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Improved observer design for heat equation with constant measurement delay via Legendre polynomials

Jin Zhang, Wen Kang, Emilia Fridman and Alexandre Seuret

Abstract—In this paper, we present improved results on observer design for 1D heat equation. We first introduce an observer under delayed spatially point measurements that leads to an error heat equation with time-delay. Inspired by recent developments in the area of delayed ODEs, we propose novel Lyapunov functionals based on the Legendre polynomials. Then, new Bessel-Legendre (BL) inequalities are provided to derive sufficient stability conditions in the form of linear matrix inequalities (LMIs) that are parameterized by the degree of the polynomials. Finally, a numerical example illustrates the efficiency of the results that allow to enlarge the value of delay preserving the stability by more than 20%.

I. INTRODUCTION

Estimation of partial differential equations (PDEs) is becoming an active research topic [1]. It is of interest to design observers using delayed measurements. Constructive conditions in the form of LMIs for estimation of PDEs under delayed measurements that are applicable to the performance (e.g. exponential decay rate) analysis have been presented in [2], [3], [4].

To enlarge the delay size, the concept of chain of sub-observers was recently extended to heat equation in [5]. However, construction of these observers involves serious computational complexity when solving chain of sub-observers in the form of PDEs. The objective of the current note is derivation of less conservative LMI conditions for the stability analysis of the delayed heat equation. Such conditions will allow to reduce the order of chain of sub-observers.

In the case of ODEs with time-delay, Jensen inequality [6], [7] and Wirtinger-based integral inequality [8] were usually employed. Several contributions to derive less conservative integral inequalities for time-delay systems were provided in [9], [10]. Recently, a novel integral inequality so-called Bessel-Legendre (BL) inequality that encompasses Jensen inequality and Wirtinger-based integral inequality as particular cases was introduced in [11]. The latter paper presented a hierarchy of LMI conditions that are competitive with [6],

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[7], [8] in terms of conservatism and of complexity. Stability analysis of a coupled ODE-heat PDE was presented in [12] via a new BL inequality.

Following the idea of [11], we provide new BL inequalities by using Legendre polynomials and their properties. Then, we propose novel Lyapunov functionals to derive sufficient stability conditions in the form of LMIs that are parameterized by the degree of the polynomials. Finally, a numerical example illustrates that our LMIs lead to an improvement over 20%. This implies that our results allow to reduce the order of chain of sub-observers considered in [5] by more than 20%.

Notation. Throughout the paper, the subindexes denote the corresponding partial derivatives, $\text{He}(X)$ denotes the expression $X + X^T$, $L^2(0, l)$ stands for the Hilbert space of square integrable scalar functions $z(x)$ on $(0, l)$ with the norm $\|z(x)\|_{L^2(0, l)} = \sqrt{\int_0^l z^2(x) dx}$, $H^1(0, l)$ is the Sobolev space of absolutely continuous scalar functions $z : [0, l] \rightarrow \mathbb{R}$ with $\frac{dz}{dx} \in L^2(0, l)$, and $H^2(0, l)$ is the Sobolev space of scalar functions $z : [0, l] \rightarrow \mathbb{R}$ with absolutely continuous $\frac{dz}{dx}$ and with $\frac{d^2z}{dx^2} \in L^2(0, l)$.

We now present a useful inequality:

Lemma 1: (Wirtinger Inequality [13]) For given scalars $a < b$, consider a scalar function $g \in \mathcal{H}^1(a, b)$ such that $g(a) = g(b) = 0$. Then the following inequality holds:

$$\int_a^b g^2(x) dx \leq \frac{(b-a)^2}{\pi^2} \int_a^b \left[\frac{dg(x)}{dx} \right]^2 dx. \quad (1)$$

II. PROBLEM FORMULATION

A. System description

Consider a semilinear reaction-diffusion equation

$$z_t(x, t) = \frac{\partial}{\partial x} [a(x)z_x(x, t)] + f(z(x, t), x, t), \quad (2)$$

$$t \geq 0, \quad x \in [0, l], \quad l > 0$$

under the Dirichlet boundary conditions

$$z(0, t) = z(l, t) = 0, \quad (3)$$

where $z(x, t) \in \mathbb{R}$ is the state with initial condition $z(x, 0) = z_0(x)$, a and f are functions of class C^1 . These functions satisfy the inequalities $a > a_0 > 0$ and

$$\phi_m \leq f_z(z, x, t) \leq \phi_M, \quad \forall (z, x, t) \in \mathbb{R} \times [0, l] \times [0, \infty), \quad (4)$$

where a_0 , ϕ_m and ϕ_M are known bounds.

