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On the necessity of sufficient LMIs conditions for time-delay systems arising from Legendre approximation [★]

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Abstract

This work is dedicated to the stability analysis of time-delay systems with a single constant delay. To answer this problem, Lyapunov-Krasovskii methods have been widely used in the literature and numerous sufficient conditions of stability are proposed and expressed as linear matrix inequalities (LMIs). These conditions being only sufficient, these contributions are often criticized because of the lack of information regarding the reduction of the conservatism. Recently, scalable methods have been investigated using Bessel-Legendre inequality or orthogonal polynomial-based inequalities. The interest of these methods relies on their hierarchical structure. However, the convergence is still an open question that will be answered for the first time in this paper. The objective of this paper is thus to prove that, the stability of a time-delay system implies the feasibility of the scalable LMIs provided in Seuret et al., at a sufficiently large order of the Bessel-Legendre inequality and an estimation of this order is provided analytically.

Key words: Time-delay systems; Stability conditions; Lyapunov analysis; LMI; Polynomial approximation.

1 Introduction

Stability analysis of time-delay systems has been at the heart of many research works for several decades [6,12,19,26]. Indeed, this class of systems can represent many physical phenomena of engineering interest and also proposes many theoretical problems related to their intrinsic infinite dimensional nature. Thus, many methods to establish their stability properties have been developed and among them the use of Lyapunov functionals remains one of the most popular techniques.

In the framework of linear systems with delays, as for finite dimensional systems, Lyapunov-Krasovskii functionals can be constructed to obtain necessary and sufficient conditions as proposed by [19]. However, even if the necessary condition of stability is simple to test, the sufficient condition of stability is still numerically complicated to verify, because the order which guarantees the stability can be very high.

Therefore, simpler functionals have been proposed which, combined with convex optimization algorithms,

lead to only sufficient conditions (see for instance [5,7,15,31]). This line of research has given rise to numerous works, which all aim at reducing the inherent conservatism aiming at recovering the necessity using, for instance, discretized functionals [12], delay partitioning methods [14]. Among them, an approach based on state extension has been proposed in [27]. The Lyapunov functional therein depends directly on the projections of the state of the delay system onto Legendre polynomials. This approach led to linear matrix inequality (LMI) stability conditions which are proven to have a hierarchical structure, arising from the use of the Bessel-Legendre inequality. Similar approach based on orthogonal polynomials [22,21] dissipative inequalities [4] have been also considered in the literature. The size of this LMI increases with the number of Legendre polynomials N taken under consideration. This result drastically reduces the pessimism introduced by the LMI approaches. However, to the best of our knowledge, the proof of convergence of the LMI to a necessary and sufficient condition of stability is still missing. This corresponds to answering the following questions.

- If a time-delay system is stable, does there exist a minimal order N^* from which the LMI stability condition is necessary satisfied?
- Is it possible to estimate this order N^* ?

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This contribution provides an affirmative answer to that questions by proving the necessity of the proposed Bessel-Legendre LMI conditions for time-delay systems, for sufficiently large orders.

Returning to the original problem, this paper retrieves the complete Lyapunov-Krasovskii functional [19] in order to provide an approximated Lyapunov functional. The main idea is to keep track of the necessity condition by managing the remainder part as in [9–11]. Note that similar manipulations are also followed via sum of square techniques [1,23,24,28]. Here, the approximation is made on Legendre orthogonal polynomials. Based on strong regularity conditions satisfied by the Lyapunov matrix [17,18], convergence properties of Legendre approximation [29,30] will be the centerpiece of the demonstration. Then, by leaving free the different coefficient in the associated approximated functional, we fall back to the LMIs given in [27].

To sum up, this contribution fills the gap between the sufficiency of the LMI conditions based on an extended Lyapunov functional on Legendre polynomials and their necessity. To reach this conclusion, two intermediate results are presented. Firstly, we prove that the approximated Lyapunov functional issued from Legendre approximation converges towards the complete Lyapunov functional and likewise for their time derivatives. Secondly, we formulate a new necessary and sufficient condition of stability based on this approximated Lyapunov functional. Thus, we obtain the necessity of sufficient LMI conditions from a minimal order N^* , which can be given numerically.

The paper is organized as follows. In Section 2, time-delay systems are rewritten as an interconnection between a transport equation and a finite-dimensional system and the necessary and sufficient condition of existence of a complete Lyapunov functional is presented. Then, the convergence properties of Legendre approximation are recalled in Section 3. These results are applied to the complete Lyapunov functional in order to approximate it and its time derivatives in Section 4. Lastly, Section 5 presents a by-product of this approximation, which concerns the necessity of sufficient LMI conditions of stability is derived.

Notations : Throughout the paper, \mathbb{N} (\mathbb{N}^*), $\mathbb{R}^{m \times p}$ and \mathbb{S}^m denote the sets of natural numbers (excluding zero), real matrices of size $m \times p$ and symmetric matrices of size m , respectively. For any square matrix $M \in \mathbb{R}^{m \times m}$, the transpose of M is denoted M^\top and $\mathcal{H}(M)$ stands for $M + M^\top$. Furthermore, for any M in \mathbb{S}^m , its minimal and maximal eigenvalues are denoted $\sigma(M)$ and $\bar{\sigma}(M)$. The 2-norm of matrix M in $\mathbb{R}^{m \times p}$ is $|M| = \sqrt{\bar{\sigma}(M^\top M)}$. The vector $u = \text{vec}(M)$ in $\mathbb{R}^{mp \times 1}$ collocates the columns of matrix M . Moreover, notation \otimes represents the Kronecker product. The set of square-integrable functions

from (a, b) to $\mathbb{R}^{m \times p}$ is noted $\mathcal{L}_2(a, b; \mathbb{R}^{m \times p})$. Let also $\mathcal{H}_1(a, b; \mathbb{R}^{m \times p})$ the set of functions F , such that F and F' are in $\mathcal{L}_2(a, b; \mathbb{R}^{m \times p})$ and $\mathcal{C}_\infty(a, b; \mathbb{R}^{m \times p})$ the set of smooth functions. et finally $\mathcal{C}_{pw}(a, b; \mathbb{R}^{m \times p})$ the set of piecewise continuous functions with a finite number of discontinuity points and, for each continuity interval, finite left and right-hand side limits. Lastly, denote the norm $|(x, z)| = \sqrt{|x|^2 + \int_0^1 |z(\theta)|^2 d\theta}$, for any (x, z) in $\mathbb{R}^{n_x} \times \mathcal{L}_2(0, 1; \mathbb{R}^{n_z})$.

2 Necessary and sufficient stability condition for time-delay systems

2.1 Stability of time-delay systems

Consider a time-delay system given by

$$\dot{x}(t) = Ax(t) + A_d x(t-h), \quad \forall t \geq 0, \quad (1a)$$

$$x(t) = \varphi(t) \in \mathcal{C}_{pw}(-h, 0; \mathbb{R}^{n_x}), \quad \forall t \in [-h, 0], \quad (1b)$$

where the single delay $h > 0$ and matrices A, A_d in $\mathbb{R}^{n_x \times n_x}$ are constant and known. The trivial solution of (1) is said to be uniformly globally exponentially stable (UGES) if there exist $\kappa \geq 1$ and $\mu > 0$ such that for any piecewise continuous function φ , the inequality $|x(t)| \leq \kappa e^{-\mu t} \sup_{[-h, 0]} |\varphi|$ holds, for all $t \geq 0$.

Without loss of generality, let decompose A_d into a product BC , with $B, C^\top \in \mathbb{R}^{n_x \times n_z}$ being full rank matrices, with n_z the rank of matrix A_d . Introducing the state $z(t, \theta) = Cx(t + (\theta - 1)h)$, the time-delay system (1) can be rewritten as an ordinary differential equation interconnected with a transport equation as

$$\begin{cases} \dot{x}(t) = Ax(t) + Bz(t, 0), & \forall t \geq 0, \\ h \frac{\partial}{\partial t} z(t, \theta) = \frac{\partial}{\partial \theta} z(t, \theta), & \forall \theta \in (0, 1), \forall t \geq 0, \\ z(t, 1) = Cx(t), & \forall t \geq 0, \end{cases} \quad (2a)$$

$$\begin{cases} x(0) = \varphi(0), \\ z(0, \theta) = C\varphi((\theta - 1)h), & \forall \theta \in (0, 1). \end{cases} \quad (2b)$$

The trivial solution of (2) is said to be UGES if there exist $\kappa \geq 1$ and $\mu > 0$ such that for any piecewise continuous functions φ , the following inequality holds

$$\left| \left(x, \sqrt{hz} \right) \right| (t) \leq \kappa e^{-\mu t} \sup_{[-h, 0]} |\varphi|, \quad \forall t \geq 0. \quad (3)$$

This interconnected model has been largely used in [20]. Indeed, such representation simplifies and facilitates the understanding and the calculations leaning on significant signals. Moreover, both systems share the same stability properties as stated below.

Proposition 1 *System (1) is UGES in the sense of Lyapunov-Krasovskii if and only if system (2) is UGES in the sense of (3).*

Proof: First, notice that the solution $(x(t), z(t))$ of (2) is equal the solution $(x(t), Cx(t+(\cdot-1)h))$ of (1), for any $t \geq 0$, since they satisfy the same dynamics and initial conditions. The path from (2) to (1) is then obvious because $|x(t)|^2 \leq \left| \begin{pmatrix} x(t) \\ \sqrt{h}z(t) \end{pmatrix} \right|^2$ holds, for any $t \geq 0$. Assuming now that system (1) is UGES, we have

$$\begin{aligned} |x(t)| &\leq \kappa e^{-\mu t} \sup_{[-h,0]} |\varphi|, \quad \forall t \geq 0, \\ |x(t)| &\leq \sup_{[-h,0]} |\varphi|, \quad \forall t \in [-h, 0]. \end{aligned} \quad (4)$$

Moreover, the following inequality holds

$$\left| \begin{pmatrix} x(t) \\ \sqrt{h}z(t) \end{pmatrix} \right|^2 \leq (1+h|C^\top C|) \sup_{[-h,0]} |x(t+\cdot)|^2. \quad (5)$$

This yields

$$\left| \begin{pmatrix} x(t) \\ \sqrt{h}z(t) \end{pmatrix} \right| \leq \sqrt{1+h|C^\top C|} \kappa e^{-\mu t} \sup_{[-h,0]} |\varphi|, \quad \forall t \geq 0, \quad (6)$$

which is the definition (3) of UGES for (2). \square

In the sequel, as supported by this proposition, the stability analysis is pursued for system (2).

