Frequency delay-dependent stability criterion for time-delay systems thanks to Fourier-Legendre remainders

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May 2021

Abstract

This paper investigates the stability of a linear finite-dimensional system interconnected to a single delay operator. From robust approaches, to derive delay-dependent frequency tests, a characterization of the delay behavior is required. Based on approximation methods, one describes the transported signal by lumped parameters. More precisely, by the use of the first Fourier-Legendre polynomials coefficients, we split the delay block into a finite-dimensional part interconnected to a specific infinite-dimensional residual part. Two models are investigated with residuals related to two Fourier-Legendre remainders of the delayed transfer function. The main contribution is to highlight that the finite-dimensional models based on the first Legendre coefficients are proven to be related to Padé approximations and are recognized to be more and more accurate as the dimension increases. Interestingly, this modeling allows computing in an accurate manner the root locus of time-delay systems. Furthermore, as a by-product of this result, taking into account the infinite-dimensional remainders to keep track of the initial time-delay system, stability criteria are proposed by $\mathcal{H}_\infty$ analysis. Considering both infinite-dimensional remainders as bounded delay-free uncertainties, the small-gain theorem provides a new sufficient condition of stability for retarded time-delay systems, which can be implemented as a delay-dependent frequency-sweeping test. Our results are illustrated on several academic examples.

Keywords: Time-delay systems, Partial differential equations, Model approximation, Spectral analysis, Stability analysis, $\mathcal{H}_\infty$ analysis, Robust stability.

1 Introduction

Material or information transfer between several dynamical physical systems can often be modeled by the interconnection of an overall system and a delay element. Such systems are called time-delay systems.
Stabilizing or destabilizing effects introduced by the delay have been widely investigated in the literature.[6, 21, 42, 20, 38, 15] The behavior of linear time-delay systems is provided by the roots of the associated characteristic equation [34, 40], which is given analytically by Lambert W functions [33]. Numerically, the spectrum location is indeed estimated with approximated finite-dimensional models of the infinite-dimensional part. Padé approximations[17] are recognized to be the most used tools to approximate the delay block and traduce the behavior of time-delay systems. Multiple other ways to design these approximated models can also be found in the literature such as pseudo-spectral collocation [8, 12], spectral least square [47] or spectral tau [22, 23] methods, to cite a few. Each spectral technique can take into account Fourier, Chebyshev or Legendre approximation methods. One obtains therefore accurate characteristic roots on compact sets of the complex plane and plots good approximations of the solution. Note that numerical comparative studies on the accuracy [48] have been conducted and conclude that models based on Legendre coefficients are efficient most of the time. Surprisingly, none of these approximated models are used to deal with stability. The connection with stability analysis of the original time-delay system is effectively not straighforward.[13] Indeed, if roots are located on the imaginary axis, the convergence properties are not usable to assess the stability.
There exists multiple ways to analyze the stability properties of time-delay systems, multiple paths may be used. [38, 15, 45] In the frequency domain, one determines explicitly the imaginary crossing of the poles. [31, 46] Especially, matrix pencil approach [37, 32] examines and calculates analytically the imaginary axis eigenvalues of the retarded matrix delay systems. In addition, application of Nyquist theorem [35] also lead to some conditions, which can be checked on Mikhailov diagrams. In the time domain, several fundamental results are based on the necessary and sufficient condition of existence of a complete Lyapunov-Krasovskii functional. [27] Since the computation of the involved delay Lyapunov matrix is an hard task, discretization procedures have been developed (see discretized Lyapunov functional [19] based for instance on Legendre polynomials coefficients [43]). The recent work [18] also solves the delay equations verified by the Lyapunov matrix-function thanks to polynomial approximation. [25]

*This research is related to ANR ODISSE (ANR-19-CE48-0004-01).

These competitive but complementary methods can be used and provide tangible results on stability of time-delay systems.
Otherwise, some robust approaches can be pursued to make the link between stability and approximation schema. Originally, the delay element e^{-hs} was embedded into a unit norm-bounded unstructured uncertainty. Using the small gain theorem, it results a classical delay-independent condition. In order to refine the results, a method proposed in many recent papers consists in extracting from the delay transfer function a delay-dependent finite-dimensional system. The infinite-dimensional remainder is then embedded into an unstructured uncertainty. [13, 7] The use of classical tools like small gain theorem, μ-analysis or integral quadratic constraints allows to construct conservative stability tests. [49] Notice that the candidate filters or constraints are not easy to find. There are often related to a specific Padé remainder transfer function [28] or to Bessel inequality on polynomial orthogonal basis [43, 2].
This paper proposes to merge spectral and robust approaches to ease the stability analysis of time-delay systems. The main contributions are listed below.

- In the foreground, two augmented time-delay systems are proposed and express the dynamics of the original one. The finite-dimensional part is issued from a spectral method on Legendre polynomials basis and the infinite-dimensional part is coming from the remainder. Such interconnections allow us to formulate the relation with Padé approximations.
- In the background, the stability of time-delay systems is formulated in terms of frequency-sweeping tests based on $\mathcal{H}_\infty$ analysis. This is made possible by handling these augmented systems and removing high-pass filters associated to the Fourier-Legendre remainders.

Section 2 is dedicated to the problem statement. Then, technical tools on the use of Legendre polynomials are given in Section 3. From there, Section 4 proposes the remodeling of the transported part on the first Legendre polynomials coefficients. These models are nice to the extent that the delay turns out to be approximated by the well-known $(n-1|n)$ and $(n|n)$ Padé approximants and that a natural filter is obtained. Afterwards, two augmented time-delay systems are constructed in Section 5. Based on Padé properties, we recall that the eigenvalues of our finite-dimensional part can approximate the expected eigenvalues on compact sets. Lastly, Section 6 deals with stability analysis in the light of the application of the small-gain theorem on the whole interconnection between the approximated model, the high-pass filter and the bounded remainder. Finally, in Section 7, examples are shown off to highlight the effectiveness of our modeling to approximate eigenvalues and deal with stability of time-delay systems with respect to the delay.

Notations: In this paper, the set of natural, real, non-negative real, complex numbers and $n \times m$ real matrices are respectively denoted $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{C}$ and $\mathbb{R}^{n\times m}$. Moreover, I_n is the identity matrix of size n, $\operatorname{diag}(d_1, ..., d_n)$ is the diagonal matrix defined by its diagonal coefficients $(d_1, ..., d_n)$ and M^T denotes the transpose of matrix M. For any square matrix M, $\det(M)$, $\operatorname{adj}(M)$, $\operatorname{tril}(M)$ and $M \succ 0$ stands respectively for its determinant, adjugate matrix, lower triangular part and symmetry positive definiteness. For $M \in \mathbb{R}^{n\times n}$, its characteristic polynomial is $\chi_M(s) = \det(sI_n - M)$. In addition, $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ denotes the block matrix whereas $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ denotes the usual form of a realization of a linear system. Notation $|\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)|_{\mathcal{H}_\infty}$ refers to the $\mathcal{H}_\infty$ norm of the system defined as the maximal singular value for any frequency. Furthermore, for any analytic function F_1 and F_2, $F_1(s) = \underset{s\to 0}{O}\big(F_2(s)\big)$ means that the ratio $\frac{F_1}{F_2}(s)$ is finite for s tends to 0. Throughout this paper, variables in capital letters represent the Laplace transforms associated with variables in non-capital letters. Finally, $\mathcal{L}^2(0, 1; \mathbb{R})$ represents the set of square-integrable functions from normalized interval $(0, 1)$ to $\mathbb{R}$ with its associated dot product $\langle z_1|z_2\rangle = \int_0^1 z_1(\theta)z_2(\theta)d\theta$ and $\mathcal{C}^\infty(E_1, E_2)$ refers to the set of smooth functions from E_1 to E_2.

2 Problem statement

Consider a linear time-delay system with a single constant delay $h > 0$, modeled as an interconnection between an ordinary differential equation (1a) and a transport partial differential equation (1b) given by

$$\begin{cases} \dot{x}(t) = Ax(t) + B_d u(t), \\ y(t) = C_d x(t), \end{cases} \quad \forall t \in \mathbb{R}_{\geq 0}, \tag{1a}$$

$$\begin{cases} h\dfrac{\partial}{\partial t}z(t,\theta) = -\dfrac{\partial}{\partial \theta}z(t,\theta), \ \forall \theta \in [0,1], \\ \qquad z(t,0) = y(t), \\ \qquad u(t) = z(t,1), \end{cases} \quad \forall t \in \mathbb{R}_{\geq 0}, \tag{1b}$$

where matrices A, B_d and C_d respectively belongs to $\mathbb{R}^{m\times m}$, $\mathbb{R}^{m\times 1}$ and $\mathbb{R}^{1\times m}$ and are assumed to be constant and known.

Remark 1 *Note that B_d and C_d could also belong to $\mathbb{R}^{m\times l}$ and $\mathbb{R}^{l\times m}$ for $l \in \mathbb{N}$. For the sake of simplicity, we have chosen $l = 1$ since it can be easily extended to the general case by appealing to the Kronecker product. The case $l = 2$ is used in section 7 for Example 3.*

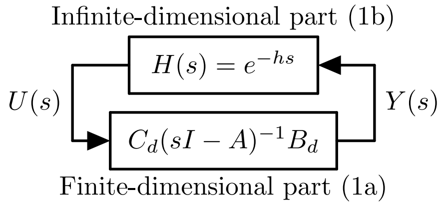


Figure 1: Block diagram of time-delay system (1).