As in [2], we assume that the points $0 = x_0 < x_1 < \dots < x_N = l$ divide $[0, l]$ into N sub-intervals, whose widths of each

interval are not necessarily equal but are upper bounded by some constant $\Delta > 0$:

$$0 < x_{j+1} - x_j = \Delta_j \leq \Delta. \quad (5)$$

It is assumed that N sensors are placed in the middle of each interval $[x_j, x_{j+1}]$:

$$\bar{x}_j = \frac{x_{j+1} + x_j}{2}, \quad j = 0, \dots, N-1, \quad (6)$$

and that the measurement is delayed by a constant delay $h > 0$. Then, delayed spatially point measurements of the state are provided by N sensors distributed over the whole domain $[0, l]$:

$$y_j(t) = \begin{cases} 0, & \text{if } t < h, \\ z(\bar{x}_j, t - h), & \text{if } t \geq h, \end{cases} \quad j = 0, \dots, N-1. \quad (7)$$

Our objective is to construct an observer for (2) under the Dirichlet boundary conditions (3) by employing delayed spatially point measurements (7), and to formulate improve stability conditions for the exponential convergence of the estimation error in terms of LMIs.

B. Observer design

We suggest a nonlinear observer of the form

$$\begin{aligned} \hat{z}_t(x, t) &= \frac{\partial}{\partial x} [a(x)\hat{z}_x(x, t)] + f(\hat{z}(x, t), x, t) \\ &\quad + K(y_j(t) - \hat{z}(\bar{x}_j, t - h)), \\ t \geq 0, \quad x_j \leq x < x_{j+1}, \quad j &= 0, \dots, N-1 \end{aligned} \quad (8)$$

under the Dirichlet boundary conditions

$$\hat{z}(0, t) = \hat{z}(l, t) = 0, \quad (9)$$

where $\hat{z}(x, t) = 0, t \in [-h, 0]$ and $K > 0$ is the scalar observer gain.

For the estimation error $e(x, t) = z(x, t) - \hat{z}(x, t)$, via (2) and (8) we obtain the PDE

$$\begin{aligned} e_t(x, t) &= \frac{\partial}{\partial x} [a(x)e_x(x, t)] + \phi \cdot e(x, t) - Ke(x, t - h) \\ &\quad + K \int_{\bar{x}_j}^x e_\xi(\xi, t - h) d\xi, \\ t \geq 0, \quad x_j \leq x < x_{j+1}, \quad j &= 0, \dots, N-1 \end{aligned} \quad (10)$$

under the Dirichlet boundary conditions

$$e(0, t) = e(l, t) = 0, \quad (11)$$

and with initial condition $e(x, t) = z_0(x), t \in [-h, 0]$.

Here $\phi \cdot e(x, t) = f(z(x, t), x, t) - f(z(x, t) - e(x, t), x, t)$ with

$$\phi = \phi(z, e, x, t) = \int_0^1 \frac{\partial}{\partial z} f(z + (\theta - 1)e, x, t) d\theta. \quad (12)$$

From (4), it follows that

$$\phi_m \leq \phi \leq \phi_M, \quad \forall (z, e, x, t) \in \mathbb{R} \times \mathbb{R} \times [0, l] \times [0, \infty). \quad (13)$$

C. Well-posedness of (10)

Let $H = L^2(0, l)$ be the Hilbert space with the norm $\|\cdot\|_{L^2(0, l)}$ and with the scalar product $\langle \cdot, \cdot \rangle$. We define an unbounded linear operator $A : \mathcal{D}(A) \subset H \rightarrow H$ as follows:

$$\begin{cases} A = -\frac{\partial}{\partial x} \left[a(x) \frac{\partial}{\partial x} \right], \\ \mathcal{D}(A) = H^2(0, l) \cap H_0^1(0, l), \end{cases} \quad (14)$$

where $H_0^1(0, l) = \{z \in H^1(0, l) : z(0) = z(l) = 0\}$. It is well-known that A is a dissipative operator, and A generates an exponentially stable semigroup. The domain $H_1 = \mathcal{D}(A) = A^{-1}H$ forms another Hilbert space with the graph inner product $\langle x, y \rangle_1 = \langle Ax, Ay \rangle, \forall x, y \in H_1$. The domain $\mathcal{D}(A)$ is dense in H . Operator $-A$ is positive implying that its square root $(-A)^{\frac{1}{2}}$ with $H_{\frac{1}{2}} = \mathcal{D}((-A)^{\frac{1}{2}}) = H_0^1(0, l)$ is well defined. The norm of $H_{\frac{1}{2}}$ is endowed by the induced inner product:

$$\begin{aligned} \langle u, v \rangle_{\frac{1}{2}} &= \langle (-A)^{\frac{1}{2}} u, (-A)^{\frac{1}{2}} v \rangle, \quad \forall u, v \in H_{\frac{1}{2}}, \\ \|f\|_{H_{\frac{1}{2}}} &= \left[\int_0^l |f'(x)|^2 dx \right]^{\frac{1}{2}}, \quad \forall f \in H_{\frac{1}{2}}. \end{aligned}$$

