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On the necessity of sufficient LMI conditions for time-delay systems arising from Legendre approximation [★]

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Abstract

This work is dedicated to the stability analysis of time-delay systems with a single constant delay using the Lyapunov-Krasovskii theorem. This approach has been widely used in the literature and numerous sufficient conditions of stability are proposed and expressed as linear matrix inequalities (LMI). The main criticism of the method that is often pointed out is that these LMI conditions are only sufficient, and there is the lack of information regarding the reduction of the conservatism. Recently, scalable methods have been investigated using Bessel-Legendre inequality or orthogonal polynomial-based inequalities. The interest of these methods relies on their hierarchical structure with a guarantee of reduction of the level of conservatism. However, the convergence is still an open question that will be answered for the first time in this paper. The objective of this paper is thus to prove that the stability of a time-delay system implies the feasibility of the scalable LMI provided in Seuret et al., at a sufficiently large order of the Bessel-Legendre inequality. Moreover, the proposed method is even able to provide an estimation of this order analytically, giving rise to necessary and sufficient LMI for the stability of time-delay systems.

Key words: Time-delay systems; Stability conditions; Lyapunov analysis; LMI; Polynomial approximation.

1 Introduction

Time-delay systems represent a wide class of dynamical systems arising in many applications in electronics, biology, transport, etc... Their interest in automatic control is natural since they propose many challenging theoretical problems related to their intrinsic infinite dimensional nature [25,33]. In particular, their stability analysis has been at the heart of many research works for several decades [8,15,21,30,37]. Thus, many methods to establish their stability properties have been developed and, among them, the use of Lyapunov-Krasovskii functionals remains one of the most popular techniques because of their inherent robustness.

In the context of linear time-delay systems, several conditions have been obtained over the last two decades by application of the Lyapunov-Krasovskii theorem leading generally to sum of square constraints [27,28,34] or to linear matrix inequalities (LMI) [7,17,36]. These conditions, combined with semi-definite convex optimization programs, led to only sufficient conditions. This line of research has given rise to numerous works, which all aim

at reducing the inherent conservatism and at recovering the necessity using, for instance, discretized functionals [14,15] or delay partitioning methods [3,13]. Among them, an approach based on state extension has been proposed in [32]. The Lyapunov functional therein depends directly on the projections of the state of the delay system onto Legendre polynomials. This approach led to LMI stability conditions, which benefit from a particular hierarchical structure, arising from the use of the Bessel-Legendre inequality. Similar approach based on orthogonal polynomials [22,26] and dissipative inequalities [1,6] have been also considered in the literature. The size of this LMI increases with the number of Legendre polynomials n taken under consideration but their conservatism drastically reduces as n increases on example. However, to the best of our knowledge, the proof of convergence of the LMI to a necessary and sufficient LMI condition of stability is still missing.

Towards this direction, as for linear finite dimensional systems, there exists a converse Lyapunov-Krasovskii theorem for linear systems with a constant delay. Indeed, it is possible to build a Lyapunov-Krasovskii functional for stable linear time-delay systems, see for instance [21] for a larger overview on the problem. While this method has been only seen as a theoretical contributions, the authors of [23,24] paved the way to use the

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converse Lyapunov-Krasovskii theorem to derive not sufficient but necessary stability conditions for time-delay systems through approximation. Among them, the authors of [5] provided a necessary stability conditions for linear time-delay systems. This method was then extended to various classes of delay systems, see for instance [10,11]. More interestingly, their methods have led to necessary and sufficient stability conditions in [4,12]. The sufficiency part of their necessary condition has been obtained by estimating an upper bound of the approximation error due to the discretization process of the Lyapunov matrix. It is worth noting that these conditions are not LMI but only a test of positive definiteness of a given matrix, which makes their method efficient from computational reasons. However, even though the necessary condition of stability is simple to test, the sufficient condition of stability is still numerically complicated to verify, because the order assessing the sufficiency can be very high.

The objective of the present paper follows the same spirit of deriving necessary and sufficient stability conditions of time-delay systems but with a different objective. Indeed, the objective is here to use the approach developed in [12,14] to demonstrate that the sufficient LMI conditions arising from the Bessel-Legendre inequality are asymptotically necessary. This corresponds to answering the following questions.

- If a time-delay system is stable, does there exist a minimal order N^* from which some LMI stability condition is necessary satisfied?
- Is it possible to estimate this order N^* ?

The paper is organized as follows. Section 2 formulates the problem of stability analysis of linear time-delay systems. After presenting a brief summary of the properties of Legendre polynomials, a sufficient scalable LMI condition for time-delay systems is presented. Then, Section 3 states the converse theorem of this first result, which ensures the existence of an order N^* for which the LMI stability conditions must be verified for stable linear systems. Section 4 goes beyond the convergence result by providing an estimation of the order N^* of the polynomial approximation for which the LMI conditions are guaranteed. Finally, the theoretical results are evaluated on academic examples leading to several discussions. The paper ends with a conclusion and several appendixes gathering the highly technical proofs required to derived the main results of this paper.

Notations : Throughout the paper, \mathbb{N} (\mathbb{N}^*), $\mathbb{R}^{m \times p}$ and \mathbb{S}^m (\mathbb{S}_+^m) denote the sets of natural numbers (excluding zero), real matrices of size $m \times p$ and symmetric matrices of size m (positive definite), respectively. For any x in \mathbb{R} , $\lceil x \rceil$ stands for the least integer greater than or equal to x . For any square matrix $M \in \mathbb{R}^{m \times m}$, $M(p, q)$ denotes the entries of M located at the p^{th} row and q^{th} column, M^\top denotes the transpose of M and $\mathcal{H}(M)$ stands for

$M + M^\top$. Furthermore, for any M in \mathbb{S}^m , its minimal and maximal eigenvalues are denoted $\underline{\sigma}(M)$ and $\bar{\sigma}(M)$. The 2-norm of matrix M in $\mathbb{R}^{m \times p}$ is $|M| = \sqrt{\bar{\sigma}(M^\top M)}$. The vector $u = \text{vec}(M)$ in $\mathbb{R}^{mp \times 1}$ collocates the columns of matrix M and the inverse of this operation is denoted vec^{-1} and is such that $\text{vec}^{-1}(\text{vec}(M)) = M$. Moreover, notation \otimes represents the Kronecker product. The set of square-integrable functions from (a, b) to $\mathbb{R}^{m \times p}$ is noted $\mathcal{L}_2(a, b; \mathbb{R}^{m \times p})$. Let finally $\mathcal{C}_{pw}(a, b; \mathbb{R}^{m \times p})$ the set of piecewise continuous functions with a finite number of discontinuity points and, for each continuity interval, finite left and right-hand side limits.

2 A sufficient LMI stability condition for time-delay systems

2.1 System data

Consider a time-delay system given by

$$\dot{x}(t) = Ax(t) + A_d x(t-h), \quad \forall t \geq 0, \quad (1a)$$

$$x(t) = \varphi(t) \in \mathcal{C}_{pw}(-h, 0; \mathbb{R}^{n_x}), \quad \forall t \in [-h, 0], \quad (1b)$$

where the single delay $h > 0$ and matrices A, A_d in $\mathbb{R}^{n_x \times n_x}$ are constant and known. Without loss of generality, let decompose A_d into a product BC as

$$A_d = BC \text{ with } |C| = 1, \quad (2)$$

with $B, C^\top \in \mathbb{R}^{n_x \times n_z}$ being full column rank matrices, with n_z the rank of matrix A_d . Along the paper, notation $x_t(\theta) = x(t+\theta)$, for all (t, θ) in $\mathbb{R}^+ \times [-h, 0]$ is used.

Definition 1 (GES) *The trivial solution $x(t) \equiv 0$ of system (1) is globally exponentially stable (GES), if there exist $\kappa \geq 1$ and $\mu > 0$ such that the solution to (1) generated by any initial condition $\varphi \in \mathcal{C}_{pw}(-h, 0; \mathbb{R}^{n_x})$, denoted as $x(t, \varphi)$ verifies*

$$|x(t, \varphi)| \leq \kappa e^{-\mu t} \sup_{[-h, 0]} |\varphi|, \quad \forall t \geq 0. \quad (3)$$

Several ways of assessing GES of linear systems have been provided in the literature as mentioned in the introduction. Here, the contribution focuses on conditions arising from the application of the Lyapunov-Krasovskii theorem, adapted to the linear case.

Theorem 1 *Let $\varepsilon_1, \varepsilon_2$ and ε_3 be positive scalars. If there exists a continuous and differentiable functional $\mathcal{V} : \mathcal{C}_{pw}(-h, 0; \mathbb{R}^{n_x}) \rightarrow \mathbb{R}_+$ such that, for any φ in $\mathcal{C}_{pw}(-h, 0; \mathbb{R}^{n_x})$, inequalities*

$$(i) \quad \varepsilon_1 |\varphi(0)|^2 \leq \mathcal{V}(\varphi) \leq \varepsilon_2 \sup_{[-h, 0]} |\varphi|^2 \text{ hold and}$$

(ii) $\dot{\mathcal{V}}(\varphi) \leq -\varepsilon_3 |\varphi(0)|^2$ holds for all $t \geq 0$, where $\dot{\mathcal{V}}$ denotes here the derivative of \mathcal{V} along the trajectories of (1).

Then, the solution $x(t, 0) \equiv 0$ is GES for system (1).

Such a functional is called a Lyapunov-Krasovskii functional.

The main underlying idea of this theorem is to determine a positive definite functional \mathcal{V} , such that its derivative with respect to time along the trajectories of the system (1) is negative definite. The main problem within the application of this theorem is to design a suitable functional and then to provide some conditions that guarantee its positive definiteness and the negative definiteness of its derivative.

The derivation of stability conditions using Lyapunov-Krasovskii functionals usually involves quite elaborate developments. To give an idea of the procedure involved in this approach and to provide a glimpse of its technical flavor, we present here some basics on the procedure to follow in order to derive asymptotic stability criteria for time-delay systems expressed in terms of linear matrix inequality (LMI). The procedure follows three steps.

Step 1. Propose a candidate Lyapunov-Krasovskii functional \mathcal{V} , based on the structure of the system.

Step 2. Compute the derivative of the functional along the trajectories of the system.

Step 3. Apply over-approximation technique to $\dot{\mathcal{V}}$ and derive a stability condition expressed in terms of LMI.

Among the numerous method employed in the literature, we will focus here on the method based on the application of the Bessel-Legendre integral inequality [31,32], which is scalable with respect to the degree of the Legendre polynomial to be considered and provides interestingly LMI with a hierarchical structure. Before stating this theorem, let us first recall the main definitions and some characteristics about these polynomials.