Remark 2 *By noting that $z(t,\theta) = C_d x(t - \theta h)$, system (1) can be seen as a time-delay system where z is the infinite-dimensional state function considered over the normalized interval $[0,1]$ instead of $[-h,0]$, leading to the usual formulation $\dot{x}(t) = Ax(t) + B_d C_d x(t-h)$. Therefore, the results presented in this paper cover the large class of retarded time-delay systems.*

Remark 3 *Assuming f absolutely continuous from $[-h,0]$ to $\mathbb{R}^n$, system (1), satisfying the initial conditions $x(0) = f(0)$ and $z(0,\theta) = C_d f(-h\theta)$ for any $\theta \in [0,1]$, has a single solution (x,z) which belongs to continuous functions in time from $\mathbb{R}_{\geq 0}$ to $\mathbb{R}^n \times \mathcal{L}^2(0,1;\mathbb{R})$.[11]*

System (1) is depicted by Figure 1 in Laplace domain where $s \in \mathbb{C}$ is the Laplace variable. The infinite-dimensional part (1b) is driven by the transport equation and one obtains $G(s,\theta) = e^{-\theta h s} I_m$, the transfer function between $Y(s)$ and $Z(s,\theta)$. Throughout the paper, we will use indifferently the following notation

$$H(s) = G(s,1) = e^{-hs}, \tag{2}$$

for the transfer function between $Y(s) = Z(s,0)$ and $U(s) = Z(s,1)$.

The goal of the paper is to understand and to characterize the behavior of such time-delay systems and especially its stability. For that, the main issue is to approximate the infinite-dimensional part (1b), related to transport phenomenon, in a specific manner in order to get to underlying results on stability of time-delay systems. In the sequel, one recalls in the preliminaries section, some tools, which will be used to develop approximated models of the delay element.

3 Preliminaries

3.1 Padé approximants

Let an analytic function $H \in \mathcal{C}^\infty(\mathbb{C},\mathbb{C})$ be expanded in a Maclaurin series.

Definition 1 *The rational approximation with numerator $\mathcal{N}_p(s) = \sum_{i=0}^{p} a_i s^i$ at order p and denominator $\mathcal{D}_q(s) = \sum_{i=0}^{q} b_i s^i$ at order q is called $(p|q)$ Padé approximant of function $H(s)$ if*

$$H(s) - \frac{\mathcal{N}_p(s)}{\mathcal{D}_q(s)} = \underset{s\to 0}{O}(s^{p+q+1}). \tag{3}$$

Padé approximants can be seen as a generalization of Taylor expansion with a ratio of two polynomials given as power series.[5] Actually, Padé approximations are often used to solve numerically nonlinear fractional partial differential equation [39] and some extensions such as Padé-Chebyshev [26] or Padé-Legendre [9] approximations have been proposed to improve the solution.
In the rest of the paper, we focus on the $(n-1|n)$ and $(n|n)$ Padé approximations of the transfer function $H(s) = e^{-hs}$.

3.2 Legendre polynomials

Define Legendre polynomials l_k, for any $k \in \mathbb{N}$, by

$$l_k : \begin{cases} [0,1] \to \mathbb{R} \\ \theta \mapsto \displaystyle\sum_{i=0}^{k} (-1)^i \frac{(k+i)!}{(k-i)!(i!)^2}\theta^i. \end{cases} \tag{4}$$

The orthogonal family $(l_k)_{k\in\mathbb{N}}$ spans the Hilbert space $\mathcal{L}^2(0,1;\mathbb{R})$. In the following, $(l_k)_{k\in\mathbb{N}}$, as a basis of the state space of system (1b), is used to define extra-signals based on the projection of the state $z(t)$ on each Legendre polynomial.
For convenience of notation, the n first Legendre polynomials are concatenated in vector

$$\ell_n(\theta) = \begin{bmatrix} l_0(\theta) & \dots & l_{n-1}(\theta) \end{bmatrix}^T \in \mathbb{R}^n. \tag{5}$$

Some properties of Legendre polynomials [16] in terms of values at the boundaries, norm and derivation are recalled here,

$$\ell_n(0) = \mathbf{1}_n, \quad \ell_n(1) = \mathbf{1}_n^*, \quad \left\langle \ell_n \middle| \ell_n^T \right\rangle = \mathcal{I}_n^{-1}, \quad \frac{d}{d\theta}\ell_n(\theta) = -\mathbf{L}_n \ell_n(\theta), \quad \forall n \in \mathbb{N}, \tag{6}$$

with

$$\begin{array}{rclclrclcl} \mathbf{1}_n & = & \begin{bmatrix} 1 & \dots & 1 \end{bmatrix}^T & \in & \mathbb{R}^n, & \mathcal{I}_n & = & \mathrm{diag}(1, \dots, 2n-1) & \in & \mathbb{R}^{n \times n}, \\ \mathbf{1}_n^* & = & \begin{bmatrix} (-1)^0 & \dots & (-1)^{n-1} \end{bmatrix}^T & \in & \mathbb{R}^n, & \mathbf{L}_n & = & \mathrm{tril}(\mathbf{1}_n \mathbf{1}_n^T - \mathbf{1}_n^* \mathbf{1}_n^{*T})\mathcal{I}_n & \in & \mathbb{R}^{n \times n}. \end{array} \tag{7}$$

3.3 Fourier-Legendre series

Consider a function $G(s)$ in $\mathcal{C}^\infty([0,1];\mathbb{C})$, for all $s \in \mathbb{C}$. Its Fourier-Legendre series on the interval $[0,1]$ is

$$G(s, \theta) = \sum_{k=0}^{\infty} \frac{\langle l_k | G(s) \rangle}{\langle l_k | l_k \rangle} l_k(\theta), \quad \forall \theta \in [0,1].$$

Let us now define the truncated version G_n at order n in $\mathbb{N}$ as follows, for all $\theta \in [0,1]$,

$$G_n(s, \theta) = \sum_{k=0}^{n-1} \frac{\langle l_k | G(s) \rangle}{\langle l_k | l_k \rangle} l_k(\theta) = \ell_n^T(\theta) \mathcal{I}_n \langle \ell_n | G(s) \rangle. \tag{8}$$

Notice that the boundary values of $G_n(s)$ are easily expressed as

$$G_n(s, 0) = \mathbf{C}_n \langle \ell_n | G(s) \rangle, \quad G_n(s, 1) = \mathbf{C}_n^* \langle \ell_n | G(s) \rangle,$$

with

$$\mathbf{C}_n = \mathbf{1}_n^T \mathcal{I}_n, \quad \mathbf{C}_n^* = \mathbf{1}_n^{*T} \mathcal{I}_n. \tag{9}$$

In the following, our objective is to use Padé approximation combined with Legendre polynomials in order to build an approximation of $G(s, \theta) = e^{-\theta h s}$, the transfer function between $Y(s)$ and $Z(s, \theta)$.

3.4 Fourier-Legendre remainder

Define Fourier-Legendre remainder $\tilde{G}_n$ done by truncation at order n of the transfer function G on Legendre polynomials coefficients is, for all $s \in \mathbb{C}$ and $\theta \in [0,1]$,

$$\tilde{G}_n(s, \theta) = e^{-\theta h s} - \sum_{k=0}^{n-1} (2k+1) l_k(\theta) \langle l_k | G(s) \rangle. \tag{10}$$

Remark 4 *This remainder* (10) *is well-defined on the segment* $[0,1]$ *with respect to* θ *by the fact that* $G(s)$ *belongs to* $\mathcal{C}^\infty([0,1],\mathbb{C})$ *for any* s *in* $\mathbb{C}$ *and* $G_n(s)$ *converges uniformly on any compact subset of* $\mathbb{C}$.

Lemma 1 *For any* $n \in \mathbb{N}$*, the Fourier-Legendre error* $\tilde{G}_n$ *defined by* (10) *satisfies the following statement,*

$$\tilde{G}_n(s, \theta) = \underset{s \to 0}{O}(s^n), \quad \forall \theta \in [0,1].$$

Proof : The statement of Lemma 1 means that $\tilde{G}_n(s, \theta)$ has its $n-1$ first derivatives with respect to s evaluated at $s = 0$ equal to zero. Notice that error (10) can be rewritten as

$$\tilde{G}_n(s, \theta) = \sum_{k=n}^{\infty} (2k+1) l_k(\theta) \int_0^1 e^{-\tau h s} l_k(\tau) d\tau.$$

Then, since G is smooth and G_n converges uniformly, p successive derivations of $\tilde{G}_n$ with respect to s give

$$\frac{\partial^p}{\partial s^p} \tilde{G}_n(s, \theta) = \sum_{k=n}^{\infty} (2k+1) l_k(\theta) \int_0^1 (-h\tau)^p e^{-\tau h s} l_k(\tau) d\tau.$$

Evaluating it at $s = 0$ and by the fact that τ^p can be decomposed on p first Legendre polynomials, one obtains zero for $p < n$. □

This error is close to zero for s near zero and allows us to expect that it leads to accurate models, which satisfy the definition of the Padé approximations (3).

3.5 A technical lemma

Recall the matrix inversion lemma, also called Woodbury matrix identity. For any vectors u, v in $\mathbb{R}^n$ and non singular matrix M in $\mathbb{R}^{n\times n}$, it states that

$$1 - v^T(M + uv^T)^{-1}u = (1 + v^T M^{-1}u)^{-1}. \tag{11}$$

Remind also the matrix determinant lemma, for any matrices defined above,

$$\det(M + uv^T) = \det(M)(1 + v^T M^{-1}u). \tag{12}$$

Derived from (11),(12), a usefull lemma can then be given.