Note that $H_{\frac{1}{2}}$ norm is equivalent to the inherent norm $\|\cdot\|_{H_1}$ of Sobolev space $H^1(0, l)$. Then, we have $H_1 \subset H_{\frac{1}{2}} \subset H$, densely and with continuous embedding. All relevant materials on fractional operator degrees can be found in [14] (see pp. 81-83).

We will use the step method for solution of time-delay systems [15], [16]. While being viewed over the time segment $[0, h]$, system (10) can be rewritten as the differential equation given by

$$\begin{cases} \frac{d}{dt} e(\cdot, t) = Ae(\cdot, t) + F(e(\cdot, t)), \\ e(x, \theta) = z_0(x), \quad \theta \in [-h, 0] \end{cases} \quad (15)$$

subject to

$$F(e(\cdot, t)) = \phi \cdot e(x, t) + Kz_0(\bar{x}_j), \quad x \in [x_j, x_{j+1}). \quad (16)$$

Since $\phi \in C^1$ and ϕ is bounded, it follows from (16) that F is nonlinear but is globally Lipschitz continuous. Thus, by Theorem 3.3.3 of [17], we obtain that for any $z_0 \in H_0^1(0, l)$, (15) has a unique strong solution $e(\cdot, t) \in H_0^1(0, l)$ on the interval $[0, h]$.

The same line of reasoning is applied step-by-step to the time segments $[h, 2h], [2h, 3h], \dots$ (see, e.g. [16]). Following this procedure, we find that $e(\cdot, t)$ is continuous differentiable on the point $h, 2h, \dots$. Therefore, there exists a unique strong solution $e(\cdot, t)$ of (10) for all $t \geq 0$ with the initial condition $e(x, \theta) = z_0(x) \in H_0^1(0, l), \theta \in [-h, 0]$.

III. MAIN RESULTS

In this section, we will derive improved stability conditions for the delayed heat equation (10) via new BL inequalities. First, let us recall the definition of Legendre polynomials.

A. Legendre polynomials

As in [11], we consider here shifted Legendre polynomials over interval $[-h, 0]$

$$\mathcal{L}_k(s) = (-1)^k \sum_{i=0}^k p_i^k \left(\frac{s+h}{h}\right)^i, \quad \forall k \in \mathbb{N}_0 \quad (17)$$

with $p_i^k = (-1)^i \binom{k}{i} \binom{k+i}{i}$. The notation $\binom{k}{i}$ refers to the binomial coefficients given by $\frac{k!}{(k-i)!i!}$. Note that the Legendre polynomials described by (17) form an orthogonal sequence since the inner product satisfies

$$\int_{-h}^0 \mathcal{L}_k(s) \mathcal{L}_i(s) ds = \frac{h}{2k+1} \delta_{ki}, \quad \forall k, i \in \mathbb{N}_0, \quad (18)$$

where δ_{ki} is the Kronecker delta (that equals to 1 if $k = i$ and to 0 otherwise). The boundary values are as follows:

$$\mathcal{L}_k(0) = 1, \quad \mathcal{L}_k(-h) = (-1)^k, \quad \forall k \in \mathbb{N}_0. \quad (19)$$

Moreover, the following relation holds:

$$\frac{d}{ds} \mathcal{L}_k(s) = \begin{cases} 0, & \text{if } k = 0, \\ \sum_{i=0}^{k-1} \frac{2i+1}{h} (1 - (-1)^{k+i}) \mathcal{L}_i(s), & \text{if } k \geq 1. \end{cases} \quad (20)$$

Let us introduce the projection components of the error heat equation (10) onto Legendre polynomials. These components are given by

$$\Omega_k(x, t) = \int_{-h}^0 \mathcal{L}_k(s) e(x, t+s) ds, \quad k \in \mathbb{N}_0. \quad (21)$$

Differentiating $\Omega_k(x, t)$ with respect to t leads to

$$\frac{\partial}{\partial t} \Omega_k(x, t) = \int_{-h}^0 \mathcal{L}_k(s) \frac{\partial}{\partial t} e(x, t+s) ds, \quad k \in \mathbb{N}_0. \quad (22)$$

