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► To cite this version:

Mathieu Bajodek, Frédéric Gouaisbaut, Alexandre Seuret. Necessary and sufficient stability condition for time-delay systems arising from Legendre approximation. 2021. hal-03435028v1

HAL Id: hal-03435028

<https://laas.hal.science/hal-03435028v1>

Preprint submitted on 18 Nov 2021 (v1), last revised 26 Jul 2023 (v3)

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Necessary and sufficient stability condition for time-delay systems arising from Legendre approximation

Bajodek, M. and Gouaisbaut, F. and Seuret, A.

Abstract— Recently, sufficient conditions of stability or instability for time-delay systems have been proven to be necessary. In this way, a remarkable necessary and sufficient condition has then been developed by Gomez et al. It is presented as a simple test of positive definiteness of a matrix issued from the Lyapunov matrix. In this paper, an extension of this result is presented. Without going into details, the uniform discretization of the state has been replaced by projections on the first Legendre polynomials. Like Gomez et al., based on convergence arguments, the necessity is obtained in finite order, which can be calculated analytically. Compared to them, by relying on the fast convergence rate of Legendre approximation, the required order to ensure stability has been reduced. Thanks to this major modification, as shown in the example section, it is possible to find stable regions for low orders and unstable ones for even smaller orders.

I. INTRODUCTION

As soon as time processing or analog-to-digital converters interfere in between dynamical systems, a delay is introduced. To take into consideration this latency and to analyze the stability of time-delay systems, numerous numerical methods have been deployed [19]. To enumerate some of them, which provide an exact indication on the stability properties, one can look at D-partition [20] issued from modulus-argument calculation is the simplest one to implement. Approximated models issued from pseudospectral [1] or tau-method [16] are also often plebiscited. Indeed, besides that bifurcation analysis leads to stability regions, an outline of the root locus can be drawn. To avoid case by case study, frequency delay-dependant techniques have also been developed. The \mathcal{H}_∞ analysis provides nice stability results [18] and foments the design of controllers [7], [21]. Furthermore, stability areas can be inferred by quasi-polynomials approaches and the set up of Mikhaïov diagrams [22]. Lastly, in time domain, it is well-known that the existence of the Lyapunov-Krasovskii functional (LKF) is a necessary and sufficient condition of stability [17] but that the sufficiency is not numerically tractable. Recently, this has been made feasible by discretization of the LKF [5]. Applied to time-delay systems with integral [23], neutral [13] or multiple delays [14] types, numerical necessary and sufficient conditions have been presented as scalable criteria of positive definiteness of a certain matrix \mathcal{K}_{r^*} defined in [12].

In this paper, one focuses on this last test. From one side, the necessity is directly obtained by construction of the LKF (see [6]). From the other side, the sufficiency is obtained asymptotically (see [9]). In practice, if the positiveness is satisfied up to a certain order r^* [11], then the stability is assessed. Nevertheless, these orders seem extremely large and pessimistic. Then, is it possible to extend the methodology developed by Mondié et al [11] to other support basis? Do the numerical complexity of the numerical test can be reduced? In that direction, we investigated another way to approximate the LKF by projection of the infinite-dimensional state on the first Legendre polynomials. The selection of Legendre polynomials is not harmless insofar promising sufficient stability results have been obtained for low orders in the linear matrix inequality framework [24]. By taking

the benefits of Legendre approximation, in particular the fast convergence rate in $\frac{\rho^n}{n!}$, this work explains how the necessary and sufficient criterion of stability is preserved. With this improved technique, the required orders for the new positivity test is drastically reduced. Thus, it is possible to predict the stability of time-delay systems in a faster and easier way.

The article is organized as follows. In Section II, based on the complete LKF, the lemmas used in [11], [12] to prove the necessity and sufficiency are recalled. To complete these preliminaries, Section III is dedicated to the convergence lemma occurring when performing Legendre approximations. Then, our new numerical necessary and sufficient condition of stability is exposed in Section IV. The last section is dedicated to the evaluation and comparison of our contribution.