2.2 Existence of a complete Lyapunov functional

It is well known that the stability of system (1) can be established thanks to the use of the so-called complete Lyapunov-Krasovskii functional proposed in [19]. This Lyapunov-Krasovskii functional, originally defined for (1), can also be introduced for system (2) on the normalized interval $[0, 1]$. All along this article, for any positive definite matrices P, Q, R, S in $\mathbb{S}^{n_x}, \mathbb{S}^{n_x}, \mathbb{S}^{n_z}, \mathbb{S}^{n_z}$, we define this complete Lyapunov functional \mathcal{V} by

$$\mathcal{V}(x, z) = \int_0^1 \int_0^1 \begin{bmatrix} x \\ z(\theta_1) \\ z(\theta_2) \end{bmatrix}^\top \Pi(\theta_1, \theta_2) \begin{bmatrix} x \\ z(\theta_1) \\ z(\theta_2) \end{bmatrix} d\theta_1 d\theta_2, \quad (7)$$

where Π is a matrix function given by

$$\Pi(\theta_1, \theta_2) = \begin{bmatrix} P+U(0) & \frac{h}{2}F_1(\theta_1) & \frac{h}{2}F_1(\theta_2) \\ * & \frac{1}{2}(\theta_1 R+2hS) & \frac{h^2}{2}F_2(\theta_2-\theta_1) \\ * & * & \frac{1}{2}(\theta_2 R+2hS) \end{bmatrix}, \quad (8)$$

for all θ_1, θ_2 in $(0, 1)^2$. Functions F_1, F_2 are given by

$$\begin{aligned} F_1(\theta) &= U(\theta)B, \quad \forall \theta \in [0, 1], \\ F_2(\theta) &= B^\top U(\theta)B, \quad \forall \theta \in [-1, 1], \end{aligned} \quad (9)$$

and depend on the Lyapunov matrix U defined from $[-1, 1]$ to $\mathbb{R}^{n_x \times n_x}$ as in [6] by

$$\begin{cases} U(\theta) = \text{vec}^{-1} \left([I_{n_x}^2 \ 0] e^{\theta \mathcal{M}} \mathcal{N}^{-1} \begin{bmatrix} -\text{vec}(W) \\ 0 \end{bmatrix} \right) & \text{if } \theta \leq 0, \\ U(\theta) = U^\top(-\theta) & \text{if } \theta > 0, \end{cases} \quad (10)$$

where matrices

$$\begin{aligned} W &= Q + C^\top (hR + S)C, \\ \mathcal{M} &= h \begin{bmatrix} -A^\top \otimes I_{n_x} & -A_d^\top \otimes I_{n_x} \\ I_{n_x} \otimes A_d^\top & I_{n_x} \otimes A^\top \end{bmatrix}, \\ \mathcal{N} &= \begin{bmatrix} A^\top \otimes I_{n_x} + I_{n_x} \otimes A^\top & A_d^\top \otimes I_{n_x} \\ I_{n_x}^2 & 0 \end{bmatrix} + \begin{bmatrix} I_{n_x} \otimes A_d^\top & 0 \\ 0 & -I_{n_x}^2 \end{bmatrix} e^{-\mathcal{M}}. \end{aligned} \quad (11)$$

Note that the construction of this Lyapunov matrix U by vectorization technique can be found in [6].

Remark 1 *Before going any further, it is important to note that this functional cannot be defined if matrix \mathcal{N} is singular. As explained in [19], this condition is equivalent to the Lyapunov condition, i.e. there is no eigenvalue of (2) such that its opposite is also eigenvalue. It therefore excludes all systems which has eigenvalues on the imaginary axis.*

In the next paragraph, under the Lyapunov condition, we extend the well-known results on the existence of a complete Lyapunov-Krasovskii functional for the system (1) to the necessity and sufficiency of the existence of the proposed Lyapunov functional (7) for the system (2).

2.3 Necessary and sufficient stability condition

The following lemma states that the existence of such a functional (7) is a necessary and sufficient condition of stability for system (2).

Lemma 1 *System (2) is UGES if and only if there exist $\alpha, \beta > 0$ such that the Lyapunov functional defined by (7) satisfies*

$$\begin{cases} \mathcal{V}(x, z) \geq \alpha \left| \begin{pmatrix} x \\ \sqrt{h}z \end{pmatrix} \right|^2, & (12a) \\ \dot{\mathcal{V}}(x, z) \leq -\beta \left| \begin{pmatrix} x \\ [z(0)] \\ \sqrt{h}z \end{pmatrix} \right|^2, & (12b) \end{cases}$$

for all (x, z) in $\mathbb{R}^{n_x} \times \mathcal{L}_2(0, 1; \mathbb{R}^{n_z})$.

Proof : For the sake of simplicity, the time argument has been discarded in the following proof. Firstly, the sufficiency of the proposed lemma is easily obtained. From (12b), inequality $\dot{\mathcal{V}}(x, z) \leq -\beta \left| \begin{bmatrix} x \\ \sqrt{hz} \end{bmatrix} \right|^2$ holds. Thanks to the positive quadratic lower bound for \mathcal{V} and negative quadratic upper bound for $\dot{\mathcal{V}}$, there exist $\kappa \geq 1$ and $\mu > 0$ such that

$$\begin{aligned} \left| \begin{bmatrix} x \\ \sqrt{hz} \end{bmatrix} \right| (t) &\leq \kappa e^{-\mu t} \left| \begin{bmatrix} x \\ \sqrt{hz} \end{bmatrix} \right| (0), \\ &\leq \sqrt{1 + h|C^\top C|} \kappa e^{-\mu t} \sup_{[-h, 0]} |\varphi|, \quad \forall t \geq 0, \end{aligned}$$

so that we recover the definition of UGES for system (2). Secondly, concerning the necessity, let introduce \mathcal{W} as

$$\begin{aligned} \mathcal{W}(x, z) &= \mathcal{V}(x, z) - \int_0^1 \begin{bmatrix} x \\ z(\theta) \end{bmatrix}^\top \begin{bmatrix} P & 0 \\ 0 & hS \end{bmatrix} \begin{bmatrix} x \\ z(\theta) \end{bmatrix} d\theta, \\ &= \int_0^1 \int_0^1 \begin{bmatrix} x \\ z(\theta_1) \\ z(\theta_2) \end{bmatrix}^\top \begin{bmatrix} U(0) & \frac{h}{2} F_1(\theta_1) & \frac{h}{2} F_1(\theta_2) \\ * & \frac{1}{2}(\theta_1 R + hS) & \frac{h}{2} F_2(\theta_2 - \theta_1) \\ * & * & \frac{1}{2}(\theta_2 R + hS) \end{bmatrix} \begin{bmatrix} x \\ z(\theta_1) \\ z(\theta_2) \end{bmatrix} d\theta_1 d\theta_2. \end{aligned}$$

By construction, differentiating \mathcal{W} along the trajectories of (2) leads to

$$\dot{\mathcal{W}}(x, z) = - \int_0^1 \begin{bmatrix} x \\ z(\theta) \\ z(0) \end{bmatrix}^\top \begin{bmatrix} Q & 0 & 0 \\ 0 & hR & 0 \\ 0 & 0 & S \end{bmatrix} \begin{bmatrix} x \\ z(\theta) \\ z(0) \end{bmatrix} d\theta.$$

Then, integrating the above from t to ∞ , we obtain

$$\lim_{t \rightarrow \infty} \mathcal{W}(x(t), z(t)) - \mathcal{W}(x(t), z(t)) \leq 0, \quad \forall t \geq 0.$$

Furthermore, as system (2) is exponentially stable, $(x, z)(t) \xrightarrow[t \rightarrow \infty]{} 0$, then inequality $\mathcal{W}(x, z) \geq 0$ holds which leads to (12a) with $\alpha = \min(\underline{\sigma}(P), \underline{\sigma}(S)) > 0$. Likewise, we also obtain

$$\dot{\mathcal{V}}(x, z) = \int_0^1 \begin{bmatrix} x \\ z(\theta) \\ z(0) \end{bmatrix}^\top \begin{bmatrix} \mathcal{H}(PA) + C^\top SC - Q & 0 & PB \\ 0 & -hR & 0 \\ * & 0 & -2S \end{bmatrix} \begin{bmatrix} x \\ z(\theta) \\ z(0) \end{bmatrix} d\theta,$$

When matrix $H = \begin{bmatrix} \mathcal{H}(PA) + C^\top SC - Q & PB \\ * & -2S \end{bmatrix} \prec 0$, i.e. by choosing $\underline{\sigma}(Q)$ large enough, we directly certify that (12b) holds with $\beta = \min(\underline{\sigma}(H), \underline{\sigma}(R)) > 0$. \square

Remark 2 Notice that the free matrices P, Q, R, S can be chosen arbitrary as non negative multiples of the identity matrix, with Q sufficiently large. A possible way to fix it could be

$$\begin{aligned} P &= I_{n_x}, \quad R = I_{n_z}, \quad S = \frac{|B|}{|C|} I_{n_z}, \\ Q &= (1 + |A + A^\top| + |B||C|) I_{n_x}, \end{aligned} \quad (13)$$

which yields

$$\alpha = \beta = \min \left(1, \frac{|B|}{|C|} \right). \quad (14)$$

This particular selection will show its interest hereafter.

2.4 Problem statement

Thanks to Lemma 1, it is possible to assess stability of system (2) with inequalities (12). Nevertheless, condition (12a) cannot be verified numerically. To face the problem, two approaches can be pursued to develop a tractable criterion to assess inequality (12a).

From one hand, an approximation of the Lyapunov matrix Π given by (8) lead to numerically tractable necessity conditions [3] of stability for system (2). Recently, the authors of [9–11] have proven that this result also leads to a sufficient condition for large enough approximation orders. From the other hand, many sufficient stability conditions based on Lyapunov-Krasovskii functionals have been proposed [7,13]. It consists in leaving free the Lyapunov matrix Π given by (8). For instance, when matrix function Π is seen as a polynomial with free coefficients, a hierarchical sufficient LMI condition of stability for system (2) is obtained [16,24,27]. However, if system (2) is stable, nobody knows if these LMI conditions are satisfied. The necessity of these criteria has never been proven. The main goal of this work is then to provide the asymptotic necessity of sufficient LMI condition issued from Bessel-Legendre inequalities.