By using (19) and (20), and integrating by parts, we have

$$\begin{aligned} \frac{\partial}{\partial t} \Omega_k(x, t) &= \mathcal{L}_k(s) e(x, t+s) \Big|_{s=-h}^0 \\ &\quad - \int_{-h}^0 \left[\frac{d}{ds} \mathcal{L}_k(s) \right] e(x, t+s) ds \\ &= e(x, t) + (-1)^{k+1} e(x, t-h) \\ &\quad - \int_{-h}^0 \left[\frac{d}{ds} \mathcal{L}_k(s) \right] e(x, t+s) ds \\ &= \Gamma_n(k) \eta_n(x, t), \quad k = 0, \dots, n, \quad n \in \mathbb{N}_0, \end{aligned} \quad (23)$$

where

$$\Gamma_n(k) = \begin{cases} [1, 0, (-1)^{k+1}, 0], & \text{if } n = 0, \\ [1, 0, (-1)^{k+1}, 0, \gamma_{nk}^0, \dots, \gamma_{nk}^{n-1}], & \text{if } n \geq 1, \end{cases} \quad (24)$$

$$\gamma_{nk}^i = \begin{cases} -\frac{2i+1}{h} (1 - (-1)^{k+i}), & \text{if } i \leq k, \\ 0, & \text{if } i > k, \end{cases}$$

and

$$\eta_n(x, t) = \begin{cases} \left[e(x, t), e_t(x, t), e(x, t-h), \int_{\bar{x}_j}^x e_\xi(\xi, t-h) d\xi \right]^T, & \text{if } n = 0, \\ \left[e(x, t), e_t(x, t), e(x, t-h), \int_{\bar{x}_j}^x e_\xi(\xi, t-h) d\xi \right. \\ \left. \Omega_0(x, t), \dots, \Omega_{n-1}(x, t) \right]^T, & \text{if } n \geq 1. \end{cases} \quad (25)$$

The relation given by (23) is very useful to derive BL inequality (see e.g. the proof of Lemma 3) and also for the later stability analysis.

B. Bessel-Legendre inequalities

Based on the Legendre polynomials and an application of Bessel's inequality [18], we obtain the following lemma:

Lemma 2: Let $e \in L^2([-h, 0]; L^2(0, l))$, $h > 0$ and $l > 0$. The integral inequality

$$\int_0^l \int_{-h}^0 e^2(x, t+s) ds dx \geq \frac{1}{h} \sum_{k=0}^n (2k+1) \int_0^l \Omega_k^2(x, t) dx \quad (26)$$

holds for all $n \in \mathbb{N}_0$, where $\Omega_k(x, t)$, $k = 0, \dots, n$ are given by (21).

Proof: Consider a function $e \in L^2([-h, 0]; L^2(0, l))$, and define the error vector $\varepsilon_n(x, t+s)$, $n \in \mathbb{N}_0$ given by

$$\varepsilon_n(x, t+s) = e(x, t+s) - \frac{1}{h} \sum_{k=0}^n (2k+1) \mathcal{L}_k(s) \Omega_k(x, t). \quad (27)$$

From its definition, $\varepsilon_n(x, t+s)$ is in $L^2([-h, 0]; L^2(0, l))$ and the integral $\int_0^l \int_{-h}^0 \varepsilon_n^2(x, t+s) ds dx$ exists. From the orthogonal property of the Legendre polynomials, via (21) we easily get

$$\begin{aligned} \int_0^l \int_{-h}^0 \varepsilon_n^2(x, t+s) ds dx &= \int_0^l \left\{ \int_{-h}^0 e^2(x, t+s) ds \right. \\ &\quad - \frac{2}{h} \sum_{k=0}^n (2k+1) \left[\int_{-h}^0 \mathcal{L}_k(s) e(x, t+s) ds \right] \Omega_k(x, t) \\ &\quad \left. + \frac{1}{h^2} \sum_{k=0}^n (2k+1)^2 \int_{-h}^0 \mathcal{L}_k^2(s) \Omega_k^2(x, t) ds \right\} dx \\ &= \int_0^l \left\{ \int_{-h}^0 e^2(x, t+s) ds - \frac{1}{h} \sum_{k=0}^n (2k+1) \Omega_k^2(x, t) \right\} dx. \end{aligned} \quad (28)$$

Noting that the left hand side of the latter equation is positive definite, we finally arrive at (26), which concludes the proof. \blacksquare

For the delay-dependent analysis of delayed heat equation (10), we are interested to derive a lower bound of $\int_0^l \int_{-h}^0 e_s^2(x, t+s) ds dx$. The next lemma addresses this particular problem:

Lemma 3: Let $e_s \in L^2([-h, 0]; L^2(0, l))$, $h > 0$ and $l > 0$. The integral inequality

$$\int_0^l \int_{-h}^0 e_s^2(x, t+s) ds dx \geq \frac{1}{h} \sum_{k=0}^n (2k+1) \int_0^l [\Gamma_n(k) \eta_n(x, t)]^2 dx \quad (29)$$

holds for all $n \in \mathbb{N}_0$, where $\Gamma_n(k)$, $k = 0, \dots, n$ and $\eta_n(x, t)$ are given by (24) and (25).

