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Abstract

This paper deals with the analysis reaction diffusion equation, which can be found in many applications such as in pharmaceutic or chemistry fields. The particularity of the present paper is to study the interconnection of this class of infinite-dimensional systems to a finite-dimensional systems. In this situation, stability is not straightforward to assess any more and one needs to look for dedicated tools to provide accurate numerical tests. Here, the objective is to provide a Lyapunov analysis leading to efficient and scalable stability criteria. This is made possible thanks to the Legendre orthogonal basis which allows building accurate Lyapunov functionals. Indeed this functional is expressed thanks to the state of the finite-dimensional system, the first Fourier-Legendre coefficients and the remainder of the Fourier-Legendre expansion of the infinite-dimensional state. Using this representation, efficient formulation of the Bessel and Wirtinger inequalities are provided leading to sufficient stability conditions expressed in terms of linear matrix inequalities. Numerical examples finally illustrate the accuracy and the potential of the stability result.

Key words: Partial differential equations, Lyapunov stability analysis, Wirtinger and Bessel inequalities, Linear matrix inequalities.

1 Introduction

Robust stability of linear systems has been widely studied since several decades [8,28,30]. In general, the objectives therein are to ensure the stability and performances of linear systems interconnected to several classes of uncertainties, such as unknown parameters [13,25] or nonlinearities [3,36]. These contributions, among many others provide milestones to assess robust stability for a wide class of problems. Among these problems, a lot of attention has been paid to the case of linear systems subject to time-delay uncertainties [11,16]. Nevertheless, this class of systems differs notably from the afore mentioned classes of uncertainties since introducing a delay modify the nature of the systems, giving rise to a class of infinite-dimensional systems. Recently, a lot of attention has indeed been paid to time-delay systems [11,39], and many dedicated tools related to accuracy integral inequalities have been provided [17,26,27,32,33]. More interestingly, these tools have been recently used to assess robust stability of linear systems subject to uncertainties arising from the interconnection with partial differential equations [2,14,35], to cite only a few.

The interest of this latter class of infinite-dimensional systems is motivated by a wide class of applications such as in pharmaceutic or chemistry fields [9] and dedicated tools have been provided. Stability analysis of a sole partial differential equation is already a hard task [4,5,10]. The study is generally carried out by studying the eigenvalues of the infinite dimensional operator if the boundary condition are suitable or by the design of a Lyapunov functional [20,31]. The task becomes drastically complicated if one considers that the partial differential equation is coupled to a finite-dimensional system via its boundary conditions. The calculations of the eigenvalues is obviously non longer relevant and the Lyapunov functional should be adapted to take into account the ordinay differential equation. It is tough task, which has only been studied recently as [1,2,6,14,35]. To rule on the stability of interconnected ordinary-partial differential equations, input-to-state [19,23,24] or Lyapunov [12,29,38] approaches can be followed. For generic cases, quadratic constraints and complete Lyapunov functionals lead to criteria solved by semi-definite programming [7,34,37].

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The objectives of this paper is to provide stability conditions for a class of linear ordinary differential equation coupled with a reaction-diffusion partial differential equation using Robin boundary conditions, the latter being considered for instance in [21]. It is worth noting that stability of this class of systems can be studied using the generic approach provided in [6]. However, this generic solution sometimes lacks of having a deep understanding of the system. Hence, the contribution here is strongly related to [2], even though a reaction term was not considered and other boundary conditions were selected. The novelty of this paper comes from the model transformation arising from the Fourier-Legendre coefficients and remainder, which eases the expression and understanding of the analysis. In particular, Bessel and Wirtinger inequalities can be efficiently rewritten so that its application is simpler compared to [2]. A quadratic Lyapunov function based on this transformed model is provided and leads to an efficient and scalable stability conditions expressed in terms of a linear matrix inequality. Numerical results demonstrate the effectiveness of this approach.

Notations : In this paper, the set of positive integers, real numbers, real positive numbers, matrices of size $n \times m$ and of symmetric positive definite matrices of size n are respectively denoted $\mathbb{N}$, $\mathbb{R}$ $\mathbb{R}_{\geq 0}$, $\mathbb{R}^{n\times m}$ and $\mathbb{S}^n_+$, respectively. The identiy matrix of dimension n is denoted by I_n and $\mathrm{diag}(d_0, ..., d_n)$ stands for the diagonal matrix whose coefficients are $(d_0, ..., d_n)$. For any matrix M, $M^{i,j}$ refers to the coefficient located on the i^{th} row and j^{th} column. For any square matrix M, the transpose matrix is denoted $M^\top$, $\mathcal{H}(M) = M + M^\top$ and $M \succ 0$ means that M is symmetric positive definite. For any square matrix M, $\sigma(M)$ denotes the spectrum of M. Moreover, if M is symmetric, $\underline{\sigma}(M)$ and $\bar{\sigma}(M)$ stands for its lower and larger eigenvalues, respectively. For any scalars $a < b$, the space of square integrable functions $\mathcal{L}^2(a, b; \mathbb{R})$ is associated to the scalar product $\langle z_1|z_2\rangle = \int_a^b z_1(\theta)z_2(\theta)\mathrm{d}\theta$ and the induced norm $\|z\|^2 = \int_a^b z^2(\theta)\mathrm{d}\theta$. Set $\mathcal{H}^1(a, b; \mathbb{R})$ stands for the set of functions z, such that z and $\partial_\theta z$ are in $\mathcal{L}^2(a, b; \mathbb{R})$. With a light abuse of notations, the notation for inner product $\langle z_1|z_2\rangle$ will be used when z_1 and z_2 are vector functions. Therefore, for any z_1 in $\mathcal{L}^2(a, b; \mathbb{R}^n)$ and z_2 in $\mathcal{L}^2(a, b; \mathbb{R}^m)$, notation $\langle z_1|z_2\rangle$ stands for the matrix defined by $\int_a^b z_1(\theta)z_2^\top(\theta)\mathrm{d}\theta$. Consequently, the following equality holds $\langle z_1|z_2\rangle = \langle z_2|z_1\rangle^\top$.

2 Problem formulation

2.1 System modeling

Consider the following system composed of an ordinary differential equation interconnected with a reaction-diffusion partial differential equation with Robin boundary conditions

$$\begin{cases} \dot{x}(t) = Ax(t) + B\left[\begin{smallmatrix} z(t,a) \\ z(t,b) \end{smallmatrix}\right], & \text{(1a)} \\ \partial_t z(t,\theta) = (\delta\partial_{\theta\theta} + \lambda)z(t,\theta),\ \forall\theta\in(a,b), & \text{(1b)} \\ \left[\begin{smallmatrix} \partial_\theta z(t,a) \\ \partial_\theta z(t,b) \end{smallmatrix}\right] = Cx(t) + D\left[\begin{smallmatrix} z(t,a) \\ z(t,b) \end{smallmatrix}\right], & \text{(1c)} \end{cases}$$

for any $t \in \mathbb{R}_{\geq 0}$. Coefficients $\delta > 0$, $\lambda \in \mathbb{R}$, and matrices $A \in \mathbb{R}^{n_x\times n_x}$, $B \in \mathbb{R}^{n_x\times 2}$, $C \in \mathbb{R}^{2\times n_x}$ and diagonal matrix $D \in \mathbb{R}^{2\times 2}$ are supposed to be constant and known.