2.2 Definition of Legendre polynomials

The Legendre polynomials widely used in polynomial approximation theory [2] are defined over the normalized interval $[0, 1]$ as

$$\forall k \in \mathbb{N}, \quad l_k(\theta) = \sum_{j=0}^k \binom{k}{j} \binom{k+j}{j} (\theta-1)^j, \quad (4)$$

where $\binom{k}{j}$ stands for the binomial coefficients. The main motivation for employing these polynomials comes from their nice properties that are summarized below. Before going into the details, for any order n and n_z in \mathbb{N}^* and $\theta \in [0, 1]$, let us introduce the following

notation

$$\ell_n(\theta) = \begin{bmatrix} l_0(\theta)I_{n_z} & l_1(\theta)I_{n_z} & \dots & l_{n-1}(\theta)I_{n_z} \end{bmatrix}^\top \in \mathbb{R}^{nn_z \times n_z}, \quad (5)$$

Property 1 The Legendre polynomials given in (4) represent an orthogonal sequence with respect to the inner product $\int_0^1 \phi^\top(\theta)\psi(\theta)d\theta$, for any ϕ, ψ in $\mathcal{L}_2(0, 1; \mathbb{R}^{n_z})$. The following equality holds for all n in \mathbb{N}^* , and for all symmetric matrix $R \in \mathbb{S}_+^{n_z}$

$$\int_0^1 \ell_n(\theta)R^{-1} \ell_n^\top(\theta)d\theta = \mathcal{R}_n^{-1} \in \mathbb{R}^{nn_z \times nn_z}, \quad (6)$$

where $\mathcal{R}_n = \mathcal{I}_n^0 \otimes R$, with

$$\mathcal{I}_n^0(p, q) = \begin{cases} 2p-1 & \text{if } p=q, \\ 0 & \text{otherwise.} \end{cases}$$

Proof : The proof can be found in [9]. \square

In particular, we will consider over the paper notation $\mathcal{I}_n^p = \mathcal{I}_n^0 \otimes I_p$, for any integer p . To ease the reading of the paper and where no confusion is possible, we will omit the lower script ‘ p ’ and only use notation \mathcal{I}_n .

Furthermore, the Legendre polynomials have been introduced because they allows formulating efficient integral inequalities, which are stated here

Lemma 1 Let $x \in \mathcal{L}_2(-h, 0; \mathbb{R}^{n_z})$ and $R \in \mathbb{S}_+^{n_z}$ a positive definite matrix. The integral inequalities

$$\int_{-h}^0 x^\top(\theta)Rx(\theta)d\theta \geq \frac{1}{h} \xi_n(x)^\top \mathcal{R}_n \xi_n(x), \quad (7a)$$

$$\int_{-h}^0 (\theta+h)x^\top(\theta)Rx(\theta)d\theta \geq \frac{1}{2} \xi_n(x)^\top \tilde{\mathcal{R}}_n \xi_n(x), \quad (7b)$$

hold, for all $n \in \mathbb{N}^*$, where $\xi_n(x) = \int_{-h}^0 \ell_n\left(\frac{\theta+h}{h}\right)x(\theta)d\theta$ and where matrix \mathcal{R}_n is defined in Property 1 and matrix $\tilde{\mathcal{R}}_n \in \mathbb{R}^{nn_z \times nn_z}$ is defined as follows

$$\tilde{\mathcal{R}}_n = \mathcal{J}_n^0 \otimes R, \quad \mathcal{J}_n^0(p, q) = \begin{cases} 2p-1 & \text{if } p=q, \\ \min(p, q) & \text{if } |p-q|=1, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, if x is a polynomial function of degree $n-1$ over $(-h, 0)$, both inequalities become equality.

Proof : The proof of these two inequalities can be found in [32] for the first one and in [16] for the second one. \square

2.3 Sufficient LMI condition

We are now in position to state a stability theorem for system (1) based on the previous developments.

Theorem 2 *For a given constant delay h , if there exist an order n in \mathbb{N}^* and matrices (\mathbf{P}_n, R, S) in $\mathbb{S}^{n_x + nn_z} \times \mathbb{S}_+^{n_z} \times \mathbb{S}_+^{n_z}$ such that the following LMI holds*

$$\Phi_n^+ = \mathbf{P}_n + \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{h}\mathcal{S}_n + \frac{1}{2}\tilde{\mathcal{R}}_n \end{bmatrix} \succ 0, \quad (8a)$$

$$\Phi_n^- = \begin{bmatrix} \mathcal{H}(\mathbf{P}_n \mathbf{A}_n) + \Phi_n^0(S, R) & \mathbf{P}_n \mathbf{B}_n \\ * & -S \end{bmatrix} \prec 0, \quad (8b)$$

where

$$\begin{aligned} \Phi_n^0(S, R) &= \begin{bmatrix} C^\top(S + hR)C & 0 \\ 0 & -\frac{1}{h}\mathcal{R}_n \end{bmatrix}, \\ \mathbf{A}_n &= \begin{bmatrix} A & 0 \\ \ell_n(1)C & -\frac{1}{h}\mathcal{L}_n^{n_z} \end{bmatrix}, \quad \mathbf{B}_n = \begin{bmatrix} B \\ -\ell_n(0) \end{bmatrix}, \\ \ell_n(1) &= \begin{bmatrix} I_{n_z} \\ \vdots \\ I_{n_z} \end{bmatrix}, \quad \ell_n(0) = \begin{bmatrix} I_{n_z} \\ \vdots \\ (-1)^{n-1}I_{n_z} \end{bmatrix} \in \mathbb{R}^{nn_z}, \\ \mathcal{L}_n^{n_z} &= \mathcal{L}_n^0 \otimes I_{n_z} \in \mathbb{R}^{nn_z \times nn_z}, \\ \mathcal{L}_n^0(p, q) &= \begin{cases} (2q-1)(1-(-1)^{p+q}) & \text{if } p \geq q, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

then, the trivial solution of system (1) is GES.

Proof: For a given integer $n \geq 0$, consider the functional defined as follows

$$\begin{aligned} \mathcal{V}_n(x_t) &= \begin{bmatrix} x_t(0) \\ \xi_n(x_t) \end{bmatrix}^\top \mathbf{P}_n \begin{bmatrix} x_t(0) \\ \xi_n(x_t) \end{bmatrix} \\ &+ \int_{-h}^0 x_t^\top(\theta) C^\top (S + (\theta + h)R) C x_t(\theta) d\theta, \end{aligned} \quad (9)$$

where matrices \mathbf{P}_n , S and R are solution to the LMI stated in the theorem and where the augmented vector ξ_n is defined as follows

$$\xi_n(x_t) = \int_{-h}^0 \ell_n \left(\frac{\theta + h}{h} \right) C x_t(\theta) d\theta. \quad (10)$$

Applying the Bessel inequalities presented in Lemma 1 to both integral terms yields

$$\mathcal{V}_n(x_t) \geq \begin{bmatrix} x_t(0) \\ \xi_n(x_t) \end{bmatrix}^\top \Phi_n^+ \begin{bmatrix} x_t(0) \\ \xi_n(x_t) \end{bmatrix}.$$

Therefore, if condition (8a) is verified, then there exists a sufficiently small $\varepsilon_1 > 0$ such that $\mathcal{V}_n(x_t) \geq \varepsilon_1 |x_t(0)|^2$. In addition, since \mathcal{V} is quadratic with respect to x_t , we can take $\varepsilon_2 = \bar{\sigma}(\mathbf{P}_n) + \bar{\sigma}(S) + h\bar{\sigma}(R)$ so that inequality $\mathcal{V}_n(x_t) \leq \varepsilon_2 \sup_{[-h,0]} |x_t|$ holds.

As in [31], computing the derivative of the functionals along the trajectories of the system yields

$$\begin{aligned} \dot{\mathcal{V}}_n(x_t) &= 2 \begin{bmatrix} x_t(0) \\ \xi_n(x_t) \end{bmatrix}^\top \mathbf{P}_n \begin{bmatrix} \dot{x}_t(0) \\ \dot{\xi}_n(x_t) \end{bmatrix} - x_t^\top(-h) C^\top S C x_t(-h) \\ &+ x_t^\top(0) C^\top (S + hR) C x_t(0) - \int_{-h}^0 x_t^\top(\theta) C^\top R C x_t(\theta) d\theta. \end{aligned} \quad (11)$$

Then, the dynamics (1a) and an integration by parts of $\dot{\xi}_n(x_t)$ provides an expression $\begin{bmatrix} \dot{x}_t(0) \\ \dot{\xi}_n(x_t) \end{bmatrix}$ with respect to $x_t(0)$, $\xi_n(x_t)$ and $x_t(-h)$. It ensures that

$$\begin{bmatrix} \dot{x}_t(0) \\ \dot{\xi}_n(x_t) \end{bmatrix} = \mathbf{A}_n \begin{bmatrix} x_t(0) \\ \xi_n(x_t) \end{bmatrix} + \mathbf{B}_n C x_t(-h).$$

Re-injecting this expression into (11) and applying the Bessel inequality yields

$$\dot{\mathcal{V}}_n(x_t) \leq \begin{bmatrix} x_t(0) \\ \xi_n(x_t) \\ C x_t(-h) \end{bmatrix}^\top \Phi_n^- \begin{bmatrix} x_t(0) \\ \xi_n(x_t) \\ C x_t(-h) \end{bmatrix}.$$

Therefore, if condition (8b) is satisfied, then there exists a sufficiently small ε_3 such that $\dot{\mathcal{V}}_n(x_t) \leq -\varepsilon_3 |x_t(0)|^2$, which concludes the proof. \square

Remark 1 *Note that the proposed theorem has a couple of differences with respect to the original conditions presented in [31] or in [32]. The first one is due to the application of inequality (7b), which can also be seen as double integral inequality. This will have a fundamental importance in the next developments. The second one is related to the decomposition of A_d as the product of matrices $B \in \mathbb{R}^{n_x \times n_z}$ and $C \in \mathbb{R}^{n_z \times n_x}$ of ranks $n_z \leq n_x$. This manipulation allows reducing the complexity of the LMI by reducing both the number of rows and columns but also the number of decision variables when A_d is singular.*

As in [31] or in [32], the previous stability condition form a hierarchy with respect to n . This aspect is presented formally in the following lemma.

Lemma 2 *If there exists an integer $N \in \mathbb{N}^*$, for which a solution to LMI (8) exists, then there also exists a solution to the same problem for any integer $n \geq N$.*

Proof : The proof uses similar arguments to the ones provided in [32, Theorem 7]. A glimpse of the proof consists in introducing $\mathbf{P}_{n+1} = \begin{bmatrix} \mathbf{P}_n & 0 \\ 0 & 0 \end{bmatrix}$, so that $\mathcal{V}_{n+1}(x_t) = \mathcal{V}_n(x_t)$ and so that one can exhibit a solution to the LMI problem at order $n + 1$ based on the solution at order n . The details of the proof are omitted but strongly relies on the structure of Φ_n^+ and Φ_n^- . \square

To sum up the results presented so far, Theorem 2 presents scalable LMI conditions for the stability (GES) of linear time-delay systems. These conditions depend on an order $n \in \mathbb{N}^*$, corresponding to the degree of the Legendre polynomial considered in the construction of the Lyapunov-Krasovskii functional. Moreover, it is demonstrated that when n increases, the conservatism can only be reduced. It is however legitimate to wonder if a converse result can be proven. This direction refers to the possibility of assessing the satisfaction of these LMI conditions for a sufficiently large n , when the system under consideration is known a priori to be GES for a given delay h . Apart the convergence, ie. the existence of an order N^* for which the LMI condition is guaranteed, an interesting underlying question is to estimate analytically such a bound. The next developments aim at providing a solution to these two problems. More specifically, the next section presents a converse theorem with first a proof of existence of N^* , while the latter address the problem of estimating order N^* .