Proof: By using Lemma 2, we obtain

$$\int_0^l \int_{-h}^0 e_s^2(x, t+s) ds dx \geq \frac{1}{h} \sum_{k=0}^n (2k+1) \int_0^l \left[\frac{\partial}{\partial t} \Omega_k(x, t) \right]^2 dx. \quad (30)$$

Replacing $\frac{\partial}{\partial t} \Omega_k(x, t)$ by its expression (23) using the vectors $\Gamma_n(k)$, $k = 0, \dots, n$ and $\eta_n(x, t)$ yields (29). This completes the proof. \blacksquare

C. Stability analysis

Guided by these new BL inequalities, we consider an augmented Lyapunov functional as follows

$$V_n(t) = V_{P_n}(t) + V_{p_2}(t) + V_{s_1}(t) + V_{s_2}(t) + V_r(t), \quad n \in \mathbb{N}_0, \quad (31)$$

where

$$\begin{aligned} V_{P_n}(t) &= \int_0^l \zeta_n^T(x,t) P_n \zeta_n(x,t) dx, \\ V_{p_2}(t) &= p_2 \int_0^l a(x) e_x^2(x,t) dx, \\ V_{s_1}(t) &= s_1 \int_0^l \int_{t-h}^t e^{2\alpha(s-t)} e^2(x,s) ds dx, \\ V_{s_2}(t) &= s_2 \int_0^l \int_{t-h}^t e^{2\alpha(s-t)} e_x^2(x,s) ds dx, \\ V_r(t) &= hr \int_0^l \int_{t-h}^t e^{2\alpha(s-t)} (s-t+h) e_s^2(x,s) ds dx \end{aligned}$$

with

$$\zeta_n(x,t) = \begin{cases} e(x,t), & \text{if } n=0, \\ [e(x,t), \Omega_0(x,t), \dots, \Omega_{n-1}(x,t)]^T, & \text{if } n \geq 1 \end{cases}$$

and matrix $P_n \in \mathbb{R}^{(n+1) \times (n+1)}$, scalars $p_2 > 0$, $s_1 > 0$, $s_2 > 0$, $r > 0$ and decay rate $\alpha > 0$. The terms V_{s_1} and V_r are the extensions of the standard Lyapunov functionals [15] for delay-dependent analysis, whereas the term V_{s_2} is introduced to compensate $\int_0^l e_x(x,t-h) dx$. For $n=0$, V_n coincides with the Lyapunov functional introduced in [2].

We now present the following stability results with an arbitrary $n \in \mathbb{N}_0$:

Theorem 1: Given scalars $\phi_M \geq \phi_m$, $a_0 > 0$, $h > 0$, $l > 0$, $\Delta > 0$ and $\alpha > 0$, let there exist $P_n \in \mathbb{R}^{(n+1) \times (n+1)}$ and scalars $p_1 > 0$, $p_2 > 0$, $s_1 > 0$, $s_2 > 0$, $r > 0$ and $\mu > 0$ satisfying the following LMIs

$$s_2 < 2a_0(p_1 - \alpha p_2), \quad \mu < e^{-2\alpha h} s_2, \quad (32)$$

and

$$\Phi_n(\phi_m) + U_n \leq 0, \quad \Phi_n(\phi_M) + U_n \leq 0, \quad \Theta_n > 0, \quad (33)$$

where

$$\begin{aligned} \Phi_n(\phi) &= \text{He}(G_n^T P_n H_n + D_n^T R_n(\phi)) + 2\alpha G_n^T P_n G_n \\ &\quad + W_n - e^{-2\alpha h} r \sum_{k=0}^n (2k+1) \Gamma_n^T(k) \Gamma_n(k), \\ U_n &= \text{diag} \left\{ -\frac{(2a_0(p_1 - \alpha p_2) - s_2)\pi^2}{l^2}, 0, \right. \\ &\quad \left. -\frac{(e^{-2\alpha h} s_2 - \mu)\pi^2}{l^2}, 0_{1 \times n} \right\}, \\ \Theta_n &= \begin{cases} P_n, & \text{if } n=0, \\ P_n + \frac{1}{h} e^{-2\alpha h} s_1 \text{diag}\{0, 1, \dots, 2n-1\}, & \text{if } n \geq 1 \end{cases} \end{aligned} \quad (34)$$