Remark 1 *Under the initial condition $(x(0), z(0, \theta))$ in $\mathbb{R}^{n_x} \times \mathcal{H}^1(a, b; \mathbb{R})$ which verify the compatibility condition imposed by the boundary condition* (1c)*, system* (1) *is well-posed. It admits a continuous and unique solution $(x(t), z(t, \theta))$ in $\mathbb{R}^{n_x} \times \mathcal{H}^1(a, b; \mathbb{R})$. The proof follows the same arguments highlighted in [21] working on reaction-diffusion equation with Robin boundary conditions.*

Because of the interconnection with the dynamics (1a), the boundary condition of the reaction-diffusion process is no more null nor periodic [22]. Therefore, the eigenbasis of the overall coupled system cannot be given analytically and the behavior of the system becomes difficult to describe. There is then a need for providing efficient tools for the stability analysis of such a class of ordinary-partial differential systems.

2.2 Equilibrium point

As a first step and before studying stability of such a class of systems, it is important to characterize the equilibrium of (1). More particularly, one has to understand under which condition, system (1) admits a unique equilibrium. This is formulated in the next proposition.

Proposition 2 *System* (1) *admits a unique equilibrium, $(\bar{x}, \bar{z}) = (0, 0)$ if and only if*

$$\det\Omega = \det\begin{bmatrix} A & BE_\lambda \\ C & DE_\lambda - E_\lambda F_\lambda \end{bmatrix} \neq 0, \tag{2}$$

where matrices E_λ and F_λ are given by

$$\begin{cases} E_\lambda = \left[\begin{smallmatrix} \cosh(\tilde{\lambda}a) & \sinh(\tilde{\lambda}a) \\ \cosh(\tilde{\lambda}b) & \sinh(\tilde{\lambda}b) \end{smallmatrix}\right], & F_\lambda = \left[\begin{smallmatrix} 0 & \tilde{\lambda} \\ \tilde{\lambda} & 0 \end{smallmatrix}\right], & \textit{if } \lambda < 0; \\ E_\lambda = \left[\begin{smallmatrix} a & 1 \\ b & 1 \end{smallmatrix}\right], & F_\lambda = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right], & \textit{if } \lambda = 0, \\ E_\lambda = \left[\begin{smallmatrix} \cos(\tilde{\lambda}a) & \sin(\tilde{\lambda}a) \\ \cos(\tilde{\lambda}b) & \sin(\tilde{\lambda}b) \end{smallmatrix}\right], & F_\lambda = \left[\begin{smallmatrix} 0 & \tilde{\lambda} \\ -\tilde{\lambda} & 0 \end{smallmatrix}\right], & \textit{if } \lambda > 0, \end{cases}$$

with $\tilde{\lambda} = \sqrt{|\lambda|/\delta}$.

Proof : Let $(\bar{x}, \bar{z})$ be an equilibrium of system (1), meaning that the following relations hold.

$$\begin{cases} A\bar{x} + B\begin{bmatrix} \bar{z}(a) \\ \bar{z}(b) \end{bmatrix} = 0, & \text{(3a)} \\ (\delta\partial_{\theta\theta} + \lambda)\bar{z}(\theta) = 0, & \text{(3b)} \\ \qquad \begin{bmatrix} \partial_\theta \bar{z}(a) \\ \partial_\theta \bar{z}(b) \end{bmatrix} = C\bar{x} + D\begin{bmatrix} \bar{z}(a) \\ \bar{z}(b) \end{bmatrix}. & \text{(3c)} \end{cases}$$

By integration of the differential equation (3b), one obtains that

$$\bar{z}(\theta) = \begin{cases} \left[\cosh(\sqrt{|\lambda|/\delta}\theta) \ \sinh(\sqrt{|\lambda|/\delta}\theta) \right]\left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right], & \text{if } \lambda < 0, \\ \left[\theta \ 1 \right]\left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right], & \text{if } \lambda = 0, \\ \left[\cos(\sqrt{|\lambda|/\delta}\theta) \ \sin(\sqrt{|\lambda|/\delta}\theta) \right]\left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right], & \text{if } \lambda > 0, \end{cases}$$

where $\left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right]$ in $\mathbb{R}^2$ to be fixed. By computing $\partial_\theta \bar{z}$ and re-injecting this expression into (3a) and (3c) it yields $\Omega\left[\begin{smallmatrix} \bar{x} \\ \alpha \\ \beta \end{smallmatrix}\right] = 0$. Hence, system (3) admits a unique solution leading to the trivial equilibrium $(\bar{x}, \bar{z}) = (0, 0)$ if and only if $\det(\Omega) \neq 0$. $\square$

2.3 Objectives

The aim of this paper is to propose a stability criterion for the equilibrium point $(0, 0)$ expressed in terms of linear matrix inequalities. From Lyapunov arguments, one provides a scalable stability analysis for these coupled ordinary-partial differential equations. More specifically, the analysis is elaborated thanks to an accurate Lyapunov functional, which is build using Legendre polynomials. Contrary to a previous study provided in [2], the analysis will be performed through the introduction of the remainder of Fourier-Legendre series, allowing to simplify some technical aspects. Indeed, the use of the remainder allows to rewrite the Wirtinger and the Bessel-Legendre inequalities in a simpler manner compared to the formulation presented in [2] or [33]. Wirtinger's inequality can also be adapted and transformed even if that is neither one nor two boundary conditions set to zero here.

3 Legendre remainder and inequalities

After recalling the basics of Legendre polynomials and some of their properties, this section provides the definition of the Fourier-Legendre coefficients and remainder. In a last step, several relevant inequalities will be presented.

3.1 Fourier-Legendre coefficients and remainder

Legendre polynomials, denoted as l_k for any positive integer k, are given by (see [15])

$$l_k(\theta) = \sum_{i=0}^{k} (-1)^i \frac{(k+i)!}{(i!)^2(k-i)!}\left(\frac{b-\theta}{b-a}\right)^i, \ \forall \theta \in [a, b]. \tag{4}$$

The orthogonal family $\{l_k\}_{k\in\mathbb{N}}$ spans $\mathcal{L}^2(a, b; \mathbb{R})$. For writing comfort, let introduce the notation ℓ_n for any $n \in \mathbb{N}$, which gathers the $n+1$ first Legendre polynomials in vector formulation, that is

$$\ell_n(\theta) = \begin{bmatrix} l_0(\theta) & \dots & l_n(\theta) \end{bmatrix}^\top \in \mathbb{R}^{n+1}. \tag{5}$$

Recall also some important properties of Legendre polynomials [15], that will be useful along the paper, that are

$$\langle \ell_n | \ell_n \rangle = \mathcal{I}_n^{-1}, \quad \partial_\theta \ell_n(\theta) = \mathcal{L}_n \mathcal{I}_n \ell_n(\theta), \tag{6}$$

where matrices $\mathcal{L}_n$ and $\mathcal{I}_n$ are square matrices of dimension $n+1$, given by

$$\begin{aligned} \mathcal{I}_n &= \frac{1}{b-a}\mathrm{diag}(1, \dots, 2n+1), \\ \mathcal{L}_n^{i,j} &= \begin{cases} \left(1 - (-1)^{i+j}\right), & j \le i-1, \\ 0, & j \ge i, \end{cases} \quad \forall i, j \in [\![1, n+1]\!]. \end{aligned} \tag{7}$$

Functions ℓ_n can be easily evaluated at the boundaries of the interval $[a, b]$ and are given by

$$\ell_n(a) = \left[1 \ -1 \ \dots \ (-1)^n \right]^\top, \quad \ell_n(b) = \left[1 \ 1 \ \dots \ 1 \right]^\top. \tag{8}$$

In addition to the previous properties, one emphasizes a property on the matrix $\mathcal{L}_n$, which will be of the highest interest in the next developments.