3 Necessity of the LMI condition: Existence of an order

The converse result of Theorem 2 is formally stated.

Theorem 3 *If the trivial solution of system (1) is GES for a given delay h , then there exist an order N^* in \mathbb{N}^* and matrices (\mathbf{P}_{N^*}, R, S) in $\mathbb{S}_+^{n_x + N^* n_z} \times \mathbb{S}_+^{n_x} \times \mathbb{S}_+^{n_z}$ such that inequalities $\Phi_{N^*}^+ \succ 0$ and $\Phi_{N^*}^- \prec 0$ in (8) hold.*

The proof is decomposed in five subsections respectively based on the steps listed below.

Step 1. *To ensure the existence of a complete Lyapunov-Krasovskii functional which can be expressed analytically with respect to a Lyapunov matrix function.*

Step 2. *To use the properties of convergence of Legendre polynomial series to approximate the Lyapunov matrix function and its derivatives.*

Step 3. *To make the link between \mathcal{V}_n introduced in (9) and the Legendre approximation of the complete functional.*

Step 4. *To apply the approximation results to obtain a new necessary condition based on functional \mathcal{V}_n .*

Step 5. *To recover the LMI condition given in Theorem 2.*

3.1 Existence and properties of complete functionals

It is well known that the stability of system (1) can be established thanks to the use of the so-called complete Lyapunov-Krasovskii functional, firstly introduced in [18] and fully described in [21]. All along this article, for any positive definite matrices (W_1, W_2, W_3) in $\mathbb{S}_+^{n_x} \times \mathbb{S}_+^{n_x} \times \mathbb{S}_+^{n_z}$, we define the complete Lyapunov-Krasovskii functional \mathcal{V} by

$$\begin{aligned} \mathcal{V}(x_t) = & x_t^\top(0)U(0)x_t(0) + 2x_t^\top(0) \int_{-h}^0 U(\theta+h)BCx_t(\theta)d\theta \\ & + \int_{-h}^0 \int_{-h}^0 x_t^\top(\theta_1)(BC)^\top U(\theta_2 - \theta_1)BCx_t(\theta_2)d\theta_1 d\theta_2 \\ & + \int_{-h}^0 x_t^\top(\theta)C^\top(W_2 + (\theta+h)W_3)Cx_t(\theta)d\theta, \end{aligned} \quad (12)$$

where U is a matrix function from $[-h, h]$ to $\mathbb{R}^{n_x \times n_x}$ and is called the Lyapunov matrix. Interestingly, referring to [21, Section 2.10], a method is provided to build analytically this matrix. Its expression is therefore given by

$$\begin{cases} U(\theta) = \text{vec}^{-1} \left([I_{n_x} \ 0] e^{\theta \mathcal{M}} \mathcal{N}^{-1} \begin{bmatrix} -\text{vec}(W) \\ 0 \end{bmatrix} \right) & \text{if } \theta \leq 0, \\ U(\theta) = U^\top(-\theta) & \text{if } \theta > 0, \end{cases} \quad (13)$$

where matrices

$$\begin{aligned} W &= W_1 + C^\top(W_2 + hW_3)C, \\ \mathcal{M} &= \begin{bmatrix} -A^\top \otimes I_{n_x} & -A_d^\top \otimes I_{n_x} \\ I_{n_x} \otimes A_d^\top & I_{n_x} \otimes A^\top \end{bmatrix}, \\ \mathcal{N} &= \begin{bmatrix} A^\top \otimes I_{n_x} + I_{n_x} \otimes A^\top & A_d^\top \otimes I_{n_x} \\ I_{n_x} & 0 \end{bmatrix} + \begin{bmatrix} I_{n_x} \otimes A_d^\top & 0 \\ 0 & -I_{n_x} \end{bmatrix} e^{-h\mathcal{M}}. \end{aligned} \quad (14)$$

and where ‘vec’ is the operator that transforms a matrix in $\mathbb{R}^{n_x \times n_x}$ to a vector in $\mathbb{R}^{n_x^2}$ that collects all the columns of this matrix and ‘vec⁻¹’ denotes the inverse operator.

Remark 2 *Before going any further, it is important to note that this functional cannot be defined if matrix \mathcal{N} is singular. As mentioned in [21, Theorem 2.10], this condition is equivalent to the Lyapunov condition, i.e. there is no eigenvalue of (1) such that its opposite is also eigenvalue. This is not an issue since it excludes a set of systems which are intrinsically unstable.*

The complete functional \mathcal{V} has been built so that its derivative with respect to time along the trajectories of the system yields

$$\begin{aligned} \dot{\mathcal{V}}(x_t) = & -x_t^\top(0)W_1x_t(0) - x_t^\top(-h)C^\top W_2Cx_t(-h) \\ & - \int_{-h}^0 x_t^\top(\theta)C^\top W_3Cx_t(\theta)d\theta, \end{aligned} \quad (15)$$

Recall that W_1, W_2 and W_3 are arbitrary symmetric positive definite matrices. The following theorem holds.

Theorem 4 System (1) is GES for the delay h if and only if \mathcal{V} defined by (12) is a Lyapunov-Krasovskii functional for this system.

Proof : The proof can be found for instance in [15, Theorem 5.19] or in [21]. \square

In addition, the Lyapunov matrix satisfy several properties. The first property is related to the fact that U is continuous over $[-h, h]$ but its derivative, U' , is not only continuous over $[-h, 0) \cup (0, h]$. Indeed, U' has a discontinuity at 0, which is measured as follows.

Property 2 ([21]) The Lyapunov matrix U verifies the following properties

- (i) The Lyapunov matrix U is infinitely differentiable on $[0, h]$ (and $[-h, 0]$) and is continuous at 0 with

$$U(0) = U^\top(0).$$

- (ii) The derivative of the Lyapunov matrix is infinitely differentiable on $[-h, 0) \cup (0, h]$ and a unique discontinuity at 0 so that

$$\Delta U'(0) := \lim_{\epsilon \rightarrow 0} (U'(\epsilon) - U'(-\epsilon)) = W.$$

These regularity conditions satisfied by the Lyapunov matrix have been used for \mathcal{H}_2 [19] or \mathcal{H}_∞ [20] analysis. They will also be at the heart of the derivation of the convergence results.

3.2 Polynomial approximation of the Lyapunov matrix and convergence of the approximation errors

In this section, the objective is to understand how to relate the Lyapunov-Krasovskii functional defined in (9) and the complete one in (12). To do so, the main idea is to exploit the terms of (12) that are expressed in $U(\theta+h)B$ and $B^\top U(\theta_2 - \theta_1)B$. The Legendre polynomial approximation of these functions at any order n in \mathbb{N}^* writes

$$\begin{aligned} U(\theta+h)B &= \mathbf{U}_{1,n} \ell_n \left(\frac{\theta+h}{h} \right) + \tilde{U}_{1,n}(\theta), \quad \forall \theta \in [-h, 0], \\ B^\top U(\theta)B &= \mathbf{U}_{2,n} \ell_n \left(\frac{\theta+h}{2h} \right) + \tilde{U}_{2,n}(\theta), \quad \forall \theta \in [-h, h]. \end{aligned} \quad (16)$$

In this decomposition, the constant matrices $\mathbf{U}_{1,n}$ and $\mathbf{U}_{2,n}$ have been selected as the orthogonal projection of $U(\theta+h)B$ and $B^\top U(\theta)B$, respectively, on the n first Legendre polynomials ℓ_n . Their expressions are given by

$$\begin{aligned} \mathbf{U}_{1,n} &= \frac{1}{h} \left(\int_{-h}^0 U(\theta+h)B \ell_n^\top \left(\frac{\theta+h}{h} \right) d\theta \right) \mathcal{I}_n, \\ \mathbf{U}_{2,n} &= \frac{1}{2h} \left(\int_{-h}^h B^\top U(\theta)B \ell_n^\top \left(\frac{\theta+h}{2h} \right) d\theta \right) \mathcal{I}_n. \end{aligned} \quad (17)$$

Functions $\tilde{U}_{1,n}(\theta)$ and $\tilde{U}_{2,n}(\theta)$ can be interpreted as the approximation errors of the orthogonal projections, which verify

$$\begin{aligned} \int_{-h}^0 \tilde{U}_{1,n}(\theta) \ell_n^\top \left(\frac{\theta+h}{h} \right) d\theta &= \underbrace{\int_{-h}^0 U(\theta+h)B \ell_n^\top \left(\frac{\theta+h}{h} \right) d\theta}_{=\mathbf{U}_{1,n}(\mathcal{I}_n/h)^{-1}} \\ &- \mathbf{U}_{1,n} \underbrace{\int_{-h}^0 \ell_n \left(\frac{\theta+h}{h} \right) \ell_n^\top \left(\frac{\theta+h}{h} \right) d\theta}_{=(\mathcal{I}_n/h)^{-1}} = 0, \end{aligned}$$

where we have used (6) on $[-h, 0]$. Similarly, the same calculations ensure that error $\tilde{U}_{2,n}$ is orthogonal to the n first Legendre polynomials considered over $[-h, h]$, i.e. $\int_{-h}^h \tilde{U}_{2,n}(\theta) \ell_n^\top \left(\frac{\theta+h}{2h} \right) d\theta = 0$.

The next developments aim at demonstrating the uniform convergence of the polynomial approximation. Following the theory of polynomial approximation (see for instance [2] and references therein), it results that $\tilde{U}_{1,n}$ and $\tilde{U}_{2,n}$ given in (16) converge to zero in the sense of the \mathcal{L}_2 norm. This is actually a by-product of Bessel inequality. This implies that the approximation errors converges to zero almost everywhere on their domain of definition, as n tends to infinity. Nevertheless, uniform convergence properties can even be obtained using the regularity of the Lyapunov matrix.

The following lemma assesses the uniform convergence that will be used in the proof of Theorem 3.

Lemma 3 Consider the Lyapunov matrix U for the time-delay systems (1) defined for any positive definite matrices (W_1, W_2, W_3) . The following statements hold.

- (i) The approximated Legendre function $\mathbf{U}_{1,n} \ell_n \left(\frac{\theta+h}{h} \right)$ converges uniformly to $U(\theta+h)B$, when n tends to infinity, on the closed interval $[0, h]$. More precisely, for any $n \geq 4$, the following inequality holds

$$\sup_{\theta \in [0, h]} \left| \tilde{U}_{1,n}(\theta) \right| \leq \bar{u}_{1,n} := \frac{\varrho_1}{\sqrt{n-3}} |W|, \quad (18)$$

where

$$\varrho_1 = \sqrt{n_x} \left(\frac{\pi}{2} \right)^{\frac{3}{2}} e^{h|\mathcal{M}|} |\mathcal{M}^2 \mathcal{N}^{-1}| h^2 |B|, \quad (19)$$

and matrices \mathcal{M} , \mathcal{N} and W are given by (14).