Notations : Throughout the paper, \mathbb{N} and $\mathbb{R}^{m \times p}$ and \mathbb{S}^m denote the set of natural numbers, real matrices of size $m \times p$ and symmetric matrices of size m , respectively. For any square matrix $M \in \mathbb{R}^{m \times m}$, the transpose of M is denoted M^\top and $\mathcal{H}(M)$ is equal to $M + M^\top$. For any matrix $M \in \mathbb{S}^m$, $M \succ 0$ means that M is positive definite (i.e. the eigenvalues of M are strictly positive). Furthermore, for any matrix M in $\mathbb{R}^{m \times p}$, the 2-norm of M is $|M| = \sqrt{\bar{\sigma}(M^\top M)}$, where $\bar{\sigma}$ defines the maximal eigenvalue. The vector $u = \text{vec}(M)$ in \mathbb{R}^{mp} collocates the columns of matrix M . Moreover, notation δ_{jk} represents the Kronecker delta, symbol \otimes denotes the Kronecker product, matrix $\begin{bmatrix} M_1 & M_2 \\ * & M_3 \end{bmatrix}$ stands for $\begin{bmatrix} M_1 & M_2 \\ M_2^\top & M_3 \end{bmatrix}$, I_m is the identity matrix of size m and $\lceil r \rceil$ is the ceiling part of the real number r . The set of piece-wise continuous functions from $[-h, 0]$ to \mathbb{R}^m with a finite number of discontinuity points and, for each continuity interval, finite right-hand-side and left-hand-side limits is noted $\mathcal{C}_{pw}(-h, 0; \mathbb{R}^m)$. For any function φ in such space, the induced norm is $\|\varphi\| = \sup_{[-h, 0]} |\varphi(\tau)|$. Denote also $\mathcal{C}_\infty(-h, 0; \mathbb{R}^m)$, the set of

smooth functions from $[-h, 0]$ to \mathbb{R}^m . Lastly, the standard notation $x_t : \begin{cases} [-h, 0] \rightarrow \mathbb{R}^m \\ \tau \mapsto x_t(\tau) = x(t + \tau) \end{cases}$ is used for the state function.

II. LYAPUNOV NECESSARY AND SUFFICIENT STABILITY CONDITION FOR TIME-DELAY SYSTEMS

A. Time-delay system and Lyapunov-Krasovskii functional

Consider a time-delay system given by

$$\dot{x}(t) = Ax(t) + A_d x(t - h), \quad \forall t \geq 0, \quad (1)$$

where the delay $h > 0$ and matrices A, A_d in $\mathbb{R}^{m \times m}$ are constant and known. Such a system is initialized by $x_0 = \varphi$ in $\mathcal{C}_{pw}(-h, 0; \mathbb{R}^m)$ and, for any $t \geq 0$, the solution x_t belongs to $\mathcal{C}_{pw}(-h, 0; \mathbb{R}^m)$.

The trivial solution of system (1) is said to be exponentially stable if there exist $\kappa, \mu > 0$ such that $\|x_t\| \leq \kappa e^{-\mu t} \|x_0\|$, for any $t \geq 0$.

In order to study the stability of system (1), let recall the Lyapunov-Krasovskii functional introduced in [17]:

$$V(\varphi) = \int_{-h}^0 \int_{-h}^0 \begin{bmatrix} \varphi(0) \\ \varphi(\tau_1) \\ \varphi(\tau_2) \end{bmatrix}^\top \Pi(\tau_1, \tau_2) \begin{bmatrix} \varphi(0) \\ \varphi(\tau_1) \\ \varphi(\tau_2) \end{bmatrix} d\tau_1 d\tau_2, \quad (2)$$

for any $\varphi \in \mathcal{C}_{pw}(-h, 0; \mathbb{R}^m)$, where matrix

$$\Pi(\tau_1, \tau_2) = \begin{bmatrix} \frac{U(0)}{h^2} & \frac{1}{2h}U^\top(h + \tau_1)A_d & \frac{1}{2h}U^\top(h + \tau_2)A_d \\ * & \frac{1}{2h}I_m & \frac{1}{2}A_d^\top U(\tau_1 - \tau_2)A_d \\ * & * & \frac{1}{2h}I_m \end{bmatrix}, \quad (3)$$

and where the Lyapunov matrix U in $\mathbb{R}^{m \times m}$ is given by $U = \text{vec}^{-1}(\mathcal{U})$ since the vector $\mathcal{U} = \text{vec}(U)$ is given analytically by