Lemma 2 *For any $u \in \mathbb{R}^n$ with a non-zero first component, $v \in \mathbb{R}^n$ not equal to the zero vector and $L \in \mathbb{R}^{n\times n}$ a strictly lower triangular matrix such that* $\operatorname{rank}(L) = n - 1$*, one obtains*

$$1 - v^T(sI_n + L + uv^T)^{-1}u = \underset{s\to 0}{O}(s^n). \tag{13}$$

Proof : The matrix inversion lemma (11) applied to vectors u, v and matrix $M = sI_n + L$ gives

$$1 - v^T(sI_n + L + uv^T)^{-1}u = (1 + v^T(sI_n + L)^{-1}u)^{-1},$$

and the matrix determinant lemma (12) leads to

$$1 - v^T(sI_n + L + uv^T)^{-1}u = \frac{\det(sI_n + L)}{\det(sI_n + L + uv^T)}.$$

Then, since L is strictly lower triangular, we have

$$\det(sI_n + L) = \det(sI_n) = s^n.$$

and, because L has non-zero under diagonal coefficients and under the hypothesis done on vectors u, v, matrix $L+uv^T$ has full rank which means $\det(sI_n+L+uv^T) \neq 0$ in a neighborhood of $s = 0$, which concludes the proof. $\square$

4 Modeling of the delay element

In this part, the objective is to split the delay transfer function in an adequate manner to reach delay-dependent stability results. The finite-dimensional part is chosen to be a nice approximation of the delay and, on the contrary of what it is usually done in spectral methods, the remainder is conserved and structured in order to use robust analysis and to establish stability results.

4.1 Naive Padé modeling

An intuitive approach in Laplace domain is to take Padé rational approximations $\mathcal{P}_{(p_n|q_n)}(s) = \frac{\mathcal{N}_{p_n}(s)}{\mathcal{D}_{q_n}(s)}$ of the transfer $H(s) = e^{-hs}$. Indices p_n and q_n are positive integers which are given in function of $n \in \mathbb{N}$. These approximations are as accurate as required on any compact subset of $\mathbb{C}$ if the limit of $\frac{p_n}{q_n}$ for $n \to \infty$ is finite.

Proposition 1 *The delay transfer function $H(s) = e^{-hs}$ can be split into two parts*

$$H(s) = \mathcal{P}_{(p_n|q_n)}(s) + \left(H(s) - \mathcal{P}_{(p_n|q_n)}(s)\right), \tag{14}$$

and $\tilde{\mathcal{P}}_{(p_n|q_n)}(s) = H(s) - \mathcal{P}_{(p_n|q_n)}(s)$ is the $(p_n|q_n)$ Padé remainder to be evaluated hereafter.

This splitting is depicted on Figure 2. To take advantages of the relevance of these approximated models, one knows that the choice of the error put aside is not inconsequential. Indeed, focusing on $(n-1|n)$ or $(n|n)$, errors $\tilde{\mathcal{P}}_{(p_n|q_n)}(s)$ are upper bounded [30] and lead to some conservative results. Along the frequencies denoted ω, the modulus of both errors $\tilde{\mathcal{P}}_{(n-1|n)}(j\omega)$ and $\tilde{\mathcal{P}}_{(n|n)}(j\omega)$ are depicted in Figure 3. In particular, $|\tilde{\mathcal{P}}_{(n|n)}|_{\mathcal{H}_\infty} = \frac{1}{2}$ for any $n \in \mathbb{N}$. Hence, the errors transfer function can be embedded into a norm-bounded uncertainty[7]. By interconnection with the ordinary differential equation part, the small-gain theorem can then be applied. Nevertheless, in practice, this naive model is too conservative, not suitable and does not lead to an efficient stability criterion. A deeply fit of the remainder is then recommended and implies to design filters. Indeed, one notes that the slopes in low frequencies are respectively $40n$dB and $(40n + 20)$dB by decade. But, such candidate filters [28], closely related to a specific Padé remainder transfer function, are difficult to find.

Guided by spectral methods [48] and stability works [43, 2] with Legendre polynomials, we have chosen to bypass the design of adequate filters by the use of a description of the delay element thanks to the n first Legendre polynomials coefficients. For $n \in \mathbb{N}$, it forms a particular state $\langle \ell_n | z(t) \rangle$ denoted hereafter as $\mathbf{z}_n(t)$. On this state space, two models are designed by considering Fourier-Legendre remainders at order n (see subsection 4.2) and

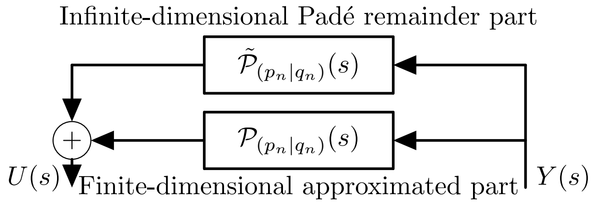
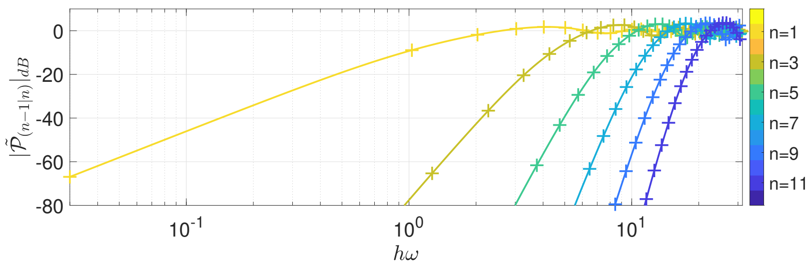
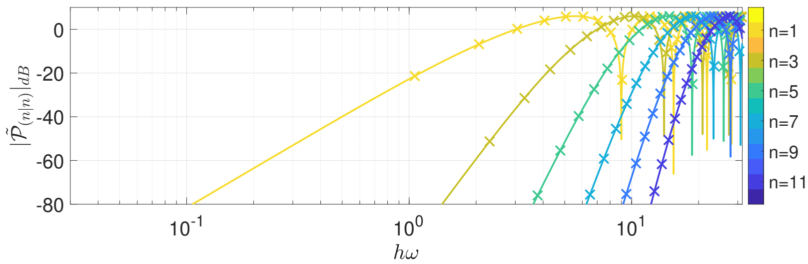


Figure 2: Naive modeling of the delay element (1b).



(a) First error transfer function.



(b) Second error transfer function.

Figure 3: Modulus of $\tilde{\mathcal{P}}_{(n-1|n)}$ on the top and $\tilde{\mathcal{P}}_{(n|n)}$ on the bottom with respect to the frequencies.

$n+1$ (see subsection 4.3). The associated error transfers $\tilde{H}_n$ and $\tilde{H}_n^\flat$ are simply a linear combination between e^{-hs} and Legendre polynomials coefficients of transfer $G(s)$. It relies on Fourier-Legendre remainders of the delayed transfer functions $G(s,1)$ and $G(s,0)$. Then, the resulting state form representations obtained reveal to give the $(n-1|n)$ and $(n|n)$ Padé approximated models and to provide suitable filters. Including the interconnection with the finite-dimensional system (1a), these new decompositions of transfer function H involve, by a small-gain approach, stability results for time-delay systems.

4.2 A first modeling with Fourier-Legendre remainder at order n

To derive information from the delay, thanks to the n first Legendre polynomials coefficients collocated in $\mathbf{z}_n$, one proposes to study the Fourier-Legendre remainder at order n of transfer function $H(s) = G(s,1)$, denoted $\tilde{H}_n(s)$ given by

$$\tilde{H}_n(s) = \tilde{G}_n(s,1) = e^{-hs} - \mathbf{C}_n^* \langle \ell_n | G(s) \rangle . \tag{15}$$

Proposition 2 *The delay transfer function $H(s) = e^{-hs}$ satisfies the following decomposition*

$$H(s) = \mathbf{C}_n^*(hsI_n - \mathbf{A}_n)^{-1}\mathbf{1}_n + \left(1 - \mathbf{C}_n^*(hsI_n - \mathbf{A}_n)^{-1}\mathbf{1}_n^*\right)\tilde{H}_n(s), \tag{16}$$

where the left finite-dimensional part $\begin{pmatrix} h^{-1}\mathbf{A}_n & \mathbf{1}_n \\ h^{-1}\mathbf{C}_n^* & 0 \end{pmatrix}$ *is a realization of the* $(n-1|n)$ *Padé approximation* $\mathcal{P}_{(n-1|n)}(s)$ *of the delay. Matrix* $\mathbf{C}_n^*$ *is defined in* (9) *and*

$$\mathbf{A}_n = -(\mathbf{L}_n + \mathbf{1}_n^* \mathbf{C}_n^*). \tag{17}$$

Proof : Using an integration by parts, the dynamics of the n first Legendre coefficients $\mathbf{z}_n(t)$ lead to

$$h\dot{\mathbf{z}}_n(t) = -\left\langle \ell_n \middle| \frac{\partial}{\partial\theta} z(t) \right\rangle = \left\langle \frac{d}{d\theta}\ell_n \middle| z(t) \right\rangle - \ell_n(1)z(t,1) + \ell_n(0)z(t,0).$$

The properties of Legendre polynomial coefficients given in (6) ensure that the previous equation can be rewritten

$$h\dot{\mathbf{z}}_n(t) = -\mathbf{L}_n\mathbf{z}_n(t) - \mathbf{1}_n^* z(t,1) + \mathbf{1}_n z(t,0). \tag{18}$$

The signal $z(t,1)$ can be approximated by its the truncated Fourier-Legendre series at order n. The remainder on the side $\theta = 1$ is called $\epsilon_n(t)$ and we have

$$u(t) = z(t,1) = \mathbf{C}_n^*\mathbf{z}_n(t) + \underbrace{z(t,1) - \mathbf{C}_n^*\mathbf{z}_n(t)}_{\epsilon_n(t)}.$$