- (ii) The approximated Legendre function $\frac{1}{h} \mathbf{U}_{1,n} \ell_n' \left(\frac{\theta+h}{h} \right)$ converges uniformly to $U'(\theta+h)B$, when n tends to infinity, on the closed interval $[0, h]$. More precisely, for any $n \geq 6$, the following inequality

holds

$$\sup_{\theta \in [0, h]} \left| \tilde{U}'_{1,n}(\theta) \right| \leq \bar{u}_{2,n} := \frac{\varrho_2}{\sqrt{n-5}} |W|, \quad (20)$$

where

$$\varrho_2 = \frac{1}{2} \sqrt{n_x} \left(\frac{\pi}{2} \right)^{\frac{3}{2}} e^{h|\mathcal{M}|} |\mathcal{M}^4 \mathcal{N}^{-1}| h^3 |B|, \quad (21)$$

and matrices \mathcal{M} , \mathcal{N} and W are given by (14).

(iii) The approximated Legendre function $\mathbf{U}_{2,n} \ell_n \left(\frac{\theta+h}{2h} \right)$ converges uniformly to $B^\top U(\theta)B$, when n tends to infinity, on the closed interval $[-h, h]$.

More precisely, for any $n \geq 4$, the following inequality holds

$$\sup_{\theta \in [-h, h]} \left| \tilde{U}_{2,n}(\theta) \right| \leq \bar{u}_{3,n} := \frac{\varrho_3}{\sqrt{n-3}} |W|, \quad (22)$$

where

$$\varrho_3 = \sqrt{2\pi} (1 + \sqrt{n_x} \pi h e^{h|\mathcal{M}|} |\mathcal{M}^2 \mathcal{N}^{-1}|) h |B|^2, \quad (23)$$

and matrices \mathcal{M} , \mathcal{N} and W are given by (14).

Proof: The proof follows the arguments provided in [35, Theorem 2.5] and is based on the regularity properties of the Lyapunov matrix U . Because of its technicality, the proof is postponed to Appendix C, for the sake of readability. \square

The previous lemma provides uniform upper bounds on the error done by three approximations related to the Lyapunov matrix U . This lemma extensively uses the properties of the Lyapunov matrix U on its domain of definition. These uniform convergence results are only presented as properties of U and its polynomial approximations but addresses specific problems of functional analysis of continuous and discontinuous functions.

All these upper bounds depend explicitly on the arbitrary symmetric positive definite matrices (W_1, W_2, W_3) through the term $|W|$, which is intuitively relevant.

As a final comments, Lemma 3 implies the following corollary.

Corollary 1 *For any positive scalar $\eta > 0$, there exists an integer N_η^* such that*

$$\max(\bar{u}_{1,n}, \bar{u}_{2,n}, \bar{u}_{3,n}) \leq \eta, \quad \forall n \geq N_\eta^*. \quad (24)$$

Proof: Since the upper bounds of approximation errors tend to 0 as n tends to infinity, the existence of such N_η^* for any $\eta > 0$ is guaranteed. \square

3.3 Construction of \mathbf{P}_n , S and R

Lemma 4 *Consider any positive definite matrices (W_1, W_2, W_3) in $\mathbb{S}_+^{n_x} \times \mathbb{S}_+^{n_z} \times \mathbb{S}_+^{n_x}$ and assume that the following inequality holds*

$$\Psi_n(\theta) := \begin{bmatrix} W_1 + \mathcal{H}(\tilde{U}_{1,n}(0)C) & -\tilde{U}_{1,n}(-h) & \Psi_n^1(\theta) \\ * & W_2 & \Psi_n^2(\theta) \\ * & * & \frac{1}{h} W_3 \end{bmatrix} \succ 0, \quad (25)$$

for a given n in \mathbb{N}^* and for all θ in $[-h, 0]$ with

$$\begin{aligned} \Psi_n^1(\theta) &= A^\top \tilde{U}_{1,n}(\theta) - \tilde{U}'_{1,n}(\theta) + C^\top \tilde{U}_{2,n}(\theta), \\ \Psi_n^2(\theta) &= B^\top \tilde{U}_{1,n}(\theta) - \tilde{U}_{2,n}(\theta + h), \end{aligned}$$

and with $\tilde{U}_{1,n}$, $\tilde{U}'_{1,n}$ and $\tilde{U}_{2,n}$ are the approximation errors of the Lyapunov matrix U of system (1) generated with matrices (W_1, W_2, W_3) .

Then, having system (1) GES for the delay h ensures that there exists $\varepsilon > 0$ such that the candidate functional \mathcal{V}_n , given in (9) with matrices

$$\begin{aligned} \mathbf{P}_n &= \begin{bmatrix} U(0) & \mathbf{U}_{1,n} \\ * & \mathbf{T}_n \end{bmatrix}, \quad S = W_2, \quad R = W_3, \\ \mathbf{T}_n &= \int_{-h}^0 \int_{-h}^0 \frac{\mathcal{I}_n}{h} \ell_n \left(\frac{\theta_1+h}{h} \right) \mathcal{U}_{2,n}(\theta_2 - \theta_1) \ell_n^\top \left(\frac{\theta_2+h}{h} \right) \frac{\mathcal{I}_n}{h} d\theta_1 d\theta_2, \\ \mathcal{U}_{2,n}(\theta) &= \mathbf{U}_{2,n} \ell_n \left(\frac{\theta+h}{2h} \right), \end{aligned} \quad (26)$$

where $\mathbf{U}_{1,n}$, $\mathbf{U}_{2,n}$ are given in (17), verifies

$$\dot{\mathcal{V}}_n(\varphi) \geq \varepsilon \left(|\varphi(0)|^2 + h \int_{-h}^0 |C\varphi(\theta)|^2 d\theta \right), \quad (27a)$$

$$\dot{\mathcal{V}}_n(\varphi) \leq -\varepsilon \left(|\varphi(0)|^2 + h \int_{-h}^0 |C\varphi(\theta)|^2 d\theta + |C\varphi(-h)|^2 \right), \quad (27b)$$

for all φ in $\mathcal{C}_{pw}(-h, 0; \mathbb{R}^{n_x})$.

Proof: The proof is divided into two parts, each one being dedicated to each inequality in (27). Let us start with the proof of inequality (27b). Re-injecting the expression of $U(\theta+h)B$ and $B^\top U(\theta)B$ and using their approximation given by (16) into the complete functional

\mathcal{V} leads to

$$\begin{aligned}
\mathcal{V}(x_t) &= x_t^\top(0)U(0)x_t(0) \\
&+ 2x_t^\top(0)\mathbf{U}_{1,n}\int_{-h}^0 \ell_n\left(\frac{\theta+h}{h}\right)Cx_t(\theta)d\theta \\
&+ \int_{-h}^0 \int_{-h}^0 x_t^\top(\theta_1)C^\top \mathbf{U}_{2,n}\ell_n\left(\frac{\theta_2-\theta_1+h}{2h}\right)Cx_t(\theta_2)d\theta_1d\theta_2 \\
&+ \int_{-h}^0 x_t^\top(\theta)C^\top(W_2 + (\theta+h)W_3)Cx_t(\theta)d\theta \\
&+ 2x_t^\top \int_{-h}^0 \tilde{U}_{1,n}(\theta)Cx_t(\theta)d\theta \\
&+ \int_{-h}^0 \int_{-h}^0 x_t^\top(\theta_1)C^\top \tilde{U}_{2,n}(\theta_2-\theta_1)Cx_t(\theta_2)d\theta_1d\theta_2.
\end{aligned} \tag{28}$$

Since $U_{2,n}(\theta_2-\theta_1) = \mathbf{U}_{2,n}\ell_n\left(\frac{\theta_2-\theta_1+h}{2h}\right)$ belongs to $\mathbb{R}^{n_z \times n_z}$ and is a polynomial function of degree $n-1$ in both θ_1 and θ_2 , it can be decomposed using the basis of Legendre polynomials $\ell_n\left(\frac{\theta_1+h}{h}\right)$ and $\ell_n\left(\frac{\theta_2+h}{h}\right)$. Thanks to the orthogonality of Legendre polynomials, this decomposition gives

$$\mathbf{U}_{2,n}\ell_n\left(\frac{\theta_2-\theta_1+h}{2h}\right) = \ell_n^\top\left(\frac{\theta_1+h}{h}\right) \mathbf{T}_n \ell_n\left(\frac{\theta_2+h}{h}\right),$$

where matrix \mathbf{T}_n is the symmetric matrix given in (26). Note that the symmetry of \mathbf{T}_n is ensured by the symmetry of $\tilde{U}_{2,n}$ highlighted in Property 4 that has been postponed in Appendix B in order to ease the reading. Hence, using the same augmented vector as in the proof of Theorem 2, i.e. $\xi_n(x_t) = \int_{-h}^0 \ell_n\left(\frac{\theta+h}{h}\right)Cx_t(\theta)d\theta$, functional \mathcal{V} reduces to the following expression

$$\begin{aligned}
\mathcal{V}(x_t) &= \begin{bmatrix} x_t(0) \\ \xi_n(x_t) \end{bmatrix}^\top \begin{bmatrix} U(0) & \mathbf{U}_{1,n} \\ * & \mathbf{T}_n \end{bmatrix} \begin{bmatrix} x_t(0) \\ \xi_n(x_t) \end{bmatrix} \\
&+ \int_{-h}^0 x_t^\top(\theta)C^\top(W_2 + (\theta+h)W_3)Cx_t(\theta)d\theta \\
&+ 2x_t^\top(0) \int_{-h}^0 \tilde{U}_{1,n}(\theta)Cx_t(\theta)d\theta \\
&+ \int_{-h}^0 \int_{-h}^0 x_t^\top(\theta_1)C^\top \tilde{U}_{2,n}(\theta_2-\theta_1)Cx_t(\theta_2)d\theta_1d\theta_2.
\end{aligned} \tag{29}$$

Therefore, by selecting \mathbf{P}_n , S and R as in (26), functional \mathcal{V}_n in (9) is retrieved and we obtain

$$\begin{aligned}
\mathcal{V}_n(x_t) &= \mathcal{V}(x_t) - 2x_t^\top(0) \int_{-h}^0 \tilde{U}_{1,n}(\theta)Cx_t(\theta)d\theta \\
&- \int_{-h}^0 \int_{-h}^0 x_t^\top(\theta_1)C^\top \tilde{U}_{2,n}(\theta_2-\theta_1)Cx_t(\theta_2)d\theta_1d\theta_2.
\end{aligned}$$

Differentiating the previous expression leads to

$$\begin{aligned}
\dot{\mathcal{V}}_n(x_t) &= \dot{\mathcal{V}}(x_t) - 2\dot{x}_t^\top(0) \int_{-h}^0 \tilde{U}_{1,n}(\theta)Cx_t(\theta)d\theta \\
&- 2x_t^\top(0) \int_{-h}^0 \tilde{U}_{1,n}(\theta)C\dot{x}_t(\theta)d\theta \\
&- \int_{-h}^0 \int_{-h}^0 \dot{x}_t^\top(\theta_1)C^\top \tilde{U}_{2,n}(\theta_2-\theta_1)Cx_t(\theta_2)d\theta_1d\theta_2 \\
&- \int_{-h}^0 \int_{-h}^0 x_t^\top(\theta_1)C^\top \tilde{U}_{2,n}(\theta_2-\theta_1)C\dot{x}_t(\theta_2)d\theta_1d\theta_2.
\end{aligned}$$

