists in the association of a finite set of dynamics with a switching rule that designates at each time which is the active one among them [28]. This is a wide particular class of hybrid systems [23], that allows to model numerous applications in various fields as embedded systems, electromechanics, biology, or networked control systems and to characterize complex and not intuitive behaviors. Among them, there are discrete-time switched systems, that can be considered in their own or potentially generated by continuous-time switched systems with a switching rule that is piecewise constant according to a given sampling period [27]. It is interesting to emphasize that in the last case, linear or affine modes remain linear or affine by discretization. This kind of systems have generated a rich study concerning their stability to the origin [12,36], stabilization [21] or stabilizability [18].

Stabilizing Switched Affine Systems (SAS's) to a desired position, that is not in the set of equilibrium points of the modes, has motivated a large collection of contributions, especially in the continuous-time framework: global quadratic stabilization via a min-switching strategy [10], use of dynamic programming to select the mode to activate and also the time to switch [31], globally stabilizing min-switching strategy taking into account a cost to minimize [15], local stabilization without requiring the existence of a Hurwitz combination of the linear parts [25], practical stabilization with minimum dwell-time guarantees [1], robust stabilization to an unknown equilibrium point [2,3]. We can emphasize also contributions in the discrete-time domain leading to the practical stabilization via different types of Lyapunov functions: quadratic forms [14], switched quadratic functions based on Lyapunov-Metzler inequalities [17] or thanks to multiple shifted quadratic functions [32].

Besides equilibrium points, dynamical systems may have as asymptotic behaviors, self-sustained oscillations, or limit cycles: that are closed and isolated trajectories [34, Section 7]. Their studies have been initiated by Poincaré and are generically related to the ω-limit sets (set of accumulation points of the trajectories) and Poincaré-Bendixson theorem. For hybrid or switched systems, limit cycles have been mainly investigated in the continuous-time domain, motivated by switched circuits [4,24,26,30]. The Poincaré-Bendixson theorem has been extended for hybrid systems [33] and the ω-limit sets of hybrid systems have been also investigated [11]. The main difficulty is to determine the switching times related to a limit cycle [20,22]. There are only few contributions on limit cycles for discrete-time SAS's: one case study is provided in [29], based on prac-

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tical stability [38] or clock- and state-dependent switched Lyapunov functions [16] and also the preliminary result on multiple shifted quadratic Lyapunov functions putting in evidence the notion of limit cycles [32].
This paper is focused on limit cycles for nominal and uncertain discrete-time SAS's, and their stabilizability thanks to state-dependent switching laws. In summary the contributions are:

- A rigorous definition of limit cycles in the framework of discrete-time SAS's.
- An original pure state-feedback min-switching control strategy allows to obtain an autonomous system, which differs to the time-dependent one presented in [16].
- An exponential stabilization criteria, using a min-switching control law, expressed in terms of Linear Matrix Inequalities (LMI) is proven.
- A robust stabilization criteria for the uncertain case.

The paper is organized as follows. The investigated system is presented in Section 2. Limit cycles are rigorously defined in Section 3, where a necessary and sufficient condition for their existence is provided. Sections 4 and 5 deal with the stabilization to a limit cycle, in the nominal and uncertain cases. Families of optimization problems are presented in Section 6. Several illustrations are provided in Section 7 before concluding remarks in Section 8.

Notations: Throughout the paper, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{R}$ the real numbers, $\mathbb{R}^n$ the n-dimensional Euclidean space, $\mathbb{R}^{n\times m}$ the set of all real $n\times m$ matrices and $\mathbb{S}^n$ the symmetric matrices in $\mathbb{R}^{n\times n}$. For any n and m in $\mathbb{N}$, I_n and $\mathbf{0}_n$ denote the identity and null matrices of $\mathbb{R}^{n\times n}$, respectively. Notation $1_{n\times m}$ stands for the matrix in $\mathbb{R}^{n\times m}$ whose entries are all 1. For any matrix M of $\mathbb{R}^{n\times n}$, the notation $M \succ 0$, ($M \prec 0$) means that M is symmetric positive (negative) definite. For any suitable matrices $A = A^\top, B, C = C^\top$, notation $\left[\begin{smallmatrix} A & B \\ * & C \end{smallmatrix}\right]$ stands for the symmetric matrix $\left[\begin{smallmatrix} A & B \\ B^\top & C \end{smallmatrix}\right]$. $\|x\|$ denotes the Euclidean norm of x. For any $M \succ 0$ in $\mathbb{S}_n$ and any vector $x \in \mathbb{R}^n$, we denote $\|x\|_M = \sqrt{x^\top M x}$, the weighted norm and $\lambda_m(M)$ and $\lambda_M(M)$ denote its minimal and maximal eigenvalues respectively of M. For any vector $h \in \mathbb{R}^n$, we denote the shifted ellipsoid $\mathcal{E}(M,h) = \{x \in \mathbb{R}^n,\ (x-h)^\top M(x-h) \leq 1\}$.

2 Problem formulation

Consider the discrete-time SAS given by

$$\begin{cases} x^+ = A_\sigma x + B_\sigma, \\ \sigma \in u(x) \subset \mathbb{K}, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, which adopts the following notation $x^+ = x_{k+1}$ and $x = x_k$ and $x_0 \in \mathbb{R}^n$. Likewise, $\sigma \in \mathbb{K} := \{1, 2, .., K\}$ characterizes the active mode. Finally, $A_i \in \mathbb{R}^{n\times n}$ and $B_i \in \mathbb{R}^{n\times 1}$ are the matrices of mode $i \in \mathbb{K}$.
The particularity of this class of systems relies on its control action, which is performed through the selection of the mode σ, requiring a particular attention. The objective here is to design a suitable set valued map u that ensures the convergence of the state trajectories to a set to be characterized accurately. Indeed, it is well-known that stabilizing system (1) to a single equilibrium cannot be achieved in general [38]. Its nonlinear nature, due to the affine term imposes to relax the control objectives to an acceptable stability result such as practical stability in [14], for instance.
In this paper, the objective is to go deeper into the analysis and design of SAS's and to characterize their steady-state behavior, i.e. limit cycles. Compared to [4] or [16], the following improvements have been performed. First, a distinction between *periodic state-trajectories*, as stated in [16], *state-limit cycles*, and *hybrid limit cycles* is provided. Then, a necessary and sufficient condition for their existence regardless of their stability will be presented, which is not the case in [16]. More interestingly, we will show that the LMI condition provided here, which is the same as the one in [9] for periodic systems or more recently [16] for SAS's, ensures both the existence of a hybrid limit cycle and its stability. Finally, an extension to the case of uncertain systems is conducted, enhancing then the notion of robust limit cycles. To do so, we propose to follow our preliminary results provided in [32] where a different Lyapunov function inspired from [17] is included to better understand the attractor of the resulting control Lyapunov function. An important by-product of the following analysis is that the usual underlying necessary condition consisting in the existence of a Schur stable convex combination of matrices A_i is not required anymore, which, apart from [16], has been rarely completed in the literature.

3 Limit cycles of switched affine systems

This section focuses on the time-varying steady-states of SAS's, which seem to have a natural convergence to a repeated sequence, behaving as a limit cycle in discrete time. Limit cycles represent the stationary state of sustained oscillations, that depend exclusively on the parameters of the system and their intrinsic properties. The notion of limit cycles [34,35] is adapted here to discrete-time SAS's.

3.1 Definitions and notations

Let us first denote as $\mathbb{H}^n$, the set of functions from $\mathbb{N}$ to $\mathbb{K}\times\mathbb{R}^n$. The following definition of hybrid limit cycle is formulated.

Definition 1 (Hybrid limit cycle) *A hybrid limit cycle for system* (1) *or limit cycle in short, is a closed and isolated hybrid trajectory* $s \in \mathbb{H}^n$.
More formally, this means that $k \mapsto s_k = (\sigma_k, x_k)$ *is an* N*-periodic solution to* (1)*, which verifies*

$$\exists\kappa(s) > 0,\ s.t.\ \forall \tilde{s} \in \mathbb{T}^n(s),\quad \min_{\delta\in\mathbb{N}}\left(\sup_{k\in\mathbb{N}} \|s_k - \tilde{s}_{k+\delta}\|\right) > \kappa(s),$$

where $\mathbb{T}^n(s) \subset \mathbb{H}^n$ *is the set of* N*-periodic functions of* $\mathbb{H}^n$ *(i.e. with the same period as* s*), that are solution to* (1) *but* *are not shifted version of s*

$$\mathbb{T}^n(s) = \left\{ \tilde{s} \in \mathbb{H}^n, \begin{cases} (i) & \tilde{x}_{k+1} = A_{\tilde{\sigma}_k}\tilde{x}_k + B_{\tilde{\sigma}_k}, & \forall k \in \mathbb{N}, \\ (ii) & \tilde{s}_{k+N} = \tilde{s}_k, & \forall k \in \mathbb{N}, \\ (iii) & \min_{\delta\in\mathbb{N}} \left(\sup_{k\in\mathbb{N}} \|s_k - \tilde{s}_{k+\delta}\| \right) \neq 0. \end{cases} \right\}.$$

Studying limit cycles is usually performed using Poincaré map and Poincaré-Bendixson theorem. However, the results are not easy to generalize for high dimensions nor for hybrid systems [22,33]. Another approach, taking the point of view of closed-loop dynamics, recovers more classical definitions of autonomous differential equations/inclusions [35]. Compared to [16], the notion of hybrid limit cycles is introduced. It is composed of both the state trajectory x_k and the control input σ_k. Here, 'closed' refers to periodic nature of this trajectory (avoiding heteroclinic orbits [34, Ex.6.6.1]) and 'isolated' means that there exists a neighborhood of this trajectory, which does not contain another periodic solution generated by the same switching law. This is highlighted in the definition of $\mathbb{T}^n(s)$, which excludes the solutions to (1) (e.g. (i)) that are N-periodic (e.g. (ii) and where N is the minimal period of s) and that are a shifted version of s (e.g. (iii)).