with

$$\begin{aligned} G_n &= \begin{bmatrix} 1 & 0_{1 \times 3} & 0_{1 \times n} \\ 0_{n \times 1} & 0_{n \times 3} & I_n \end{bmatrix}, \\ H_n &= [F_n^T \quad \Gamma_n^T(0) \quad \dots \quad \Gamma_n^T(n-1)]^T, \\ F_n &= [0 \quad 1 \quad 0_{1 \times (n+2)}], \\ D_n &= [p_1 \quad p_2 \quad 0_{1 \times (n+2)}], \\ R_n(\phi) &= [\phi \quad -1 \quad -K \quad K \quad 0_{1,n}], \\ W_n &= \text{diag} \left\{ s_1, h^2 r, -e^{-2\alpha h} s_1, -\mu \frac{\pi^2}{\Delta^2}, 0_{1 \times n} \right\}, \end{aligned}$$

and $\Gamma_n(k)$ ($k=0, \dots, n$) given by (24). Then the delayed heat equation (10) under the Dirichlet boundary conditions (11) is exponentially stable with a decay rate $\alpha > 0$.

Proof: Choose the Lyapunov function $V_n(t)$ given by (31). Using Lemma 2 gives a lower bound of the term V_{s_1} as follows

$$\begin{aligned} V_{s_1}(t) &\geq e^{-2\alpha h} s_1 \int_0^l \int_{-h}^0 e^2(x,t+s) ds dx \\ &\geq \frac{s_1}{h} e^{-2\alpha h} \sum_{k=0}^{n-1} (2k+1) \int_0^l \Omega_k^2(x,t) dx. \end{aligned} \quad (35)$$

Thus, the positive definiteness of V_n results from the condition $\Theta_n > 0$ given by (33).

Differentiating $V_n(t)$ along (10), via (23) we obtain

$$\begin{aligned} \dot{V}_n(t) + 2\alpha V_n(t) &= 2 \int_0^l \zeta_n^T(x,t) P_n \dot{\zeta}_n(x,t) dx \\ &\quad + 2\alpha \int_0^l \zeta_n^T(x,t) P_n \zeta_n(x,t) dx, \end{aligned} \quad (36)$$

where

$$\dot{\zeta}_n(x,t) = \begin{bmatrix} \dot{e}(x,t) \\ \dot{\Omega}_0(x,t) \\ \vdots \\ \dot{\Omega}_{n-1}(x,t) \end{bmatrix} = \begin{bmatrix} F_n \eta_n(x,t) \\ \Gamma_n(0) \eta_n(x,t) \\ \vdots \\ \Gamma_n(n-1) \eta_n(x,t) \end{bmatrix} = H_n \eta_n(x,t).$$

For the term V_{p_2} , We have

$$\begin{aligned} \dot{V}_{p_2}(t) + 2\alpha V_{p_2}(t) &= 2p_2 \int_0^l a(x) e_x(x,t) e_{xt}(x,t) dx \\ &\quad + 2\alpha p_2 \int_0^l a(x) e_x^2(x,t) dx. \end{aligned} \quad (37)$$

For the first term of the right hand side of (37), we employ the descriptor method [15], where the right-hand side of the following expression is added to $\dot{V}_n(t) + 2\alpha V_n(t)$:

$$\begin{aligned} 0 &= 2 \int_0^l [p_1 e(x,t) + p_2 e_t(x,t)] \left\{ \frac{\partial}{\partial x} [a(x) e_x(x,t)] \right. \\ &\quad \left. + \phi e(x,t) - K e(x,t-h) - e_t(x,t) \right\} dx \\ &\quad + 2K \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} [p_1 e(x,t) + p_2 e_t(x,t)] \\ &\quad \times \int_{\bar{x}_j}^x e_{\xi}(\xi, t-h) d\xi dx \end{aligned} \quad (38)$$

with some scalar $p_1 > 0$. This avoids substitution of $e_t(x,t)$ from (10) into the right-hand side of (36).