Since \mathcal{V} is the complete functional, its derivative is expressed using W_1 , W_2 , W_3 . Several integrations by parts lead to the following expression of $\dot{\mathcal{V}}_n$

$$\begin{aligned}
\dot{\mathcal{V}}_n(x_t) &= -x_t^\top(0)W_1x_t(0) - x_t^\top(-h)C^\top W_2Cx_t(-h) \\
&- \int_{-h}^0 x_t^\top(\theta)C^\top W_3Cx_t(\theta)d\theta \\
&- 2(Ax_t(0) + BCx_t(-h))^\top \int_{-h}^0 \tilde{U}_{1,n}(\theta)Cx_t(\theta)d\theta \\
&- 2x_t^\top(0) \left(\tilde{U}_{1,n}(0)Cx_t(0) - \tilde{U}_{1,n}(-h)Cx_t(-h) \right) \\
&+ 2x_t^\top(0) \int_{-h}^0 \tilde{U}'_{1,n}(\theta)Cx_t(\theta)d\theta \\
&- x_t^\top(0) \int_{-h}^0 C^\top \underbrace{(\tilde{U}_{2,n}(\theta) + \tilde{U}_{2,n}^\top(-\theta))}_{=2\tilde{U}_{2,n}(\theta)}Cx_t(\theta)d\theta \\
&+ x_t^\top(-h) \int_{-h}^0 C^\top \underbrace{(\tilde{U}_{2,n}(\theta+h) + \tilde{U}_{2,n}^\top(-h-\theta))}_{=2\tilde{U}_{2,n}(\theta+h)}Cx_t(\theta)d\theta \\
&+ \int_{-h}^0 \int_{-h}^0 x_t^\top(\theta_1)C^\top \tilde{U}'_{2,n}(\theta_2-\theta_1)Cx_t(\theta_2)d\theta_1d\theta_2, \\
&- \int_{-h}^0 \int_{-h}^0 x_t^\top(\theta_1)C^\top \tilde{U}'_{2,n}(\theta_2-\theta_1)Cx_t(\theta_2)d\theta_1d\theta_2.
\end{aligned}$$

We first notice that the two last terms of the previous expression are opposite and thus sum up to zero. Moreover, Property 4 given in Appendix B helps reducing the expression of the terms that depend on $\tilde{U}_{2,n}$. Re-ordering the previous expression yields

$$\dot{\mathcal{V}}_n(x_t) = -\frac{1}{h} \int_{-h}^0 \begin{bmatrix} x_t(0) \\ Cx_t(-h) \\ hCx_t(\theta) \end{bmatrix}^\top \Psi_n(\theta) \begin{bmatrix} x_t(0) \\ Cx_t(-h) \\ hCx_t(\theta) \end{bmatrix} d\theta, \tag{30}$$

where Ψ_n is given in (25). Therefore if matrix Ψ_n is positive definite there exists a sufficiently small $\varepsilon > 0$ such that $\Psi_n(\theta) \succ \varepsilon I_{3n_z}$ which yields to (27b).

In order to ensure inequality (27a), let us introduce the following functional

$$\mathcal{W}(x_t) = \mathcal{V}_n(x_t) - \varepsilon \left(|x_t(0)|^2 + h \int_{-h}^0 |Cx_t(\theta)|^2 d\theta \right), \tag{31}$$

where ε is a positive scalar to be determined and where \mathcal{V}_n is the candidate functional given in (9) with the same selection of matrices \mathbf{P}_n , S and R as in (26) associated to (W_1, W_2, W_3) in $\mathbb{S}_+^{n_x} \times \mathbb{S}_+^{n_z} \times \mathbb{S}_+^{n_z}$. Differentiating \mathcal{W} along the trajectories of the system, yields

$$\begin{aligned} \dot{\mathcal{W}}(x_t) &= \dot{\mathcal{V}}_n(x_t) - 2\varepsilon x_t^\top(0)(Ax_t(0) + BCx_t(-h)) \\ &\quad - \varepsilon h (|Cx_t(0)|^2 - |Cx_t(-h)|^2), \end{aligned} \quad (32)$$

which can be rewritten as

$$\dot{\mathcal{W}}(x_t) = -\frac{1}{h} \int_{-h}^0 \begin{bmatrix} x_t(0) \\ Cx_t(-h) \\ hCx_t(\theta) \end{bmatrix}^\top \Psi_n^\varepsilon(\theta) \begin{bmatrix} x_t(0) \\ Cx_t(-h) \\ hCx_t(\theta) \end{bmatrix} d\theta, \quad (33)$$

where

$$\Psi_n^\varepsilon(\theta) = \Psi_n(\theta) + \varepsilon \begin{bmatrix} \mathcal{H}(A) + C^\top C & B & 0 \\ * & -I_{n_z} & 0 \\ * & * & 0 \end{bmatrix}, \quad \forall \theta \in [-h, 0].$$

Therefore, if Ψ_n is positive definite, then there exists a sufficiently small $\varepsilon > 0$ such that Ψ_n^ε and consequently, $\dot{\mathcal{W}}(x_t)$ are negative definite. Integrating then $\dot{\mathcal{W}}(x_t)$ from t to infinity yields

$$\lim_{T \rightarrow \infty} \mathcal{W}(x_T) - \mathcal{W}(x_t) = -\mathcal{W}(x_t) \leq 0,$$

where we have used the assumption that system (1) is GES meaning that $\mathcal{W}(x_T) \xrightarrow{T \rightarrow \infty} 0$. Hence, (31) ensures that there exists a sufficiently small $\varepsilon > 0$ such that (27a) holds, which concludes the proof. \square

3.4 New necessary condition for functional \mathcal{V}_n

The next statement provides a convergence result on the functional \mathcal{V}_n . To do so, it suffices to ensure that matrix Ψ_n is positive definite, which still has to be proven.

Corollary 2 *If the trivial solution of system (1) is GES, then there exist an order N^* in \mathbb{N}^* and matrices (\mathbf{P}_n, S, R) in $\mathbb{S}^{n_x + nn_x} \times \mathbb{S}_+^{n_z} \times \mathbb{S}_+^{n_z}$ such that the associated Lyapunov functional \mathcal{V}_n given in (9) satisfies (27), for any $n \geq N^*$.*

Proof: As a first step to the proof of convergence, let us emphasize the structure of matrix Ψ_n and recall that matrices (W_1, W_2, W_3) in $\mathbb{S}_+^{n_x} \times \mathbb{S}_+^{n_z} \times \mathbb{S}_+^{n_z}$ have been fixed to build the complete functional \mathcal{V} . Therefore, matrix Ψ_n can be decomposed as the sum of a block diagonal positive definite matrices that is independent of n and of a matrix whose entries are all expressed using the approximation errors $\bar{U}_{1,n}$, $\bar{U}'_{1,n}$ and $\bar{U}_{2,n}$, which can be made uniformly arbitrarily small in light of Lemma 3.

To do so, we will use the following equivalence results. For any matrix X in $\mathbb{R}^{p \times q}$ such that $X^\top X \preceq |X|^2 I_q$, then inequality $\begin{bmatrix} |X| I_p & X \\ X^\top & |X| I_q \end{bmatrix} \succeq 0$ holds. Using this inequality, the following lower bounds of Ψ_n is derived

$$\Psi_n(\theta) \succeq \begin{bmatrix} \mu_{1,n} I_{n_x} & 0 & 0 \\ 0 & \mu_{2,n} I_{n_z} & 0 \\ 0 & 0 & \mu_{3,n} I_{n_z} \end{bmatrix},$$

with

$$\begin{aligned} \mu_{1,n} &= \underline{\sigma}(W_1) - (1 + |A| + 2|C|)\bar{u}_{1,n} - \bar{u}_{2,n} - |C|\bar{u}_{3,n}, \\ \mu_{2,n} &= \underline{\sigma}(W_2) - (1 + |B|)\bar{u}_{1,n} - \bar{u}_{3,n}, \\ \mu_{3,n} &= \frac{1}{h} \underline{\sigma}(W_3) - (|A| + |B|)\bar{u}_{1,n} - \bar{u}_{2,n} - (1 + |C|)\bar{u}_{3,n}, \end{aligned} \quad (34)$$

where $\bar{u}_{1,n}$, $\bar{u}_{2,n}$ and $\bar{u}_{3,n}$ are the upper bounds of the approximation errors of the polynomial approximation given by (18), (20) and (22), respectively.

Then, Corollary 1 ensures that all the negative terms can be made arbitrarily small as n increases, while the first terms are positive and independent of n . In particular, there exists N^* in \mathbb{N}^* such that inequalities (27) hold, for all $n \geq N^*$, which concludes the proof. \square

Note that such result could also be obtained with other approximation methods. For instance, it is also the case of piece-wise approximation as demonstrated in [14].

3.5 Necessary LMI condition

The next and final step of the proof consists in demonstrating that the previous corollary imposes the satisfaction of the sufficient LMI in Theorem 2.

Proof of Theorem 3: Assume that system (1) is GES. Thanks to Corollary 2, there exist N^* such that inequalities (27) hold for any $n \geq N^*$. This means that, for such an n , there exists matrices (\mathbf{P}_n, S, R) in $\mathbb{S}^{n_x + nn_x} \times \mathbb{S}_+^{n_z} \times \mathbb{S}_+^{n_z}$ given by (26) such that the associated functional \mathcal{V}_n verifies (27) for all φ in $\mathcal{C}_{pw}(-h, 0; \mathbb{R}^{n_x})$. In particular, consider φ as a function expressed as follows

$$\varphi(\theta) = \begin{bmatrix} \delta_0(\theta) I_{n_x} & \frac{1}{h} C^\dagger f_n^\top(\theta) \mathcal{I}_n \delta_{-h}(\theta) C^\dagger \end{bmatrix} \begin{bmatrix} \varphi_0 \\ \varphi_n \\ \varphi_h \end{bmatrix}, \quad (35)$$

with $C^\dagger = C^\top (CC^\top)^{-1}$ in $\mathbb{R}^{n_x \times n_z}$ is the right pseudo-inverse of C and with

$$f_n(\theta) = \ell_n \left(\frac{\theta + h}{h} \right) - \ell_n(1)\delta_0(\theta) - \ell_n(0)\delta_{-h}(\theta).$$

In this formulation, vector $\begin{bmatrix} \varphi_0 \\ \varphi_n \\ \varphi_h \end{bmatrix}$ in $\mathbb{R}^{n_x + n_z(n+1)}$ is arbitrary and δ_{θ_0} is zero everywhere except at θ_0 , where is

it equal to 1, i.e.

$$\delta_{\theta_0}(\theta) = \begin{cases} 1 & \text{for } \theta = \theta_0, \\ 0 & \text{otherwise.} \end{cases}$$

Note that such a function φ has been selected so that φ at the boundary of $[-h, 0]$ is given by $\varphi(0) = \varphi_0$, $\varphi(-h) = C^\dagger \varphi_{-h}$, and in the interval $(-h, 0)$ by $\varphi(\theta) = \frac{1}{h} C^\dagger \ell_n \left(\frac{\theta+h}{h} \right) \mathcal{I}_n \varphi_n$, for all θ in $(-h, 0)$, which is a polynomial function of θ of degree $n-1$.