For a given hybrid limit cycle s_k, both the projected switching signal trajectory $\mathbb{N} \to \mathbb{K}$, $k \mapsto \sigma_k$ and the projected state trajectory $\mathbb{N} \to \mathbb{R}^n$, $k \mapsto x_k$ are periodic functions. This gives rise to the next definitions.

Definition 2 (Cycle) *For a hybrid limit cycle $k \mapsto s_k = (\sigma_k, x_k)$, the cycle refers to the $k \mapsto \sigma_k$ function, denoted v. A limit cycle being a closed trajectory, there exists N in $\mathbb{N}\backslash\{0\}$, such that $v(\ell + N) = v(\ell), \forall \ell \in \mathbb{N}$.*
In addition, notations N_v and $\mathbb{D}_v$ stand for the minimum period and the minimal domain of v, defined as follows

$$\begin{aligned} N_v &= \min N \in \mathbb{N}\backslash\{0\} \text{ s.t. } v(\ell + N) = v(\ell), \forall \ell \in \mathbb{N}, \\ \mathbb{D}_v &= \{1, 2, \ldots, N_v\}. \end{aligned}$$

Definition 3 (Set of cycles) *Let C defined as*
$C := \{v : \mathbb{N} \to \mathbb{K},\ s.t.\ \exists N \in \mathbb{N}\backslash\{0\},\ \forall \ell \in \mathbb{N},\ v(\ell + N) = v(\ell)\}$ *being the set of cycles (periodic functions) from $\mathbb{N}$ to $\mathbb{K}$. Moreover, C_N denotes the set of cycles that are N-periodic* $C_N := \{v : \mathbb{N} \to \mathbb{K},\ s.t.\ \forall \ell \in \mathbb{N},\ v(\ell + N) = v(\ell)\}$.

To ease the readability, we introduce the following modulo notation: $\lfloor i \rfloor_v = ((i-1) \bmod N_v) + 1$, for any $i \in \mathbb{N}$, $i \geq 1$. In particular, $\lfloor i \rfloor_v = i$, for any $i = 1, \ldots, N_v$ and $\lfloor N_v + 1 \rfloor_v = 1$.

Definition 4 (State limit cycle) *For a hybrid limit cycle $k \mapsto s_k = (\sigma_k, x_k)$, the state limit cycle refers to the $k \mapsto x_k$. In the following, we will denote ρ_i, $i \in \mathbb{D}_v$ (v being the associated cycle) as the ordered vectors.*

Definition 5 (Attractor) *A compact set $\mathcal{A} \subset \mathbb{R}^n$ is said to be an attractor of system* (1) *if it is attractive and invariant.*

The next paragraph exhibits conditions for the existence of limit cycles for switched affine systems.

3.2 Necessary and sufficient conditions of existence

To characterize the limit cycles of system (1), we take advantage of the associated periodic switching law and benefits of the discrete-time (linear) periodic system literature, see for instance [7,9,13,37] or the survey [8]. A necessary and sufficient condition to the existence of a limit cycle for a given cycle v is provided in the following lemma, that generalizes [29] to the case of an arbitrary number of modes and an arbitrary period N_v.

Lemma 1 *A cycle $v \in C$ generates a unique limit cycle for system* (1) *if and only if* 1 *is not an eigenvalue of matrix $(\Phi_v(0))^M$, for all $M \in \mathbb{N}\backslash\{0\}$, where $\Phi_v(\ell)$ is the monodromy matrix at time $\ell \in \mathbb{N}$, defined by $\Phi_v(\ell) := \prod_{\iota=\ell+1}^{\ell+N_v} A_{v(\iota)} = A_{v(\ell+N_v)}A_{v(\ell+N_v-1)}\ldots A_{v(\ell+1)}$, $\forall \ell \in \mathbb{N}$. Moreover, the state-limit cycle is given by $\rho := (I_{nN_v} - \mathbb{A}_v)^{-1}\mathbb{B}_v$, where*

$$\mathbb{A}_v = \begin{bmatrix} 0 & \ldots & 0 & A_{v(N_v)} \\ A_{v(1)} & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & A_{v(N_v-1)} & 0 \end{bmatrix},\ \mathbb{B}_v = \begin{bmatrix} B_{v(N_v)} \\ B_{v(1)} \\ \vdots \\ B_{v(N_v-1)} \end{bmatrix},\ \rho = \begin{bmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_{N_v} \end{bmatrix}.$$

Proof. A cycle $v \in C$ is associated with a unique limit cycle for the switched affine system (1) if and only if there exists a unique sequence of N_v vectors $\{\rho_i\}_{i\in\mathbb{D}_v}$ such that,

$$\rho_{\lfloor i+1 \rfloor_v} = A_{v(i)}\rho_i + B_{v(i)}, \quad \forall i \in \mathbb{D}_v, \tag{2}$$

which is illustrated on the schematic representation shown on Fig. 1. In this figure, the attractor refers to $\mathcal{A}_v = \{\rho_1\} \bigcup \{\rho_2\} \bigcup \{\rho_3\}$, the state limit cycle $k \mapsto x_k = \rho_{\lfloor k+\delta \rfloor_v}$, with $\delta \in \{1, 2, 3\}$. Finally, the hybrid limit cycle is $k \mapsto s_k = (\sigma_k, x_k) = (v(k + \delta), \rho_{\lfloor k+\delta \rfloor_v})$, with $\delta \in \{1, 2, 3\}$. Relations (2) can be reformulated into the following equation, by using a cyclic augmented representation inspired by [19,37]:

$$(I_{nN_v} - \mathbb{A}_v)\rho = \mathbb{B}_v, \tag{3}$$

where $\mathbb{A}_v$, $\mathbb{B}_v$ and ρ are defined in the lemma. Let us first recall that matrix $\mathbb{A}_v \in \mathbb{R}^{nN_v \times nN_v}$ is closely related to the monodromy matrix at time $\ell \in \mathbb{N}$, $\Phi_v(\ell)$. For discrete-time periodic systems, the spectrum of the monodromy matrix does not depend on the time ℓ (see [5, Section 3.1]). These eigenvalues are called characteristic multipliers. Moreover the spectrum of $\mathbb{A}_v$ is the set of all N_v-roots of the n eigenvalues of $\Phi_v(0)$ (see [7, page 322, Section 3.2] or [37, Th.4]).
Sufficiency: Assume that 1 is not an eigenvalue of matrix $(\Phi_v(0))^M = \Phi_v^M(0)$, where M is any strictly positive integer. In particular 1 is not an eigenvalue of $\Phi_v(0)$ and we infer that matrix $(I_{nN_v} - \mathbb{A}_v)$ is nonsingular. It follows that ρ defined in Lemma 1 is a periodic solution of (2). This ensures that a periodic solution in the neighborhood of this solution shares the same cycle v. The period of the associated hybrid solution is necessarily a multiple of N_v. Then, it is $\tilde{M}N_v$-periodic, with

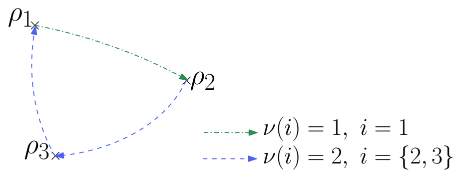


Fig. 1. Schematic representation of a cycle ν of period $N_\nu = 3$, for a system (1) with $K = 2$ modes. Here, we have $\nu : \{1, 2, 2\}$.