That implies

$$\begin{cases} h\dot{\mathbf{z}}_n(t) = \mathbf{A}_n\mathbf{z}_n(t) + \mathbf{1}_n y(t) - \mathbf{1}_n^*\epsilon_n(t), \\ \quad u(t) = \mathbf{C}_n^*\mathbf{z}_n(t) + \epsilon_n(t). \end{cases} \tag{19}$$

In the Laplace domain, the error is defined by $\mathcal{E}_n(s) = \tilde{H}_n(s)Y(s)$, where the error transfer function is given by (15). It is the remainder of the truncated Fourier-Legendre series of G with $\theta = 1$ at order n. System (19) leads to

$$\begin{cases} \mathbf{Z}_n(s) = (hsI_n - \mathbf{A}_n)^{-1}(\mathbf{1}_n - \mathbf{1}_n^*\tilde{H}_n(s))Y(s), \\ \; U(s) = \mathbf{C}_n^*\mathbf{Z}_n(s) + \tilde{H}_n(s)Y(s). \end{cases}$$

The transfer function $H(s) = e^{-hs}$ from $Y(s)$ to $U(s)$ can then be rewritten as the sum given by (16).
Thanks to Lemma 1, we already have $\tilde{H}_n(s) = \underset{s\to 0}{O}(s^n)$. Then, considering $\mathbf{A}_n$ given in (17), by application of Lemma 2 with vectors $u = \mathbf{1}_n^*$, $v^T = \mathbf{C}_n^*$ and matrix $L = \mathbf{L}_n$ which satisfy the expected assumptions, we find

$$H(s) - \mathbf{C}_n^*(hsI_n - \mathbf{A}_n)^{-1}\mathbf{1}_n = \underset{s\to 0}{O}(s^{2n}).$$

According to the definition of the Padé approximations given in (3) and knowing that $\chi_{\mathbf{A}_n}(hs)$ is a polynomial of degree n and that $\mathbf{C}_n^*\mathrm{adj}(hsI_n - \mathbf{A}_n)\mathbf{1}_n$ is a polynomial of degree $n-1$ with respect to s, we prove that $H_n(s) = \mathbf{C}_n^*(hsI_n - \mathbf{A}_n)^{-1}\mathbf{1}_n$ is a $(n-1|n)$ Padé approximant of the transport equation $H(s) = e^{-hs}$. $\square$

Remark 5 *The same proposition was also proven by induction.[3] Note that the proposed proof is quite different and easier to generalize to other transfer functions H.*

4.3 A second modeling with Fourier-Legendre remainder at order $n+1$

To describe more precisely the delay, based on tau-models with Legendre polynomials, Fourier-Legendre remainders at order $n+1$ are now taken into consideration. This second error transfer function based on Fourier-Legendre remainders of the delayed transfer functions $G(s,1)$ and $G(s,0)$ is called $\tilde{H}_n^\flat$ and is equal to

$$\tilde{H}_n^\flat(s) = \tilde{G}_{n+1}(s,1) - (-1)^n\tilde{G}_{n+1}(s,0). \tag{20}$$

It is constructed naturally in the proof of the following proposition thanks to information taken from the boundary condition.

Proposition 3 *The delay transfer function function $H(s) = e^{-hs}$ satisfies this new decomposition*

$$H(s) = \mathbf{C}_n^\flat(hsI_n - \mathbf{A}_n^\flat)^{-1}\mathbf{B}_n^\flat + \mathbf{D}_n^\flat + \left(1 - \mathbf{C}_n^\flat(hsI_n - \mathbf{A}_n^\flat)^{-1}\mathbf{1}_n^*\right)\tilde{H}_n^\flat(s), \tag{21}$$

where system $\begin{pmatrix} h^{-1}\mathbf{A}_n^\flat & \mathbf{B}_n^\flat \\ h^{-1}\mathbf{C}_n^\flat & \mathbf{D}_n^\flat \end{pmatrix}$ is a realization of the $(n|n)$ Padé approximation $\mathcal{P}_{(n|n)}(s)$ of the delay and where the infinite-dimensional residual part $\tilde{H}_n^\flat(s)$ is given by (20). Matrices of this representation are defined by

$$\mathbf{A}_n^\flat = -\left(\mathbf{L}_n + \mathbf{1}_n^*\mathbf{C}_n^\flat\right), \quad \mathbf{B}_n^\flat = \mathbf{1}_n - \mathbf{D}_n^\flat\mathbf{1}_n^*, \quad \mathbf{C}_n^\flat = \mathbf{C}_n^* - \mathbf{D}_n^\flat\mathbf{C}_n, \quad \mathbf{D}_n^\flat = (-1)^n. \tag{22}$$

Proof : To obtain a more accurate model than the one proposed in Proposition 2, one takes into account one more Legendre coefficient. Since we have, for $\theta = 0$,

$$z(t,0) = y(t) = \mathbf{C}_{n+1}\mathbf{z}_{n+1}(t) + \epsilon_{n+1}^*(t),$$

the additional coefficient z_n is evaluated from the boundary condition as follows

$$(2n+1)z_n(t) = y(t) - \mathbf{C}_n\mathbf{z}_n(t) - \epsilon_{n+1}^*(t).$$

That involves a novel output approximation given by

$$\begin{aligned} z(t,1) &= \mathbf{C}_{n+1}^*\mathbf{z}_{n+1}(t) + \epsilon_{n+1}(t), \\ &= \mathbf{C}_n^*\mathbf{z}_n(t) + \mathbf{D}_n^\flat(2n+1)z_n(t) + \epsilon_{n+1}(t), \\ &= \mathbf{C}_n^\flat\mathbf{z}_n(t) + \mathbf{D}_n^\flat z(t,0) + \epsilon_n^\flat(t), \end{aligned}$$

taking $\epsilon_n^\flat(t) = \epsilon_{n+1}(t) - \mathbf{D}_n^\flat \epsilon_{n+1}^*(t)$. This error signal, combination of the errors at order $n+1$ given at each side of the segment $[0,1]$, can be seen as the inverse Laplace transform of $\tilde{H}_n^\flat(s)Y(s)$ with

$$\tilde{H}_n^\flat(s) = \tilde{G}_{n+1}(s,1) - \mathbf{D}_n^\flat \tilde{G}_{n+1}(s,0),$$

as suggested in (20). By putting aside this error $\epsilon_n^\flat(t)$ done on $u(t) = z(t,1)$, the dynamical equation (18) leads to

$$\begin{cases} h\dot{\mathbf{z}}_n(t) = \mathbf{A}_n^\flat \mathbf{z}_n(t) + \mathbf{B}_n^\flat y(t) - \mathbf{1}_n^* \epsilon_n^\flat(t), \\ u(t) = \mathbf{C}_n^\flat \mathbf{z}_n(t) + \mathbf{D}_n^\flat y(t) + \epsilon_n^\flat(t). \end{cases} \tag{23}$$

The Laplace transform of such system (23) gives

$$\begin{cases} \mathbf{Z}_n(s) = (hsI_n - \mathbf{A}_n^\flat)^{-1}(\mathbf{B}_n^\flat - \mathbf{1}_n^* \tilde{H}_n^\flat(s))Y(s), \\ U(s) = \mathbf{C}_n^\flat \mathbf{Z}_n(s) + (\mathbf{D}_n^\flat + \tilde{H}_n^\flat(s))Y(s). \end{cases}$$

The transfer function $H(s)$ between $Y(s)$ and $U(s)$ can then be cut into two parts as formulated by (21). Thanks to Lemma 2 with $u = \mathbf{1}_n^*$, $v^T = \mathbf{C}_n^\flat$ and $L = \mathbf{L}_n$ from the structure of $\mathbf{A}_n^\flat$ and Lemma 1 applied at order $n+1$, one obtains

$$H(s) - \mathbf{C}_n^\flat (hsI_n - \mathbf{A}_n^\flat)^{-1} \mathbf{B}_n^\flat - \mathbf{D}_n^\flat = \underset{s\to 0}{O}(s^{2n+1}).$$

Therefore, according to (3) and because we know that $\chi_{\mathbf{A}_n^\flat}(hs)$ is a polynomial of degree n and that $\mathbf{C}_n^\flat \mathrm{adj}(hsI_n - \mathbf{A}_n^\flat)\mathbf{B}_n^\flat + \chi_{\mathbf{A}_n^\flat}(hs)\mathbf{D}_n^\flat$ is a polynomial of degree n, the transfer function $H_n^\flat(s)$ given by $\mathbf{C}_n^\flat(hsI_n - \mathbf{A}_n^\flat)^{-1}\mathbf{B}_n^\flat + \mathbf{D}_n^\flat$ is a $(n|n)$ Padé approximant of the expected transfer function $H(s) = e^{-hs}$. □

Remark 6 *Older works [1] deal theoretically with the link between Padé realization and Legendre orthogonal polynomials but never focus on the error part and the potential underlying applications.*