Integration by parts and substitution of the Dirichlet boundary conditions (11) lead to

$$\begin{aligned}
& 2 \int_0^l [p_1 e(x,t) + p_2 e_t(x,t)] \frac{\partial}{\partial x} [a(x) e_x(x,t)] dx \\
&= 2a(x) [p_1 e(x,t) + p_2 e_t(x,t)] e_x(x,t) \Big|_{x=0}^l \\
&\quad - 2 \int_0^l a(x) [p_1 e_x(x,t) + p_2 e_{xt}(x,t)] e_x(x,t) dx \\
&= -2p_1 \int_0^l a(x) e_x^2(x,t) dx - 2p_2 \int_0^l a(x) e_x(x,t) e_{xt}(x,t) dx.
\end{aligned} \tag{39}$$

To “compensate” the cross terms in (38) with $\int_{\bar{x}_j}^x e_\xi(\xi, t-h) d\xi$, we here apply S-procedure. Namely, application of Wirtinger’s inequality (1) yields

$$\begin{aligned}
& \int_{x_j}^{x_{j+1}} \left[\int_{\bar{x}_j}^x e_\xi(\xi, t-h) d\xi \right]^2 dx \\
&= \int_{\bar{x}_j}^{x_{j+1}} \left[\int_{\bar{x}_j}^x e_\xi(\xi, t-h) d\xi \right]^2 dx \\
&\quad + \int_{x_j}^{\bar{x}_j} \left[\int_{\bar{x}_j}^x e_\xi(\xi, t-h) d\xi \right]^2 dx \\
&\leq \frac{\Delta^2}{\pi^2} \int_{x_j}^{x_{j+1}} e_x^2(x, t-h) dx, \quad j = 0, \dots, N-1.
\end{aligned}$$

Then the following inequality

$$\begin{aligned}
& \mu \int_0^l e_x^2(x, t-h) dx \\
&\quad - \mu \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} \frac{\pi^2}{\Delta^2} \left[\int_{\bar{x}_j}^x e_\xi(\xi, t-h) d\xi \right]^2 dx \geq 0
\end{aligned} \tag{40}$$

holds for some constant $\mu > 0$.

Moreover, we have

$$\begin{aligned}
\dot{V}_{s_1}(t) + 2\alpha V_{s_1}(t) &= s_1 \int_0^l e^2(x, t) dx \\
&\quad - e^{-2\alpha h} s_1 \int_0^l e^2(x, t-h) dx,
\end{aligned} \tag{41}$$

and

$$\begin{aligned}
\dot{V}_{s_2}(t) + 2\alpha V_{s_2}(t) &= s_2 \int_0^l e_x^2(x, t) dx \\
&\quad - e^{-2\alpha h} s_2 \int_0^l e_x^2(x, t-h) dx.
\end{aligned} \tag{42}$$

Further by using Lemma 3, we obtain

$$\begin{aligned}
\dot{V}_r(t) + 2\alpha V_r(t) &= h^2 r \int_0^l e_t^2(x, t) dx \\
&\quad - hr \int_0^l \int_{t-h}^t e^{2\alpha(s-t)} e_s^2(x, s) ds dx \\
&\leq h^2 r \int_0^l e_t^2(x, t) dx \\
&\quad - e^{-2\alpha h} r \sum_{k=0}^n (2k+1) \int_0^l [\Gamma_n(k) \eta_n(x, t)]^2 dx.
\end{aligned} \tag{43}$$

Combining (36), (37), (41)-(43) together, adding the right hand side of (38) to $\dot{V}_n(t) + 2\alpha V_n(t)$, and applying S-

procedure with (40), we have

$$\begin{aligned}
\dot{V}_n(t) + 2\alpha V_n(t) &\leq \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} \eta_n^T(x, t) \Phi_n(\phi) \eta_n(x, t) dx \\
&\quad + s_2 \int_0^l e_x^2(x, t) dx \\
&\quad - 2(p_1 - \alpha p_2) \int_0^l a(x) e_x^2(x, t) dx \\
&\quad - (e^{-2\alpha h} s_2 - \mu) \int_0^l e_x^2(x, t-h) dx,
\end{aligned} \tag{44}$$

where $\Phi_n(\phi)$ is given by (34).

Note that the first inequality of (32) implies $p_1 - \alpha p_2 > 0$ since $s_2 > 0$ and $a_0 > 0$. Taking into account $a > a_0$, we further arrive at

$$\begin{aligned}
\dot{V}_n(t) + 2\alpha V_n(t) &\leq \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} \eta_n^T(x, t) \Phi_n(\phi) \eta_n(x, t) dx \\
&\quad - [2a_0(p_1 - \alpha p_2) - s_2] \int_0^l e_x^2(x, t) dx \\
&\quad - (e^{-2\alpha h} s_2 - \mu) \int_0^l e_x^2(x, t-h) dx \\
&\leq \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} \eta_n^T(x, t) (\Phi_n(\phi) + U_n) \eta_n(x, t) dx.
\end{aligned} \tag{45}$$

Here U_n is given by (34). The latter follows from Wirtinger’s inequality (1).