To achieve this objective, it is useful to merge both approaches. Indeed, the Legendre approximation of the complete Lyapunov functional (7) is used at an arbitrary order n in \mathbb{N} to keep track of its necessity properties. Then, the necessity of the LMIs issued from the original Lyapunov-Krasovskii functional [27] will appear quite naturally. The steps followed in the body of the article are given below.

- Section 3: The convergence properties of Legendre approximations are recalled and applied to the Lyapunov matrix functions F_1, F_2 in (9) and its derivatives.
- Section 4: A Legendre approximation of the Lyapunov matrices given by (9) is proposed. Based on the nice convergence properties and after some calculations, we formulate Lemma 6, which states that the Lyapunov remainder functional and its derivatives with respect to the time converge to zero, as n tends to infinity.
- Section 5: The approximated Lyapunov functional inherits the same necessary and sufficient conditions as the complete Lyapunov functional. Then, leaving free certain matrices, a new LMI criterion of stability is presented and stand as a necessary and sufficient condition of stability for time-delay systems. The proof of sufficiency of this LMI follows the logical path argued in [27] and the necessity comes from the existence

the approximated Lyapunov functional, for some fixed Lyapunov matrices.

3 Legendre approximation of Lyapunov matrices and convergence properties

3.1 Definition of Legendre polynomials

The Legendre polynomials widely used in polynomial approximation theory [2] are defined over the normalized interval $[0, 1]$ as

$$\forall k \in \mathbb{N}, \quad l_k(\theta) = (-1)^k \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{k+j}{j} \theta^j, \quad (15)$$

where $\binom{k}{j}$ stands for the binomial coefficient.

The main idea for employing these polynomials comes from their nice properties that are summarized below. Before going into the details, for any order $n \in \mathbb{N}^*$, size $p \in \mathbb{N}^*$ and $\theta \in [0, 1]$, let us introduce the following notation

$$\ell_n(\theta) = \left[l_0(\theta)I_p \ l_1(\theta)I_p \ \dots \ l_{n-1}(\theta)I_p \right]^\top \in \mathbb{R}^{np \times p}, \quad (16)$$

Property 1 *The Legendre polynomials given in (15) represent an orthogonal sequence with respect to the inner product $\int_0^1 \phi^\top(\theta)\psi(\theta)d\theta$, for any ϕ, ψ in $\mathcal{L}_2(0, 1; \mathbb{R}^p)$. The following equality holds for all n in \mathbb{N}^* ,*

$$\int_0^1 \ell_n(\theta)\ell_n^\top(\theta)d\theta = \begin{bmatrix} I_p & 0 & \dots & 0 \\ 0 & 3I_p & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & (2n-1)I_p \end{bmatrix}^{-1} = \mathcal{I}_n^{-1}. \quad (17)$$

Proof: The proof can be found in [8]. □

We provide some additional properties of the Legendre polynomials that will be helpful in the sequel.

Property 2 *For any integer k in \mathbb{N} , the Legendre polynomials are evaluated pointwisely by*

$$l_k(0) = (-1)^k, \quad l_k(1) = 1, \quad l'_k(1) = k(k+1), \quad (18)$$

$$l_k\left(\frac{1}{2}\right) = \begin{cases} \frac{(-1)^p (2p)!}{2^{2p} (p!)^2}, & \text{for } k = 2p, \\ 0, & \text{for } k = 2p + 1. \end{cases}$$

3.2 Legendre approximation

Consider the polynomial approximation of a function F in $\mathcal{L}_2(a, b; \mathbb{R}^{m \times p})$. For any approximation order $n \in \mathbb{N}^*$,

the function F is rewritten as

$$\forall \theta \in [a, b], \quad F(\theta) = \mathbf{F}_n^\top \mathcal{I}_n \ell_n \left(\frac{\theta - a}{b - a} \right) + \tilde{F}_n(\theta), \quad (19)$$

where \mathbf{F}_n is a matrix in $\mathbb{R}^{np \times m}$ representing the coordinates of F on the basis generated by the n first Legendre polynomials and \tilde{F}_n is the approximation error. They are defined by

$$\mathbf{F}_n^\top = \frac{1}{b-a} \int_a^b F(\theta) \ell_n^\top \left(\frac{\theta - a}{b - a} \right) d\theta, \quad (20)$$

$$\tilde{F}_n(\theta) = F(\theta) - \mathbf{F}_n^\top \mathcal{I}_n \ell_n \left(\frac{\theta - a}{b - a} \right). \quad (21)$$

Since Legendre polynomials are orthogonal with respect to the inner product under consideration, this manipulation can be interpreted as the Gram-Schmidt normalization of F with respect to the n first Legendre polynomials. Indeed, it can be easily verified that the resulting function \tilde{F}_n is orthogonal to the n first Legendre polynomials and represents the remainder of the Legendre approximation.

3.3 Convergence of the Legendre remainders and its derivatives

Following the theory of polynomial approximation (see for instance [2] and references therein), it results that \tilde{F}_n given in (21) converges to zero in the sense of the $\mathcal{L}_2(a, b; \mathbb{R}^{m \times p})$ norm. This is actually a by-product of Bessel inequality. This implies that \tilde{F}_n converges to zero almost everywhere on (a, b) , as n tends to infinity. Nevertheless, uniform convergence properties on $[a, b]$ can even be straighten under some regularity conditions.

3.3.1 Case of smooth functions

Consider a smooth function F defined on the closed interval $[0, 1]$, without loss of generality. Then, as formulated in the following lemmas, we obtain the uniform convergence of $|\tilde{F}_n|$ and its derivatives $|\tilde{F}'_n|$ towards zero.

Lemma 2 *Consider a function $F \in \mathcal{C}_\infty(0, 1; \mathbb{R}^{m \times p})$ with the additional assumption that $\max_{[0,1]} |F'''(\theta)| \leq \rho$.*

Then, the truncated Legendre series $\mathbf{F}_n^\top \mathcal{I}_n \ell_n(\theta)$ converges uniformly to $F(\theta)$, when n tends to infinity, on the closed interval $[0, 1]$. Moreover, for all $n \geq 4$, the following inequality holds

$$\max_{[0,1]} |\tilde{F}_n(\theta)| \leq \frac{\left(\frac{\pi}{2}\right)^{\frac{3}{2}} \rho}{\sqrt{n-3}}, \quad (22)$$

where \tilde{F}_n is the approximation error given by (21).

Proof: The proof follows the arguments provided in [29, Theorem 2.5] and is given in Appendix A.1. \square

Lemma 3 Consider a function $F \in \mathcal{C}_\infty(0, 1; \mathbb{R}^{m \times p})$ with the additional assumption that $\max_{[0,1]} |F'''(\theta)| \leq \rho'$.

Then, the truncated Legendre series $\mathbf{F}_n^\top \mathcal{I}_n \ell'_n(\theta)$ converges uniformly to $F'(\theta)$, when n tends to infinity, on the closed interval $[0, 1]$. More precisely, for all $n \geq 6$, the following inequality holds

$$\max_{[0,1]} |\tilde{F}'_n(\theta)| \leq \frac{60 \left(\frac{\pi}{2}\right)^{\frac{3}{2}} \rho'}{\sqrt{n-5}}, \quad (23)$$

where \tilde{F}'_n is the derivatives of the approximation error given by (21).

Proof: The proof is given in Appendix A.2. \square

Remark 3 Note that the convergence rate could be improved on the strength of an existing upper bound for $|F^{(k)}|$, for any $k \in \mathbb{N}$. For more details in that direction, one can refer to [29] and references therein.

3.3.2 Case of continuous functions

Let now consider weaker regularity conditions. Function F is assumed to be continuous on $[-1, 1]$ with continuous derivatives on $[-1, 0)$ and $(0, 1]$ but has a discontinuity at 0 and one notes

$$\bar{f} = \frac{1}{2} |F'(0^+) - F'(0^-)| > 0.$$

As highlighted by [30] and formulated in the following lemma, the uniform convergence can be maintained.

Lemma 4 Consider a continuous function F from $[-1, 1]$ to $\mathbb{R}^{m \times p}$ with continuous derivatives on subintervals $[-1, 0) \cup (0, 1]$ which satisfies $\sup_{(-1,1) \setminus \{0\}} |F''(\theta)| \leq \bar{\rho}$.

Then, the truncated Fourier-Legendre series $\mathbf{F}_n^\top \mathcal{I}_n \ell_n(\frac{\theta+1}{2})$ converges uniformly to $F(\theta)$, when n tends to infinity, on the closed interval $[-1, 1]$. More precisely, for all $n \geq 4$, the following inequality holds

$$\max_{[-1,1]} |\tilde{F}_n(\theta)| \leq \frac{4 \left(\left(\frac{\pi}{2}\right)^{\frac{3}{2}} \bar{\rho} + \frac{\epsilon}{\pi} \bar{f} \right)}{\sqrt{n-3}}, \quad (24)$$

where \tilde{F}_n is the approximation error given by (21).

Proof: The proof is given in Appendix A.3. \square

Remark 4 Under the condition of discontinuity of the derivatives of F , the proposed convergence rate in $\frac{1}{\sqrt{n}}$ is tight and cannot be improved (see [30] for more details).

3.4 Application to the Lyapunov matrix

Consider now the Legendre approximation of the Lyapunov matrix $U(\theta)B$ on $[0, 1]$ as well as the one of $B^\top U(\theta)B$ on $[-1, 1]$. According to (20), the corresponding Legendre coefficients are noted

$$\begin{aligned} \mathbf{F}_{1,n}^\top &= \int_0^1 U(\theta)B \ell_n^\top(\theta) d\theta, \\ \mathbf{F}_{2,n}^\top &= \frac{1}{2} \int_{-1}^1 B^\top U(\theta)B \ell_n^\top\left(\frac{\theta+1}{2}\right) d\theta. \end{aligned} \quad (25)$$

The associated errors calculated using (21) with, from one side, $F := F_1$ and, from the other side, $F := F_2$. We are now in position to introduce the Legendre remainders, denoted $\tilde{F}_{1,n}, \tilde{F}_{2,n}$, defined as follows

$$\begin{aligned} \tilde{F}_{1,n}(\theta) &= F_1(\theta) - \mathbf{F}_{1,n}^\top \mathcal{I}_n \ell_n(\theta), \quad \forall \theta \in [0, 1], \\ \tilde{F}_{2,n}(\theta) &= F_2(\theta) - \mathbf{F}_{2,n}^\top \mathcal{I}_n \ell_n\left(\frac{\theta+1}{2}\right), \quad \forall \theta \in [-1, 1], \end{aligned} \quad (26)$$

where ℓ_n, \mathcal{I}_n are given by (16),(17) with $p = n_z$.