$$\mathcal{U}(\tau) = \begin{cases} [I_{m^2} \ 0]e^{\tau M}N^{-1} \begin{bmatrix} -\mathcal{I}_m \\ 0 \end{bmatrix} & \text{if } \tau \geq 0, \\ [0 \ I_{m^2}]e^{(h+\tau)M}N^{-1} \begin{bmatrix} -\mathcal{I}_m \\ 0 \end{bmatrix} & \text{if } \tau < 0, \end{cases} \quad (4)$$

with \mathcal{I}_m the vector which stacks the columns of I_m and with

$$M = \begin{bmatrix} A^\top \otimes I_m & A_d^\top \otimes I_m \\ -I_m \otimes A_d^\top & -I_m \otimes A^\top \end{bmatrix}, \quad (5)$$

$$N = \begin{bmatrix} A^\top \otimes I_m + I_m \otimes A^\top & A_d^\top \otimes I_m \\ I_{m^2} & 0 \end{bmatrix} + \begin{bmatrix} I_m \otimes A_d^\top & 0 \\ 0 & -I_{m^2} \end{bmatrix} e^{hM}.$$

Remark 1: Notice that the Lyapunov matrix U exists if and only if matrix N is non singular. As explained in [17], such a limitation is called the Lyapunov condition and excludes all systems which have eigenvalues s_1, s_2 satisfying $|s_1 + s_2| = 0$.

B. Necessary and sufficient stability condition

Under the Lyapunov condition, the authors of [11], [17] provide lemmas, respectively, on a sufficient and necessary condition for exponential stability of time-delay system (1). They are recalled below.

Lemma 1: Let system (1) be exponentially stable, then there exists $\eta > 0$ such that,

$$V(\varphi) \geq \eta \left(|\varphi(0)|^2 + \frac{1}{h} \int_{-h}^0 |\varphi(\tau)|^2 d\tau \right), \quad \varphi \in \mathcal{C}_{pw}(-h, 0; \mathbb{R}^m), \quad (6)$$

with the Lyapunov-Krasovskii functional V defined by (2).

Proof: As in [15, Theorem 5.19], we introduce

$$W(\varphi) = V(\varphi) - \int_{-h}^0 \begin{bmatrix} \varphi(0) \\ \varphi(\tau) \end{bmatrix}^\top \begin{bmatrix} \frac{\epsilon}{h}I_m & 0 \\ 0 & \frac{1}{2}I_m \end{bmatrix} \begin{bmatrix} \varphi(0) \\ \varphi(\tau) \end{bmatrix} d\tau. \quad (7)$$

Functional V is built so that the derivative along trajectories x_t of system (1) gives

$$\dot{W}(x_t) = - \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}^\top \begin{bmatrix} \epsilon \mathcal{H}(A) + \frac{1}{2}I_m & \epsilon A_d \\ * & \frac{1}{2}I_m \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix} d\tau, \quad (8)$$

for which there exists a sufficiently small $\epsilon > 0$ such that $\dot{W}(x_t) \leq 0$. Then, integrating from 0 to ∞ and assuming exponential stability yields $W(x_0) \geq 0$, for any initial conditions $x_0 = \varphi$ in $\mathcal{C}_{pw}(-h, 0; \mathbb{R}^m)$. Thus, (6) holds with $\eta = \min(\epsilon, h)$. ■

Lemma 2: Assume that system (1) has an eigenvalue with a strictly positive real part. Then,

$$\exists \varphi \in \mathcal{S}, \quad V(\varphi) \leq -\eta_0 = -\frac{e^{-2rh}}{4r} \cos^2(b_0) < 0, \quad (9)$$

where \mathcal{S} stands for the compact set given by

$$\mathcal{S} = \left\{ \varphi \in \mathcal{C}_\infty(-h, 0; \mathbb{R}^m); \begin{array}{l} |\varphi(0)| = 1, \forall k \in \mathbb{N} \\ \max_{[-h, 0]} |\varphi^{(k)}(\tau)| \leq r^k \end{array} \right\}, \quad (10)$$

with a system dependent parameter r given by

$$r = |A| + |A_d|, \quad (11)$$

and with b_0 is the unique root on $[0, \frac{\pi}{2}]$ of

$$g(b) = \sin^4(b) \left((hr)^2 + b^2 \right) - (hr)^2 = 0.$$

Proof: The proof is similar to the one given in appendix in [11]. The only difference relies on the definition of set \mathcal{S} that has been extended to \mathcal{C}_∞ instead of \mathcal{C}_1 . Let denote $s_0 = \alpha + i\beta$ be the eigenvalue with positive real part of system (1). According to [10], there exists a vector $C = C_1 + iC_2$ such that $|C_2| \leq |C_1| = 1$, $C_2^\top C_1 = 0$ and that