Since $\Phi_n(\phi)$ given by (34) is affine in $\phi \in [\phi_m, \phi_M]$, the feasibility of (33) implies the feasibility of $\Phi_n(\phi) + U_n < 0$ for all $\phi \in [\phi_m, \phi_M]$. Thus, we have $\dot{V}_n(t) + 2\alpha V_n(t) \leq 0$ implying the exponential convergence of $V_n(t)$ with a decay rate $\alpha > 0$. ■

Remark 1: By following arguments of [11], [19], we can easily show that the derived stability conditions can form a hierarchy of LMI conditions. This will be illustrated via the later example.

IV. NUMERICAL SIMULATIONS

Consider the following 1D heat equation:

$$\begin{aligned}
z_t(x, t) &= z_{xx}(x, t) + 1.02\pi^2 z(x, t), \\
t &\geq 0, \quad x \in [0, 1]
\end{aligned} \tag{46}$$

under the Dirichlet boundary conditions (3). Choose $z(x, 0) = \sin(\pi x)$ and $x_{j+1} - x_j \equiv \Delta$ with $\Delta = 0.02$ implying the number of sub-intervals $N = 50$. The heat equation above is unstable (see Fig. 1).

We suggest the following observer:

$$\begin{aligned}
\hat{z}_t(x, t) &= \hat{z}_{xx}(x, t) + 1.02\pi^2 \hat{z}(x, t) \\
&\quad + (z(\bar{x}_j, t-h) - \hat{z}(\bar{x}_j, t-h)), \\
t &\geq 0, \quad x_j \leq x < x_{j+1}, \quad j = 0, \dots, 49
\end{aligned} \tag{47}$$

under the Dirichlet boundary conditions (9). Here \bar{x}_j ($j = 0, \dots, 49$) are given by (6) and $h > 0$ is a constant delay.

LMIs of Theorem 1 with various values of n and $\alpha = 0$ lead to the maximal allowable delay h_{\max} preserving the stability shown in Table I. As expected, better results are obtained as the degree of the polynomial n increases, but at the price of additional decision variables. Moreover, the

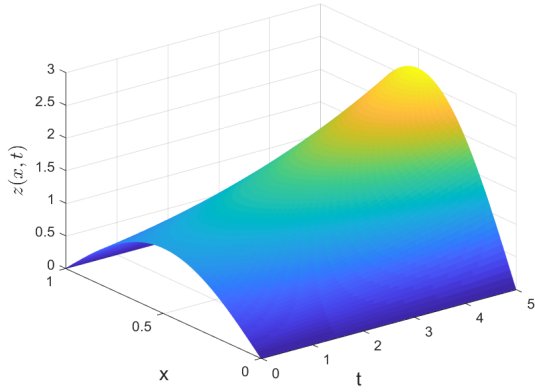


Fig. 1. The state $z(x,t)$.

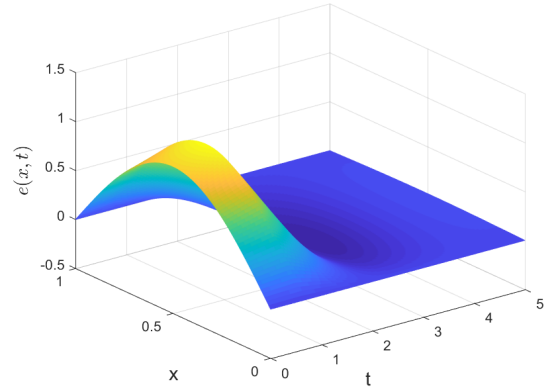


Fig. 2. The error $e(x,t)$.

TABLE I

MAXIMAL ALLOWABLE DELAY h_{\max} VIA THEOREM 1.

Theorem 1	h_{\max}	Number of variables
$n = 0$	0.5215	7
$n = 1$	0.6226	9
$n = 2$	0.6349	12
$n = 3$	0.6389	16
$n = 4$	0.6421	21
$n = 5$	0.6440	27
$n = 6$	0.6449	34
$n = 7$	0.6450	42
$n \geq 8$	0.6451	$0.5n^2 + 1.5n + 7$

maximal allowable delay h_{\max} remains as 0.6451 when $n \geq 8$. Clearly, an improvement over 20% is achieved by using our new BL inequalities. This implies that our result allows to reduce the order of chain of sub-observers considered in [5] by more than 20%.

The numerical simulations show that the error heat equation under the Dirichlet boundary conditions (11) and with constant delay $h = 0.6451$ is stable (see Fig. 2).

V. CONCLUSIONS

In this paper, an observer was introduced for 1D heat equation under delayed spatially point measurements. The stability analysis of the resulting error heat equation with time-delay was performed via the newly proposed BL inequalities. A numerical example illustrated the efficiency of the results. One of the directions for the future research is extension of the obtained results to the KdVB equation with time-delay [20].

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