In order to use Lemmas 2, 3 and 4 and prove the uniform convergence of the remainder to zero, we firstly verify that the Lyapunov matrix U defined by (10) satisfies the regularity assumptions presented above. Looking at the definition of U in (10), this function is obviously \mathcal{C}^∞ on each subintervals $[-1, 0)$ and $(0, 1]$ but not on $[-1, 1]$, since U' has a discontinuity at 0. More precisely, we are now in position to state the following property.

Property 3 The Lyapunov matrix U associated to W defined in (10) satisfies

$$\sup_{(-1,1)} |U''(\theta)| = \max_{[0,1]} |U''(\theta)| \leq \rho |W|, \quad (27)$$

$$\max_{[0,1]} |U''''(\theta)| \leq \rho' |W|, \quad (28)$$

$$\bar{f} = \frac{1}{2} |U'(0^+) - U'(0^-)| = \frac{1}{2} |W|, \quad (29)$$

with parameters

$$\rho = \sqrt{n_x} e^{|\mathcal{M}|} |\mathcal{M}^2 \mathcal{N}^{-1}|, \quad \rho' = \sqrt{n_x} e^{|\mathcal{M}|} |\mathcal{M}^4 \mathcal{N}^{-1}|, \quad (30)$$

and matrices \mathcal{M}, \mathcal{N} given by (11).

Proof: Thanks to the equivalence of matrix norms, inequalities $|M^\top| = |M| \leq |M^v| \leq \sqrt{p} |M|$ hold, for

any M in $\mathbb{R}^{p \times p}$. Then, for all θ in $[0, 1]$, we have

$$\begin{aligned} |U^{(k)}(\theta)| &\leq |[I_{n_x} \ 0] e^{\theta \mathcal{M}} \mathcal{M}^k \mathcal{N}^{-1} [-W^v] |, \\ &\leq |e^{\theta \mathcal{M}}| |\mathcal{M}^k \mathcal{N}^{-1}| |W^v|, \\ &\leq \sqrt{n_x} |e^{\theta \mathcal{M}}| |\mathcal{M}^k \mathcal{N}^{-1}| |W|. \end{aligned}$$

Moreover, as $e^{\theta \mathcal{M}} = \sum_{k=0}^{\infty} \frac{(\theta \mathcal{M})^k}{k!}$, one obtains

$$|e^{\theta \mathcal{M}}| \leq \sum_{k=0}^{\infty} \left| \frac{(\theta \mathcal{M})^k}{k!} \right| \leq \sum_{k=0}^{\infty} \frac{|\theta|^k |\mathcal{M}|^k}{k!} \leq \sum_{k=0}^{\infty} \frac{|\mathcal{M}|^k}{k!} = e^{|\mathcal{M}|},$$

which yields the results (27),(28). Finally, the boundary conditions verified by the Lyapunov matrix U , ensures as in [19] that

$$U'(0^+) - U'(0^-) = - \left(\frac{d}{d\theta} U^\top(0^-) + \frac{d}{d\theta} U(0^-) \right) = W,$$

which provides the value of \bar{f} given by (29). \square
Thanks to this property, we are able to apply the previous results on F_1, F_2 to assess the following lemma.

Lemma 5 *The Legendre remainders $\tilde{F}_{1,n}, \tilde{F}_{2,n}$ of the functions F_1, F_2 described by (26) satisfy, for all $\varepsilon > 0$,*

$$\begin{cases} \max_{[0,1]} |\tilde{F}_{1,n}(\theta)| \leq \varepsilon, & \forall n \geq n_1(\varepsilon), \\ \max_{[0,1]} |\tilde{F}'_{1,n}(\theta)| \leq \varepsilon, & \forall n \geq n'_1(\varepsilon), \\ \max_{[-1,1]} |\tilde{F}_{2,n}(\theta)| \leq \varepsilon, & \forall n \geq n_2(\varepsilon), \end{cases} \quad (31)$$

where the orders n_1, n'_1 and n_2 can be calculated according to

$$n_1(\varepsilon) = 3 + \left\lceil \frac{\pi^3 \rho^2}{8\varepsilon^2} |B|^2 |W|^2 \right\rceil, \quad (32)$$

$$n'_1(\varepsilon) = 5 + \left\lceil \frac{450\pi^3 \rho'^2}{\varepsilon^2} |B|^2 |W|^2 \right\rceil, \quad (33)$$

$$n_2(\varepsilon) = 3 + \left\lceil \frac{16 \left(\left(\frac{\pi}{2} \right)^{\frac{3}{2}} \rho + \frac{\varepsilon}{2\pi} \right)^2}{\varepsilon^2} |B|^4 |W|^2 \right\rceil, \quad (34)$$

with parameters ρ, ρ' given in (30).

Proof : Lemmas 2 and 3 with $F := F_1$ yield

$$\begin{aligned} \max_{[0,1]} |\tilde{F}_{1,n}(\theta)| &\leq \frac{\left(\frac{\pi}{2} \right)^{\frac{3}{2}} \rho |B| |W|}{\sqrt{n-3}}, \\ \max_{[0,1]} |\tilde{F}'_{1,n}(\theta)| &\leq \frac{60 \left(\frac{\pi}{2} \right)^{\frac{3}{2}} \rho' |B| |W|}{\sqrt{n-5}}, \end{aligned}$$

Then, similarly, Lemma 4 with $F := F_2$ leads to

$$\max_{[-1,1]} |\tilde{F}_{2,n}(\theta)| \leq \frac{4 \left(\left(\frac{\pi}{2} \right)^{\frac{3}{2}} \rho + \frac{\varepsilon}{2\pi} \right) |B|^2 |W|}{\sqrt{n-3}}.$$

Therefore, inequalities (31) hold from orders n_1, n'_1 and n_2 given respectively by (32), (33), (34). \square

Remark 5 *It is easy to see from (32)-(34) that selecting small ε implies having large $n(\varepsilon)$.*

At this stage, we have built approximations of F_1, F_2 such that the remainder of their approximation and their derivatives converge uniformly to zero, as n tends to infinity. We are now in position to build an approximation of the complete Lyapunov functional in (7), thanks to $\mathbf{F}_{1,n}, \mathbf{F}_{2,n}$.

4 Legendre approximation of the complete Lyapunov functional

The Legendre approximation detailed in the previous section, in particular the expansion (19), allows us to define an approximated Lyapunov functional of the complete Lyapunov functional, given in (7).

4.1 Approximated Lyapunov functional

Introduce the approximated Lyapunov functional candidate

$$\mathcal{V}_n(x, z) = \int_0^1 \int_0^1 \begin{bmatrix} x \\ z(\theta_1) \\ z(\theta_2) \end{bmatrix}^\top \Pi_n(\theta_1, \theta_2) \begin{bmatrix} x \\ z(\theta_1) \\ z(\theta_2) \end{bmatrix} d\theta_1 d\theta_2, \quad (35)$$

where Π_n is an approximation of Π in (8), defined using $\mathbf{F}_{1,n}$ and $\mathbf{F}_{2,n}$ as follows

$$\begin{aligned} \Pi_n(\theta_1, \theta_2) = & \\ & \begin{bmatrix} P + U(0) & \frac{h}{2} \mathbf{F}_{1,n}^\top \mathcal{I}_n \ell_n(\theta_1) & \frac{h}{2} \mathbf{F}_{1,n}^\top \mathcal{I}_n \ell_n(\theta_2) \\ * & \frac{1}{2} (\theta_2 R + 2hS) & \frac{h^2}{2} \ell_n^\top(\theta_1) \mathcal{I}_n \mathbf{T}_n \mathcal{I}_n \ell_n(\theta_2) \\ * & * & \frac{1}{2} (\theta_2 R + 2hS) \end{bmatrix}, \end{aligned} \quad (36)$$

and where $\mathbf{T}_n = \sum_{k=0}^{n-1} (2k+1) T_k \otimes \mathbf{F}_{2,n,k}$ such that, for any k in $\{0, \dots, n-1\}$, matrix $T_k \in \mathbb{S}^{n_z}$ is given by

$$T_k = \int_0^1 \int_0^1 \ell_n(\theta_1) l_k \left(\frac{\theta_2 - \theta_1 + 1}{2} \right) \ell_n^\top(\theta_2) d\theta_1 d\theta_2,$$

and that $\mathbf{F}_{2,n,k}$ is the $(k+1)^{th}$ Legendre coefficient of F_2 given by (25).

It remains to show that this Lyapunov functional candidate for system (2) respects the two conditions (12a)-(12b). To do so, we firstly prove that this candidate constitutes an approximated of the complete Lyapunov functional. Then, one proves also that its derivative along the trajectories remains an approximation of the derivative of the complete Lyapunov functional. Gathering these two facts allows us to conclude.

4.2 Lyapunov functional remainder

Introduce the remainder $\tilde{\mathcal{V}}_n(x, z) = \mathcal{V}(x, z) - \mathcal{V}_n(x, z)$, which can be written as

$$\begin{aligned} \tilde{\mathcal{V}}_n(x, z) = & \underbrace{2hx^\top \int_0^1 \tilde{F}_{1,n}(\theta)z(\theta)d\theta}_{\mathcal{V}_{1,n}(x,z)} \\ & + \underbrace{h^2 \int_0^1 \int_0^1 z^\top(\theta_1)\tilde{F}_{2,n}(\theta_2 - \theta_1)z(\theta_2)d\theta_1d\theta_2}_{\tilde{\mathcal{V}}_{2,n}(x,z)}. \end{aligned} \quad (37)$$

where the Legendre remainders $\tilde{F}_{1,n}, \tilde{F}_{2,n}$ are expressed in (26).