an integer $\tilde{M} \geq 1$. Let us introduce the $\tilde{M}N_\nu$-periodic cycle $\tilde{\nu}$ defined by $\tilde{\nu}(\ell) = \nu(\lfloor \ell \rfloor_\nu)$, $\forall \ell \geq 1$. We denote by $\tilde{\rho} \in \mathbb{R}^{n\tilde{M}N_\nu}$, such a solution to $(1_{\tilde{M}\times 1} \otimes \rho - \tilde{\rho}) \in \mathrm{Ker}(I_{n\tilde{M}N_\nu} - \mathbb{A}_{\tilde{\nu}})$. The eigenvalues of $\mathbb{A}_{\tilde{\nu}}$ are the $\tilde{M}N_\nu$-roots of $\Phi_{\tilde{\nu}}(0) = \Phi_\nu^{\tilde{M}}(0)$. Hence, $\mathbb{A}_{\tilde{\nu}}$ admits 1 as eigenvalue if and only if 1 is also an eigenvalue of $\Phi_\nu^{\tilde{M}}(0)$. If 1 is not an eigenvalue of matrix $\Phi_\nu^M(0)$ where M is any strictly positive integer, this kernel is reduced to the null vector and the two hybrid trajectories are the same. The periodic solution induced by ν exists and the limit cycle is unique.
Necessity: The proof is performed by contraposition. Assume that there exists a positive integer M such that $\Phi_\nu^M(0)$ admits 1 as eigenvalue. Let us set M as the minimum integer satisfying this. If $M = 1$, (3) has no solution or an infinite number of solutions, whether or not $\mathbb{B}_\nu$ belongs to $\mathrm{Span}\,(I_{nN_\nu} - \mathbb{A}_\nu)$. That implies that there does not exist a periodic solution of period N_ν or it is not isolated. If $M > 1$, then there exists a solution $\rho \in \mathbb{R}^{nN_\nu}$. We build now an M-repetition of this periodic solution, that is $1_{M\times 1} \otimes \rho$. Due to the assumption, there exists a non trivial vector $\check{\rho} \in \mathrm{Ker}\,(I_{nMN_\nu} - \mathbb{A}_{\bar{\nu}}) \subset \mathbb{R}^{nMN_\nu}$, with $\bar{\nu}$ being the MN_ν-periodic cycle defined by $\bar{\nu}(\ell) = \nu(\lfloor \ell \rfloor_\nu)$, $\forall \ell \geq 1$. Therefore, for any small scalar $\kappa > 0$, $1_{M\times 1} \otimes \rho + \frac{1}{2}\kappa\check{\rho}$ is related to a periodic solution in the neighborhood of ρ, which is not isolated, and therefore concludes the proof. □

A simpler but only sufficient test is presented below.

Corollary 1 *A cycle $\nu \in C$ generates a unique limit cycle for system* (1) *if either $\Phi_\nu(0)$ or $\mathbb{A}_\nu$ is Schur stable.*

Proof. If $\Phi_\nu(0)$ or $\mathbb{A}_\nu$ are Schur stable, they do not have any eigenvalue on the unitary disc, which proves the result. □

3.3 Additional properties

When stabilizing of SAS's, it is usual to perform a change of coordinates in order to locate the reference position at the origin. The following proposition holds.

Proposition 1 *Assume that a cycle ν generates a limit cycle for system* (1)*, with the components of the state limit cycle denoted $\{\rho_i\}_{i\in\mathbb{D}_\nu}$. Then, for any nonsingular matrix T and any vector $w \in \mathbb{R}^n$, $\{T\rho_i + w\}_{i\in\mathbb{D}_\nu}$ are the components of the state limit cycle associated to the same cycle for the same system* (1) *but expressed in the new coordinates $z = Tx + w$.*

Proof. Simple manipulations of (2) conclude the proof. □

Another useful property to ensure an equivalent class of switching laws related to the attractor is provided here.

Corollary 2 *Consider a cycle $\nu \in C$ and its limit cycle $\{\rho_i\}_{i\in\mathbb{D}_\nu}$. If there exist $(i_0, i_1) \in \mathbb{D}_\nu^2$, $i_1 > i_0$, such that $\rho_{i_0} = \rho_{i_1}$, then $\{\rho_i\}_{i\in\mathbb{D}_\nu}$ is the union of two closed trajectories associated with cycles of periods strictly less than N_ν.*

Proof. Based on the cycle ν, the proof is obtained by designing the two following cycles ν_1 and ν_2. ν_1 is defined as a $N_{\nu_1} = (i_1 - i_0)$-periodic cycle given by $\nu_1(\ell) = \nu(i_0 - 1 + \ell)$, $\ell = 1, \cdots, (i_1 - i_0)$ and is associated with the periodic trajectory $\{\rho_{i_0}, \rho_{i_0+1}, \cdots, \rho_{i_1}\}$. ν_2 is defined as a $N_{\nu_2} = (N - i_1 + i_0)$-periodic cycle given by $\nu_2(\ell) = \nu(i_1 - 1 + \ell)$, $\ell = 1, \cdots, (N - i_1 + i_0)$ and is associated with the periodic trajectory $\{\rho_{i_1}, \rho_{i_1+1}, \cdots \rho_{N_\nu}, \rho_1, \cdots, \rho_{i_0}\}$. □

4 Stabilization to a limit cycle

4.1 Stabilization and Control Lyapunov function

This section presents the first stabilization theorem to a limit cycle defined by a given cycle $\nu \in C$.

Theorem 1 *For a given cycle ν in C, assume that there exist matrices $\{P_i\}_{i\in\mathbb{D}_\nu}$ in $\mathbb{S}^n$, such that*

$$P_i \succ 0, \quad A_{\nu(i)}^\top P_{\lfloor i+1 \rfloor_\nu} A_{\nu(i)} - P_i \prec 0, \quad \forall i \in \mathbb{D}_\nu. \tag{4}$$

Then, the following statements hold:
(i) Cycle ν generates a unique limit cycle for system (1).
(ii) $\mathcal{A}_\nu := \bigcup_{i\in\mathbb{D}_\nu}\{\rho_i\}$, with ρ_i solution to (2) *is globally exponentially stable for system* (1) *with the switching control law*

$$u(x) = \left\{ \nu(\theta),\ \theta \in \underset{i\in\mathbb{D}_\nu}{\mathrm{argmin}}\,(x-\rho_i)^\top P_i\,(x-\rho_i) \right\} \subset \mathbb{K}. \tag{5}$$

(iii) If $\rho_i \neq \rho_j$ for any $i \neq j$ in $\mathbb{D}_\nu$, then the switching signal σ resulting from the closed-loop system (1),(5) *converges ultimately to a shifted version of ν.*

Proof. The proof is split into three parts.
Proof of (i): Introduce $\bar{P} = \mathrm{diag}\,(P_i)_{i=1,\dots,N_\nu}$ solution to (4). Simple calculations show that

$$\mathbb{A}_\nu^\top \bar{P} \mathbb{A}_\nu - \bar{P} = \mathrm{diag}\left(A_{\nu(i)}^\top P_{\lfloor i+1 \rfloor_\nu} A_{\nu(i)} - P_i\right)_{i=1,\dots,N_\nu} \prec 0.$$

Hence, matrix $\mathbb{A}_\nu$ is Schur stable, which, according to Corollary 1, implies that (2) thus admits a unique solution.

Proof of (ii): Consider the Lyapunov function candidate defined as follows

$$V(x) = \min_{i\in\mathbb{D}_\nu} \|x-\rho_i\|_{P_i}^2 = \min_{i\in\mathbb{D}_\nu}(x-\rho_i)^\top P_i (x-\rho_i), \quad \forall x \in \mathbb{R}^n, \tag{6}$$

where vectors ρ_i's are solution to (2). Since $\mathbb{D}_\nu$ is bounded and $P_i > 0$, inequality

$$0 \le c_1 d^2_{\mathcal{A}_\nu}(x) \le V(x) \le c_2 d^2_{\mathcal{A}_\nu}(x), \qquad (7)$$

holds with $c_1 = \min_{i\in\mathbb{D}_\nu} \lambda_m(P_i) > 0$ and $c_2 = \max_{i\in\mathbb{D}_\nu} \lambda_M(P_i) > 0$ where $d_{\mathcal{A}_\nu}(x) = \min_{i\in\mathbb{D}_\nu} \|x - \rho_i\|$ defines the distance of a vector x in $\mathbb{R}^n$ to $\mathcal{A}_\nu$. Moreover, ΔV writes

$$\begin{aligned}\Delta V(x) := V(x^+) - V(x) &= \min_{j\in\mathbb{D}_\nu} \|x^+ - \rho_j\|^2_{P_j} - \min_{i\in\mathbb{D}_\nu} \|x^+ - \rho_i\|^2_{P_i} \\ &= \min_{j\in\mathbb{D}_\nu} \|x^+ - \rho_j\|^2_{P_j} - \|x^+ - \rho_\theta\|^2_{P_\theta}.\end{aligned}$$