4.4 Synthesis on the two Legendre-based modeling

By the use of the first Legendre polynomials coefficients, two models have been presented in order to approximate finely the delay behavior. Especially, the finite-dimensional parts turn out to be related to the well-known $(n-1|n)$ and $(n|n)$ Padé approximants. From n coefficients,

$$H_n(s) := \left(\begin{array}{cc} h^{-1}\mathbf{A}_n & \mathbf{1}_n \\ h^{-1}\mathbf{C}_n & 0 \end{array}\right), \quad H_n^\flat(s) := \left(\begin{array}{cc} h^{-1}\mathbf{A}_n^\flat & \mathbf{B}_n^\flat \\ h^{-1}\mathbf{C}_n^\flat & \mathbf{D}_n^\flat \end{array}\right), \tag{24}$$

are respectively equal to $\mathcal{P}_{(n-1|n)}(s)$ and $\mathcal{P}_{(n|n)}(s)$. Moreover, our Legendre-based models extract from the Padé remainders the finite-dimensional filters

$$W_n(s) := \left(\begin{array}{cc} h^{-1}\mathbf{A}_n & \mathbf{1}_n^* \\ -h^{-1}\mathbf{C}_n^* & 1 \end{array}\right), \quad W_n^\flat(s) := \left(\begin{array}{cc} h^{-1}\mathbf{A}_n^\flat & \mathbf{1}_n^* \\ -h^{-1}\mathbf{C}_n^\flat & 1 \end{array}\right). \tag{25}$$

The leftover infinite-dimensional parts are simply Fourier-Legendre remainders at orders n and $n+1$ and they can also be given on the state form representation as

$$\tilde{H}_n(s) := \left(\begin{array}{cc} -h^{-1}\mathbf{L}_n & \mathbf{1}_n - \mathbf{1}_n^* e^{-hs} \\ -h^{-1}\mathbf{C}_n^* & e^{-hs} \end{array}\right), \quad \tilde{H}_n^\flat(s) := \left(\begin{array}{cc} -h^{-1}\mathbf{L}_n & \mathbf{1}_n - \mathbf{1}_n^* e^{-hs} \\ -h^{-1}\mathbf{C}_n^\flat & -\mathbf{D}_n^\flat + e^{-hs} \end{array}\right). \tag{26}$$

To sum up, the transport phenomenon has been modeled as shown in Figure 4. In the next section, by having the same realization state, finite-dimensional approximated part H_n and filter W_n (resp. $H_n^\flat$ and $W_n^\flat$) are merged into the same delay-dependent finite-dimensional block. By interconnection with (1a), two augmented systems equivalent to (1) are finally constructed. Modal and frequency analysis of time-delay systems is then investigated in the next seciton.

5 Modeling of time-delay systems

5.1 Augmented time-delay systems

Focusing on time-delay system (1), it is now possible to split it to have a finite-dimensional part which, increasing its order n, incorporates a more precise description of the behavior of the whole system. System (1) can then be rewritten as the interconnection depicted in Figure 5 where the finite-dimensional part can be given by (27) or (29). These models are simply a Redheffer product of the finite dimensional part (1a) with each state representations (19) and (23) proposed above. From one side, one obtains

$$\begin{bmatrix} \dot{\xi}_n(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \mathbb{A}_n & \mathbb{B}_n \\ \mathbb{C}_n & 0 \end{bmatrix} \begin{bmatrix} \xi_n(t) \\ \epsilon_n(t) \end{bmatrix}, \tag{27}$$

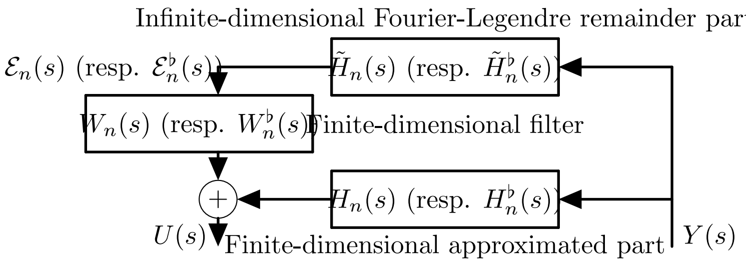


Figure 4: Modeling of the delay element (1b) by the use of Legendre polynomials.

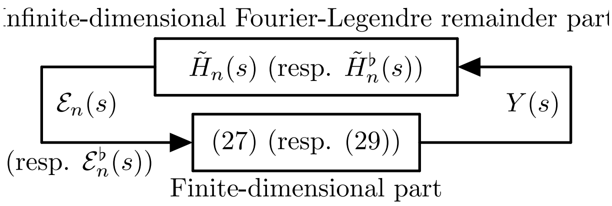


Figure 5: Block diagram of the redesigned time-delay system.

with $\xi_n(t) = \begin{bmatrix} x(t) \\ h\mathbf{z}_n(t) \end{bmatrix}$ and

$$\mathbb{A}_n = \begin{bmatrix} A & h^{-1}B_d\mathbf{C}_n^* \\ \mathbf{1}_n C_d & h^{-1}\mathbf{A}_n \end{bmatrix}, \ \mathbb{B}_n = \begin{bmatrix} B_d \\ -\mathbf{1}_n^* \end{bmatrix}. \ \mathbb{C}_n = \begin{bmatrix} C_d & 0 \end{bmatrix}. \tag{28}$$

From the other side, one gets

$$\begin{bmatrix} \dot{\xi}_n(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \mathbb{A}_n^\flat & \mathbb{B}_n \\ \mathbb{C}_n & 0 \end{bmatrix} \begin{bmatrix} \xi_n(t) \\ \epsilon_n^\flat(t) \end{bmatrix}, \tag{29}$$

where $\mathbb{B}_n$ and $\mathbb{C}_n$ are given in (28) and where

$$\mathbb{A}_n^\flat = \begin{bmatrix} A + B_d\mathbf{D}_n^\flat C_d & h^{-1}B_d\mathbf{C}_n^\flat \\ \mathbf{B}_n^\flat C_d & h^{-1}\mathbf{A}_n^\flat \end{bmatrix}. \tag{30}$$

Remark 7 *Note that model* (29) *is identical to the one developed by Legendre-tau method [22] based on Galerkin approximations. It is recognized to be very accurate to approximate solutions of system* (1)*.*

The interconnected infinite-dimensional part is represented by its transfer function given by (15) (resp. (20)), closely linked with Fourier-Legendre remainder of the transported transfer function G on each extremities.
These two augmented time-delay systems (27),(29) have good properties to deal with spectrum analysis and assess stability of time-delay systems.

5.2 Approximation of the characteristic roots

Putting aside the infinite-dimensional part, both finite-dimensional models given by the state matrices $\mathbb{A}_n$ and $\mathbb{A}_n^\flat$ can be used to approximate the characteristic roots of the original time-delay system (1). As increasing n, these models are able to formulate more and more precisely the behavior of the infinite-dimensional system thanks to the information on the transported signal contained into the additional states $\mathbf{z}_n(t)$. Indeed, the following theorem proves that a certain number of the $m+n$ eigenvalues of $\mathbb{A}_n$ (resp. $\mathbb{A}_n^\flat$) can approximate as close as desired the characteristic roots of system (1) increasing n.

Theorem 1 *For $R > 0$, if system* (1) *contains K characteristic roots with multiplicities $\nu^*_{k\in\{1,\dots,K\}}$ into the open ball $\mathcal{B}(0,R)$, then $\sum_{k=1}^{K}\nu_k^*$ eigenvalues of $\mathbb{A}_n$ (resp. $\mathbb{A}_n^\flat$) converges towards them. More precisely,*

$$\forall r \in (0, r^*), \quad \exists n^* \in \mathbb{N}; \quad \forall n \geq n^*, \quad \max|s_n - s^*| \leq r, \tag{31}$$

with s^* the vector which contains the K expected roots repeated by their multiplicity and s_n the vector with the corresponding approximated eigenvalues.

Proof : Define χ_n and χ which belong to $\mathcal{C}^\infty(\mathbb{C},\mathbb{C})$ as

$$\forall s \in \mathbb{C}, \begin{cases} \chi_n(s) = \det\big((sI_n - A)\mathcal{D}_{q_n}(s) - B_d C_d \mathcal{N}_{p_n}(s)\big), \\ \chi(s) = \det\big((sI_n - A)e^{\frac{hs}{2}} - B_d C_d e^{-\frac{hs}{2}}\big), \end{cases} \tag{32}$$

with $\mathcal{N}_{p_n}(s)$ and $\mathcal{D}_{q_n}(s)$ respectively being the numerator and denominator of the $(p_n|q_n)$ Padé approximation of function e^{-hs}. Here, the cases $(n-1|n)$ and $(n|n)$ are handled. Based on convergence results issued from Padé theory [4], $\chi_n(s)$ converges uniformly to $\chi(s)$ on compact sets of the complex plane.[**breda2015eig**] By application of the Hurwitz's theorem [10], the zeros of χ_n are close enough to some zeros of χ (i.e. characteristic roots of the original time-delay system), for n chosen sufficiently large, which concludes the proof. □

Remark 8 *For more details, one refers to Theorem 8 and the associated proof in [3].*

Nevertheless, even if this convergence property from Padé is interesting, it is not sufficient to assert the stability of time-delay systems. Using robust analysis, and especially the small gain theorem, the whole model is exploited in order to obtain numerically tractable stability conditions.

6 Stability analysis of time-delay systems

Based on the strong properties of the modeling proposed in the previous section, subsequent stability analysis can be obtained by a robust approach.

6.1 Structure of the infinite-dimensional Fourier-Legendre remainders

Taking into account the well-chosen remainder ϵ_n (resp. $\epsilon_n^\flat$), multiple ways to analyze the stability of the original time-delay system (1) can be investigated. Relying on the fact that the realization of the Padé approximated finite-dimensional models (27) (resp. (29)) are constructed on Legendre polynomials coefficients, orthogonal polynomial properties can be used. By considering Bessel inequality, a Lyapunov-Krasovskii approach provides sufficient condition of stability with respect to the delay. For instance, stability criterion can be proposed in term of linear matrix inequality.[43, 3] Nevertheless, a more intuitive approach consists in applying the small-gain technique. Even if it is a conservative approach, it brings a straightforward and fast delay-dependent stability criterion given in 6.2 by Theorem 2.
To do so, the infinite-dimensional remainders are embedded into a delay-free unstructured uncertainties. That implies to verify that errors $\tilde{H}_n$ (resp. $\tilde{H}_n^\flat$), the transfer functions from $Y(s)$ to $\mathcal{E}_n(s)$ (resp. $\mathcal{E}_n^\flat(s)$) given by (15) (resp. (20)), are bounded.