The derivatives of $\tilde{\mathcal{V}}_n$ along the trajectories of system (2) can be calculated with a term-by-term decomposition. Firstly, derivation of $\tilde{\mathcal{V}}_{1,n}$ leads to

$$\dot{\tilde{\mathcal{V}}}_{1,n}(x, z) = 2h \left(\dot{x}^\top \int_0^1 \tilde{F}_{1,n}(\theta)z(\theta)d\theta + x^\top \int_0^1 \tilde{F}'_{1,n}(\theta)\dot{z}(\theta)d\theta \right).$$

By noting that $h\frac{\partial}{\partial t}z = \frac{\partial}{\partial \theta}z$, an integration by parts yields

$$\begin{aligned} \dot{\tilde{\mathcal{V}}}_{1,n}(x, z) = & 2 \int_0^1 \begin{bmatrix} x \\ z(\theta) \\ z(0) \end{bmatrix}^\top \begin{bmatrix} \tilde{F}_{1,n}(1)C & \begin{pmatrix} hA^\top \tilde{F}_{1,n}(\theta) \\ -\tilde{F}'_{1,n}(\theta) \\ 0 \end{pmatrix} & -\tilde{F}_{1,n}(0) \\ 0 & \begin{pmatrix} 0 \\ hB^\top \tilde{F}_{1,n}(\theta) \\ 0 \end{pmatrix} & 0 \end{bmatrix} \begin{bmatrix} x \\ z(\theta) \\ z(0) \end{bmatrix} d\theta. \end{aligned}$$

Secondly, differentiating $\tilde{\mathcal{V}}_{2,n}$ with respect to time yields

$$\begin{aligned} \dot{\tilde{\mathcal{V}}}_{2,n}(x, z) = & h \int_0^1 \int_0^1 z^\top(\theta_1)\tilde{F}_{2,n}(\theta_2 - \theta_1)\partial_{\theta_2}z(\theta_2)d\theta_1d\theta_2 \\ & + h \int_0^1 \int_0^1 \partial_{\theta_1}z^\top(\theta_1)\tilde{F}_{2,n}(\theta_2 - \theta_1)z(\theta_2)d\theta_1d\theta_2, \end{aligned}$$

Using the inequality $(\partial_{\theta_1} + \partial_{\theta_2})\tilde{F}_{2,n}(\theta_2 - \theta_1) = 0$, an integration by parts leads to

$$\begin{aligned} \dot{\tilde{\mathcal{V}}}_{2,n}(x, z) = & h \int_0^1 z^\top(\theta)\tilde{F}_{2,n}(1 - \theta)d\theta Cx \\ & - h \int_0^1 z^\top(\theta)\tilde{F}_{2,n}(-\theta)d\theta z(0) \\ & + h(Cx)^\top \int_0^1 \tilde{F}_{2,n}(\theta - 1)z(\theta)d\theta \\ & - hz^\top(0) \int_0^1 \tilde{F}_{2,n}(\theta)z(\theta)d\theta. \end{aligned}$$

To sum up, the derivative of the Lyapunov functional remainder (37) along the trajectories of system (2) is expressed as

$$\dot{\tilde{\mathcal{V}}}_n(x, z) = \int_0^1 \begin{bmatrix} x \\ z(\theta) \\ z(0) \end{bmatrix}^\top \Delta_n(\theta) \begin{bmatrix} x \\ z(\theta) \\ z(0) \end{bmatrix} d\theta, \quad (38)$$

with

$$\Delta_n(\theta) = \begin{bmatrix} \mathcal{H}(\tilde{F}_{1,n}(1)C) & \begin{pmatrix} hA^\top \tilde{F}_{1,n}(\theta) - \tilde{F}'_{1,n}(\theta) \\ +hC^\top \frac{\tilde{F}_{2,n}(\theta-1) + \tilde{F}_{2,n}^\top(1-\theta)}{2} \end{pmatrix} & -\tilde{F}_{1,n}(0) \\ * & \begin{pmatrix} 0 \\ hB^\top \tilde{F}_{1,n}(\theta) \\ -h \frac{\tilde{F}_{2,n}(\theta) + \tilde{F}_{2,n}^\top(-\theta)}{2} \end{pmatrix} & * \\ * & & 0 \end{bmatrix}. \quad (39)$$

We are now in position to provide an upper bound for the previously mentioned functionals.

4.3 Convergence of the Lyapunov functional remainder and its time derivative

In view of the expressions of the remainder of the Lyapunov functional and its time derivatives given by (37),(38), we recognize coefficients which converge to zero as n increases. Indeed, in Section 3, we have shown that $\tilde{F}_{1,n}, \tilde{F}'_{1,n}$ and $\tilde{F}_{2,n}$ converge uniformly to zero with respect to n . This is due to the fact that functions F_1, F_2 are based on the Lyapunov matrix U , which satisfies Property 3. Therefore, the main convergence lemma can be formulated.

Lemma 6 For any $(x, z) \in \mathbb{R}^{n_x} \times \mathcal{L}_2(0, 1; \mathbb{R}^{n_z})$ solutions to system (2) and for any $\alpha, \beta > 0$, the following inequalities hold

$$\left\{ \left| \tilde{\mathcal{V}}_n(x, z) \right| \leq \frac{\alpha}{3} \left| \begin{bmatrix} x \\ \sqrt{h}z \end{bmatrix} \right|^2, \quad \forall n \geq N_1(\alpha), \quad (40a)$$

$$\left\{ \left| \dot{\tilde{\mathcal{V}}}_n(x, z) \right| \leq \frac{\beta}{3} \left| \begin{bmatrix} x \\ z(0) \\ \sqrt{h}z \end{bmatrix} \right|^2, \quad \forall n \geq N_2(\beta), \quad (40b)$$

where $\tilde{\mathcal{V}}_n$ is defined in (37). The guaranteed lower bounds of n are explicitly given by

$$N_1(\alpha) = \max \left(n_1 \left(\frac{\alpha}{6\sqrt{h}} \right), n_2 \left(\frac{\alpha}{6h} \right) \right), \quad (41)$$

$$N_2(\beta) = \max \left(n_1 \left(\frac{\beta}{9(1+2|C|+\sqrt{h}(|A|+|B|))} \right), n_2 \left(\frac{\beta}{9\sqrt{h}(1+|C|)} \right), n'_1 \left(\frac{\sqrt{h}\beta}{9} \right) \right). \quad (42)$$

where n_1, n'_1, n_2 are given by (32),(33),(34), respectively.

Proof : For the sake of simplicity, the time argument has been again removed from the proof.

Firstly, the application of Young and Cauchy-Schwartz

inequalities allows deriving the following bound on $\tilde{\mathcal{V}}_n$

$$|\tilde{\mathcal{V}}_n| \leq \int_0^1 \begin{bmatrix} x \\ z(\theta) \end{bmatrix}^\top \begin{bmatrix} \sqrt{h} \max_{[0,1]} |\tilde{F}_{1,n}(\theta)| & 0 \\ 0 & h^{\frac{3}{2}} \max_{[0,1]} |\tilde{F}_{1,n}(\theta)| \\ 0 & +h^2 \max_{[-1,1]} |\tilde{F}_{2,n}(\theta)| \end{bmatrix} \begin{bmatrix} x \\ z(\theta) \end{bmatrix} d\theta.$$

Similarly a bound of its time derivatives $\dot{\tilde{\mathcal{V}}}_n$ is obtained

$$|\dot{\tilde{\mathcal{V}}}_n| \leq \int_0^1 \begin{bmatrix} x \\ z(\theta) \end{bmatrix}^\top \begin{bmatrix} \tilde{Q}_n & 0 & 0 \\ 0 & h\tilde{R}_n & 0 \\ 0 & 0 & \tilde{S}_n \end{bmatrix} \begin{bmatrix} x \\ z(\theta) \end{bmatrix} d\theta,$$

with matrices

$$\begin{aligned} \tilde{Q}_n &= \begin{pmatrix} (1 + 2|C| + \sqrt{h}|A|) \max_{[0,1]} |\tilde{F}_{1,n}(\theta)| \\ + \frac{1}{\sqrt{h}} \max_{[0,1]} |\tilde{F}'_{1,n}(\theta)| + \sqrt{h}|C| \max_{[-1,1]} |\tilde{F}_{2,n}(\theta)| \end{pmatrix} I_{n_x}, \\ \tilde{R}_n &= \begin{pmatrix} \sqrt{h}(|A| + |B|) \max_{[0,1]} |\tilde{F}_{1,n}(\theta)| + \frac{1}{\sqrt{h}} \max_{[0,1]} |\tilde{F}'_{1,n}(\theta)| \\ + \sqrt{h}(1 + |C|) \max_{[-1,1]} |\tilde{F}_{2,n}(\theta)| \end{pmatrix} I_{n_z}, \\ \tilde{S}_n &= \begin{pmatrix} (1 + \sqrt{h}|B|) \max_{[0,1]} |\tilde{F}_{1,n}(\theta)| + \sqrt{h} \max_{[-1,1]} |\tilde{F}_{2,n}(\theta)| \end{pmatrix} I_{n_z}. \end{aligned}$$

Finally, Lemma 5 ensures the convergence and even provides orders $N_1(\alpha)$, $N_2(\beta)$ from which (40a),(40b) are satisfied. \square

Thanks to Lemma 6 which ensures nice convergence properties, the Lyapunov functional \mathcal{V}_n will be exploited to derive necessary and sufficient conditions of stability, for large enough orders.

5 Necessary and sufficient LMI condition of stability for time-delay systems

5.1 A new necessary and sufficient condition

The necessary and sufficient condition of stability for time-delay systems comes from the existence of a unique complete Lyapunov-Krasovskii functional. This corresponds to Lemma 1 exposed in the first section and relying with a complete Lyapunov functional \mathcal{V} given by (7). Hereafter, we extend this result to the approximated Lyapunov functional \mathcal{V}_n defined by (35) by Legendre approximation of (7).

Theorem 1 *System (2) is UGES if and only if there exist $\alpha, \beta > 0$ such that the Lyapunov functional defined by (35) satisfies*

$$\begin{cases} \mathcal{V}_n(x, z) \geq \frac{2\alpha}{3} \left| \begin{bmatrix} x \\ \sqrt{hz} \end{bmatrix} \right|^2, & \forall n \geq N_1(\alpha), \quad (43a) \\ \dot{\mathcal{V}}_n(x, z) \leq -\frac{2\beta}{3} \left| \begin{bmatrix} x \\ [z(0)] \end{bmatrix}, \sqrt{hz} \right|^2, & \forall n \geq N_2(\beta), \quad (43b) \end{cases}$$

for all (x, z) in $\mathbb{R}^{n_x} \times \mathcal{L}_2(0, 1; \mathbb{R}^{n_z})$.