The last expression has been obtained by noting that θ results from the control law in (5), and minimizes the quadratic term, by definition. The first term of ΔV, being the minimum of several values, is consequently less than or equal to any of them, and, in particular, to $\lfloor\theta + 1\rfloor_\nu$. This yields $\Delta V(x) \le \|x^+ - \rho_{\lfloor\theta+1\rfloor_\nu}\|^2_{P_{\lfloor\theta+1\rfloor_\nu}} - \|x - \rho_\theta\|^2_{P_\theta}$. The closed-loop system (1),(5) imposes that $x^+ - \rho_{\lfloor\theta+1\rfloor_\nu} = A_{\nu(\theta)}x + B_{\nu(\theta)} - \rho_{\lfloor\theta+1\rfloor_\nu} = A_{\nu(\theta)}(x - \rho_\theta)$, since $A_{\nu(\theta)}\rho_\theta + B_{\nu(\theta)} - \rho_{\lfloor\theta+1\rfloor_\nu} = 0$ from (2). Re-injecting this expression into the upper bound of $\Delta V(x)$ yields

$$\Delta V(x) \le (x - \rho_\theta)^\top \left(A^\top_{\nu(\theta)} P_{\lfloor\theta+1\rfloor_\nu} A_{\nu(\theta)} - P_\theta\right)(x - \rho_\theta).$$

Therefore, if matrices P_i's verify the strict inequalities (4), then there exists a small enough positive scalar $c_3 > 0$, such that $A^\top_{\nu(i)} P_{\lfloor i+1\rfloor_\nu} A_{\nu(i)} - P_i < -c_3 I_n$, for all $i \in \mathbb{D}_\nu$, so that

$$\Delta V(x) \le -c_3\|x - \rho_\theta\|^2 \le -c_3 d^2_{\mathcal{A}_\nu}(x) \le -c_3/c_2 V(x), \quad \forall x \in \mathbb{R}^n,$$

due to $\|x - \rho_\theta\| \ge d_{\mathcal{A}_\nu}(x)$ and inequality (7), which proves the global exponential convergence to attractor $\mathcal{A}_\nu$.

Proof of (iii): The idea is to prove that there exist $k_0 \in \mathbb{N}$ and $\delta \in \mathbb{K}$ such that $u(x_k) = \{\nu(k-\delta)\}$, $\sigma_k = \nu(k-\delta)$, for all $k \ge k_0$. To do so, one needs to show that there exists a sufficiently small scalar $\epsilon > 0$ (to be determined), such that having $x \in \mathcal{S}_\epsilon = \{x \in \mathbb{R}^n,\ V(x) \le \epsilon^2\}$ and $\theta \in \arg\min_{i\in\mathbb{D}_\nu} \|x^+ - \rho_i\|^2_{P_i}$ implies $\|x - \rho_{\lfloor\theta+1\rfloor_\nu}\|^2_{P_{\lfloor\theta+1\rfloor_\nu}} < \|x - \rho_j\|^2_{P_j}$, for all $j \in \mathbb{D}_\nu$, with $j \ne \lfloor\theta + 1\rfloor_\nu$, that is the solution to (5) is $\lfloor\theta + 1\rfloor_\nu$.
To sum up, k_0 is related to the time to reach the level set $\mathcal{S}_\epsilon$, which is always possible to reach thanks to the convergence of the Lyapunov function to zero. The shift δ only depends on the initial condition x_0. To prove this item, the equivalence of weighted norms ensures that

$$\exists c_{i,j} > 0,\ \forall (i, j) \in \mathbb{D}_\nu, \text{s.t. } \forall y \in \mathbb{R}^n,\ \|y\|_{P_i} \le c_{i,j}\|y\|_{P_j}, \qquad (8)$$

(for instance, select $c_{i,j} \ge \sqrt{\lambda_M(P_i)/\lambda_m(P_j)}$). Thanks to (4) and $x \in \mathcal{S}_\epsilon$, we have, with the notation $x^+ = A_{\nu(\theta)}x + B_{\nu(\theta)}$,

$$\|x^+ - \rho_{\lfloor\theta+1\rfloor_\nu}\|_{P_{\lfloor\theta+1\rfloor_\nu}} \le \|x - \rho_\theta\|_{P_\theta} \le \epsilon. \qquad (9)$$

As $x^+ - \rho_{\lfloor\theta+1\rfloor_\nu} = A_{\nu(\theta)}(x - \rho_\theta)$ and x is in $\mathcal{S}_\epsilon$, inequalities

$$\|x^+ - \rho_{\lfloor\theta+1\rfloor_\nu}\|_{P_\theta} = \|A_{\nu(\theta)}(x - \rho_\theta)\|_{P_\theta} \le \|A_{\nu(\theta)}\|_{P_\theta}\|x - \rho_\theta\|_{P_\theta} \le \|A_{\nu(\theta)}\|_{P_\theta}\epsilon \qquad (10)$$

hold, where $\|A_{\nu(\theta)}\|_{P_\theta}$ denotes the matrix norm induced by the weighted norm $\|x\|_{P_\theta} = \sqrt{x^T P_\theta x}$, for any vector x in $\mathbb{R}^n$. The triangular inequality and relations (8) yield, for all $j \in \mathbb{D}_\nu$

$$\begin{aligned}&\|\rho_{\lfloor\theta+1\rfloor_\nu} - \rho_j\|_{P_\theta} - \|A_{\nu(\theta)}\|_{P_\theta}\epsilon \le \|\rho_{\lfloor\theta+1\rfloor_\nu} - \rho_j\|_{P_\theta} - \|A_{\nu(\theta)}(x - \rho_\theta)\|_{P_\theta} \\ &\le \|\rho_{\lfloor\theta+1\rfloor_\nu} - \rho_j + A_{\nu(\theta)}(x - \rho_\theta)\|_{P_\theta} \le \|x^+ - \rho_j\|_{P_\theta} \le c_{\theta,j}\|x^+ - \rho_j\|_{P_j}.\end{aligned}$$

Since having $\rho_i \ne \rho_j$ for any $i \ne j$ in $\mathbb{D}_\nu$, it is always possible to find $\epsilon > 0$ such that $c_{\theta,j}\epsilon < \|\rho_{\lfloor\theta+1\rfloor_\nu} - \rho_j\|_{P_\theta} - \|A_{\nu(\theta)}\|_{P_\theta}\epsilon$ holds for any $j \in \mathbb{D}_\nu\backslash\{\lfloor\theta + 1\rfloor_\nu\}$. Combining both inequalities leads to $\epsilon < \|x^+ - \rho_j\|_{P_j}$, for all $j \in \mathbb{D}_\nu\backslash\{\lfloor\theta+1\rfloor_\nu\}$. Comparing this with (9) yields $\|x^+ - \rho_{\lfloor\theta+1\rfloor_\nu}\|_{P_{\lfloor\theta+1\rfloor_\nu}} \le \epsilon < \|x^+ - \rho_j\|_{P_j}$, for all $j \in \mathbb{D}_\nu\backslash\{\lfloor\theta + 1\rfloor_\nu\}$, which concludes the proof. □

Hereafter, Theorem 1 and results in Section 3.2 are commented and several important consequences are emphasized.

4.2 Feasibility of the sufficient conditions

Theorem 1 is based on the feasibility of LMIs (4), which have been already encountered in the framework of discrete-time linear periodic systems. Indeed the periodic Lyapunov lemma (see [6]) states the following result:

Lemma 2 ([9]) *For a given cycle ν, there exist positive definite matrices $\{P_i\}_{i\in\mathbb{D}_\nu}$ satisfying LMIs* (4) *if and only if the monodromy matrix $\Phi_\nu(0)$ is Schur.*

One of the main advantage of Lemma 2 is that the condition dealing with the monodromy matrix can be moved closer to the condition in Lemma 1: for a given cycle $\nu \in C$, if the monodromy matrix $\Phi_\nu(0)$ is Schur, then there exists a unique limit cycle (thanks to Lemma 1). The importance of having Schur monodromy matrices being revealed, the question is now to understand whether there exists a cycle $\nu \in C$ for a given system (1), which is associated to a stable monodromy matrix. The literature about the (periodic)-stabilizability of switched linear system provides useful conditions as for instance, [21] or more recently [18, Th. 6 and 22]).