Proposition 3 *For any integer n, matrix $\mathcal{L}_n$ verifies equality $\mathcal{L}_n + \mathcal{L}_n^\top = \mathcal{B}_n^0 \mathcal{C}_n^0$, where $\mathcal{B}_n^0$ and $\mathcal{C}_n^0$ are given by*

$$\mathcal{B}_n^0 = \left[-\ell_n(a) \ \ell_n(b) \right], \quad \mathcal{C}_n^0 = \left[\ell_n(a) \ \ell_n(b) \right]^\top. \tag{9}$$

Note that matrices $\mathcal{B}_n^0$ and $\mathcal{C}_n^0$ will be widely used in the following section.
As stated in the previous section, we will defined here the main features of this paper in the following definition.

Definition 1 *For any signal $z \in \mathcal{L}^2(a, b; \mathbb{R})$ and for integer n in $\mathbb{N}$, we define*

- *the $(n+1)$ first Fourier-Legendre coefficients of z as*

$$\zeta_n = \langle \ell_n | z \rangle \in \mathbb{R}^{n+1}, \tag{10}$$

- *the associated Fourier-Legendre remainder of z at order n as*

$$w_n(\theta) = z(\theta) - \ell_n^\top(\theta)\mathcal{I}_n\zeta_n, \quad \forall\theta \in [a,b]. \tag{11}$$

It is worth noting that w_n is also in $\mathcal{L}^2(a,b;\mathbb{R})$. The main interest for introducing this remainder is stated in the two following lemmas.

Lemma 4 *For any n in $\mathbb{N}$, the Fourier-Legendre remainder is orthogonal to ℓ_n, i.e. $\langle \ell_n|w_n\rangle = 0$.*

Proof : Thanks to the orthogonality (6) of the Legendre polynomials, re-injecting the definition of w_n into $\langle \ell_n|w_n\rangle$ yields

$$\langle \ell_n|w_n\rangle = \langle \ell_n|z\rangle - \langle \ell_n|\ell_n\rangle \mathcal{I}_n \langle \ell_n|z\rangle = \langle \ell_n|z\rangle - \langle \ell_n|z\rangle = 0,$$

which concludes the proof. □

Lemma 5 *The norm of Fourier-Legendre remainder is given by*

$$\|w_n\|^2 = \|z\|^2 - \zeta_n^\top\mathcal{I}_n\zeta_n, \quad \forall n \in \mathbb{N}. \tag{12}$$

Proof : Re-injecting the definition of w_n into $\|w_n\|$ yields

$$\begin{aligned}\|w_n\|^2 &= \|z - \ell_n^\top(\theta)\mathcal{I}_n\zeta_n\|^2 \\ &= \|z\|^2 - 2\langle \ell_n|z\rangle^\top \mathcal{I}_n\zeta_n + \zeta_n^\top\mathcal{I}_n\langle \ell_n|\ell_n\rangle\mathcal{I}_n\zeta_n.\end{aligned}$$

Thanks to the orthogonality (6) of the Legendre polynomials and recalling that $\langle \ell_n|\ell_n\rangle = \mathcal{I}_n^{-1}$, one can conclude the proof. □

In the next paragraphs, Bessel and Wirtinger inequalities are rewritten in an adequate manner to reduce their conservatism.

3.2 Bessel inequalities

Let us first recall the Bessel-Legendre inequality as stated in [2].

Lemma 6 *For any signal $z \in \mathcal{L}^2(a,b;\mathbb{R})$ and for any integer n in $\mathbb{N}$, the Bessel inequality states that the following inequality holds for any integer n in $\mathbb{N}$*

$$\|z\|^2 \geq \zeta_n^\top\mathcal{I}_n\zeta_n. \tag{13}$$

Proof : The proof is directly derived from (12), where the norm of the remainder $\|w_n\|^2$ is positive. □

The application of Lemma 6 on Fourier-Legendre remainder yields $\|w_n\|^2 \geq \langle \ell_n|w_n\rangle^\top \mathcal{I}_n \langle \ell_n|w_n\rangle = 0$, since the remainder is orthogonal to the $n+1$ first Legendre polynomials, i.e. $\langle \ell_n|w_n\rangle = 0$.

The interest of using the remainder is related to the formulation of this inequality when it is applied to $\|\partial_\theta w_n\|$ that is presented in the next lemma.

Lemma 7 *For any function z in $\mathcal{H}^1(a,b;\mathbb{R})$ and any integer n in $\mathbb{N}$, the remainder w_n of the Fourier-Legendre series of z verifies*

$$\|\partial_\theta w_n\|^2 \geq \frac{1}{b-a}\begin{bmatrix} w_n(a) \\ w_n(b)\end{bmatrix}^\top \Psi_{n+2}\begin{bmatrix} w_n(a) \\ w_n(b)\end{bmatrix}. \tag{14}$$

where, for all integer k, matrix Ψ_k is given by

$$\Psi_k = \begin{bmatrix} k^2 & (-1)^k k \\ (-1)^k k & k^2 \end{bmatrix}. \tag{15}$$

Proof : The proof is given in Appendix A.1. □

It is important to note that the previous lemma allows to express a lower bound on the derivative with respect to θ of the Fourier-Legendre series, which only depends on the evaluation of w_n at the boundary of the interval $[a,b]$. This bound does not depends on the $n+1$ first Fourier-Legendre coefficients, since we have chosen to consider the remainder only, which is orthogonal to the $n+1$ first Legendre polynomial. This will simplify many technical calculations in the next developments.

3.3 Modified Wirtinger's inequality

In the literature [18], Wirtinger's inequalities refer to inequalities which estimate the integral of the derivative function with the help of the integral of the function. These inequalities have been widely used in the context of analysis, control and observation of time-delay and reaction-diffusion systems [31]. In this paper, one uses Wirtinger's inequality of second type, stated as follows.

Lemma 8 *For any function z in $\mathcal{H}^1(a,b;\mathbb{R})$, satisfying $z(a) = z(b) = 0$, inequality $\|\partial_\theta z\| \geq \frac{\pi}{b-a}\|z\|$ holds.*

Proof : The proof is omitted but can be found in [18]. □

The next lemma is an application of the previous Wirtinger inequality to the Fourier-Legendre remainder without requiring any assumption on the boundary conditions on w_n.

Lemma 9 *For any function z in $\mathcal{H}^1(a,b;\mathbb{R})$ and for any $n \geq 1$, the Fourier-Legendre remainder w_n of z verifies*

$$\|\partial_\theta w_n\|^2 - \left(\frac{\pi}{b-a}\right)^2\|w_n\|^2 \geq \frac{1}{b-a}\begin{bmatrix} w_n(a) \\ w_n(b)\end{bmatrix}^\top \Phi_n\begin{bmatrix} w_n(a) \\ w_n(b)\end{bmatrix}, \tag{16}$$

where

$$\Phi_n = \Psi_n + \frac{2\pi^2}{4n^2-1}\begin{bmatrix} 2n & (-1)^{n+1} \\ * & 2n \end{bmatrix}. \tag{17}$$

with Ψ_n *defined in* (15).

Proof : The proof is given in Appendix A.2. □

The main advantages of using the Fourier-Legendre remainder appears in the simple formulation of the lower bound in (16). It is important to stress that the orthogonality condition $\langle \ell_n | w_n \rangle = 0$ drastically simplifies the expression and the calculations. Otherwise the expression would be much more complicated and difficult to employ.

Notice that the previous lemma can be refined when $n \geq 2$ usign the first Wirtinger inequality (under the assumption that $z(a) = z(b)$ and $\int_a^b z(\theta) d\theta = 0$). Even though it reduces the conservatism of the inequality, it has a minor impact on the numerical results.