$$(s_0 I_m - A - A_d e^{-hs_0})C = 0.$$

Consequently, we have

$$\alpha \leq |s_0| \leq |A| + |A_d| = r. \quad (12)$$

and,

$$\bar{x}(t) = e^{\alpha t} (\cos(\beta t)C_1 - \sin(\beta t)C_2), \quad \forall t \in \mathbb{R},$$

is a solution of system (1). Then, by construction, the derivatives of $V(\bar{x}_t)$ along the trajectories of (1) yields to

$$\dot{V}(\bar{x}_t) = -|\bar{x}(t-h)|^2. \quad (13)$$

After some calculations developed in [11], we also obtain that

$$V(\bar{x}_0) \leq -\frac{e^{-2\alpha h}}{4\alpha} \cos^2(b_0) \leq -\frac{e^{-2rh}}{4r} \cos^2(b_0) = -\eta_0. \quad (14)$$

Let $\varphi = \bar{x}_0$, which belongs to $\mathcal{C}_\infty(-h, 0; \mathbb{R}^m)$ and satisfies $|\varphi(0)| = |x(0)| = 1$. We finally prove by induction that $|\varphi^{(k)}(\tau)| \leq r^k e^{\alpha\tau}$, for any τ in $[-h, 0]$. Initially, $|\bar{x}(\tau)| \leq e^{\alpha\tau}$ holds on $[-h, 0]$. Then, assuming $|\bar{x}^{(k)}(\tau)| \leq r^k e^{\alpha\tau}$, since \bar{x} satisfies (1) and is infinitely differentiable, we obtain

$$\left| \bar{x}^{(k+1)}(\tau) \right| \leq |A| \left| \bar{x}^{(k)}(\tau) \right| + |A_d| \left| \bar{x}^{(k)}(\tau-h) \right| \leq r^{k+1} e^{\alpha\tau}.$$

Therefore, $\max_{[-h, 0]} |\varphi^{(k)}(\tau)| \leq r^k$ for any $k \in \mathbb{N}$. ■

III. PRELIMINARIES ON LEGENDRE POLYNOMIALS

A. Legendre approximation

Recall the Legendre polynomials considered on $[-h, 0]$ and defined as

$$\forall k \in \mathbb{N}, \quad \ell_k(\tau) = (-1)^k \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{k+j}{j} \left(\frac{\tau+h}{h} \right)^j, \quad (15)$$

where $\binom{k}{j}$ stands for the binomial coefficient [8].

These polynomials $\{\ell_k\}_{k \in \mathbb{N}}$ form an orthogonal sequence, which spans the space of square-integrable functions [8].

In the sequel, an approximation on the n first Legendre polynomials is performed. For the sake of simplicity, introduce matrix ℓ_n in $\mathbb{R}^{nm \times m}$ given by

$$\ell_n(\theta) = [\ell_0(\theta)I_m \quad \ell_1(\theta)I_m \quad \dots \quad \ell_{n-1}(\theta)I_m]^\top, \quad \forall \theta \in [-h, 0]. \quad (16)$$

For any approximation order $n \in \mathbb{N}^*$ and any function φ in $\mathcal{C}_{pw}(-h, 0; \mathbb{R}^m)$, let us decompose

$$\forall \tau \in [-h, 0], \quad \varphi(\tau) = \underbrace{\ell_n^\top(\tau)\Phi_n}_{\varphi_n(\tau)} + \tilde{\varphi}_n(\tau), \quad (17)$$

where $\varphi_n(\tau) = \ell_n^\top(\tau)\Phi_n$ is the polynomial approximation and $\tilde{\varphi}_n(\tau) = \varphi(\tau) - \varphi_n(\tau)$ is the residual error. The vector Φ_n represents the normalized n first polynomial coefficients of the function φ and is defined by

$$\Phi_n = \mathbf{I}_n \int_{-h}^0 \ell_n(\tau)\varphi(\tau)d\tau \in \mathbb{R}^{nm}, \quad (18)$$

where \mathbf{I}_n is the normalization matrix given by

$$\mathbf{I}_n = \begin{bmatrix} \frac{1}{h}I_m & 0 & \dots & 0 \\ 0 & \frac{3}{h}I_m & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & \frac{2n-1}{h}I_m \end{bmatrix}. \quad (19)$$

In the sequel, the objective is to prove that the Legendre approximation φ_n converges uniformly towards φ with respect to τ and more interestingly to quantify the convergence rate on the subset \mathcal{S} .