4.3 Comparison with [16]

This section aims at comparing Theorem 1 with respect to [16, Th. 2]. While the LMI conditions are exactly the same for a given cycle ν, the contributions are notably different. Indeed the control law given in [16, Th. 2] is

$$u(x, \nu, k) = \underset{j\in\mathbb{K}}{\operatorname{argmin}} \begin{bmatrix} x - \rho_{\lfloor k\rfloor_\nu} \\ 1 \end{bmatrix}^\top \mathcal{L}_{k,j} \begin{bmatrix} x - \rho_{\lfloor k\rfloor_\nu} \\ 1 \end{bmatrix} \subset \mathbb{K}, \qquad (11)$$

where $\mathcal{L}_{i,j} = \begin{bmatrix} A_j^\top P_{\lfloor i+1\rfloor_\nu} A_j - P_i & A_j^\top P_{\lfloor i+1\rfloor_\nu} b_{i,j} \\ * & b_{i,j}^\top P_{\lfloor i+1\rfloor_\nu} b_{i,j} \end{bmatrix}$ and with $b_{i,j} = A_j\rho_i + B_j - \rho_{\lfloor i+1\rfloor_\nu}$ and with the clock-dependent Lyapunov function

$$V(x, \nu, k) = (x - \rho_{\lfloor k\rfloor_\nu})^\top P_{\lfloor k\rfloor_\nu}(x - \rho_{\lfloor k\rfloor_\nu}), \quad \forall x \in \mathbb{R}^n. \qquad (12)$$

The authors in [16] provide sufficient conditions for stabilization to the state trajectory (and not to the hybrid trajectory including the switching law), if (4) are satisfied. Note that this control law depends on a time counter $\nu(k)$ such that at each instant time k, the control input selects the point in the cycle ν_K from an *argmin* function over all K modes. Our first result concerns the convergence to the attractor, while [16, Th. 2] provides the convergence to the state trajectory of the limit cycle. However if the assumption of item (iii) is satisfied, Theorem 1 provides a periodic solution for the hybrid trajectory with the switching law converging to $\nu(k-\delta)$. Notice that this is not the case for [16, Th. 2], even if the additional assumption on the state limit cycle holds. This value of shift, δ, depends on the initial state x_0 and possibly on the choice of the switching law in the inclusion (5). Notice also that, for $k \geq k_0$, the set $u(x_k)$ reduces to a singleton and there is a unique selection of the mode to activate. A similar result may be obtained in [16, Th. 2], when a unique mode allows to steer a ρ_i to its successor in the state trajectory, that is there exists a unique j_0 such that $b_{i,j_0} = 0$. Item (iii) of Theorem 1 emphasizes that even if the Lyapunov function does not depend on ν, the min-switching strategy recovers a shifted version of ν, as an element of hybrid trajectory to the equivalent relation of attractors.
For a given cycle ν, control law (5) aims at selecting the best mode that minimizes the quadratic term in V, looking for the best position in the cycle. Alternatively, control law (11) selects the mode that minimizes (12), evaluated a $k+1$. Hence, the computational complexity of both control laws are different, depending on the length of the cycle and on the number of modes. For instance, depending on whether $N_\nu > K$ or $N_\nu < K$, control (5) or (11) can reduce the computational cost and the transient respectively.
To sum up, both contributions are different and their use depends on the context. An important property of the min-switching algorithm (5) is that this control law does not depend on the system parameters, enabling to develop a robust control law that takes into account parametric uncertainties. Hence, the following section presents a robust control law that ensures the states to converge to a *robust limit cycle*.

5 Robust stabilization of uncertain systems

Here matrices A_σ and B_σ are assumed to be unknown and/or time-varying, with a polytopic representation given by

$$[A_\sigma, B_\sigma] \in \mathrm{Co}\left(\left[A_\sigma^\ell, B_\sigma^\ell\right]\right)_{\ell\in\mathbb{L}}, \ \forall \sigma \in \mathbb{K}, \tag{13}$$

where $\mathbb{L}$ is a bounded subset of $\mathbb{N}$ and A_σ^ℓ and B_σ^ℓ are known and constant for any $\sigma \in \mathbb{K}$ and any $\ell \in \mathbb{L}$. Note that set $\mathbb{L}$ may depend on the mode σ but is avoided here without lack of generality. The results of the previous section fail, about stabilization and stabilizability. Indeed, the main problem appears in the selection of the limit cycle solving equations (2) in the situation of uncertain and/or time-varying system matrices. Therefore, it is important to provide an alternative solution dedicated to this relevant situation from a practical point of view. The robust stabilization is formalized here.

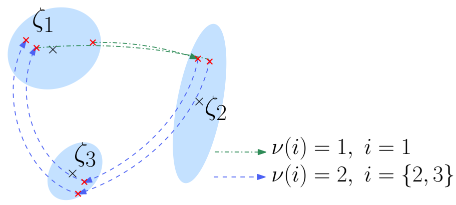


Fig. 2. Schematic representation of a robust limit cycle associated to a cycle ν of period $N_\nu = 3$, for a system (1) with $K = 2$ modes.

Theorem 2 *For a given cycle ν in C and for a parameter $\mu \in (0,1)$, assume that there exist $\{(W_i, \zeta_i)\}_{i\in\mathbb{D}_\nu}$ in $\mathbb{S}^n \times \mathbb{R}^n$ and that are solutions to the following matrix inequalities*

$$W_i \succ 0, \quad \Psi_i(A_{\nu(i)}^\ell, B_{\nu(i)}^\ell) \succ 0, \quad \forall (i,\ell) \in \mathbb{D}_\nu \times \mathbb{L}, \tag{14}$$

where Ψ_i depend on the system matrices and on the decision variables[1] $\{(W_i, \zeta_i)\}_{i\in\mathbb{D}_\nu}$ *in $\mathbb{S}^n \times \mathbb{R}^n$ and are given by*

$$\Psi_i\left(A_{\nu(i)}^\ell, B_{\nu(i)}^\ell\right) = \begin{bmatrix} (1-\mu)W_i & 0 & W_i (A_{\nu(i)}^\ell)^\top \\ * & \mu & (A_{\nu(i)}^\ell \zeta_i + B_{\nu(i)}^\ell - \zeta_{\lfloor i+1 \rfloor_\nu})^\top \\ * & * & W_{\lfloor i+1 \rfloor_\nu} \end{bmatrix}. \tag{15}$$

Then, the following statements hold:

(i) $\mathcal{S}_\nu := \bigcup_{i\in\mathbb{D}_\nu} \mathcal{E}(W_i^{-1}, \zeta_i)$ *is robustly globally exponentially stable for system* (1) *with the control law*

$$u(x) = \left\{ \nu(\theta), \ \theta \in \underset{i\in\mathbb{D}_\nu}{\mathrm{argmin}} (x-\zeta_i)^\top W_i^{-1} (x-\zeta_i) \right\} \subset \mathbb{K}. \tag{16}$$

(ii) Moreover, if $\mathcal{E}(W_i^{-1}, \zeta_i) \bigcap \mathcal{E}(W_j^{-1}, \zeta_j) = \emptyset$ *for all $i \neq j \in \mathbb{D}_\nu$, then the switching signal σ of closed-loop system* (1),(16) *converges ultimately to a shifted version of ν.*

Remark 1 Note that condition (14) becomes an LMI once parameter $\mu \in (0,1)$ is fixed. ⌟

Proof. Consider the same Lyapunov function given in (6) but with the Lyapunov matrices P_i replaced by W_i^{-1} and with vectors ρ_i's replaced by ζ_i's, solution to (14).
Proof of (i): As for Theorem 1, we can show that $\Delta V(x) \leq \|x^+ - \zeta_{\lfloor\theta+1\rfloor_\nu}\|^2_{W^{-1}_{\lfloor\theta+1\rfloor_\nu}} - \|x-\zeta_\theta\|^2_{W_\theta^{-1}}$ and we write

$$x^+ - \zeta_{\lfloor\theta+1\rfloor_\nu} = A_{\nu(\theta)}(x - \zeta_\theta) + \mathcal{B}_{\nu(\theta)}, \tag{17}$$

where $\mathcal{B}_{\nu(\theta)} = A_{\nu(\theta)}\zeta_\theta + B_{\nu(\theta)} - \zeta_{\lfloor\theta+1\rfloor_\nu}$, which are not necessarily zero, compared to the proof of Theorem 1 because vectors ζ_i's are now decision variables in inequalities (14). In order to find an alternative solution, we introduce $\chi_\theta^\top = [(W_\theta^{-1}(x-\zeta_\theta))^\top 1]$, so that $x^+ - \zeta_{\lfloor\theta+1\rfloor_\nu} = [A_{\nu(\theta)} W_\theta \ \mathcal{B}_{\nu(\theta)}]\chi_\theta$. Hence, $\Delta V(x)$ can be rewritten in a more compact form

$$\Delta V(x) \leq -\chi_\theta^\top \Phi_\theta\left(A_{\nu(\theta)}, B_{\nu(\theta)}\right)\chi_\theta, \tag{18}$$

[1] For simplicity, variables $\{(W_i, \zeta_i)\}_{i\in\mathbb{D}_\nu}$ are omitted in Ψ_i.

with $\Phi_\theta\Big(A_{\nu(\theta)}, B_{\nu(\theta)}\Big) = \begin{bmatrix} W_\theta & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} W_\theta A_{\nu(\theta)}^\top \\ \mathcal{B}_{\nu(\theta)}^\top \end{bmatrix} W_{\lfloor\theta+1\rfloor_\nu}^{-1} \begin{bmatrix} W_\theta A_{\nu(\theta)}^\top \\ \mathcal{B}_{\nu(\theta)}^\top \end{bmatrix}^\top$.