4 Modeling of an augmented system

Consider (x, z), solution to system (1) and, as previously, let denote $\zeta_n(t)$, for any $n \in \mathbb{N}$, the $n+1$ first Fourier-Legendre coefficients of $z(t)$ as

$$\zeta_n(t) = \langle \ell_n | z(t) \rangle \in \mathbb{R}^{n+1}. \tag{18}$$

These coefficients extract finite-dimensional information from $z(t)$ and we are able to define its Fourier-Legendre remainder as follows

$$w_n(t,\theta) = z(t,\theta) - \ell_n^\top(\theta)\mathcal{I}_n\zeta_n(t), \quad \forall (t,\theta) \in \mathbb{R}_{\geq 0} \times [a,b]. \tag{19}$$

We have seen in Lemma 4 that this remainder is orthogonal to the $n+1$ first Legendre polynomials for any $t \geq 0$. The objective of this section is to rewrite system (1) by exhibiting a finite-dimensional part composed of the $\xi_n = \left[\begin{smallmatrix}\zeta_n \\ x\end{smallmatrix}\right]$ and an infinite-dimensional part represented by the Fourier-Legendre remainder w_n. This is formulated in the following proposition.

Proposition 10 *If* (x, z) *is a solution of system* (1)*, then* $(\xi_n = \left[\begin{smallmatrix}\zeta_n \\ x\end{smallmatrix}\right], w_n)$ *defined by* (18),(19) *verifies the following dynamics*

$$\begin{cases} \dot{\xi}_n(t) = \begin{bmatrix} \mathbf{A}_n & \mathbf{B}_n \end{bmatrix} \begin{bmatrix} \xi_n(t) \\ \begin{bmatrix} w_n(t,a) \\ w_n(t,b) \end{bmatrix} \end{bmatrix}, & \text{(20a)} \\ \partial_t w_n(t,\theta) = (\delta\partial_{\theta\theta} + \lambda) w_n(t,\theta) & \text{(20b)} \\ \qquad - \delta \ell_n^\top(\theta)\mathcal{I}_n \begin{bmatrix} \mathbf{E}_n & \mathbf{F}_n \end{bmatrix} \begin{bmatrix} \xi_n(t) \\ \begin{bmatrix} w_n(t,a) \\ w_n(t,b) \end{bmatrix} \end{bmatrix}, & \\ \begin{bmatrix} \partial_\theta w_n(t,a) \\ \partial_\theta w_n(t,b) \end{bmatrix} = \begin{bmatrix} \mathbf{C}_n & \mathbf{D}_n \end{bmatrix} \begin{bmatrix} \xi_n(t) \\ \begin{bmatrix} w_n(t,a) \\ w_n(t,b) \end{bmatrix} \end{bmatrix}. & \text{(20c)} \end{cases}$$

where the matrices that defined this model are given by

$$\begin{aligned} \mathbf{A}_n &= \begin{bmatrix} \mathcal{A}_n & \delta\mathcal{B}_n^0 C \\ B\mathcal{C}_n^0\mathcal{I}_n & A \end{bmatrix}, & \mathbf{B}_n &= \begin{bmatrix} \delta\mathcal{B}_n^1 \\ B \end{bmatrix}, \\ \mathbf{C}_n &= \begin{bmatrix} \mathcal{C}_n^1\mathcal{I}_n & C \end{bmatrix}, & \mathbf{D}_n &= D, \\ \mathbf{E}_n &= \mathcal{B}_n^0 \begin{bmatrix} \mathcal{C}_n^1\mathcal{I}_n & C \end{bmatrix}, & \mathbf{F}_n &= \mathcal{B}_n^1, \end{aligned} \tag{21}$$

with matrices $\mathcal{B}_n^0$ *and* $\mathcal{C}_n^0$ *are given in* (9) *and*

$$\begin{aligned} &\mathcal{A}_n = \delta(\mathcal{L}_n^\top\mathcal{I}_n)^2 + \delta\mathcal{B}_n^0\mathcal{C}_n^1\mathcal{I}_n + \lambda I_{n+1}, \\ &\mathcal{B}_n^1 = \mathcal{B}_n^0 D - \mathcal{L}_n\mathcal{I}_n\mathcal{B}_n^0, \quad \mathcal{C}_n^1 = D\mathcal{C}_n^0 - \mathcal{C}_n^0\mathcal{I}_n\mathcal{L}_n^\top. \end{aligned} \tag{22}$$

Proof : For the sake of simplicity, the time argument in the following equations is omitted. The proof is also split into three parts referring to each equation in (20).

<u>Proof of (20c):</u> The objective is to rewrite the boundary condition (1c) on z as boundary conditions on w_n as in (20c), for any solution (x, z) of (1) and any n in $\mathbb{N}$. Using equation (19), one get

$$\begin{bmatrix} z(a) \\ z(b) \end{bmatrix} = \begin{bmatrix} \ell_n^\top(a) \\ \ell_n^\top(b) \end{bmatrix}\mathcal{I}_n\zeta_n + \begin{bmatrix} w_n(a) \\ w_n(b) \end{bmatrix} = \mathcal{C}_n^0\mathcal{I}_n\zeta_n + \begin{bmatrix} w_n(a) \\ w_n(b) \end{bmatrix}, \tag{23}$$

where matrix $\mathcal{C}_n^0$ is defined in (9). Furthermore, differentiating w_n in (11) with respect to θ and evaluating it at $\theta \in \{a, b\}$ yields

$$\begin{bmatrix} \partial_\theta z(a) \\ \partial_\theta z(b) \end{bmatrix} = \mathcal{C}_n^0\mathcal{I}_n\mathcal{L}_n^\top\mathcal{I}_n\zeta_n + \begin{bmatrix} \partial_\theta w_n(a) \\ \partial_\theta w_n(b) \end{bmatrix}. \tag{24}$$

The injection of equations (23),(24) into the boundary condition $\left[\begin{smallmatrix}\partial_\theta z(a) \\ \partial_\theta z(b)\end{smallmatrix}\right] = Cx + D\left[\begin{smallmatrix} z(a) \\ z(b)\end{smallmatrix}\right]$ leads to the result

$$\begin{aligned} \begin{bmatrix} \partial_\theta w_n(a) \\ \partial_\theta w_n(b) \end{bmatrix} &= \underbrace{(D\mathcal{C}_n^0 - \mathcal{C}_n^0\mathcal{I}_n\mathcal{L}_n^\top)}_{\mathcal{C}_n^1}\mathcal{I}_n\zeta_n + Cx + D\begin{bmatrix} w_n(a) \\ w_n(b) \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} \mathcal{C}_n^1\mathcal{I}_n & C \end{bmatrix}}_{\mathbf{C}_n}\underbrace{\begin{bmatrix} \zeta_n \\ x \end{bmatrix}}_{\xi_n} + \underbrace{D}_{\mathbf{D}_n}\begin{bmatrix} w_n(a) \\ w_n(b) \end{bmatrix}. \end{aligned} \tag{25}$$

<u>Proof of (20a):</u> Let us now derive an expression of the dynamics of the Fourier-Legendre coefficients. Recalling that $\zeta_n = \langle \ell_n | z \rangle$ and that z verifies (1b), we have

$$\dot{\zeta}_n = \langle \ell_n | \partial_t z \rangle = \langle \ell_n | (\delta\partial_{\theta\theta} + \lambda) z \rangle = \delta \langle \ell_n | \partial_{\theta\theta} z \rangle + \lambda\zeta_n.$$