Lemma 3 *For any $n \in \mathbb{N}$, the $\mathcal{H}_\infty$ norms of $\tilde{H}_n$ and $\tilde{H}_n^\flat$ exist.*

Proof : First, $\tilde{H}_n$ and $\tilde{H}_n^\flat$ recalled in (26) are causal transfer functions with no poles in the right half planes. By confining now to the imaginary axis with frequencies denoted ω, pair functions $|\tilde{H}_n|(\omega)$ and $|\tilde{H}_n^\flat|(\omega)$ are smooth, null in zero and have a bounded behavior as $\omega \to \infty$ ($\lim\limits_{s\to\infty}(\tilde{H}_n(s)) = 1$ and $\lim\limits_{s\to\infty}(\tilde{H}_n(s)) \leq 2$). From the extreme value theorem, both errors are upper bounded. □

Define error bounds γ_n and $\gamma_n^\flat$ such as

$$\gamma_n|\tilde{H}_n|_{\mathcal{H}_\infty} < 1, \quad \gamma_n^\flat|\tilde{H}_n^\flat|_{\mathcal{H}_\infty} < 1. \tag{33}$$

These lower bounds are computed with a precision 10^{-3} thanks to derivative-free optimization such as Nelder-Mead algorithm [29] applied to $|\tilde{H}_n|^{-1}_{\mathcal{H}_\infty}$ (resp. $|\tilde{H}_n^\flat|^{-1}_{\mathcal{H}_\infty}$) with an initial point at low frequencies.

Remark 9 *Note that $\gamma_n < 1$ and $\gamma_n^\flat < 0.5$ confirm that the minimal bounds are not reached for ω tends to ∞.*

Remark 10 *Besides, values of $|\tilde{H}_n|_{\mathcal{H}_\infty}$ and $|\tilde{H}_n^\flat|_{\mathcal{H}_\infty}$ could be upper bounded by application of the restrictive triangular inequality and Bessel inequality by $\sqrt{2n}$ and $\sqrt{4(n+1)}$, respectively. In fact, we have*

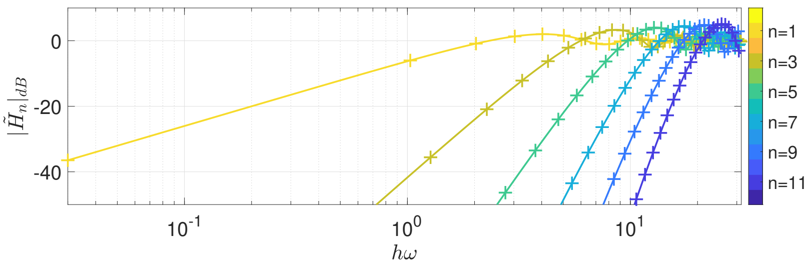
$$|\tilde{G}_n(j\omega,\theta)|^2 \leq |G(j\omega,\theta)|^2 + \sum_{k=0}^{n-1}(2k+1)\frac{|\langle l_k|G(j\omega)\rangle|^2}{\langle l_k|l_k\rangle} \leq |e^{-j\theta h\omega}|^2 + (2n-1)|e^{-j\theta h\omega}|^2 = 2n,\ \forall\omega\in\mathbb{R}.$$

However, by the use of these bounds the result would be too restrictive. Indeed, the bounds γ_n and $\gamma_n^\flat$ are much finer.

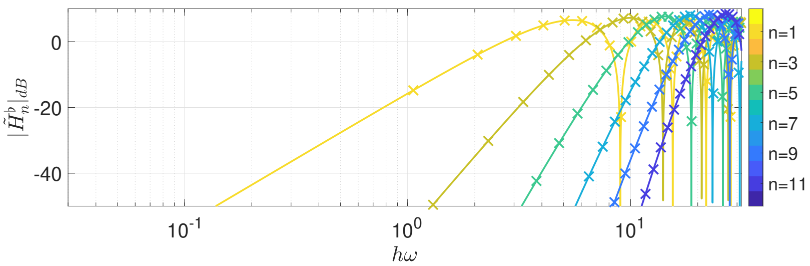
As both errors can be given in function of hs, the bounds γ_n and $\gamma_n^\flat$ are independent of the delay h and can directly be saved and shown on Table 1. Consequently, a sufficient delay-dependent stability condition based on the small-gain theorem is applied to augmented time-delay systems (27) and (29).

n	0	1	2	3	4	5	6	7	8	9	10	11	12
γ_n	1.000	0.793	0.722	0.680	0.651	0.629	0.611	0.597	0.584	0.573	0.564	0.555	0.547
$\gamma_n^\flat$	0.500	0.470	0.450	0.435	0.423	0.413	0.404	0.397	0.391	0.385	0.380	0.375	0.371

Table 1: Lower bounds of $|\tilde{H}_n|_{\mathcal{H}_\infty}^{-1}$ and $|\tilde{H}_n^\flat|_{\mathcal{H}_\infty}^{-1}$ with respect to n.



(a) First error transfer function.



(b) Second error transfer function.

Figure 6: Modulus of $\tilde{H}_n$ on the top and $\tilde{H}_n^\flat$ on the bottom with respect to the frequencies.

Remark 11 *The modulus of both errors $\tilde{H}_n$ (resp. $\tilde{H}_n^\flat$) are depicted in Figure 6. In Table 1, these errors have been roughly bounded independently of $h\omega$. However, Figure 6 shows that the error remains small in larger ranges of frequencies as the order n increases. For low frequencies, the slope is of $20n dB$ (resp. $20(n+1)dB$) by decade as Lemma 1 applied to $\tilde{H}_n$ (resp. $\tilde{H}_n^\flat$) shows. Then, the error could also be deeply fitted by using a frequency characterization and applying Kalman-Yacubovich-Popov on frequency intervals [24]. By upper bounding the error by a high-pass filter, the μ analysis can also be used to propose a tighter result. The subsequent results would be less restrictive but at the price of a much more complex algorithm than the proposed application of the small-gain theorem leading to Theorem 2. A last possible way to improve the conservatism of the result consists in better choosing the high-pass filter W_n (resp. $W_n^\flat$). It might be possible to improve the resulting by scaling other basis of $\mathcal{L}^2(0,1;\mathbb{R})$. This could be the subject of future works.*

6.2 Sufficient condition of stability by application of the small-gain theorem

By $\mathcal{H}_\infty$ analysis, a stability condition for time-delay systems with respect to the delay is formulated in the following theorem.

Theorem 2 *If the $\mathcal{H}_\infty$ norm of system (27) (resp. (29)) is lower than γ_n (resp. $\gamma_n^\flat$) then time-delay system (1) is stable.*

Proof : By application of the small-gain theorem on the augmented time-delay system (27), we directly obtain the sufficient condition of stability. Indeed, the inequality $|\left(\begin{smallmatrix}\mathbb{A}_n & \mathbb{B}_n \\ \mathbb{C}_n & 0\end{smallmatrix}\right)|_{\mathcal{H}_\infty} < \gamma_n$ implies, thanks to (33), $|\left(\begin{smallmatrix}\mathbb{A}_n & \mathbb{B}_n \\ \mathbb{C}_n & 0\end{smallmatrix}\right)|_{\mathcal{H}_\infty}|\tilde{H}_n|_{\mathcal{H}_\infty} < 1$. The proof works similarly for system (29) by replacing $\tilde{H}_n$, $\mathbb{A}_n$ and γ_n by $\tilde{H}_n^\flat$, $\mathbb{A}_n^\flat$ and $\gamma_n^\flat$, respectively. $\square$

Remark 12 *It is important to see that a necessary condition of Theorem 2 at order n is the stability of the corresponding finite-dimensional model. If $\mathbb{A}_n$ (resp. $\mathbb{A}_n^\flat$) is not stable then, the test cannot be performed.*

By reformulation of Theorem 2, calling

$$\rho_n = \frac{1}{\gamma_n}|\left(\begin{smallmatrix}\mathbb{A}_n & \mathbb{B}_n \\ \mathbb{C}_n & 0\end{smallmatrix}\right)|_{\mathcal{H}_\infty}, \quad \rho_n^\flat = \frac{1}{\gamma_n^\flat}|\left(\begin{smallmatrix}\mathbb{A}_n^\flat & \mathbb{B}_n \\ \mathbb{C}_n & 0\end{smallmatrix}\right)|_{\mathcal{H}_\infty}, \tag{34}$$

if $\rho_n < 1$ (resp. $\rho_n^\flat < 1$) then system (1) is stable.

It is also possible to extend the proposed theorem to time-delay systems subject to polytopic uncertainties. Assuming matrices A, B_d and the inverse of the delay h contained into a bounded convex polytope $\mathcal{P}$, the problem can be tackle with linear matrix inequalities as presented in the following Corollary.