Note that ΔV is not required to be negative in the whole state space, but only outside of set $\mathcal{S}_\nu$. From the definition of the Lyapunov function, the control law given in (16) ensures that $V(x) = \|x - \zeta_\theta\|^2_{W_\theta^{-1}}$. Therefore, x being outside of $\mathcal{S}_\nu$, inequality $V(x) \leq 1$ writes

$$\begin{bmatrix} x - \zeta_\theta \\ 1 \end{bmatrix}^\top \begin{bmatrix} W_\theta^{-1} & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x - \zeta_\theta \\ 1 \end{bmatrix} = \chi_\theta^\top \begin{bmatrix} W_\theta & 0 \\ 0 & -1 \end{bmatrix} \chi_\theta > 0. \qquad (19)$$

Then, the problem can be summarized as the satisfaction of $\chi_\theta^\top \Phi_\theta\left(A_{\nu(\theta)}, B_{\nu(\theta)}\right)\chi_\theta > 0$ for all x such that (19) holds. Using an S-procedure, if there exists $\mu \in (0, 1)$ where

$$\begin{bmatrix} (1-\mu)W_\theta & 0 \\ 0 & \mu \end{bmatrix} - \begin{bmatrix} W_\theta A_{\nu(\theta)}^\top \\ \mathcal{B}_{\nu(\theta)}^\top \end{bmatrix} W_{\lfloor\theta+1\rfloor_\nu}^{-1} \begin{bmatrix} W_\theta A_{\nu(\theta)}^\top \\ \mathcal{B}_{\nu(\theta)}^\top \end{bmatrix}^\top \succ 0, \qquad (20)$$

then, $\Delta V(x) < 0$ for all $x \notin \mathcal{S}_\nu$. Finally, a Schur's complement yields $\Psi_\theta\left(A_{\nu(\theta)}, B_{\nu(\theta)}\right) \succ 0$, for a fixed parameter μ, where matrix Ψ_θ is defined in (15). Since matrices $A_{\nu(\theta)}$ and $B_{\nu(\theta)}$ are uncertain, it is not yet possible to evaluate numerically these LMIs for all possible values of θ. However, since they belong to the polytopic set (13), one can define those matrices as convex combinations, with possibly time-varying weights $A_{\nu(\theta)} = \sum_{\ell\in\mathbb{L}} \lambda_\ell A^\ell_{\nu(\theta)}$ and $B_{\nu(\theta)} = \sum_{\ell\in\mathbb{L}} \lambda_\ell B^\ell_{\nu(\theta)}$, where parameters $\lambda_\ell \in [0, 1]$ and hold $\sum_{\ell\in\mathbb{L}} \lambda_\ell = 1$. Since Ψ_θ are affine with respect to $A_{\nu(\theta)}$ and $B_{\nu(\theta)}$, it follows $\Psi_\theta\left(A_{\nu(\theta)}, B_{\nu(\theta)}\right) = \sum_{\ell\in\mathbb{L}} \lambda_\ell \Psi_\theta\left(A^\ell_{\nu(\theta)}, B^\ell_{\nu(\theta)}\right) \succ 0$, which is guaranteed by conditions (14). This guarantees that $\Delta V(x)$ is negative definite outside of $\mathcal{S}_\nu$. Exponential stability is obtained thanks to the strict inequalities (14).
To conclude the proof, it remains to prove that the attractive set $\mathcal{S}_\nu$ is invariant. By noting that item (i) ensures that $\Delta V(x) + \mu(V(x) - 1) < 0$, for all $x \in \mathbb{R}^n$. This establishes

$$V(x^+) = V(x) - \mu(V(x)-1) + \Delta V(x) + \mu(V(x)-1) \leq (1-\mu)V(x) + \mu.$$

Since x belongs to $\mathcal{S}_\nu$ and μ in $(0, 1)$, $V(x^+) \leq (1-\mu) + \mu = 1$ holds true, guaranteeing that x^+ also belongs to $\mathcal{S}_\nu$.

Proof of (ii): The proof of this result is omitted because is similar to the proof of item (iii) in Theorem 1. □

A relevant byproduct of this theorem is an extension of the definition of limit cycles in equation (2) to the case of uncertain switched affine systems, which can be now expressed in terms of series of inclusions. More specifically, Theorem 2 states that the invariance of the attractor $\mathcal{S}_\nu$ ensures that

$$A^\ell_{\nu(i)}\mathcal{E}(W_i^{-1}, \zeta_i) + B^\ell_{\nu(i)} \subset \mathcal{E}(W^{-1}_{\lfloor i+1\rfloor_\nu}, \zeta_{\lfloor i+1\rfloor_\nu}), \quad \forall (i, \ell) \in \mathbb{D}_\nu \times \mathbb{L}, \qquad (21)$$

where the left-hand-side of the inclusion means, with a slight abuse of notations, that, for any $i \in \mathbb{D}_\nu$ and for all $x \in \mathcal{E}(W_i^{-1}, \zeta_i)$, vector $A^\ell_{\nu(i)}x + B^\ell_{\nu(i)}$ belongs to $\mathcal{E}(W^{-1}_{\lfloor i+1\rfloor_\nu}, \zeta_{\lfloor i+1\rfloor_\nu})$ for all $\ell \in \mathbb{L}$. This inclusion can be seen as the natural extension of (2) to uncertain systems. The union of $\mathcal{E}(W_i^{-1}, \zeta_i)$, $i \in \mathbb{D}_\nu$, can be viewed as a stable *robust limit cycle*. It is also relevant to understand how conservative the previous theorem is with respect to the nominal case, presented in Theorem 1. The following proposition is stated.

Proposition 2 *For a given cycle ν in C, consider (P_i, ρ_i) in $\mathbb{S}^n \times \mathbb{R}^n$ for $i \in \mathbb{D}_\nu$ solution to* (4) *for a nominal system. Then, $(W_i, \zeta_i) = (\beta P_i^{-1}, \rho_i)$ is solution to* (14), *for any arbitrary scalar $\beta > 0$, with a sufficiently small value of $\mu \in (0, 1)$. Moreover $\lim_{\beta\to 0^+} \mathcal{S}_\nu = \mathcal{A}_\nu$.*

Proof. For matrices P_i and vectors ρ_i solution to inequalities (4) and to Lemma 1, respectively, let us definite $(W_i, \zeta_i) = (\beta P_i^{-1}, \rho_i)$, for any scalar $\beta > 0$. Then $\mathcal{B}_{\nu(i)} = A_{\nu(i)}\rho_i + B_{\nu(i)} - \rho_{\lfloor i+1\rfloor_\nu} = 0$, as defined in (17). For any $\mu \in (0, 1)$, $\Psi_i\left(A^\ell_{\nu(i)}, B^\ell_{\nu(i)}\right) \succ 0$ in (14) is equivalent to $A_{\nu(i)}^\top P_{\lfloor i+1\rfloor_\nu} A_{\nu(i)} - P_i \prec -\mu P_i$ by application of the Schur complement. The latter inequality being true for a small enough value $\mu > 0$, thanks to the strict inequality (4). Moreover, by noting that the attractor $\mathcal{S}_\nu$ is composed by the ellipsoids given by $\mathcal{E}(P_i/\beta, \rho_i)$, for all $i \in \mathbb{D}_\nu$, it reduces to $\mathcal{A}_\nu$, as β tends to zero. □

The previous proposition states that there is no conservatism induced by Theorem 2 with respect to Theorem 1. Therefore, in the following section dealing with the introduction of optimization problems, only the LMI constraints presented in Theorem 2 (for a fixed parameter μ) will be considered for the sake of simplicity, knowing that the same optimization problem could also be presented using the LMI constraints of Theorem 1. Then, in the sequel, we will refer to the attractor only as $\mathcal{S}_\nu$ knowing that in the nominal case, $\mathcal{S}_\nu = \mathcal{A}_\nu$.

6 Optimization algorithms

The objective of this section is to include to the previous developments some additional constraints to conditions (14) aiming at selecting the decision variables $\{W_i, \zeta_i\}_{i\in\mathbb{D}_\nu}$ that optimize a given cost function, evaluated for each cycle under consideration. In practical situations such as in power converters, an additional objective to stabilization could be to drive the solutions to the system as close as possible to a desired reference position, $x_d \in \mathbb{R}^n$, referring to a desired voltage. Hence, it appears highly relevant that the cost function does not only reflect the "size" of the attractor in order to limit the amplitude of the trajectories within the attractor but also the "distance" between the reference position to the attractor.

To go further in this direction, let us introduce the ellipsoid $\mathcal{E}(Q_\nu^{-1}, h_\nu)$ defined for some positive definite matrix Q_ν in $\mathbb{S}^n$ and some shifting vector h_ν in $\mathbb{R}^n$ to be optimized for a given cycle so that $\mathcal{E}(Q_\nu^{-1}, h_\nu)$ is the "smallest" ellipsoid verifying $(\{x_d\} \cup \mathcal{S}_\nu) \subset \mathcal{E}(Q_\nu^{-1}, h_\nu)$. The following lemma helps expressing this inclusion as a matrix inequality.