Re-injecting this expression into the definition of \mathcal{V}_n in (9) and its derivative in (11) yields

$$\mathcal{V}_n(\varphi) = \begin{bmatrix} \varphi_0 \\ \varphi_n \end{bmatrix}^\top \Phi_n^+ \begin{bmatrix} \varphi_0 \\ \varphi_n \end{bmatrix}, \quad \dot{\mathcal{V}}_n(\varphi) = \begin{bmatrix} \varphi_0 \\ \varphi_n \end{bmatrix}^\top \Phi_n^- \begin{bmatrix} \varphi_0 \\ \varphi_n \end{bmatrix},$$

where we have used the equality case of Lemma 1, since φ is a polynomial of degree $n-1$ over $(-h, 0)$. Similarly, re-injecting (35) into both inequalities in (27) leads to

$$\mathcal{V}_n(\varphi) \geq \varepsilon \left(|\varphi_0|^2 + \varphi_n^\top \mathcal{I}_n \varphi_n \right) \geq \varepsilon \left| \begin{bmatrix} \varphi_0 \\ \varphi_n \end{bmatrix} \right|^2, \quad (36a)$$

$$\dot{\mathcal{V}}_n(\varphi) \leq -\varepsilon \left(|\varphi_0|^2 + \varphi_n^\top \mathcal{I}_n \varphi_n + |\varphi_h|^2 \right) \leq -\varepsilon \left| \begin{bmatrix} \varphi_0 \\ \varphi_n \\ \varphi_h \end{bmatrix} \right|^2, \quad (36b)$$

for a sufficiently small scalar $\varepsilon > 0$. Therefore, the following inequalities hold for any vector $\begin{bmatrix} \varphi_0 \\ \varphi_n \\ \varphi_h \end{bmatrix}$ in $\mathbb{R}^{n_x + (n+1)n_z}$

$$\begin{bmatrix} \varphi_0 \\ \varphi_n \end{bmatrix}^\top \Phi_n^+ \begin{bmatrix} \varphi_0 \\ \varphi_n \end{bmatrix} \geq \varepsilon \left| \begin{bmatrix} \varphi_0 \\ \varphi_n \end{bmatrix} \right|^2, \quad \begin{bmatrix} \varphi_0 \\ \varphi_n \\ \varphi_h \end{bmatrix}^\top \Phi_n^- \begin{bmatrix} \varphi_0 \\ \varphi_n \\ \varphi_h \end{bmatrix} \leq -\varepsilon \left| \begin{bmatrix} \varphi_0 \\ \varphi_n \\ \varphi_h \end{bmatrix} \right|^2.$$

Hence matrices Φ_n^+ and Φ_n^- are necessarily positive and negative definite, respectively. \square

4 Necessity of the LMI condition: Estimation of the order N^*

This section aims at providing an estimation of the order N^* for which the sufficient LMI conditions of Theorem 2 are necessarily satisfied if the system is assumed to be GES for a given delay h .

Theorem 5 *If the trivial solution of system (1) is GES for a given delay h , then there exist matrices (\mathbf{P}_{N^*}, R, S) in $\mathbb{S}^{n_x + N^* n_z} \times \mathbb{S}_+^{n_z} \times \mathbb{S}_+^{n_z}$ such that LMI (8) holds at the order $n = N^*$ given by*

$$N^* = 5 + \left\lceil \left(\left(\frac{4}{1+h^2} + |A| + |B| \right) \varrho_1 + \varrho_2 + 2\varrho_3 \right)^2 (1+h^2) \right\rceil, \quad (37)$$

where parameters $\varrho_1, \varrho_2, \varrho_3$ are defined by (19), (21), (23),

but are recalled here for consistency

$$\begin{aligned} \varrho_1 &= \sqrt{n_x} \left(\frac{\pi}{2} \right)^{\frac{3}{2}} e^{h|\mathcal{M}|} |\mathcal{M}^2 \mathcal{N}^{-1}| h^2 |B|, \\ \varrho_2 &= \frac{1}{2} \sqrt{n_x} \left(\frac{\pi}{2} \right)^{\frac{3}{2}} e^{h|\mathcal{M}|} |\mathcal{M}^4 \mathcal{N}^{-1}| h^3 |B|, \\ \varrho_3 &= \sqrt{2\pi} (1 + \sqrt{n_x} \pi h e^{h|\mathcal{M}|} |\mathcal{M}^2 \mathcal{N}^{-1}|) h |B|^2. \end{aligned}$$

Proof: The key step for the estimation of N^* appears in the proof of Corollary 2, more particularly in equations (34), that are recalled here for the sake of readability (with $|C| = 1$),

$$\begin{aligned} 0 &< \mu_{1,n} = \underline{\sigma}(W_1) - (3 + |A|) \bar{u}_{1,n} - \bar{u}_{2,n} - \bar{u}_{3,n}, \\ 0 &< \mu_{2,n} = \underline{\sigma}(W_2) - (1 + |B|) \bar{u}_{1,n} - \bar{u}_{3,n}, \\ 0 &< \mu_{3,n} = \frac{1}{h} \underline{\sigma}(W_3) - (|A| + |B|) \bar{u}_{1,n} - \bar{u}_{2,n} - 2\bar{u}_{3,n}. \end{aligned} \quad (38)$$

As matrices (W_1, W_2, W_3) are arbitrary, it is possible to select for all $\lambda > 0$

$$W_1 = \lambda \eta_1 I_{n_x}, \quad W_2 = \lambda \eta_2 I_{n_z}, \quad h W_3 = \lambda \eta_3 I_{n_z}, \quad (39)$$

where η_1, η_2 and $\eta_3 > 0$ are positive scalars such that $\eta_1 + \eta_2 + \eta_3 = 1$. This selection makes that $\underline{\sigma}(W_1) = \lambda \eta_1$, $\underline{\sigma}(W_2) = \lambda \eta_2$ and $\underline{\sigma}(W_3) = \frac{1}{h} \lambda \eta_3$ so that $|W| = \lambda$. Therefore, the upper bounds of the approximation errors given in (18), (20) and (22) verify for all $n \geq 6$,

$$\begin{aligned} \bar{u}_{1,n} &= \frac{\varrho_1}{\sqrt{n-3}} \lambda \leq \frac{\varrho_1}{\sqrt{n-5}} \lambda, \\ \bar{u}_{2,n} &= \frac{\varrho_2}{\sqrt{n-5}} \lambda, \\ \bar{u}_{3,n} &= \frac{\varrho_3}{\sqrt{n-3}} \lambda \leq \frac{\varrho_3}{\sqrt{n-5}} \lambda. \end{aligned}$$

The objective is now to evaluate the necessary condition obtained in the previous section, more particularly the ones arising from inequalities (38). We select then η_1, η_2 and η_3 as follows

$$\begin{aligned} \eta_1 &\geq \frac{\varrho_1(3 + |A|) + \varrho_2 + \varrho_3}{\sqrt{n-5}}, \\ \eta_2 &\geq \frac{\varrho_1(1 + |B|) + \varrho_3}{\sqrt{n-5}}, \\ \eta_3 &\geq \frac{\varrho_1(|A| + |B|) + \varrho_2 + 2\varrho_3}{\sqrt{n-5}} h^2. \end{aligned}$$

Recalling that $\eta_1 + \eta_2 + \eta_3 = 1$, we obtain

$$\frac{4\varrho_1 + ((|A| + |B|)\varrho_1 + \varrho_2 + 2\varrho_3)(1+h^2)}{\sqrt{n-5}} \leq 1,$$

which yields condition (37). \square

Table 1

Sufficiency: maximal allowable delay h which satisfies (8).

	$n = 1$	$n = 2$	$n = 3$	Expected
Ex. 1	0.577	0.604	0.604	$\frac{\text{atan}\sqrt{3}}{\sqrt{3}} \simeq 0.604$
Ex. 2	—	1.600	1.603	1.603

Remark 3 *It is worth noticing that the minimal order N^* given by (37) is an over estimate of the necessary order. Indeed, to get such estimate, several over bounding approximation have been performed, leading to a conservative estimation. Nevertheless, this is a first attempt in this direction, which has not been provided in the context of LMI conditions for the stability of time-delay systems. Several improvements to get a more accurate estimation is led to future works. One of the possible improvements would consider, for instance, more accurate estimation of $\varrho_1, \varrho_2, \varrho_3$ which would greatly reduce the estimation by using the fact that U is infinitely continuous on $[-h, 0]$. Another possibility to improve our estimation of N^* is to calculate the minimal order directly from inequality (25).*

5 Numerical results

The numerical application of Theorems 2 and 5 is commented and illustrated on the following academic examples corresponding to (1) with

Example 1 $A = 1$ and $A_d = -2$.

Example 2 $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $A_d = \begin{bmatrix} -1 & 0.2 \\ -0.1 & 0 \end{bmatrix}$.

Example 3 $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -10-\lambda & 10 & 0 & 0 \\ 5 & -15 & 0 & -\frac{1}{4} \end{bmatrix}$ and $A_d = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

First, we recall the efficiency of the sufficient stability condition of Theorem 2. For Examples 1 and 2, Table 1 reports the maximal allowable delay, for which LMI (8) at order $n = 1, 2, 3$ are satisfied. Figure 1a shows the stability region for Example 3 achieved by the same condition for several orders. One can see that in this table and in this figure the efficiency of Theorem 2 even for very low orders n to provide an inner approximation of the stability regions.

Thanks to Theorem 5, these LMI conditions of stability are proven to be also necessary. Therefore, the inner approximation mentioned above is supposed to converge toward the expected regions of stability as the order n increases. Table 2 reports the estimated order N^* for which the LMI become necessary. Similarly, Figure 1b shows the values of N^* given by (37) for various values of $(\lambda, h) \in [0.1, 10] \times [0, 3]$. Clearly the values of N^* computed here are very too large to propose tractable test of instability in comparison to [12]. Nonetheless, Theorem 5 provides a theoretical estimation of N^* , from which the sufficient LMI conditions becomes necessary.

Table 2

Necessity: estimated order N^* given by (37).

	$h = 0.1$	$h = 0.5$	$h = 1$	$h = 2$
Ex. 1	75	10^8	10^{10}	10^{13}
Ex. 2	6	537	10^7	10^{10}

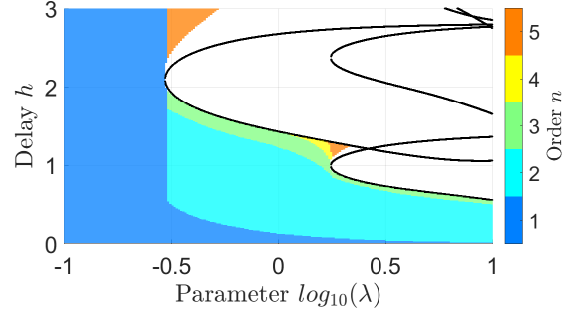
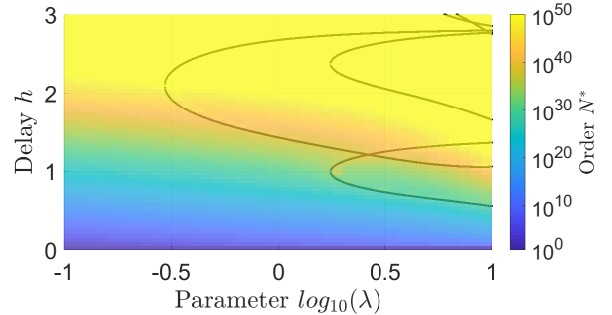
(a) Sufficiency: stability areas in the plan (λ, h) given by (8) for $n \in \{1, 2, 3, 4, 5\}$.(b) Necessity: order N^* in the plan (λ, h) given by (37).