Two successive integrations by parts on the first term of the last expression yields

$$\dot{\zeta}_n = \left(\delta(\mathcal{L}_n\mathcal{I}_n)^2 + \lambda I_{n+1}\right)\zeta_n + \delta\mathcal{B}_n^0\begin{bmatrix} \partial_\theta z(a) \\ \partial_\theta z(b) \end{bmatrix} - \delta\mathcal{L}_n\mathcal{I}_n\mathcal{B}_n^0\begin{bmatrix} z(a) \\ z(b) \end{bmatrix}, \tag{26}$$

with $\mathcal{B}_n^0$ defined in (9). Reinjecting equations (23),(24) into the previous dynamics (26) leads to

$$\dot{\zeta}_n = (\delta\mathcal{G}_n + \lambda I_{n+1})\,\zeta_n + \delta\mathcal{B}_n^0 \begin{bmatrix} \partial_\theta w_n(a) \\ \partial_\theta w_n(b) \end{bmatrix} - \delta\mathcal{L}_n\mathcal{I}_n\mathcal{B}_n^0 \begin{bmatrix} w_n(a) \\ w_n(b) \end{bmatrix}, \tag{27}$$

with

$$\mathcal{G}_n = (\mathcal{L}_n\mathcal{I}_n)^2 + \mathcal{B}_n^0\mathcal{C}_n^0\mathcal{I}_n\mathcal{L}_n^\top\mathcal{I}_n - \mathcal{L}_n\mathcal{I}_n\mathcal{B}_n^0\mathcal{C}_n^0\mathcal{I}_n = (\mathcal{L}_n^\top\mathcal{I}_n)^2,$$

where the last equality is obtained by recalling from Proposition 3 that $\mathcal{B}_n^0\mathcal{C}_n^0 = \mathcal{L}_n + \mathcal{L}_n^\top$. Then, by imposing the boundary condition (25) to the previous dynamics and by reorganization of the terms, we get

$$\begin{aligned} \dot{\zeta}_n &= \underbrace{\left(\delta(\mathcal{L}_n^\top\mathcal{I}_n)^2 + \delta\mathcal{B}_n^0\mathcal{C}_n^1\mathcal{I}_n + \lambda I_{n+1}\right)}_{\mathcal{A}_n} \zeta_n + \delta\mathcal{B}_n^0 C x \\ &\quad + \delta\underbrace{(\mathcal{B}_n^0 D - \mathcal{L}_n\mathcal{I}_n\mathcal{B}_n^0)}_{\mathcal{B}_n^1} \begin{bmatrix} w_n(a) \\ w_n(b) \end{bmatrix}. \end{aligned} \tag{28}$$

To obtain (20a), it remains to add the dynamics of the ordinary differential equation given by (1a). The dynamics of the finite dimensional state are therefore

$$\underbrace{\begin{bmatrix} \dot{\zeta}_n \\ \dot{x} \end{bmatrix}}_{\dot{\xi}_n} = \underbrace{\begin{bmatrix} \mathcal{A}_n & \delta\mathcal{B}_n^0 C \\ B\mathcal{C}_n^0\mathcal{I}_n & A \end{bmatrix}}_{\mathbf{A}_n} \underbrace{\begin{bmatrix} \zeta_n \\ x \end{bmatrix}}_{\xi_n} + \underbrace{\begin{bmatrix} \delta\mathcal{B}_n^1 \\ B \end{bmatrix}}_{\mathbf{B}_n} \begin{bmatrix} w_n(a) \\ w_n(b) \end{bmatrix}, \tag{29}$$

which corresponds to the first equation (20a).

Proof of (20b): To do so, differentiating with respect to time of the Fourier-Legendre remainder w_n given in (11) yields $\partial_t w_n(\theta) = \partial_t z(\theta) - \ell_n^\top(\theta)\mathcal{I}_n\dot{\zeta}_n$. From one side, we need to express $\partial_t z$ using the new system of coordinates, that is reflected in

$$\begin{aligned} \partial_t z(\theta) &= \delta\partial_{\theta\theta} z(\theta) + \lambda z(\theta) \\ &= (\delta\partial_{\theta\theta} + \lambda) w_n(\theta) + \left(\delta\partial_{\theta\theta}\ell_n^\top(\theta) + \lambda\ell_n^\top(\theta)\right)\mathcal{I}_n\zeta_n. \end{aligned}$$

Applying twice the differentiation rules of the Legendre polynomials in (6), the previous expression resumes to

$$\partial_t z(\theta) = (\delta\partial_{\theta\theta} + \lambda) w_n(\theta) + \ell_n^\top(\theta)\mathcal{I}_n \left(\delta(\mathcal{L}_n^\top\mathcal{I}_n)^2 + \lambda I_{n+1}\right)\zeta_n. \tag{30}$$

On the other side, the expression of $\dot{\zeta}_n$ given by (28) leads to

$$\begin{aligned} \ell_n^\top(\theta)\mathcal{I}_n\dot{\zeta}_n &= \ell_n^\top(\theta)\mathcal{I}_n \left(\delta(\mathcal{L}_n^\top\mathcal{I}_n)^2 + \lambda I_{n+1}\right)\zeta_n \\ &\quad + \delta\ell_n^\top(\theta)\mathcal{I}_n \left(\underbrace{\mathcal{B}_n^0 \begin{bmatrix} \mathcal{C}_n^1\mathcal{I}_n & C \end{bmatrix}}_{\mathbf{E}_n} \begin{bmatrix} x \\ \zeta_n \end{bmatrix} + \underbrace{\mathcal{B}_n^1}_{\mathbf{F}_n} \begin{bmatrix} w_n(a) \\ w_n(b) \end{bmatrix} \right). \end{aligned} \tag{31}$$

Thus, collecting equations (30),(31) and simplifying the term $\left(\delta(\mathcal{L}_n^\top\mathcal{I}_n)^2 + \lambda I_{n+1}\right)\zeta_n$, one recognizes the partial differential equation verified by w_n. $\square$

Remark 1 *In the new formulation, the reaction-diffusion equation, which characterizes the dynamics of w_n, is similar to the one of the original system. The only difference relies on the last term in* (20b)*. Even though, it seems at a first sight more complicated, it will appear in the next developments that this new term has no impact on the complexity of the analysis. This is due to the orthogonality of this new term and the Fourier-Legendre remainder, w_n.*

5 Stability analysis

This section is dedicated to the construction of a numerical tractable stability criterion for system (1), based on Lemmas 7 and 9 and highly related to the properties of the augmented model (20).

Theorem 11 *For a given integer $n \geq 1$ and any λ, δ satisfying $\frac{\lambda}{\delta} < \frac{\pi^2}{(b-a)^2}$, if there exists $\mathbf{P}_n \in \mathbb{S}_+^{n_x+n+1}$ such that the linear matrix inequality*

$$\Xi_n = \begin{bmatrix} \mathcal{H}(\mathbf{P}_n\mathbf{A}_n) & \mathbf{P}_n\mathbf{B}_n + \mathbf{C}_n^\top G \\ * & \mathcal{H}(\mathbf{D}_n^\top G) - \frac{(1-\kappa)\Psi_{n+2} + \kappa\Phi_n}{b-a} \end{bmatrix} \prec 0, \tag{32}$$

is satisfied, where matrices Ψ_n, Φ_n are defined in (15)*, $\mathbf{A}_n$, $\mathbf{B}_n$, $\mathbf{C}_n$ and $\mathbf{D}_n$ in* (21) *and*

$$G = \frac{1}{2}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \kappa = \max\left(0, \frac{\lambda}{\delta}\left(\frac{b-a}{\pi}\right)^2\right).$$

Then, under condition $\det(\Omega) \neq 0$ *from Proposition 2, the equilibrium* $(0,0)$ *is globally exponentially stable for system* (1)*, in the sense of the* $\mathbb{R}^{n_x} \times \mathcal{L}^2(a,b;\mathbb{R})$ *norm.*

Proof : For the sake of simplicity and compactness, we omit the time argument in the following proof. Consider the Lyapunov candidate functional

$$\mathcal{V}_n(x,z) = \underbrace{\xi_n^\top\mathbf{P}_n\xi_n}_{\mathcal{V}_n^a(x,z)} + \underbrace{(2\delta)^{-1}\|w_n\|^2}_{\mathcal{V}_n^b(x,z)}. \tag{33}$$