Lemma 3 *For some (Q, h) in $\mathbb{S}^n \times \mathbb{R}^n$, define*

$$\mathcal{K}_{Q,h}(W, \zeta, \eta) = \begin{bmatrix} \eta W & 0 & W \\ * & 1-\eta & \zeta^\top - h^\top \\ * & * & Q \end{bmatrix}, \tag{22}$$

for some matrix $0 \prec W \in \mathbb{S}^n$, a shifting vector ζ and a positive scalar η. Then, the following statements hold

(i) ζ belongs to $\mathcal{E}(Q^{-1}, h)$ if and only if $\mathcal{K}_{Q,h}(0, \zeta, 0) \geq 0$.

(ii) $\mathcal{E}(W^{-1}, \zeta)$ is included in $\mathcal{E}(Q^{-1}, h)$ if and only if there exists $\eta > 0$ such that $\mathcal{K}_{Q,h}(W, \zeta, \eta) \geq 0$.

Proof. The proof relies on the Schur complement (i) and an S-procedure (ii) and is omitted due to space limitations. □

Inclusion $(\{x_d\} \cup \mathcal{S}_\nu) \subset \mathcal{E}(Q_\nu^{-1}, h_\nu)$ is equivalent to

$$Q_\nu \succ 0,\ \mathcal{K}_{Q_\nu,h_\nu}(0,x_d,0) \geq 0,\ \mathcal{K}_{Q_\nu,h_\nu}(W_i,\zeta_i,\eta) \geq 0,\ i \in \mathbb{D}_\nu. \tag{23}$$

In the nominal case, inequalities $\mathcal{K}_{Q_\nu,h_\nu}(W_i,\zeta_i,\eta) \geq 0,\ i \in \mathbb{D}_\nu$ can be reduced to $\mathcal{K}_{Q_\nu,h_\nu}(0, \rho_i, 0) \geq 0$, for all $i \in \mathbb{D}_\nu$. A possible way to formalize the notion of a cost related to a "distance" and/or a "size", can be formulated as follows

$$J^*(\nu, x_d) := \min_{\{W_i, \zeta_i\}_{i\in\mathbb{D}_\nu}} J(\nu, x_d, \{W_i, \zeta_i\}_{i\in\mathbb{D}_\nu}) \tag{24}$$
$$\text{s.t. (14) and potential additional inequalities.}$$

where J is the cost function to be optimized and is defined as a barycenter of several families of not exhaustive costs

$$J(\nu, x_d, \{W_i, \zeta_i\}_{i\in\mathbb{D}_\nu}) = \sum_{m=1}^{4} \alpha_m J_m(\nu, x_d, \{W_i, \zeta_i\}_{i\in\mathbb{D}_\nu}), \tag{25}$$

where $\alpha_m \geq 0$ and $\sum_{m=1}^4 \alpha_m = 1$ and where J_i's are given by
(i) $J_1(\nu, x_d, \{W_i, \zeta_i\}_{i\in\mathbb{D}_\nu}) = \mathrm{Tr}\,(Q_\nu)$ with the additional inequalities given in (23), which aim at optimizing the attractor, or more precisely at evaluating the "chattering" effect when the solution reaches the attractor.
(ii) $J_2(\nu, x_d, \{W_i, \zeta_i\}_{i\in\mathbb{D}_\nu}) = \sum_{i=1}^N \mathrm{Tr}\,(W_i)$, which aims at minimizing $\mathcal{S}_\nu$, in the uncertain case.
(iii) $J_3(\nu, x_d, \{W_i, \zeta_i\}_{i\in\mathbb{D}_\nu}) = \omega_3$ with $\left[\begin{smallmatrix} \omega_3 & (\zeta_i - \zeta_j)^\top \\ * & I \end{smallmatrix}\right] \succ 0$, $\forall (i,j) \in \mathbb{D}_\nu^2,\ i \neq j$, which aims at enforcing the shifts ζ_i's to be the same value, that is to have a single shift for the ellipsoids.
(iv) $J_4(\{\zeta_i\}_{i\in\mathbb{D}_\nu}, x_d) = \omega_4$ with either $\left[\begin{smallmatrix} \omega_4 & x_d^\top - \frac{1}{N_\nu}\sum_{i\in\mathbb{D}_\nu} \zeta_i^\top \\ * & I_n \end{smallmatrix}\right] \succ 0$, which minimizes the distance between the average value of the limit cycle and the desired reference. In [16], another but similar cost function was presented with a projection matrix Γ. Here, J_4 will allow to select a limit cycle closed to a desired position that is predefined by the designer.
The optimization problem can now be properly stated.

Cycles (C)	Associated limit cycles	$J^*(\nu_i, 0)$
ν_1	$\begin{bmatrix} 1.66 \\ -0.2 \\ -0.04 \end{bmatrix}, \begin{bmatrix} 2.24 \\ -0.81 \\ -1.01 \end{bmatrix}$	2.37
ν_2	$\begin{bmatrix} 3.2 \\ 0.99 \\ -0.3 \end{bmatrix}, \begin{bmatrix} 4.49 \\ -0.81 \\ -2.8 \end{bmatrix}, \begin{bmatrix} 3.3 \\ -1.15 \\ -0.26 \end{bmatrix}, \begin{bmatrix} 2.67 \\ -0.03 \\ 0.24 \end{bmatrix}$	8.88
ν_3	$\begin{bmatrix} -0.65 \\ 1.06 \\ 0.08 \end{bmatrix}, \begin{bmatrix} 1.32 \\ 0.71 \\ -0.89 \end{bmatrix}, \begin{bmatrix} 2.42 \\ -0.64 \\ -1.63 \end{bmatrix}, \begin{bmatrix} 2 \\ -2.1 \\ -1.08 \end{bmatrix}, \begin{bmatrix} 0.59 \\ -2.48 \\ 0.39 \end{bmatrix}, \begin{bmatrix} -0.98 \\ -0.55 \\ 1.03 \end{bmatrix}$	5.17

Table 1
Limit cycles generated by a selection of cycles, which verify the assumption of Lemma 1 for system (1) with (26). The cost function (25) was computed with $\alpha = [0.5\ 0\ 0\ 0.5]$ and $x_d = [0\ 0\ 0]^\top$.

Proposition 3 *For a given bounded subset $\Omega \subset C$ and a given desired reference x_d, the optimal control law with respect to J is the one associated to $\nu^* = \mathrm{argmin}_{\nu\in\Omega} J^*(\nu, x_d)$.*

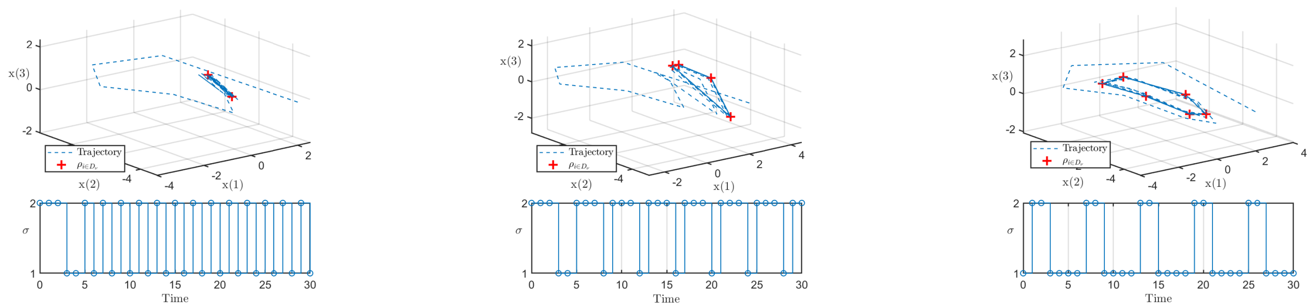
7 Numerical applications

Example 1: Consider system (1), borrowed from [14], where the matrices A_i and B_i are defined as follows

$$A_i = e^{F_i T}, \quad B_i = \int_0^T e^{F_i \tau} d\tau g_i, \quad \forall i \in \{1, 2\}, \tag{26}$$

where $T = 1$ refers to the sampling period and with $F_1 = \left[\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{smallmatrix}\right]$, $F_2 = \left[\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{smallmatrix}\right]$, $g_1 = [\,1\ 0\ 0\,]^\top$, $g_2 = [\,0\ 1\ 0\,]^\top$.
For this example, there exists a linear combination of matrices A_i which is Schur as shown in [14]. Fig. 3 shows on different graphs the limit cycles $\{\rho_i\}_{i\in\mathbb{D}_\nu}$, represented by the red crosses, obtained thanks to Lemma 1 for three different cycles. The figure also shows the trajectories of the closed-loop system, initialized at $x_0 = [2,-5,0]^\top$, with control law (5) from Theorem 1. Each trajectory converges to different limit cycles. The control signal is represented at the bottom of Fig. 3 and tends to the presumed cycle after a small transient time as pointed out in item (iii) of Theorem 1.
The results are interestingly very similar to the ones in [16] and [32]. However, the method provided in [32] is limited by the length of the cycle that needs to be equal to the number of mode. i.e. restricted to ν_1. In addition, the stabilization condition of [32] have a higher complexity since the size of each LMI increases with the number of modes. Finally compared to [16], our control law does not depend on time even though it converges to a periodic signal.
Table 1 provides informations on the cycles and their associated limit cycle as in Fig. 3. A cost function (25) has been also considered according to Section 6 with $\alpha = [\alpha_1, \alpha_2, \alpha_3, \alpha_4] = [0.5, 0, 0, 0.5]$ in (25). The cycle, for which the cost function, is minimized is ν_1 as in Table 1.