Fig. 1. Example 3.

This estimation is a by-product of our main result that has not been optimized in this paper. Some guidelines to improve this estimation have been suggested in Remark 3.

Interestingly Figure 1b shows that increasing both λ and h makes that N^* also increases very fast and reach very large values. Indeed, formula (37) shows that the order N^* grows as the delay h or the norm $|B|$ increase, respectively in $h^8 e^{2h|M|}$ and $|B|^4$. It is also worth noticing that when the parameter λ and h are getting closer to the black lines, the estimation N^* increases even more faster. These black lines correspond to the situation where some roots of the time-delay system cross the imaginary axis. This makes sense in light of Remark 2, because if characteristic roots are approaching the imaginary axis, then matrix \mathcal{N} tends to a singular matrix and $|\mathcal{N}^{-1}|$ tends to infinity. Hence, the upper bounds $\varrho_1, \varrho_2, \varrho_3$ also tend to infinity. All together, this ascertainment can be correlated with Figure 1a and can explain why some stable regions are difficult to reach with LMI conditions (8), especially for low orders n .

6 Conclusions

This paper studied the convergence of sufficient LMI conditions for the stability analysis of time-delay systems, based on the Bessel-Legendre inequalities. While the framework of Bessel-Legendre already showed its relevance regarding the hierarchical structure of the LMI, i.e. to increase the order can only reduce the conservatism, the main contribution of this paper demonstrates that this framework also offers an asymptotically necessary condition of stability for time-delay systems. In other words, it is now proven that if a time-delay system is stable, then, there exists an order N^* such that these LMI conditions are verified at least at this order. A numerical estimation of this necessary order has also been provided, ensuring then that if the LMI conditions are not verified at this order, then the system is proven to be unstable. To summarize, the LMI conditions arising from the Bessel-Legendre inequality are sufficient and asymptotically necessary.

The estimation of the order N^* provided in this paper can be numerically very large, even on some simple examples. This can be seen as a major drawback of the contribution. Providing more accurate estimations is left to future direction of research. A generalization to other approximation or discretization methods or the investigation of combined procedures could also be considered.

A Preliminaries on Legendre Polynomials for the technical proofs

In this section, several additional properties of Legendre polynomials are presented and are of high importance in the next technical developments that are required to the proofs of the main results of this paper.

Property 3 *The Legendre polynomials verify the following properties*

- (i) **Point-wise values:** *The Legendre polynomials are evaluated point wisely by*

$$l_k(0) = (-1)^k, \quad l_k(1) = 1, \quad l'_k(1) = k(k+1), \quad \forall k \in \mathbb{N}. \quad (\text{A.1})$$

- (ii) **Evenness and oddness:** *For any θ_1 and θ_2 in $[-h, 0]$ and n in \mathbb{N}^* , the following equation holds*

$$\ell_n^\top \left(\frac{h-\theta_1}{2h} \right) \mathcal{I}_n \ell_n \left(\frac{h+\theta_2}{2h} \right) = \ell_n^\top \left(\frac{h-\theta_2}{2h} \right) \mathcal{I}_n \ell_n \left(\frac{h+\theta_1}{2h} \right). \quad (\text{A.2})$$

- (iii) **Bound:** *For any θ in $[0, 1]$, the Legendre polynomials verify the following inequalities*

$$|l_k(\theta)| \leq 1, \quad \forall k \in \mathbb{N}, \quad (\text{A.3})$$

$$|l_k(\theta)| \leq \frac{1}{2} \sqrt{\frac{\pi}{2k\theta(1-\theta)}}, \quad \forall k \in \mathbb{N}^*. \quad (\text{A.4})$$

- (iv) **Derivation:** *For any θ in $[0, 1]$, the Legendre polynomials verify the following differentiation rule*

$$l_k(\theta) = \frac{1}{2(2k+1)} (l'_{k+1}(\theta) - l'_{k-1}(\theta)), \quad \forall k \in \mathbb{N}^*. \quad (\text{A.5})$$

- (v) **Bound on the derivative:** *For any θ in $[0, 1]$, the Legendre polynomials verify the following inequality*

$$|l'_k(\theta)| \leq |l'_k(1)| = k(k+1), \quad \forall k \in \mathbb{N}. \quad (\text{A.6})$$

Proof: The proofs can be found in [9] except the one of (A.4) which is given by [29, Theorem 61]. \square

B Symmetric property

In the core of the paper, we have used the following property satisfied by the Legendre approximated error $\tilde{U}_{2,n}$ of function $B^\top U B$ on the interval $[-h, h]$.

Property 4 *The approximation error $\tilde{U}_{2,n}$ defined in (16) verifies $\tilde{U}_{2,n}(\theta) = \tilde{U}_{2,n}^\top(-\theta)$, for all $\theta \in [-h, 0]$.*

Proof: From the definitions of $\tilde{U}_{2,n}$ and $\mathbf{U}_{2,n}$, we have

$$\begin{aligned} \tilde{U}_{2,n}^\top(-\theta) &= B^\top U^\top(-\theta) B \\ &- \frac{1}{2h} \left(\int_{-h}^h \ell_n^\top \left(\frac{h-\theta}{2h} \right) \mathcal{I}_n \ell_n \left(\frac{h+\theta_1}{2h} \right) B^\top U^\top(\theta_1) B d\theta_1 \right). \end{aligned}$$

Then, property (A.2) of the Legendre polynomials and since $\ell_n^\top \left(\frac{h-\theta_1}{2h} \right) \mathcal{I}_n \ell_n \left(\frac{h+\theta}{2h} \right)$ is an $n_z \times n_z$ matrix that is proportional to I_{n_z} , it commutes with $B^\top U^\top(\theta') B$, so that the previous expression writes

$$\begin{aligned} \tilde{U}_{2,n}^\top(-\theta) &= B^\top U^\top(-\theta) B \\ &- \frac{1}{2h} \left(\int_{-h}^h B^\top U^\top(\theta_1) B \ell_n^\top \left(\frac{h-\theta_1}{2h} \right) \mathcal{I}_n \ell_n \left(\frac{h+\theta}{2h} \right) d\theta_1 \right). \end{aligned}$$

Recalling that $U^\top(-\theta) = U(\theta)$ holds for all θ in $[-h, h]$ and performing the change of variable $\theta_2 = -\theta_1$, the previous expression becomes

$$\begin{aligned} \tilde{U}_{2,n}^\top(-\theta) &= B^\top U(\theta) B \\ &- \frac{1}{2h} \left(\int_{-h}^h B^\top U(\theta_2) B \ell_n^\top \left(\frac{h+\theta_2}{2h} \right) d\theta_2 \right) \mathcal{I}_n \ell_n \left(\frac{h+\theta}{2h} \right), \end{aligned}$$

which is the definition of $\tilde{U}_{2,n}(\theta)$. \square

C Proof of Lemma 3

The next properties reflect the fact that U has continuous and bounded second and fourth order derivatives on $[-h, 0)$ (or $(0, h]$). Their bound are provided therein.

Property 5 *The Lyapunov matrix U associated to W defined in (13) satisfies*

$$\sup_{\theta \in [-h, 0)} |U^{(2)}(\theta)| = \sup_{\theta \in (0, h]} |U^{(2)}(\theta)| \leq \rho |W|, \quad (\text{C.1})$$

$$\sup_{\theta \in (0, h]} |U^{(4)}(\theta)| \leq \rho' |W|, \quad (\text{C.2})$$

with parameters ρ, ρ' given by

$$\rho = \sqrt{n_x} e^{h|\mathcal{M}|} |\mathcal{M}^2 \mathcal{N}^{-1}|, \quad \rho' = \sqrt{n_x} e^{h|\mathcal{M}|} |\mathcal{M}^4 \mathcal{N}^{-1}|, \quad (\text{C.3})$$

and matrices \mathcal{M}, \mathcal{N} given by (14).

Proof : Thanks to the equivalence of matrix norms, inequalities $|M^T| = |M| \leq |\text{vec}(M)| \leq \sqrt{p} |M|$ hold, for any square M of dimension p . Then, for all θ in $[0, 1]$, we have

$$\begin{aligned} |U^{(k)}(\theta)| &\leq |[I_{n_x} \ 0] e^{\theta \mathcal{M}} \mathcal{M}^k \mathcal{N}^{-1} [-\text{vec}(W)]|, \\ &\leq |e^{\theta \mathcal{M}}| |\mathcal{M}^k \mathcal{N}^{-1}| |\text{vec}(W)|, \\ &\leq \sqrt{n_x} |e^{\theta \mathcal{M}}| |\mathcal{M}^k \mathcal{N}^{-1}| |W|. \end{aligned}$$

Moreover, recalling the definition of exponential matrices, i.e. $e^{\theta \mathcal{M}} = \sum_{k=0}^{\infty} \frac{(\theta \mathcal{M})^k}{k!}$, an upper bound of $|e^{\theta \mathcal{M}}|$ is obtained as follows

$$|e^{\theta \mathcal{M}}| \leq \sum_{k=0}^{\infty} \frac{|\theta \mathcal{M}|^k}{k!} \leq \sum_{k=0}^{\infty} \frac{|\theta|^k |\mathcal{M}|^k}{k!} \leq \sum_{k=0}^{\infty} \frac{h^k |\mathcal{M}|^k}{k!} = e^{h|\mathcal{M}|},$$

which yields the results (C.1) and (C.2). \square

C.1 Proof of item (i) of Lemma 3

Proof : The objective of the proof is to provide an upper bound of the norm of the approximation error $\tilde{U}_{1,n}$, which depends explicitly on order n . To do so, let us first rewrite the expression of this error as follows

$$\tilde{U}_{1,n}(\theta) = U(h+\theta)B - \sum_{k=0}^{n-1} \mathbf{U}_1^k l_k \left(\frac{h+\theta}{h} \right),$$

with \mathbf{U}_1^k being the projection of $U(h+\theta)B$ onto the k^{th}

Legendre polynomial, that is

$$\begin{aligned} \mathbf{U}_1^k &= \frac{2k+1}{h} \int_{-h}^0 U(h+\theta) B l_k \left(\frac{h+\theta}{h} \right) d\theta, \\ &= (2k+1) \int_0^1 U(h\theta) l_k(\theta) d\theta. \end{aligned} \quad (\text{C.4})$$

Using the differentiation rule (A.5), the previous expression can be rewritten as

$$\mathbf{U}_1^k = \frac{1}{2} \int_0^1 U(h\theta) B (l'_{k+1}(\theta) - l'_{k-1}(\theta)) d\theta. \quad (\text{C.5})$$