Assuming $\mathbf{P}_n \succ 0$, it suffices to take positive real numbers $\epsilon_1 = \min\left(\frac{1}{2n+1}\underline{\sigma}(\mathbf{P}_n), \frac{1}{2\delta}\right)$ and $\epsilon_2 = \max\left(\bar{\sigma}(\mathbf{P}_n), \frac{1}{2\delta}\right)$ to obtain

$$\begin{aligned} \mathcal{V}_n(x,z) &\geq \epsilon_1\left(x^\top x + \zeta_n^\top\mathcal{I}_n\zeta_n + \|w_n\|^2\right), \\ \mathcal{V}_n(x,z) &\leq \epsilon_2\left(x^\top x + \zeta_n^\top\mathcal{I}_n\zeta_n + \|w_n\|^2\right), \end{aligned}$$

which can be rewritten from Lemma 5 as

$$\epsilon_1\left(x^\top x + \|z\|^2\right) \leq \mathcal{V}_n(x,z) \leq \epsilon_2\left(x^\top x + \|z\|^2\right). \tag{34}$$

It remains showing that there exists $\epsilon_3 > 0$ such that

$$\dot{\mathcal{V}}_n(x,z) \le -\epsilon_3 \left(x^\top x + \|z\|^2\right). \tag{35}$$

From one part, differentiation $\mathcal{V}_n^a$ along the trajectories of the system (1) using the dynamics given by (20) yields

$$\dot{\mathcal{V}}_n^a(x,z) = \xi_n^\top \mathcal{H}(\mathbf{P}_n\mathbf{A}_n)\xi_n + 2\xi_n^\top \mathbf{P}_n\mathbf{B}_n \left[\begin{smallmatrix} w_n(a) \\ w_n(b)\end{smallmatrix}\right].$$

From the other part, from the dynamics of the Fourier-Legendre remainder in (20), we recover

$$\dot{\mathcal{V}}_n^b(x,z) = \int_a^b \partial_{\theta\theta} w_n(\theta) w_n(\theta) \mathrm{d}\theta + \frac{\lambda}{\delta}\int_a^b w_n^2(\theta)\mathrm{d}\theta.$$

Using integrations by parts, $\dot{\mathcal{V}}_n^b$ is decomposed in $\mathcal{L}^2(a,b;\mathbb{R})$ norms of signals w and $\partial_\theta w$ as

$$\begin{aligned}\dot{\mathcal{V}}_n^b(x,z) &= \tfrac{\lambda}{\delta}\|w_n\|^2 - \|\partial_\theta w_n\|^2 + 2\left[\begin{smallmatrix} \partial_\theta w_n(a) \\ \partial_\theta w_n(b)\end{smallmatrix}\right]^\top G \left[\begin{smallmatrix} w_n(a) \\ w_n(b)\end{smallmatrix}\right], \\ &= \tfrac{\lambda}{\delta}\|w_n\|^2 - \|\partial_\theta w_n\|^2 + 2\left[\begin{smallmatrix} \xi_n \\ w_n(a) \\ w_n(b)\end{smallmatrix}\right]^\top \left[\begin{smallmatrix} \mathbf{C}_n^\top \\ \mathbf{D}_n^\top\end{smallmatrix}\right] G \left[\begin{smallmatrix} w_n(a) \\ w_n(b)\end{smallmatrix}\right].\end{aligned}$$

Thereafter, the proof is split into two cases. If $\lambda < 0$, the proof is a straightforward application of Bessel-Legendre inequality at order $n+1$ given by (14). Indeed, we have

$$\dot{\mathcal{V}}_n(x,z) \le -\frac{|\lambda|}{\delta}\|w_n\|^2 + \left[\begin{smallmatrix} \xi_n \\ w_n(a) \\ w_n(b)\end{smallmatrix}\right]^\top \Xi_n \left[\begin{smallmatrix} \xi_n \\ w_n(a) \\ w_n(b)\end{smallmatrix}\right],$$

Assuming $\Xi_n \prec 0$ and taking $\epsilon_3 = \min\left(\frac{|\lambda|}{\delta}, |\bar{\sigma}(\Xi_n)|\right)$, one obtains (35), which leads to Theorem 11.

For the case $0 \le \lambda < \delta \frac{\pi^2}{(b-a)^2}$, we apply first the adapted Wirtinger inequality (16) on the Fourier-Legendre remainder w_n to obtain

$$\begin{aligned}\dot{\mathcal{V}}_n^b(x,z) &\le -(1-\kappa-\rho)\|\partial_\theta w_n\|^2 - \rho\|w_n\|^2 \\ &+2\left[\begin{smallmatrix} \xi_n \\ w_n(a) \\ w_n(b)\end{smallmatrix}\right]^\top \left[\begin{smallmatrix} \mathbf{C}_n^\top \\ \mathbf{D}_n^\top\end{smallmatrix}\right] G \left[\begin{smallmatrix} w_n(a) \\ w_n(b)\end{smallmatrix}\right] - \tfrac{\kappa+\rho}{b-a}\left[\begin{smallmatrix} w_n(a) \\ w_n(b)\end{smallmatrix}\right]^\top \Phi_n \left[\begin{smallmatrix} w_n(a) \\ w_n(b)\end{smallmatrix}\right].\end{aligned}$$

with $\kappa = \frac{\lambda}{\delta}\left(\frac{b-a}{\pi}\right)^2$ and for any sufficiently small $\rho > 0$ such that $(1-\kappa-\rho) > 0$. By application of the Bessel-Legendre inequality given by (14), we finally get

$$\begin{aligned}\dot{\mathcal{V}}_n^b(x,z) &\le -\rho\|w_n\|^2 + 2\left[\begin{smallmatrix} \xi_n \\ w_n(a) \\ w_n(b)\end{smallmatrix}\right]^\top \left[\begin{smallmatrix} \mathbf{C}_n^\top \\ \mathbf{D}_n^\top\end{smallmatrix}\right] G \left[\begin{smallmatrix} w_n(a) \\ w_n(b)\end{smallmatrix}\right] \\ &-\tfrac{1}{b-a}\left[\begin{smallmatrix} w_n(a) \\ w_n(b)\end{smallmatrix}\right]^\top ((1-\kappa-\rho)\Psi_{n+2} + (\kappa+\rho)\Phi_n)\left[\begin{smallmatrix} w_n(a) \\ w_n(b)\end{smallmatrix}\right].\end{aligned}$$

Merged with $\dot{\mathcal{V}}_n^a$, it gives that $\dot{\mathcal{V}}_n$ is upper bounded by

$$-\rho\|w_n\|^2 + \left[\begin{smallmatrix} \xi_n \\ w_n(a) \\ w_n(b)\end{smallmatrix}\right]^\top \Xi_n \left[\begin{smallmatrix} \xi_n \\ w_n(a) \\ w_n(b)\end{smallmatrix}\right] + \frac{\rho}{b-a}\left[\begin{smallmatrix} w_n(a) \\ w_n(b)\end{smallmatrix}\right]^\top \Psi_{n+2}\left[\begin{smallmatrix} w_n(a) \\ w_n(b)\end{smallmatrix}\right].$$

If the linear matrix inequality $\Xi_n \prec 0$, it is possible to take ρ small enough such that $-|\bar{\sigma}(\Xi_n)| + \frac{\rho}{b-a}\bar{\sigma}(\Psi_{n+2}) < 0$. Then, there exists $\epsilon_3 = \min(\rho, |\bar{\sigma}(\Xi_n)|) > 0$ such that the derivatives $\dot{\mathcal{V}}_n$ satisfies (35). One concludes by application of Lyapunov theorem on the exponential stability of the equilibrium point. □

Remark 12 *Having matrix $\mathbf{A}_n$ Hurwitz is a necessary condition for the feasibility of linear matrix inequality $\Xi_n \prec 0$ where Ξ_n is defined by* (32).