Proof : Assuming that system (2) is UGES, Lemma 1 ensures the existence of $\alpha, \beta > 0$ such that inequalities (12a)-(12b) hold. Thereby, one obtains

$$\begin{cases} \mathcal{V}_n(x, z) \geq \mathcal{V}(x, z) - |\tilde{\mathcal{V}}_n(x, z)| \geq \frac{2\alpha}{3} \left| \begin{bmatrix} x \\ \sqrt{hz} \end{bmatrix} \right|^2, \\ \dot{\mathcal{V}}_n(x, z) \leq \dot{\mathcal{V}}(x, z) + |\dot{\tilde{\mathcal{V}}}_n(x, z)| \leq -\frac{2\beta}{3} \left| \begin{bmatrix} x \\ [z(0)] \end{bmatrix}, \sqrt{hz} \right|^2, \end{cases} \quad (44)$$

for a sufficiently large order n , according to the convergence of the Lyapunov remainder proposed in Lemma 6. Reversely, if inequalities (43a)-(43b) hold, then

$$\begin{cases} \mathcal{V}(x, z) \geq \mathcal{V}_n(x, z) - |\tilde{\mathcal{V}}_n(x, z)| \geq \frac{\alpha}{3} \left| \begin{bmatrix} x \\ \sqrt{hz} \end{bmatrix} \right|^2, \\ \dot{\mathcal{V}}(x, z) \leq \dot{\mathcal{V}}_n(x, z) + |\dot{\tilde{\mathcal{V}}}_n(x, z)| \leq -\frac{\beta}{3} \left| \begin{bmatrix} x \\ [z(0)] \end{bmatrix}, \sqrt{hz} \right|^2. \end{cases} \quad (45)$$

Thanks to Lemma 1, one concludes that system (2) is UGES, which completes the proof. \square

Remark 6 *It is worth noting that α and β can be expressed analytically with respect to free matrices P, Q, R, S . For instance, as said in Remark 2, a possible choice of matrices leads to $\alpha = \beta = \min(1, \frac{|B|}{|C|})$.*

Remark 7 *Contrary to Lemma 1, this new necessary and sufficient condition of stability is numerically tractable by fixing P, Q, R, S as suggested in Remark 2. Inequalities (43) are convertible into a matrix positivity test, as done in [10]. In this article, this test is not exposed.*

From this intermediate theorem, which is not implementable in this format, the main goal is to derive a necessary and sufficient condition of stability in the LMI framework. By giving freedom to matrices $P, R, S, \mathbf{F}_{1,n}, \mathbf{T}_n$ in (36), the idea is to obtain a sufficient LMI condition for stability which becomes necessary for sufficiently large orders since it is possible to exhibit a Lyapunov functional from order $N^* = \max(N_1, N_2)$, which satisfies the same LMI condition. The last paragraph is then dedicated to the proof of the necessity of the LMIs conditions provided in [27].

5.2 Asymptotic necessity of sufficient LMI condition

From Theorem 1, the sufficient LMIs conditions proposed in [27] becomes necessary. Indeed, inequalities (43) are satisfied for all z in $\mathcal{L}_2(0, 1; \mathbb{R}^{n_x})$ and in particular for polynomial functions in θ . With z polynomial, the theoretical result (43) amounts to a LMI test, similar to the one obtained in [27]. We are then able to certify that, for stable time-delay systems, the LMI condition holds

for orders n greater than N^* , which can be calculated by fixing matrices P, Q, R, S .

Theorem 2 *System (2) UGES if and only if there exists an order N^* in \mathbb{N} such that for any orders $n \geq N^*$ we can find \mathbf{P}_n, R, S in $\mathbb{S}^{n_x+n_z}, \mathbb{S}^{n_x}, \mathbb{S}^{n_z}$ such that the linear matrix inequalities (46) are true:*

$$\begin{cases} \Phi_n = \mathbf{P}_n + \mathbf{I}_n \left(\frac{R}{2h} + 2S \right) + \mathbf{J}_n \left(\frac{R}{2h} \right) \succ 0, & (46a) \\ \Psi_n = \Psi_n^0 - \begin{bmatrix} \mathbf{I}_n(R) & 0 \\ 0 & 0 \end{bmatrix} \prec 0, & (46b) \end{cases}$$

where

$$\mathbf{I}_n(\Sigma) = \begin{bmatrix} 0 & 0 \\ 0 & \mathcal{I}_n(\Sigma) \end{bmatrix}, \quad \mathcal{I}_n(\Sigma) = \begin{bmatrix} \Sigma & 0 & \cdots & 0 \\ 0 & 3\Sigma & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (2n-1)\Sigma \end{bmatrix},$$

$$\mathbf{J}_n(\Sigma) = \begin{bmatrix} 0 & 0 \\ 0 & \mathcal{H}(\mathcal{J}_n(\Sigma)) \end{bmatrix}, \quad \mathcal{J}_n(\Sigma) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ \Sigma & 2\Sigma & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & (n-1)\Sigma & 0 \end{bmatrix},$$

$$\Psi_n^0 = \begin{bmatrix} \mathcal{H}(\mathbf{P}_n \mathbf{A}_n) + \mathbf{C}_n^\top (hR + 2S) \mathbf{C}_n & \mathbf{P}_n \mathbf{B}_n \\ * & -2S \end{bmatrix},$$

$$\mathbf{A}_n = \mathbf{E}_n \begin{bmatrix} A & 0 \\ \ell_n(1)C & -\mathcal{L}_n \mathcal{I}_n \end{bmatrix} \mathbf{E}_n, \quad \mathbf{E}_n = \begin{bmatrix} I_{n_x} & 0 \\ 0 & \frac{1}{\sqrt{h}} I_{n_z} \end{bmatrix},$$

$$\mathbf{B}_n = \mathbf{E}_n \begin{bmatrix} B \\ -\ell_n(0) \end{bmatrix}, \quad \mathbf{C}_n^\top = \begin{bmatrix} C^\top \\ 0 \end{bmatrix},$$

$$\mathcal{L}_{n,ij} = \begin{cases} 1 - (-1)^{i+j} & \text{if } i \geq j, \\ 0 & \text{elsewhere.} \end{cases}$$

Proof: For simplicity, the time argument is again omitted along the proof. The sufficient condition has been demonstrated in [27]. Indeed, if the LMIs (46a), (46b) are satisfied for any n , then system (2) is UGES in the sense of (3). The hierarchy condition has also been demonstrated, which means that if (46a), (46b) are true at order N^* , then it will be also true for any $n \geq N^*$.

For the necessary condition, assume that system (2) is UGES and consider \mathcal{V}_n given by

$$\begin{aligned} \mathcal{V}_n(x, z) &= \left[\sqrt{h} \int_0^1 \ell_n(\theta) z(\theta) d\theta \right]^\top \mathbf{P}_n \left[\sqrt{h} \int_0^1 \ell_n(\theta) z(\theta) d\theta \right], \\ &+ \int_0^1 z^\top(\theta) (\theta R + 2hS) z(\theta) d\theta, \end{aligned} \quad (47)$$

with matrix \mathbf{P}_n selected as follows

$$\mathbf{P}_n = \begin{bmatrix} P + U(0) & \sqrt{h} \mathbf{F}_{1,n}^\top \mathcal{I}_n \\ * & h \mathcal{I}_n \mathbf{T}_n \mathcal{I}_n \end{bmatrix}, \quad (48)$$

in order to recognize $\Pi_n(P, Q, R, S)$ given by (36) and the approximated Lyapunov functional (35). Assuming that system (2) is exponentially stable and choosing the positive definite matrices P, Q, S, R given by (13), inequality (43a) for any z in $\mathcal{L}^2(0, 1; \mathbb{R}^{n_z})$. Let now z be a function expressed as follows

$$\begin{aligned} z(\theta) &= \zeta_n^\top \mathcal{I}_n \ell_n(\theta) + (-\zeta_n^\top \mathcal{I}_n \ell_n(1) + Cx) \delta_1(\theta) \\ &+ (-\zeta_n^\top \mathcal{I}_n \ell_n(0) + z(0)) \delta_0(\theta), \end{aligned} \quad (49)$$

for any vector $\begin{bmatrix} x \\ \zeta_n \\ z(0) \end{bmatrix}$ in $\mathbb{R}^{n_x+n_z(n+1)}$ and where the function δ_{θ_0} is

$$\delta_{\theta_0}(\theta) = \begin{cases} 1 & \text{for } \theta = \theta_0, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, we have

$$\begin{aligned} \mathcal{V}_n(x, z) &= \begin{bmatrix} x \\ \sqrt{h} \zeta_n \end{bmatrix}^\top \left(\mathbf{P}_n + \mathbf{I}_n \left(\frac{R}{2h} + 2S \right) \right) \begin{bmatrix} x \\ \sqrt{h} \zeta_n \end{bmatrix} \\ &+ \int_0^1 \sqrt{h} \zeta_n^\top \mathcal{I}_n \ell_n(\theta) \left(\frac{l_1(\theta)R}{2h} \right) \sqrt{h} \ell_n^\top(\theta) \mathcal{I}_n \zeta_n d\theta. \end{aligned}$$

From Bonnet's recursion formula [8] verified by Legendre polynomials and using orthogonality of Legendre polynomials, one gets

$$\mathcal{I}_n \ell_n(\theta) l_1(\theta) = \begin{bmatrix} \mathcal{J}_n^\top \\ \vdots \\ n I_{n_z} \end{bmatrix} \ell_{n+1}(\theta) + \mathcal{J}_n \ell_n(\theta), \quad (50)$$

which leads to

$$\mathcal{V}_n(x, z) = \xi_n^\top \left(\mathbf{P}_n + \mathbf{I}_n \left(\frac{R}{2h} + 2S \right) + \mathbf{J}_n \left(\frac{R}{2h} \right) \right) \xi_n,$$

with $\xi_n = \begin{bmatrix} x \\ \sqrt{h} \zeta_n \end{bmatrix}$. According to inequality (43a) guaranteed by Theorem 1, especially for z given by (49), we obtain, for any ξ_n in $\mathbb{R}^{n_x+n_z}$ and $n \geq N_1(\alpha)$,

$$\mathcal{V}_n(x, z) = \xi_n^\top \Phi_n \xi_n \geq \frac{2\alpha}{3} |\xi_n|^2, \quad (51)$$

which means that matrix Φ_n is positive definite.