Then, an integration by parts yields

$$\mathbf{U}_1^k = \frac{h}{2} \int_0^1 U'(h\theta) B (l_{k-1}(\theta) - l_{k+1}(\theta)) d\theta, \quad (\text{C.6})$$

where we have used $l_{k+1}(0) = l_{k-1}(0)$ and $l_{k+1}(1) = l_{k-1}(1)$ as a by-product of (A.1), which cancels the first terms of the integration by parts. Repeating this operation, we get

$$\begin{aligned} \mathbf{U}_1^k &= \frac{h^2}{4(2k-1)} \int_0^1 U''(h\theta) B (l_{k-2}(\theta) - l_k(\theta)) d\theta \\ &\quad - \frac{h^2}{4(2k+3)} \int_0^1 U''(h\theta) B (l_k(\theta) - l_{k+2}(\theta)) d\theta. \end{aligned}$$

Then, Property (A.4) ensures that

$$|\mathbf{U}_1^k| \leq \frac{\sqrt{\frac{\pi}{2}} h^2 |B|}{2\sqrt{k-2}(2k-1)} \int_0^1 \frac{|U''(h\theta)|}{\sqrt{\theta(1-\theta)}} d\theta.$$

Using (C.1), $(2k-1) \geq 2(k-2)$ and $\int_0^1 \frac{d\theta}{\sqrt{\theta(1-\theta)}} = \pi$, the following upper bound is obtained

$$|\mathbf{U}_1^k| \leq \frac{\rho \left(\frac{\pi}{2}\right)^{\frac{3}{2}} h^2 |B| |W|}{2(k-2)^{\frac{3}{2}}}, \quad \forall k \geq 3.$$

Applying now (A.3) ensures that for any integer $N \geq n$ and for all θ in $[-h, 0]$

$$\left| \sum_{k=n}^N \mathbf{U}_1^k l_k \left(\frac{h+\theta}{h} \right) \right| \leq \sum_{k=n}^N |\mathbf{U}_1^k| \leq \sum_{k=n}^N \frac{\varrho_1 |W|}{2(k-2)^{\frac{3}{2}}},$$

denoting $\varrho_1 = \rho \left(\frac{\pi}{2}\right)^{\frac{3}{2}} h^2 |B|$. Finally, using an integral over estimation of the sum, we obtain

$$\left| \sum_{k=n}^N \mathbf{U}_1^k l_k \left(\frac{h+\theta}{h} \right) \right| \leq \varrho_1 |W| \left(\frac{1}{\sqrt{n-3}} - \frac{1}{\sqrt{N-3}} \right).$$

We conclude that the approximation error $\tilde{U}_1(\theta) = \sum_{k=n}^{\infty} \mathbf{U}_{1,n}^k l_k\left(\frac{\theta+h}{h}\right)$ is uniformly bounded as in (18) and, consequently, converges uniformly to zero as n tends to infinity. \square

C.2 Proof of item (ii) of Lemma 3

Proof: The objective of the proof is to demonstrate the uniform convergence towards zero of

$$\begin{aligned} \tilde{U}'_{1,n}(\theta) &= \frac{d}{d\theta} \left(U(\theta+h)B - \mathbf{U}_{1,n} \ell_n\left(\frac{\theta+h}{h}\right) \right), \\ &= U'(\theta+h)B - \frac{1}{h} \sum_{k=0}^{n-1} \mathbf{U}_1^k l'_k\left(\frac{\theta+h}{h}\right), \end{aligned}$$

with \mathbf{U}_1^k given by (C.4). Here, we repeat the process (C.5)-(C.6) four times successively to obtain

$$\mathbf{U}_1^k = \left(\frac{h}{2}\right)^4 \frac{\int_0^1 U^{(4)}(h\theta)B \left(\sum_{i=0}^4 \binom{4}{i} \alpha_{k,i} l_{k-4+2i}(\theta) \right) d\theta}{(2k-5)(2k-3)(2k-1)}$$

where $\alpha_{k,i}$ are positive coefficients whose expression is omitted for simplicity but which verify $|\alpha_{k,i}| \leq 1$. Therefore, an upper bound of the norm of \mathbf{U}_1^k can be derived using property (A.4), yielding

$$|\mathbf{U}_1^k| \leq \frac{\frac{1}{2}\sqrt{\frac{\pi}{2}}h^4|B|}{(2k-5)(2k-3)(2k-1)} \int_0^1 \frac{|U^{(4)}(h\theta)|}{\sqrt{(k-4)\theta(1-\theta)}} d\theta.$$

Using (C.2), $(2k-5) \leq 2(k-4)$ and $\int_0^1 \frac{d\theta}{\sqrt{\theta(1-\theta)}} = \pi$, the following upper bound is obtained

$$|\mathbf{U}_1^k| \leq \frac{\rho'(\frac{\pi}{2})^{\frac{3}{2}}h^4|B||W|}{2(k-4)^{\frac{3}{2}}(2k-3)(2k-1)}, \quad \forall k \geq 5. \quad (\text{C.7})$$

We denote $\varrho_2 = \frac{1}{2}\rho'(\frac{\pi}{2})^{\frac{3}{2}}h^3|B|$ and we use this upper bound to get to the result. The application of (A.6) and (C.7) give, for any integer $N \geq n \geq 5$ and for all θ in $[-h, 0]$,

$$\begin{aligned} \left| \frac{1}{h} \sum_{k=n}^N \mathbf{U}_1^k l'_k\left(\frac{\theta+h}{h}\right) \right| &\leq \sum_{k=n}^N \frac{|\mathbf{U}_1^k| k(k+1)}{h}, \\ &\leq \sum_{k=n}^N \frac{\varrho_2 |W| k(k+1)}{(k-4)^{\frac{3}{2}}(2k-3)(2k-1)}. \end{aligned}$$

Noticing that $\frac{k(k+1)}{(2k-1)(2k-3)} < \frac{1}{2}$ for all $k \geq 5$, we obtain

$$\left| \frac{1}{h} \sum_{k=n}^N \mathbf{U}_1^k l'_k\left(\frac{\theta+h}{h}\right) \right| \leq \sum_{k=n}^N \frac{\varrho_2 |W|}{2(k-4)^{\frac{3}{2}}}.$$

Finally, an integral over estimation of the sum leads to

$$\left| \frac{1}{h} \sum_{k=n}^N \mathbf{U}_1^k l'_k\left(\frac{\theta+h}{h}\right) \right| \leq \varrho_2 |W| \left(\frac{1}{\sqrt{n-5}} - \frac{1}{\sqrt{N-5}} \right).$$

We conclude that the approximation error $\tilde{U}'_{1,n}(\theta) = \frac{d}{d\theta} \sum_{k=n}^{\infty} \mathbf{U}_1^k l_k\left(\frac{\theta+h}{h}\right)$ is uniformly bounded as in (20) and converges to zero as n tends to infinity. \square

C.3 Proof of item (iii) of Lemma 3

Proof: The objective of the proof is to demonstrate the uniform convergence towards zero of

$$\begin{aligned} \tilde{U}_{2,n}(\theta) &= B^\top U(\theta)B - \mathbf{U}_{2,n} \ell_n\left(\frac{\theta+h}{2h}\right), \\ &= B^\top U(\theta)B - \sum_{k=0}^{n-1} \mathbf{U}_2^k l_k\left(\frac{\theta+h}{h}\right), \end{aligned}$$

with \mathbf{U}_2^k the k -th coefficient of matrix $\mathbf{U}_{2,n}$ given by

$$\begin{aligned} \mathbf{U}_2^k &= \frac{2k+1}{2h} \int_{-h}^h B^\top U(\theta)B l_k\left(\frac{\theta+h}{2h}\right) d\theta, \\ &= (2k+1) \int_0^1 B^\top U(h(2\theta-1))B l_k(\theta) d\theta. \quad (\text{C.8}) \end{aligned}$$

Using the differentiation rule (C.5) and integration by parts (C.6) on both intervals $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$, we obtain

$$\mathbf{U}_2^k = h \int_0^1 B^\top U'(h(2\theta-1))B (l_{k-1}(\theta) - l_{k+1}(\theta)) d\theta.$$

since $l_{k+1}(\theta) = l_{k-1}(\theta)$ for $\theta \in \{0, 1\}$ is ensured by (A.1) and since the continuity of $U(h(2\theta-1))$ at $\theta = \frac{1}{2}$ is ensured by Property 2 (i). Then, repeating this operation, we get to

$$\begin{aligned} \mathbf{U}_2^k &= -\frac{h}{2(2k-1)} B^\top \Delta U'(0)B (l_k(1/2) - l_{k-2}(1/2)) \\ &\quad + \frac{h^2}{(2k-1)} \int_0^1 B^\top U''(h(2\theta-1))B (l_{k-2}(\theta) - l_k(\theta)) d\theta \\ &\quad + \frac{h}{2(2k+3)} B^\top \Delta U'(0)B (l_{k+2}(1/2) - l_k(1/2)) \\ &\quad - \frac{h^2}{(2k+3)} \int_0^1 B^\top U''(h(2\theta-1))B (l_k(\theta) - l_{k+2}(\theta)) d\theta. \end{aligned}$$

where $\Delta U'(0) := \lim_{\epsilon \rightarrow 0} (U'(\epsilon) - U'(-\epsilon))$. Then, an upper bound of the norm of \mathbf{U}_2^k can be derived by the use of property (A.4) for $\theta \in [0, 1]$ and especially for $\theta = \frac{1}{2}$, yielding

$$|\mathbf{U}_2^k| \leq \frac{\sqrt{2\pi}h |B|^2}{\sqrt{k-2}(2k-1)} \left(|\Delta U'(0)| + h \int_0^1 \frac{|U''(h(2\theta-1))|}{\sqrt{\theta(1-\theta)}} d\theta \right).$$

Thanks to Property 2 (ii), upper bound (C.1), inequality $(2k-1) \leq 2(k-2)$ and $\int_0^1 \frac{d\theta}{\sqrt{\theta(1-\theta)}} = \pi$, we have

$$|\mathbf{U}_2^k| \leq \frac{\sqrt{2\pi}h |B|^2}{2(k-2)^{\frac{3}{2}}} (1 + \rho\pi h) |W| = \frac{\varrho_3 |W|}{2(k-2)^{\frac{3}{2}}}, \quad \forall k \geq 3.$$

where notation $\varrho_3 := \sqrt{2\pi}(1 + \rho\pi h)h |B|^2$ is introduced. For the same reasons stated in the proof of Lemma 3 (i), for any integer $N \geq n$ and for all θ in $[0, 1]$, the following upper bound is obtained

$$\left| \sum_{k=n}^N \mathbf{U}_2^k l_k \left(\frac{\theta+h}{h} \right) \right| \leq \varrho_3 |W| \left(\frac{1}{\sqrt{n-3}} - \frac{1}{\sqrt{N-3}} \right).$$

We conclude that the series $\tilde{U}_{2,n}(\theta) = \sum_{k=n}^{\infty} \mathbf{U}_{2,n}^k l_k \left(\frac{\theta+h}{h} \right)$ exists, converges to zero as n tends to infinity and that inequality (22) holds. \square

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