Corollary 1 *If it exists a unique symmetric positive definite matrix $\mathbb{P}_n$ such as $\begin{bmatrix} \mathbb{P}_n\mathbb{A}_n+\mathbb{A}_n^T\mathbb{P}_n+\mathbb{C}_n^T\mathbb{C}_n & \mathbb{P}_n\mathbb{B}_n \\ * & -\gamma_n^2 \end{bmatrix}$ are negative definite on the vertices of the convex hull formed by $\mathcal{P}$, then system* (1) *subject to these uncertainties is stable.*

Proof : Firstly, by application of Kalman-Yacubovitch-Popov lemma [41], the criterion given by Theorem 2 is equivalent to find a positive definite matrix $\mathbb{P}_n$ such as $\begin{bmatrix} \mathbb{P}_n\mathbb{A}_n+\mathbb{A}_n^T\mathbb{P}_n+\mathbb{C}_n^T\mathbb{C}_n & \mathbb{P}_n\mathbb{B}_n \\ * & -\gamma_n^2 \end{bmatrix}$ is negative definite. Then, it suffices to notice that γ_n is fixed and that matrices $\mathbb{A}_n(h, A, B_d)$ and $\mathbb{B}_n(h, A, B_d)$ are linear with respect to $\frac{1}{h}$ and parameters in matrices A, B_d to prove the statement. □

Corollary 1 and the associated proof can also be adapted to system (29) instead of system (27) replacing γ_n by $\gamma_n^\flat$.

6.3 Numerical test

The delay-dependent criterion given by Theorem 2 run numerically as described below.

- Build matrices $\mathbb{A}_n$, $\mathbb{B}_n$ and $\mathbb{C}_n$, which depend on matrices A, B_d, C_d and the delay h of the initial system.
- Compute an upper bound of the $\mathcal{H}_\infty$ norm of system (27) with Matlab function *hinfnorm*.
- Define ρ_n the delay-dependent ratio of the computed bound to γ_n, already stored in memory (see Table 1).
- Evaluate if ρ_n is strictly lower than 1. If this holds, then the initial system (1) is stable.

Replacing system (27) by (29), the error bound γ_n by $\gamma_n^\flat$ and the ratio ρ_n by $\rho_n^\flat$, the same process can be conducted.

Remark 13 *For $n = 0$, the tests are extended to $\tilde{H}_n(s) = e^{-hs}$ and $\tilde{H}_n^\flat(s) = e^{-hs} - 1$ and are delay-independent since no Legendre polynomials coefficients and delay-dependent matrices are considered in the finite-dimensional part. System* (27) *corresponds to initial Figure 1 and corresponding test is simply $|\begin{pmatrix} A & B_d \\ C_d & 0 \end{pmatrix}|_{\mathcal{H}_\infty} \leq 1$. For the second model, it is related to $|\begin{pmatrix} A+B_dC_d & B_d \\ C_d & 0 \end{pmatrix}|_{\mathcal{H}_\infty} \leq 2$.*

Note that the main objective of the paper was to highlight the links between the Legendre methods for time-delay systems and the Padé approximations. The objective was not to provide stability tests, which appear as a simple by-product of this main contribution. Nevertheless, the last example section shows that this criterion enable to reach a good precision on the intervals of stability with respect to the delay.

7 Examples

7.1 Presentation of the examples

The following time-delay systems are chosen to illustrate our results.

Example 1 *Consider* (1) *with $A = 1$, $B_d = -2$ and $C_d = 1$.*

Example 2 *Consider* (1) *with $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $B_d = \begin{bmatrix} -1 & 0.5 \\ 0 & -0.5 \end{bmatrix}$ and $C_d = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. [18] The two transported signals settled here are treated with Remark 1 invoking Kronecker products.*

Example 3 *Consider* (1) *with $A = \begin{bmatrix} 0 & 1 \\ -4 & -1 \end{bmatrix}$, $B_d = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $C_d = \begin{bmatrix} 2 & 1 \end{bmatrix}$. [31]*

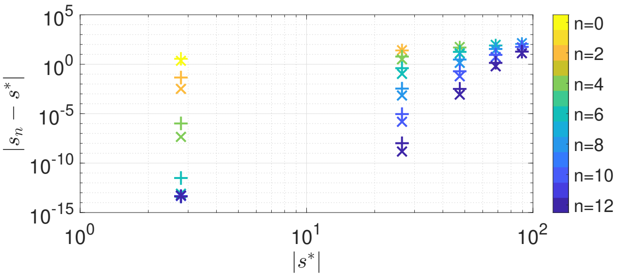
Example 4 *Consider system* (1) *with $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 0 & 0 & 0 \\ 0 & -16 & 0 & 0 \end{bmatrix}$, $B_d = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $C_d = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. [14] Once again here, the transported signals are treated thanks to Remark 1.*

7.2 Approximation of characteristic roots

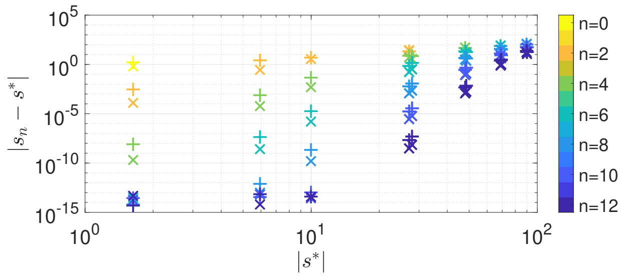
In this section, the two finite-dimensional models designed previously are investigated. To focus the study on the influence of the model and the order n on the approximation, one chooses $h = 0.3$ for each examples.

The eigenvalues of $\mathbb{A}_n$ and $\mathbb{A}_n^\flat$, called s_n, are compared with the real eigenvalues s^* computed with a precision of 10^{-15} to illustrate Theorem 1. The error done on the location of the characteristic roots in norm is depicted on Figures 7a, 7b, 7c and 7d for Example 1, 2, 3 and 4, with respect to the norm of the expected eigenvalues itself. On the figures, the dependence to the order n is paired with a color scale. Markers + and × stands respectively for the $(n-1|n)$ and $(n|n)$ Padé approximants.

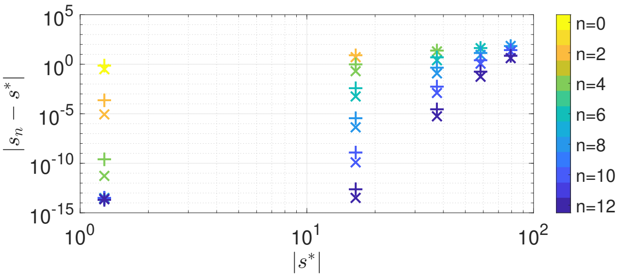
First, the approximated eigenvalues are closer and closer to the expected ones as n increases and, on compact sets $|s^*| < R$, any precision r from 1 to 10^{-15} can be reached for large enough orders n. One also remarks that the eigenvalues close to 0 in norm are approximated with smaller n than those which are far from the origin. For instance, the eigenvalues such as $|s^*| < 5$ are approximated about 10^{-5} over for any $n \geq 4$, for all the examples. However, focusing on $|s^*| < 30$, the order as to be higher than $n^* = 10$ to be have an accuracy of 10^{-5}. Furthermore,



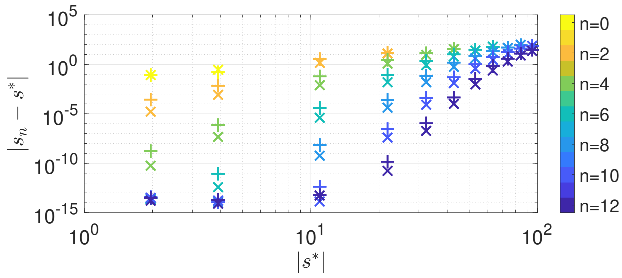
(a) Example 1 with $h = 0.3$.



(b) Example 2 with $h = 0.3$.



(c) Example 3 with $h = 0.3$.



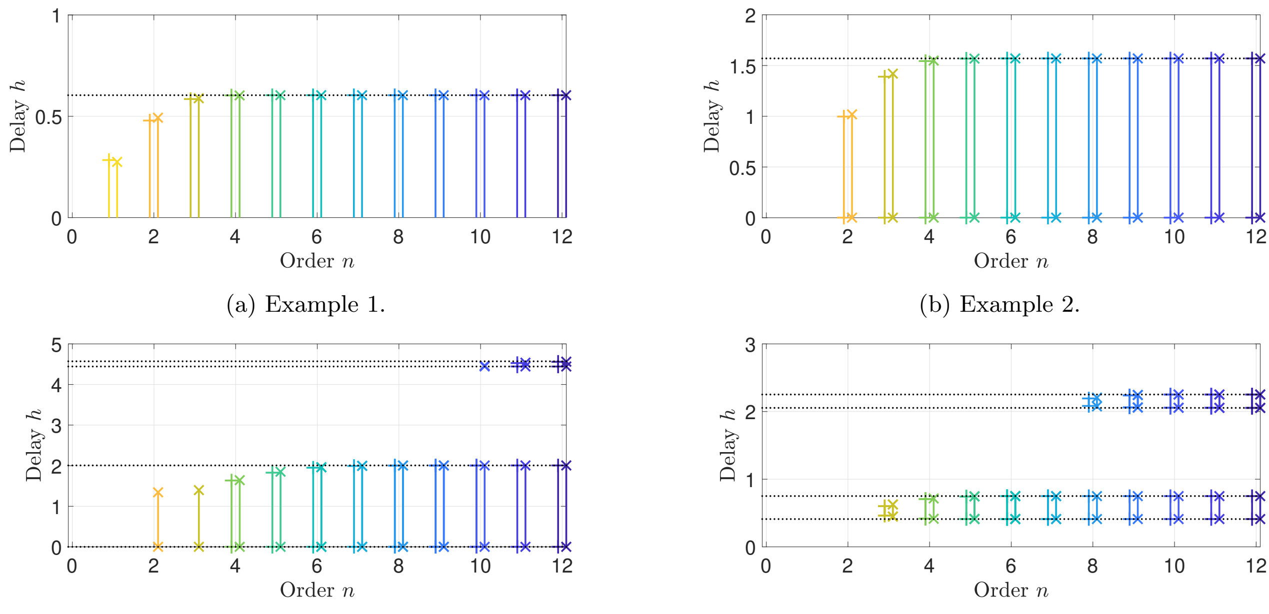
(d) Example 4 with $h = 0.3$.