Then, calculating the time derivatives of \mathcal{V}_n along the trajectories of system (2) yields

$$\begin{aligned} \dot{\mathcal{V}}_n(x, z) &= \begin{bmatrix} \sqrt{h} \int_0^1 \ell_n(\theta) z(\theta) d\theta \\ z(0) \end{bmatrix}^\top \Psi_n^0 \begin{bmatrix} \sqrt{h} \int_0^1 \ell_n(\theta) z(\theta) d\theta \\ z(0) \end{bmatrix} \\ &- \int_0^1 z^\top(\theta) R z(\theta) d\theta, \end{aligned} \quad (52)$$

Considering one more time a function z given by (49), one gets to

$$\dot{\mathcal{V}}_n(x, z) = \begin{bmatrix} x \\ \sqrt{h}\zeta_n \\ z(0) \end{bmatrix}^\top (\Psi_n^0 - [\mathbf{I}_n(R) \ 0]) \begin{bmatrix} x \\ \sqrt{h}\zeta_n \\ z(0) \end{bmatrix}. \quad (53)$$

The inequality (43b) of Theorem 1 ensures finally that

$$\dot{\mathcal{V}}_n(x, z) = \chi_n^\top \Psi_n \chi_n \leq -\frac{\beta}{2} |\chi_n|^2, \quad (54)$$

holds for all $\chi_n = \begin{bmatrix} x \\ \sqrt{h}\zeta_n \\ z(0) \end{bmatrix} \in \mathbb{R}^{n_x+n_z(n+1)}$ and $n \geq N_2(\beta)$. Thus, since matrices P, Q, R, S have been fixed according to Remark 2, we have found an order $N^* = \max(N_1(\alpha), N_2(\beta))$ with $\alpha = \beta = \min(1, \frac{|B|}{|C|})$ such that linear matrix inequalities (46) are true. \square

Remark 8 Note that the first LMI (46a) is slightly different to the one formulated in [27]. It is due on application of Bessel inequality on the term in R for the positivity of \mathcal{V}_n . This modification releases a bit the conservatism of the LMI. However, on numerical examples, the criteria still lead to same results.

Remark 9 It is also worth noticing that the order N^* can be calculated from (41),(42) because $N^* = \max(N_1(\alpha), N_2(\beta))$ with $\alpha = \beta = \min(1, \frac{|B|}{|C|})$. In practice, the necessary condition can be used as a test of instability. If the LMI is not verified at order N^* , then the time-delay system is unstable.

Remark 10 The size of the linear matrix inequality is $n_x + (n+1)n_z$ but the number of variable could be limited to $\frac{n_x(n_x+1)}{2} + n_z(n_z+1)$ with Π_n in the form proposed by (36). However, with less degree of freedom, the order n to ensure stability by the sufficient condition would be much larger.

With Theorem 2, we guarantee that, if system (2) is UGES, then LMIs (46) are necessary for sufficiently large orders. This result confirms the question of convergence raised by the hierarchic LMIs introduced in [27]. As a repercussion of the conservative inequalities proposed for the rate of convergence, we will see in the example section that the expected order N^* is at this stage too large to be exploited efficiently.

6 Numerical results

The numerical application of Theorem 2 is commented and illustrated on academic examples.

Example 1 Consider (2) with $A = 1$, $B = -2$ and $C = 1$.

Table 1

Sufficiency: maximal allowable delay h which satisfies (46).

n	1	2	3	Expected
Ex. 1	0.577	0.604	0.604	$\frac{\text{atan}\sqrt{3}}{\sqrt{3}} \simeq 0.604$
Ex. 2	0.126	1.218	1.413	1.424
Ex. 3	–	1.600	1.603	1.603

Table 2

Necessity: minimal order N^* given by (55).

	Ex. 1		Ex. 2		Ex. 3	
h	0.1	2	0.1	2	0.1	2
n_1	11	10^{12}	10^9	10^{50}	4	10^8
n_2	1336	10^{14}	10^9	10^{52}	16	10^9
n_1	2700	10^{13}	10^{12}	10^{54}	9	10^9
n_2	119916	10^{14}	10^{11}	10^{53}	1163	10^{10}
n'_1	5609	10^{17}	10^{13}	10^{58}	7	10^{12}
N^*	119916	10^{17}	10^{13}	10^{58}	1163	10^{12}

Example 2 Consider (2) with $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -11 & 10 & 0 & 0 \\ 5 & -15 & 0 & -0.25 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $C = [1 \ 0 \ 0 \ 0]$.

Example 3 Consider (2) with $A = \begin{bmatrix} 0 & 0 \\ -0.1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

First, recall the powerful result provided by the sufficient side of Theorem 2. If LMIs (46) are true, then system (2) is UGES at the origin. For several orders n , the maximal allowable delay, which satisfies the LMI (46) is reported in Table 1. Thanks to the degree of freedom in \mathbf{P}_n we clearly see that stability is ensured for very low orders n .

Then, we have proven in this contribution that these LMI conditions of stability are also necessary. For several delays h , in Table 2, we calculate the order N^* as proposed in Remark 9 by

$$N^* = \max \left(\begin{array}{l} n_1(\frac{\alpha}{6\sqrt{h}}), n_2(\frac{\alpha}{6h}), \\ n_1(\frac{\beta}{9(1+2|C|+\sqrt{h}(|A|+|B|))}), \\ n_2(\frac{\beta}{9\sqrt{h}(1+|C|)}), n'_1(\frac{\sqrt{h}\beta}{6}) \end{array} \right), \quad (55)$$

for which it is established that, if system (2) is UGES, LMIs (46) are satisfied. Compared to [10], these minimal orders are too large to propose tractable test of instability. However, we are now in position to give N^* , from which the LMIs are necessary in addition to be sufficient. Moreover, this upper bound N^* of the necessary and sufficient order of the LMIs could be greatly reduced using the fact that U is infinitely continuous on $[0, 1]$ as suggested in Remark 3.

7 Conclusions

In this paper, the convergence of scalable LMI conditions for the stability analysis of time-delay systems, based on the Bessel-Legendre inequalities, has been studied. It is well-known that LMI conditions are only sufficient. The framework of Bessel-Legendre, already showed its relevance regarding the hierarchical structure of the LMIs, i.e. to increase the order can only reduce the conservatism. The main contribution of this paper is to demonstrate that this framework also offers an asymptotically necessary condition of stability for time-delay systems. In other words, it is now proven that if a time-delay system is stable, then, there exists an order N^* such that the LMIs are verified at least at this order. Reversely, if the LMIs conditions are not verified at this order, then the system is proven to be unstable.

The estimation of the order N^* provided in this paper is very large, even on simple examples. Eventhough this can be seen as a major drawback, the convergence of the LMI is the main achievement of this paper. Further studies would win at reducing the evaluation of these thresholds and to make it more tractable.

A Proofs of uniform convergence of Legendre approximations

A.1 Proof of Lemma 2

Proof: The objective of the proof is to demonstrate the uniform convergence towards zero of

$$\tilde{F}_n(\theta) = F(\theta) - \mathbf{F}_n^\top \mathcal{I}_n \ell_n(\theta) = F(\theta) - \sum_{k=0}^{n-1} A_k l_k(\theta),$$

with

$$A_k = (2k+1) \int_0^1 F(\theta) l_k(\theta) d\theta. \quad (\text{A.1})$$

Firstly, we show that the series $\sum_{k=n}^{\infty} A_k l_k(\theta)$ exists. Secondly, we prove that the sequence of functions \tilde{F}_n converges uniformly to zero when n tends to infinity. Since Legendre polynomials satisfy $|l_k(\theta)| \leq 1$, for all $\theta \in [0, 1]$, we have

$$\left| \sum_{k=n}^N A_k l_k(\theta) \right| \leq \sum_{k=n}^N |A_k|, \quad \forall \theta \in [0, 1].$$

Hence, let us now find an upper bound of A_k given in (A.1). Recalling from [8] that $l_k = \frac{1}{2(2k+1)}(l'_{k+1} - l'_{k-1})$ for all $k \geq 1$, an integration by parts yields

$$\begin{aligned} A_k &= \frac{1}{2} \int_0^1 F(\theta) (l'_{k+1}(\theta) - l'_{k-1}(\theta)) d\theta, \\ &= \frac{1}{2} \int_0^1 F'(\theta) (l_{k-1}(\theta) - l_{k+1}(\theta)) d\theta, \end{aligned}$$

where the boundary terms vanish insofar as $l_{k+1}(0) = l_{k-1}(0)$ and $l_{k+1}(1) = l_{k-1}(1)$. Repeating this operation, coefficient A_k can be rewritten as, for all $k \geq 2$,

$$\begin{aligned} A_k &= \frac{1}{4(2k-1)} \int_0^1 F''(\theta) (l_{k-2}(\theta) - l_k(\theta)) d\theta \\ &\quad - \frac{1}{4(2k+3)} \int_0^1 F''(\theta) (l_k(\theta) - l_{k+2}(\theta)) d\theta. \end{aligned}$$

Moreover, an additional property of Legendre polynomials taken from [25, Theorem 61] states that inequality $|l_k(\theta)| \leq \frac{1}{2} \sqrt{\frac{\pi}{2k\theta(1-\theta)}}$ holds for all $\theta \in (0, 1)$. Hence, this yields

$$\begin{aligned} |A_k| &\leq \frac{1}{(2k-1)} \int_0^1 |F''(\theta)| |l_{k-2}(\theta)| d\theta, \\ &\leq \frac{\sqrt{\frac{\pi}{2}}}{2(2k-1)\sqrt{k-2}} \int_0^1 \frac{|F''(\theta)|}{\sqrt{\theta(1-\theta)}} d\theta. \end{aligned}$$

Therefore, thanks to the assumption $\sup_{(0,1)} |F''(\theta)| \leq \rho$, this result provides the upper bound

$$|A_k| \leq \frac{(\frac{\pi}{2})^{\frac{3}{2}} \rho}{(2k-1)\sqrt{k-2}} \leq \frac{(\frac{\pi}{2})^{\frac{3}{2}} \rho}{2(k-2)^{\frac{3}{2}}}. \quad (\text{A.2})$$

By integral test for convergence, the sum

$$\sum_{k=n}^N |A_k| \leq \int_{n-1}^{N-1} \frac{(\frac{\pi}{2})^{\frac{3}{2}} \rho}{2(s-2)^{\frac{3}{2}}} ds \leq -\frac{(\frac{\pi}{2})^{\frac{3}{2}} \rho}{\sqrt{N-3}} + \frac{(\frac{\pi}{2})^{\frac{3}{2}} \rho}{\sqrt{n-3}},$$

is bounded as N tends to infinity. We define the sequence of functions $\tilde{F}_n(\theta) = \lim_{N \rightarrow \infty} \sum_{k=n}^N A_k l_k(\theta)$ and identify a uniform upper bound

$$\left| \tilde{F}_n(\theta) \right| \leq \frac{(\frac{\pi}{2})^{\frac{3}{2}} \rho}{\sqrt{n-3}}, \quad \forall \theta \in [0, 1],$$

which concludes the proof. \square

A.2 Proof of Lemma 3

Proof: The objective of the proof is to demonstrate the uniform convergence towards zero of

$$\tilde{F}'_n(\theta) = F'(\theta) - \mathbf{F}_n^\top \mathcal{I}_n \ell'_n(\theta) = F'(\theta) - \sum_{k=0}^{n-1} A_k l'_k(\theta),$$

with A_k given by (A.1). The aim is to show that the series $\sum_{k=n}^{\infty} A_k l'_k(\theta)$ exists and converges uniformly to zero with respect to θ as n tends to infinity. The proof is similar

to the previous one by using $|l'_k(\theta)| \leq |l'_k(1)| = k(k+1)$ to obtain

$$\left| \sum_{k=n}^N A_k l'_k(\theta) \right| \leq \sum_{k=n}^N k(k+1) |A_k|, \quad \forall \theta \in [0, 1].$$