Remark 13 *Notice also that* (33) *is also a Lyapunov functional for system* (20).

Remark 14 *Matrices $\mathbf{E}_n$ and $\mathbf{F}_n$ are not involved in the linear matrix inequality condition of Theorem 11. This is due to the orthogonality of the last term in* (20b) *as already mentioned in Remark 1.*

Remark 15 *Compared to generalized linear matrix inequality formulations based on sum of squares [7,34], the result is condensed, more appropriate to the application and numerical burden are improved. This optimization is due to the transformations made to obtain our linear matrix inequality, highly correlated to the system under study. Nevertheless, to the best of our knowledge, we are not aware of stability condition adressing this particular class of system and this is the reason why no further comparison is presented.*

6 Application to numerical examples

In this section, we will consider two examples, both of them with $\delta = 1$ and $b - a = 1$. The other parameters are given below.

Example 16 *Consider system* (1) *with*

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -10-K & 10 & 0 & 0 \\ 5 & -15 & 0 & -0.25\end{bmatrix},\ B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0\end{bmatrix},\ C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ K & 0 & 0 & 0\end{bmatrix}.$$

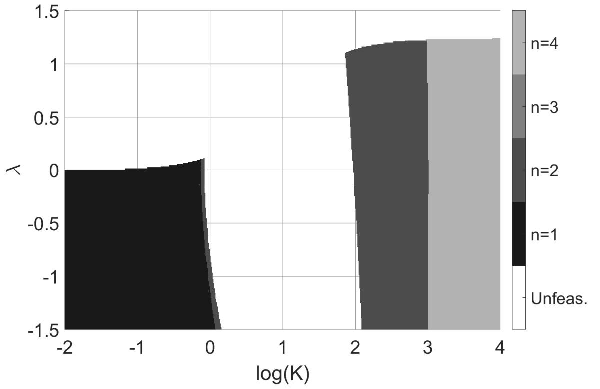
$D = 0$ and $K \in [10^{-2}, 10^4]$.

Example 17 *Consider system* (1) *with*

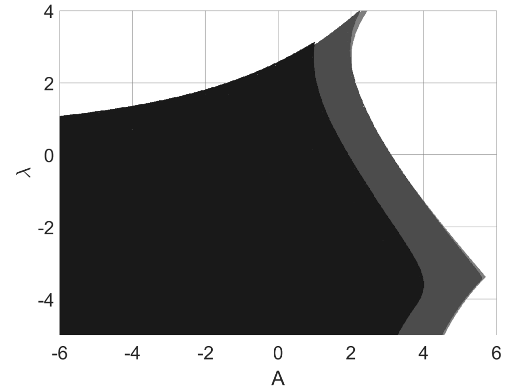
$$A \in [-6, 6],\quad B = [\,10\ 0\,],\quad C = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right],\quad D = 0, \tag{36}$$

$$A = -1,\quad B = [\,K\ 0\,],\quad K \in [10^{-2}, 10^4]. \tag{37}$$

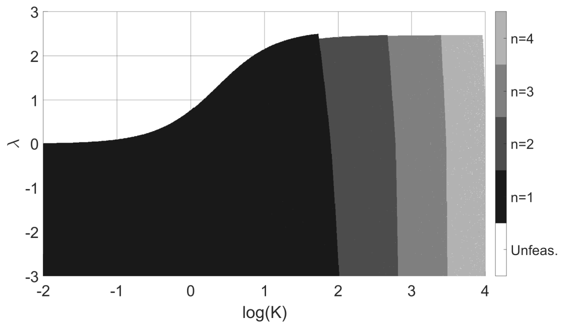
First of all, Fig. 1 illustrates the stability areas with respect to parameters A, K, λ provided by Theorem 11. For each example, an indication on the range of reaction parameter λ which stabilizes the interconnected system is obtained. For instance, Fig. 1a and 1c show that the stability region is $\lambda < 0$ for $K \to 0$. Indeed, the case



(a) Example 16.



(b) Example 17 with (36).



(c) Example 17 with (37).

Fig. 1. Stability regions guaranteed by Theorem 11 for several values of order n.

$K = 0$ amounts to have the partial differential equation in cascade with the ordinary differential equation. Then, the eigenvalues of the coupled system are contained into $\sigma(A) \cup \left\{ \lambda - k\delta \left(\frac{\pi}{b-a} \right)^2 \right\}_{k \in \mathbb{N}}$. In addition, Fig. 1b emphasizes that system (36) with $A = -1$ is stable for any $\lambda < 2.15$. Similar calculations could be done when modifying parameter δ, but is not presented here.

Secondly, it is worth noticing that neither stability of the ordinary differential equation nor of the partial differential equation is required for the stability of the overall interconnection. For Example 17, it is even possible to have both equation independently unstable and a stable interconnection (see $A = 1$, $B = 10$ and $\lambda = 1$).

Lastly, the feasibility of linear matrix inequality (32) is determined with feasp function and tested from $n = 1$ to $n = 4$. Notice that higher orders are required to detect stable area with large values of parameter K.

7 Conclusions

In this paper we have presented a scalable stability condition for a linear finite-dimensional system interconnected to a reaction-diffusion partial differential equation with Robin boundary conditions. The method emphasizes the role of the Fourier-Legendre coefficients and of the remainder of the Fourier-Legendre series. Thanks to this modeling, an efficient formulation of the Wirtinger and Bessel inequalities has been provided leading then, together with a Lyapunov approach, to a stability test expressed in terms of linear matrix inequalities. This approach has been evaluated on several numerical example demonstrating its potential.

Further works would consider for instance other boundary conditions and more generally other partial differential equations. Another direction for future research would be to introduce delay through a transport differential equation. Last but not least, this paper can be seen as a milestone for the design of finite-dimensional stabilizing controllers for the reaction-diffusion equation, enlightening even more the role of the Fourier-Legendre coefficients.

A Appendix

A.1 Proofs of Lemma 7

Proof : Thanks to the Bessel-Legendre inequality (13) at order $n + 1$, the following inequality holds

$$\|\partial_\theta w_n\|^2 \geq \langle \mathbf{l}_{n+1} | \partial_\theta w_n \rangle^\top \mathcal{I}_{n+1} \langle \mathbf{l}_{n+1} | \partial_\theta w_n \rangle . \qquad \text{(A.1)}$$

In addition, performing an integration by parts yields

$$\langle \mathbf{l}_{n+1} | \partial_\theta w_n \rangle = \mathbf{l}_{n+1}(b) w_n(b) - \mathbf{l}_{n+1}(a) w_n(a) - \langle \partial_\theta \mathbf{l}_{n+1} | w_n \rangle .$$

Then, we recall that w_n is the remainder of the Fourier-Legendre series, which is consequently orthogonal to the $n+1$ first Legendre polynomials. Therefore, the last term of the previous equality is zero ($\langle \partial_\theta \mathbf{l}_{n+1} | w_n \rangle = 0$) so that

$$\|\partial_\theta w_n\|^2 \geq \begin{bmatrix} w_n(a) \\ w_n(b) \end{bmatrix}^\top \begin{bmatrix} -\mathbf{l}_{n+1}^\top(a) \\ \mathbf{l}_{n+1}^\top(b) \end{bmatrix} \mathcal{I}_{n+1} \begin{bmatrix} -\mathbf{l}_{n+1}^\top(a) \\ \mathbf{l}_{n+1}^\top(b) \end{bmatrix}^\top \begin{bmatrix} w_n(a) \\ w_n(b) \end{bmatrix} .$$