Example 2: Consider system (1) with $A_1 = \left[\begin{smallmatrix} 0.3 & 0.1 \\ 0.1 & 0.7 \end{smallmatrix}\right]$, $A_2 = \left[\begin{smallmatrix} 0.9 & 0.5 \\ 0 & -0.8 \end{smallmatrix}\right]$, $A_3 = \left[\begin{smallmatrix} 0.4 & 0 \\ 1 & -0.1 \end{smallmatrix}\right]$, and $A_4 = A_1 + 0.2\left[\begin{smallmatrix} 1 & 0 \\ -1 & 0 \end{smallmatrix}\right]$ to illustrate the comparison made in Section 4.3. The affine terms are built so that (ν, ρ) are $\nu = \{1, 2, 3\}$ and vectors $\rho = \{[0,1]^\top, [1,1]^\top, [-1,0]^\top\}$. Hence, we have $B_1 = B_4 = -A_1\rho_1 + \rho_2$, $B_2 = -A_2\rho_2 + \rho_3$, and $B_3 = -A_3\rho_3 + \rho_1$.
Fig. 4 shows the state trajectory (x, σ) in a phase plane. The trajectories of system (1) with the switching control law (5)



(a) $\nu_1 = \{1, 2\}$ (b) $\nu_2 = \{1, 2, 2, 2\}$ (c) $\nu_3 = \{1, 1, 1, 1, 2, 2\}$

Fig. 3. Evolution of the state variables and of the control input for three different cycles ν_1, ν_2 and ν_3 with the limit cycles (red crosses).

seems to have a better convergence rate. However, one must notice the differences on the switching signals. Whereas the given limit cycle should only involve three of the system functioning modes, the mode associated to the matrices A_4 and B_4 is selected periodically instead of the mode 1 with the control law from [16, Th.2]. Hence, this example exposes the remark made in Section 4.3 concerning the possible cases where at least two modes can steer one vector ρ_i to its successor. Control (5) provides for this example a better result than control (11), as illustrated in Fig. 4c.

Example 3: Consider system (1) with

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}, \; A_2 = 4\sqrt{2} \begin{bmatrix} \cos\left(\frac{\Pi}{4}\right) & -\sin\left(\frac{\Pi}{4}\right) \\ \sin\left(\frac{\Pi}{4}\right) & \cos\left(\frac{\Pi}{4}\right) \end{bmatrix} \tag{27}$$

and with $B_1 = [0.01, 0]^\top$ and $B_2 = [0, 0]^\top$. This example is adapted from [18], in which affine terms have been included. As noticed in [18], the monodromy matrix with cycle $\nu = \{1^{11}, 2^2\}$ is Schur, guaranteeing then, from Lemma 2, that a solution to Theorem 1 exists. However, there does not exist a linear combination of A_1, A_2 that is Schur stable. Fig. 5 depicts the evolution of the state variables converging to the limit cycle associated to ν, as well as the state-space partition where the dark areas are the regions where mode 2 is activated.

To illustrate the uncertain case, consider now

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \pm \kappa_1 \delta_{max} \end{bmatrix}, \; B_1 = \begin{bmatrix} 0.01 \\ \pm\kappa_2 \delta_{max} \end{bmatrix}. \tag{28}$$

where $(\kappa_1, \kappa_2) = 10^{-3}(10, 5)$ and where δ_{max} is a parameter. Matrices A_2 and B_2 remain the same. Fig. 6 shows the different attractors $\mathcal{S}_\nu$ obtained for various values of δ_{max} after performing a gridding procedure on parameter $\mu \in (0, 1)$ to optimally solve conditions (2) minimizing the cost function J with $\alpha = [0, 1, 0, 0]$. As expected, the size of the attractor grows with δ_{max}. Vectors ζ_i's, solution to Theorem 2, are very close to the limit cycle $\{\rho_i\}_{i \in \mathbb{D}_\nu}$ of the nominal case (for example, with $\delta_{max} = 0.01$, $\| \rho_i - \zeta_i \|_\infty \leq 10^{-3}$), which illustrates Proposition 2. The attractor obtained for the two smallest values of δ_{max} is the union of disjoint ellipsoids, so that the control law converges ultimately to a periodic law, verifying item (ii) of Theorem 2. Lastly, a trajectory of the closed-loop system affected by a perturbation of amplitude $\delta_{max} = 0.01$ is plotted on Fig. 7 (in a semi-log scale). The figure shows the convergence of the state into the attractor and to the limit cycle.

8 Conclusion

A new solution to the problem of stabilizing SAS's has been derived based on a control Lyapunov function. For an a priori selected sequence of modes, LMI conditions, related to existing results on periodic systems, ensure the convergence to a limit cycle characterized by system's matrices and this sequence. Interestingly, this solution is suitable to be extended to the uncertain case, where the notion of limit cycles was necessarily modified. Then, keys to embed optimization problems have been provided, especially to select the optimal cycle or limit cycle that minimizes a given cost function. Finally, several examples have been presented, emphasizing the potential of the contributions.

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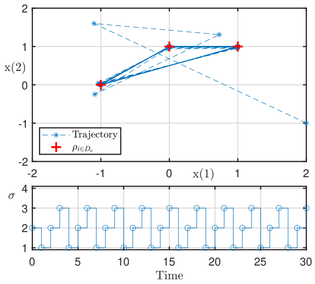
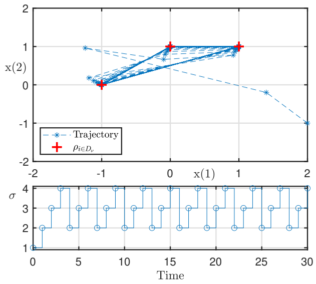
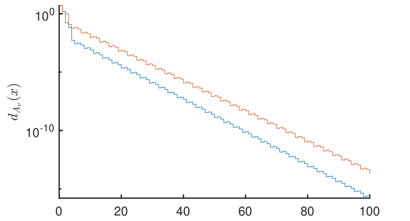


(a) Example 2 with (5).



(b) Example 2 with [16, Th.2].



(c) Evolution of the distance to the attractor $d_{\mathcal{A}_\nu}(x)$ for Example 2 with control law (5) and the control law from [16].

Fig. 4. Comparison of the solutions to Example 2 with the control law obtained with Theorem 1 (left) and with [16, Th.2] and the distance to the same attractor $\mathcal{A}_\nu$. The solution of the closed-loop systems for Example 2 (blue dashed line) has been obtained for $\nu = \{1, 2, 3\}$ with their associated state vector ρ (red crosses) of the limit cycle and the figures at the bottom show the switching control law in each case.

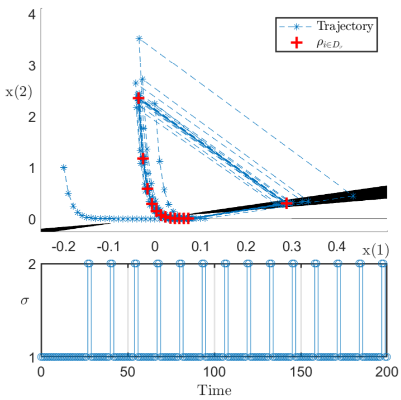


Fig. 5. Example 3: State trajectory and the switching control law.

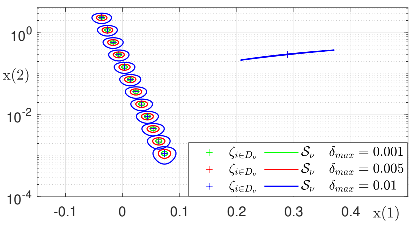


Fig. 6. Example 3: Attractors obtained for several values of δ_{max} for system (27) subject to uncertainties (28) in a logarithmic scale.

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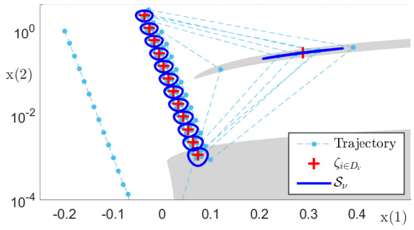

Fig. 7. Example 3: Trajectories of (27)-(28) obtained for a perturbation $\delta_{max} = 0.01$. The grey areas correspond to the regions of the state space, where the control law imposes to select mode 2.

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