Figure 7: Error done on the eigenvalues with respect to the order n.

the eigenvalues obtained with the $(n|n)$ Padé approximant are more accurate than the ones computed with $(n-1|n)$. More generally, the $(n|n)$ Padé model which is equivalent to Legendre-tau method is mainly used in practice and has a reliable numerical precision. [48] Hence, for Example 1, an error smaller than 10^{-2} is obtained from order $n^* = 2$ for the $(n|n)$ approximated model and from $n^* = 4$ for the $(n-1|n)$ approximated model. As aforementioned, the $(n|n)$ Padé approximant goes faster to the expected eigenvalues than the $(n-1|n)$ Padé approximant since $H_n^\flat(s) = \mathcal{P}_{(n|n)}(s)$ is closer to $H(s)$ than $H_n(s) = \mathcal{P}_{(n-1|n)}(s)$.
Comparisons with characteristic root computation techniques have been pursued [3] and, on studied examples, the error made by approximation is quite similar and even better than the pseudo-spectral method[8]. In addition, in the case where the model is stable, a frequency analysis of the error has been made to propose a sufficient condition of stability for time-delay systems. In the following, this delay-dependent criterion given by Theorem 2 is tested.

7.3 Illustration of the small gain theorem

In this part, using γ_n and $\gamma_n^\flat$ computed in Table 1, Theorem 2 is applied to each examples.
On Figures 8 for Examples 1, 2, 3 and 4, if the stability condition is respected at order n, then the area is colored. As previously, $+$ and $\times$ markers refers to Theorem 2 applied with $(n-1|n)$ Padé and $(n|n)$ Padé approximants, respectively. For the computation, because the $\mathcal{H}_\infty$ norm of $\begin{pmatrix} \mathbb{A}_n(h) & \mathbb{B}_n \\ \mathbb{C}_n & 0 \end{pmatrix}$ and $\begin{pmatrix} \mathbb{A}_n^\flat(h) & \mathbb{B}_n \\ \mathbb{C}_n & 0 \end{pmatrix}$ are continuous in h, the search of the upper and lower bound of the intervals of stability is done by dichotomy at a precision of 10^{-3}. In addition, the function *hinfnorm*, which is used to upper bound the $\mathcal{H}_\infty$ norm of system (27) (resp. (29)), has been settled to ensure a precision of 10^{-3} on the peak value, same tolerance as for γ_n (resp. $\gamma_n^\flat$). To better understand the results, the ratios which have to be lower than 1 are collected on Tables 2a, 2b, 2c and 2d for Example 1 with $h = 0.280$, Example 2 with $h = 1$, Example 3 with $h = 2.006$ and Example 4 with $h = 0.714$, respectively. Focusing on the first pocket of stability of Example 2, one shows that the analytical bound [31] $h = 2.006$ has been recovered from order $n = 11$. The process times are finally compared with other existing methods in Table 3 for given delays chosen in the second and third pocket of stability of Examples 3 and 4.
Firstly, the expected intervals of stability with respect to the delay, recalled in horizontal dotted lines on Figures 8a, 8b, 8c and 8d, is found more and more precisely as n increases. High values of the order n are required to evaluate the stability for larger delays h. In addition, looking at Table 2, one can see that the first stability criterion which uses $\mathbb{A}_n$ and γ_n needs most of the time higher values n that the second one which uses $\mathbb{A}_n^\flat$ and $\gamma_n^\flat$. This is not true for any delay as we can see for $h = 0.280$ on Table 2a or Figure 8a. Furthermore, for $n = 0$, both tests are never verified since it corresponds to a delay-independent frequency test, as recalled in Remark 13. It is also interesting to see that sometimes, at least for $n = 1$, the small-gain test cannot be performed due to the instability of the finite-dimensional part (see Remark 12). For example, in Table 2d, $\rho_1 = \infty$ because $\mathbb{A}_1$ is unstable. In any case, even if no hierarchy is guaranteed as for sufficient conditions on linear matrix inequality framework[44], the presented small-gain theorem seems to lead to a growing area of stability contrary to standard small-gain theorem applied directly on Padé approximant errors ($H - H_n$ or $H - H_n^\flat$). Notice though that counterexamples for a strict hierarchy can be exhibited. Invoking now necessary and sufficient results [18], our result is weaker in the sense that



(a) Example 1.

(b) Example 2.

(c) Example 3.

(d) Example 4.

Figure 8: Allowable delays guaranteed by Theorem 2. For a given order n, the information on the left (marker $+$) is related to $\rho_n < 1$ and on the right (marker $\times$) to $\rho_n^\flat < 1$.

the necessity has not been proven. However, in terms of the order of discretization, Table 2b highlights that our method works as soon as $n = 2$ instead of $n \simeq 500$, which is the necessary and sufficient order calculated in [18]. Moreover, as emphasized by Corollary 1, our methodology can be maintained for parameters uncertainties. Lastly, even if there is no guarantee that the entire set of stability is reached, our numerical results and Padé convergence properties are encouraging. A proof of convergence is an arduous task and is kept for future works.
Table 3 finally collects the computation time spent to achieve an order for which the stability test given by Theorem 2 is satisfied, for a given delay h and increasing by unitary steps the order from $n = 0$. This frequency criteria takes less than 0.06s for all the tested delays. Compared to Lyapunov techniques [43], higher orders need to be tested. But, the computation time is up to ten times better than the linear matrix inequality proposed in Theorem 5 [44]. To conclude, these tables and figures strengthen the potential of the proposed method to assess the stability of time-delay systems in a fast and easy way.

8 Conclusions

In this paper, we have designed models for time-delay systems by interconnecting cleverly a finite-dimensional system with an infinite-dimensional system. The finite-dimensional part is constructed by adding some new states depending on the approximation of the delay element. It includes projections of the distributed state on the first Legendre polynomials. Taking Fourier-Legendre truncation on input and output bounds, these especial models at order n turn out to be realizations of the $(n-1|n)$ and $(n|n)$ Padé approximants. From there, one ensures that the characteristic roots of retarded time-delay systems can be approximated as accurately as required with the proposed models. Compared to Padé approximations of the delay, the Fourier-Legendre remainder induce quite natural candidate filters which add precious information to study the stability of the time-delay system. Then, using an upper bound of this well-chosen infinite-dimensional part and the small-gain theorem, a simple delay-dependent stability condition is given. Both statements confirm the effectiveness of models based on Legendre polynomials coefficients and are finally illustrated on four examples.
Based on the same framework, a generalization to other coupling between an ordinary and a partial differential equation such as cross transport, diffusion or wave phenomena is forthcoming. This research could also be extended to multiple and time-varying delays. Afterwards, by $\mathcal{H}_\infty$ synthesis, controllers and observers could be easier to design with early-lumping techniques [36].

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n	0	1	2	3	4	5	6	7	8	9	10	11	12
ρ_n	∞	0.981	0.248	0.110	0.089	0.072	0.057	0.052	0.046	0.040	0.037	0.034	0.031
$\rho_n^\flat$	4.000	1.028	0.221	0.130	0.109	0.081	0.071	0.064	0.053	0.050	0.046	0.041	0.039

(a) Example 1 with $h = 0.280$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
ρ_n	∞	1.526	1.007	0.212	0.186	0.153	0.113	0.107	0.094	0.080	0.077	0.070	0.063
$\rho_n^\flat$	2.108	2.674	0.943	0.261	0.238	0.166	0.148	0.134	0.106	0.105	0.095	0.083	0.081

(b) Example 2 with $h = 1$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
ρ_n	1.426	1.493	1.157	2.282	6.291	21.97	92.44	237.2	75.69	11.75	1.578	0.191	0.121
$\rho_n^\flat$	∞	1.124	2.531	11.81	48.74	212.6	685.8	325.9	58.69	8.841	1.181	0.166	0.142

(c) Example 3 with $h = 2.006$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
ρ_n	∞	∞	12.946	4.154	1.206	0.219	0.080	0.069	0.060	0.053	0.048	0.044	0.041
$\rho_n^\flat$	∞	13.218	12.496	3.758	0.957	0.169	0.098	0.083	0.072	0.066	0.060	0.054	0.050

(d) Example 4 with $h = 0.714$.

Table 2: Ratio ρ_n (resp. $\rho_n^\flat$) of the upper bound of the $\mathcal{H}_\infty$ norm of systems (27) (resp. (29)) to γ_n (resp. $\gamma_n^\flat$).

Delay	$h = 2.006$		$h = 4.450$		$h = 4.751$	
Method	Order	Time	Order	Time	Order	Time
Theorem 2 with (27)	11	0.05s	11	0.05s	11	0.05s
Theorem 2 with (29)	11	0.05s	10	0.05s	15	0.06s
Theorem 5 [44]	4	0.10s	6	0.20s	6	0.21s

(a) Example 3.

Delay	$h = 0.714$		$h = 2.150$		$h = 3.750$	
Method	Order	Time	Order	Time	Order	Time
Theorem 2 with (27)	5	0.02s	8	0.04s	14	0.05s
Theorem 2 with (29)	4	0.02s	8	0.04s	13	0.05s
Theorem 5 [44]	2	0.05s	5	0.45s	9	3.2s

(b) Example 4.

Table 3: Process times to assess the stability.

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