To complete the proof, it remains to compute the components of the matrix. Hence, let us first note that $\mathbf{l}_{n+1}^\top(a)\mathcal{I}_{n+1}\mathbf{l}_{n+1}(a) = \mathbf{l}_{n+1}^\top(b)\mathcal{I}_{n+1}\mathbf{l}_{n+1}(b)$. Then, performing the matrix multiplication, we have

$$\mathbf{l}_{n+1}^\top(a)\mathcal{I}_{n+1}\mathbf{l}_{n+1}(a) = \sum_{k=0}^{n+1} \frac{(2k+1)}{b-a} = \frac{(n+2)^2}{b-a}.$$

Similarly, we have

$$\mathbf{l}_{n+1}^\top(a)\mathcal{I}_{n+1}\mathbf{l}_{n+1}(b) = \sum_{k=0}^{n+1}(-1)^k \frac{(2k+1)}{b-a} = (-1)^{n+1}\frac{(n+2)}{b-a},$$

which yields the result. □

A.2 Proof of Lemma 9

Proof : The proof follows the idea developed in [32], to derive the Wirtinger-based integral inequality. Let us introduce function $\tilde{w}_n$, defined for all $\theta \in [a,b]$ such that

$$\partial_\theta \tilde{w}_n(\theta) = \partial_\theta w_n(\theta) - \mathbf{l}_{n-1}^\top(\theta)\mathcal{I}_{n-1}\langle \mathbf{l}_{n-1}|\partial_\theta w_n\rangle.$$

Integrating this function and performing several simplifications, function $\tilde{w}_n$ is given by

$$\begin{aligned}\tilde{w}_n(\theta) &= w_n(\theta) - l_{n-1}(\theta)(\tfrac{w_n(b)-(-1)^n w_n(a)}{2}) \\ &\quad - l_n(\theta)(\tfrac{w_n(b)-(-1)^{n+1}w_n(a)}{2}).\end{aligned} \tag{A.2}$$

First, one has to verify the assumptions of the Wirtinger inequality in Lemma 8, that is $\tilde{w}_n(a) = \tilde{w}_n(b) = 0$. Recalling that $l_{n-1}(b) = l_n(b) = 1$, evaluating $\tilde{w}_n(b)$ writes

$$\begin{aligned}\tilde{w}_n(b) &= w_n(b) - \tfrac{w_n(b)-(-1)^n w_n(a)}{2} - \tfrac{w_n(b)-(-1)^{n+1}w_n(a)}{2}, \\ &= w_n(b) - w_n(b) = 0.\end{aligned}$$

Similarly, recalling that $l_{n-1}(a) = -l_n(a) = (-1)^{n-1}$, we have $\tilde{w}_n(a) = w_n(a) - w_n(a) = 0$.
Therefore, Wirtinger's inequality [18] states that, under the two previous boundary conditions, the inequality $\|\partial_\theta \tilde{w}_n\| \geq \frac{\pi}{b-a}\|\tilde{w}_n\|$ holds. It remains to commute $\|\partial_\theta \tilde{w}_n\|$ and $\|\tilde{w}_n\|$. We first note that

$$\begin{aligned}\|\partial_\theta \tilde{w}_n\|^2 &= \|\partial_\theta w_n(\theta) - \mathbf{l}_{n-1}^\top(\theta)\mathcal{I}_{n-1}\langle \mathbf{l}_{n-1}|\partial_\theta w_n\rangle\|^2, \\ &= \|\partial_\theta w_n\|^2 - 2\langle \mathbf{l}_{n-1}|\partial_\theta w_n\rangle^\top \mathcal{I}_{n-1}\langle \mathbf{l}_{n-1}|\partial_\theta w_n\rangle \\ &\quad + \langle \mathbf{l}_{n-1}|\partial_\theta w_n\rangle^\top \mathcal{I}_{n-1}\langle \mathbf{l}_{n-1}|\mathbf{l}_{n-1}\rangle \mathcal{I}_{n-1}\langle \mathbf{l}_{n-1}|\partial_\theta w_n\rangle.\end{aligned}$$

Using the ortogonality of the Legendre polynomials given in (6), we have

$$\begin{aligned}\|\partial_\theta \tilde{w}_n\|^2 &= \|\partial_\theta w_n\|^2 - 2\langle \mathbf{l}_{n-1}|\partial_\theta w_n\rangle^\top \mathcal{I}_{n-1}\langle \mathbf{l}_{n-1}|\partial_\theta w_n\rangle \\ &\quad + \langle \mathbf{l}_{n-1}|\partial_\theta w_n\rangle^\top \mathcal{I}_{n-1}\mathcal{I}_{n-1}^{-1}\mathcal{I}_{n-1}\langle \mathbf{l}_{n-1}|\partial_\theta w_n\rangle, \\ &= \|\partial_\theta w_n\|^2 - \langle \mathbf{l}_{n-1}|\partial_\theta w_n\rangle^\top \mathcal{I}_{n-1}\langle \mathbf{l}_{n-1}|\partial_\theta w_n\rangle.\end{aligned}$$

The last term of the previous expression has already been computed in (A.1), and we have

$$\|\partial_\theta \tilde{w}_n\|^2 = \|\partial_\theta w_n\|^2 - \frac{1}{b-a}\begin{bmatrix} w_n(a) \\ w_n(b)\end{bmatrix}^\top \Psi_n \begin{bmatrix} w_n(a) \\ w_n(b)\end{bmatrix}. \tag{A.3}$$

On the other hand, the norm of $\tilde{w}_n$ can be computed as follows. For the sake of simplicity, let us introduce the following notations

$$\omega_{n,a} = \tfrac{w_n(b)-(-1)^n w_n(a)}{2}, \quad \omega_{n,b} = \tfrac{w_n(b)-(-1)^{n+1}w_n(a)}{2},$$

so that we have

$$\begin{aligned}\|\tilde{w}_n\|^2 &= \|w_n(\theta) - l_{n-1}(\theta)\omega_{n,a} - l_n(\theta)\omega_{n,b}\|^2, \\ &= \|w_n\|^2 - 2\langle l_{n-1}|w_n\rangle^\top \omega_{n,a} - 2\langle l_n|w_n\rangle^\top \omega_{n,b} \\ &\quad + \begin{bmatrix}\omega_{n,a} \\ \omega_{n,b}\end{bmatrix}^\top \begin{bmatrix}\langle l_{n-1}|l_{n-1}\rangle & \langle l_{n-1}|l_n\rangle \\ \langle l_n|l_{n-1}\rangle & \langle l_n|l_n\rangle\end{bmatrix}\begin{bmatrix}\omega_{n,a} \\ \omega_{n,b}\end{bmatrix}.\end{aligned}$$

Due to the orthogonality the Legendre polynomials and of $\langle \ell_n|w_n\rangle = 0$, the previous equality simplifies to

$$\|\tilde{w}_n\|^2 = \|w_n\|^2 + \begin{bmatrix}\omega_{n,a} \\ \omega_{n,b}\end{bmatrix}^\top \begin{bmatrix}\frac{b-a}{2n-1} & 0 \\ 0 & \frac{b-a}{2n+1}\end{bmatrix}\begin{bmatrix}\omega_{n,a} \\ \omega_{n,b}\end{bmatrix},$$

which can be rewritten as

$$\|\tilde{w}_n\|^2 = \|w_n\|^2 + \tfrac{2(b-a)}{4n^2-1}\begin{bmatrix} w_n(a) \\ w_n(b)\end{bmatrix}^\top \begin{bmatrix} 2n & (-1)^{n+1} \\ * & 2n\end{bmatrix}\begin{bmatrix} w_n(a) \\ w_n(b)\end{bmatrix}. \tag{A.4}$$

The proof is concluded by merging the two expressions given in (A.3) and (A.4) into $\|\partial_\theta \tilde{w}_n\| \geq \frac{\pi}{b-a}\|\tilde{w}_n\|$. □

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