Then, compared to Appendix A.1, it suffices to repeat twice an integration by parts to obtain

$$k(k+1) |A_k| \leq k(k+1) \frac{\left(\frac{\pi}{2}\right)^{\frac{3}{2}}}{(k-4)^{\frac{7}{2}}} \rho' \leq \frac{30 \left(\frac{\pi}{2}\right)^{\frac{3}{2}}}{(k-4)^{\frac{3}{2}}}.$$

Performing once more a comparison series-integral allows concluding that

$$\left| \tilde{F}'_n(\theta) \right| \leq \frac{60 \left(\frac{\pi}{2}\right)^{\frac{3}{2}}}{\sqrt{n-5}}, \quad \forall \theta \in [0, 1].$$

Thus, the sequence of functions \tilde{F}'_n converge uniformly to zero in the interval $[0, 1]$. \square

A.3 Proof of Lemma 4

Proof: The objective of the proof is to demonstrate the uniform convergence towards zero of

$$\tilde{F}_n(\theta) = F(\theta) - \mathbf{F}_n^\top \mathcal{I}_n \ell_n \left(\frac{\theta+1}{2}\right) = F(\theta) - \sum_{k=0}^{n-1} A_k l_k \left(\frac{\theta+1}{2}\right),$$

with

$$A_k = \frac{2k+1}{2} \int_{-1}^1 F(\theta) l_k \left(\frac{\theta+1}{2}\right) d\theta. \quad (\text{A.3})$$

One aims at showing that the series $\sum_{k=n}^{\infty} A_k l_k \left(\frac{\theta+1}{2}\right)$ exists and converges uniformly to zero with respect to θ as n tends to infinity. Since Legendre polynomials satisfy $|l_k(\theta)| \leq 1$, we have

$$\left| \sum_{k=n}^N A_k l_k \left(\frac{\theta+1}{2}\right) \right| \leq \sum_{k=n}^N |A_k|, \quad \forall \theta \in [-1, 1].$$

Then, likewise in Appendix A.1, two integrations by parts on both semi-intervals $(-1, 0)$ and $(0, 1)$ lead to

$$\begin{aligned} A_k &= \frac{1}{2k-1} \int_{-1}^1 F''(\theta) \left(l_{k-2} \left(\frac{\theta+1}{2}\right) - l_k \left(\frac{\theta+1}{2}\right) \right) d\theta \\ &\quad - \frac{1}{2k+3} \int_{-1}^1 F''(\theta) \left(l_k \left(\frac{\theta+1}{2}\right) - l_{k+2} \left(\frac{\theta+1}{2}\right) \right) d\theta \\ &\quad + \frac{1}{2k-1} \left(l_{k-2} \left(\frac{1}{2}\right) - l_k \left(\frac{1}{2}\right) \right) \left(\frac{F'(0^+) - F'(0^-)}{2} \right) \\ &\quad - \frac{1}{2k+3} \left(l_k \left(\frac{1}{2}\right) - l_{k+2} \left(\frac{1}{2}\right) \right) \left(\frac{F'(0^+) - F'(0^-)}{2} \right). \end{aligned}$$

Inequality $|l_k(\frac{1}{2})| \leq \frac{e}{\pi\sqrt{k}}$ holds for $k \in \mathbb{N}^*$ and implies

$$|A_k| \leq \frac{2 \left(\left(\frac{\pi}{2}\right)^{\frac{3}{2}} \bar{\rho} + \frac{e}{\pi} \bar{f} \right)}{(k-2)^{\frac{3}{2}}}.$$

Thus, for all $\theta \in [-1, 1]$, we deduce that

$$\left| \tilde{F}_n(\theta) \right| = \lim_{N \rightarrow \infty} \left(\sum_{k=n}^N A_k l_k(\theta) \right) \leq \frac{4 \left(\left(\frac{\pi}{2}\right)^{\frac{3}{2}} \bar{\rho} + \frac{e}{\pi} \bar{f} \right)}{\sqrt{n-3}},$$

an uniform upper bound which converges to zero. \square

References

- [1] M. Ahmadi, G. Valmorbidia, and A. Papachristodoulou. Dissipation inequalities for the analysis of a class of PDEs. *Automatica*, 66:163–171, 2016.
- [2] J.P. Boyd. *Chebyshev and Fourier Spectral Methods*. Dover Books on Mathematics. Dover Publications, 2001.
- [3] A. Ergorov and S. Mondié. Necessary conditions for the stability of multiple time-delay systems via the delay Lyapunov matrix. In *IFAC Proceedings Volumes*, volume 46, pages 12–17, 2013.
- [4] Q. Feng and SK. Nguang. Stabilization of uncertain linear distributed delay systems with dissipativity constraints. *Systems & Control Letters*, 96:60–71, 2016.
- [5] E. Fridman. New Lyapunov–Krasovskii functionals for stability of linear retarded and neutral type systems. *Systems & control letters*, 43(4):309–319, 2001.
- [6] E. Fridman. *Introduction to Time-Delay Systems : analysis and control*. Systems and Control. Birkäuser, 2014.
- [7] E. Fridman. Tutorial on Lyapunov-based methods for time-delay systems. *European Journal of Control*, 20:271–283, 2014.
- [8] W. Gautschi. Orthogonal polynomials, quadrature, and approximation: computational methods and software (in Matlab). *Lecture Notes in Mathematics*, 1883:1–77, 2006.
- [9] M.A. Gomez, A. Ergorov, and S. Mondie. Necessary stability conditions for neutral type systems with a single delay. *IEEE Trans. on Automatic Control*, 62(9):4691–4697, 2016.
- [10] M.A. Gomez, A. Ergorov, and S. Mondie. Necessary and sufficient stability condition by finite number of mathematical operations for time-delay systems of neutral type. *IEEE Trans. on Automatic Control*, 66(6):2802–2808, 2021.
- [11] M.A. Gomez, A.V. Ergorov, and S. Mondie. Lyapunov matrix based necessity and sufficient stability condition by finite number of mathematical operations for retarded type systems. *Automatica*, 108(108475), 2019.
- [12] K. Gu, V. Kharitonov, and J. Chen. *Stability of Time-Delay Systems*. Birkhäuser, Boston, USA, 2003.
- [13] K. Gu and S-I. Niculescu. *Stability analysis of time-delay systems : a Lyapunov approach*. In advanced topics in control systems theory. Springer, London, 2006.
- [14] Q-L. Han. A discrete delay decomposition approach to stability of linear retarded and neutral systems. *Automatica*, 45(2):517–524, 2009.

- [15] Y. He, M. Wu, J.-H. She, and G.-P. Liu. Delay-dependent robust stability criteria for uncertain neutral systems with mixed delays. *Systems & Control Letters*, 51(1):57–65, 2004.
- [16] Y.B. Huang, Y. He, J. An, and M. Wu. Polynomial-type Lyapunov–Krasovskii functional and Jacobi–Bessel inequality: further results on stability analysis of time-delay systems. *IEEE Transactions on Automatic Control*, 66(6):2905–2912, 2021.
- [17] E. Jarlebring and W. Michiels. Characterizing and computation \mathcal{H}_2 norm of time delay systems by solving the delay Lyapunov equation. *IEEE Transactions on Automatic Control*, 56:814 – 825, 2011.
- [18] V. Kharitonov and A. Zhabko. Lyapunov-Krasovskii approach to the robust stability of time-delay systems. *Automatica*, 39:15–20, 2002.
- [19] V.L. Kharitonov. *Time-Delay Systems: Lyapunov Functionals and Matrices*. Control engineering. Birkhäuser, 2013.
- [20] M. Krstic. *Delay compensation for nonlinear, adaptive, and PDE systems*. Springer, 2009.
- [21] SY. Lee, JM. Park, and PG. Park. Bessel summation inequalities for stability analysis of discrete-time systems with time-varying delays. *International Journal of Robust and Nonlinear Control*, 29(2):473–491, 2019.
- [22] PG. Park, WI. Lee, and SY. Lee. Auxiliary function-based integral inequalities for quadratic functions and their applications to time-delay systems. *Journal of the Franklin Institute*, 352(4):1378–1396, 2015.
- [23] M. Peet. A dual to Lyapunov’s second method for linear systems with multiple delays and implementation using SOS. *IEEE Trans. on Automatic Control*, 64(3):944–959, 2019.
- [24] M. Peet, A. Papachristodoulou, and S. Lall. Positive forms and stability of linear time-delay systems. *SIAM J. Control Optim.*, 47(6):3237–3258, 2009.
- [25] E. D. Rainville. *Special functions*. The Macmillan Co., 1960.
- [26] J-P. Richard. Time delay systems: an overview of some recent advances and open problems. *Automatica*, 39:1667–1694, 10 2003.
- [27] A. Seuret and F. Gouaisbaut. Hierarchy of LMI conditions for the stability analysis of time-delay systems. *Systems and Control Letters*, 81:1–7, 2015.
- [28] G. Valmorbidia, M. Ahmadi, and A. Papachristodoulou. Stability analysis for a class of partial differential equations via semidefinite programming. *IEEE Trans. on Automatic Control*, 61(6):1649–1654, 2016.
- [29] H. Wang and S. Xiang. On the convergence rates of Legendre approximation. *Mathematical of Computation*, 2012.
- [30] M. Webb, V. Coppé, and D. Huybrechs. Pointwise and uniform convergence of Fourier extensions. *Constructive Approximation*, 52:139–175, 2019.
- [31] S. Xu and J. Lam. A survey of linear matrix inequality techniques in stability analysis of delay systems. *International Journal of Systems Science*, 39(12):1095–1113, 2008.