B. Convergence of the Legendre remainder

In light of polynomial approximation theory [2], [4], it results an important convergence lemma.

Lemma 3: For any function φ in \mathcal{S} , the approximation error $\tilde{\varphi}_n$ in (17) verifies, for any $\varepsilon > 0$,

$$\|\tilde{\varphi}_n\| \leq \varepsilon, \quad \forall n \geq \mathcal{N}(\varepsilon), \quad (20)$$

where $\mathcal{N}(\varepsilon)$ is given by

$$\mathcal{N}(\varepsilon) = \max \left(4, 1 + \left\lceil \frac{hr}{2} e^{1+\mathcal{W}\left(-\frac{\ln(\mu\varepsilon)}{(\frac{hr}{2})e}\right)} \right\rceil \right), \quad (21)$$

with

$$\rho = \max \left(2, \left\lceil \frac{hr}{2} \right\rceil \right), \quad \mu = \left(\frac{e}{\rho} \right)^\rho \left(\frac{hr}{2} \right)^{\rho-2} \left(\sqrt{\frac{\pi}{2}} \pi \right)^{-1}, \quad (22)$$

and with the Lambert function [3] $\mathcal{W} : \begin{cases} \mathbb{R}_+ \rightarrow \mathbb{R}_+ \\ z \mapsto \mathcal{W}(z) = y \end{cases}$ where y is uniquely defined by the relation $ye^y = z$.

Proof: According to Theorem 2.5 in [25], the Legendre approximation error $\tilde{\varphi}_n$ is bounded by

$$|\tilde{\varphi}_n(\tau)| \leq \frac{\sqrt{\frac{\pi}{2(n-1-k)}} \pi \left(\frac{hr}{2} \right)^{k+1}}{(k-1)(n-\frac{3}{2}) \dots (n-k+\frac{1}{2})}, \quad (23)$$

$$\leq \frac{\left(\frac{e}{\rho} \right)^\rho \left(\frac{hr}{2} \right)^{k+\rho-1}}{\mu(n-2) \dots (n-k)}, \quad (24)$$

for any $n \geq k+2$ and $k \geq 2$. Then, in order to obtain the tightest upper bound, the maximal allowable order k which satisfies

$$\frac{hr}{n-2} < \dots < \frac{hr}{n-k} \leq 1,$$

is selected. For instance, for small value of $\frac{hr}{2} \in [0, 2]$, the best order is $k = n-2$. More generally, k is selected as $\max(2, n-\rho)$. Nevertheless, since the expected n is large enough, the case $n-\rho < 2$ (i.e. $k=2$) does not occur and we consider then $k = n-\rho$. For such order k , applying the logarithm to (24) leads to

$$\begin{aligned} \ln(\mu |\tilde{\varphi}_n(\tau)|) &\leq \rho \ln \left(\frac{e}{\rho} \right) + (n-1) \ln \left(\frac{hr}{2} \right) - \sum_{\rho}^{n-2} \ln k, \\ &\leq \rho \ln \left(\frac{e}{\rho} \right) + (n-1) \ln \left(\frac{hr}{2} \right) - \int_{\rho}^{n-1} \ln x dx, \\ \ln(\mu |\tilde{\varphi}_n(\tau)|) &\leq -(n-1) \ln \left(\frac{n-1}{(\frac{hr}{2})e} \right). \end{aligned} \quad (25)$$

Moreover, for any $n \geq \mathcal{N}(\varepsilon)$, the following inequality

$$\left(\frac{n-1}{(\frac{hr}{2})e} \right) \ln \left(\frac{n-1}{(\frac{hr}{2})e} \right) \geq \mathcal{W} \left(-\frac{\ln(\mu\varepsilon)}{(\frac{hr}{2})e} \right) e^{\mathcal{W} \left(-\frac{\ln(\mu\varepsilon)}{(\frac{hr}{2})e} \right)} = -\frac{\ln(\mu\varepsilon)}{(\frac{hr}{2})e}, \quad (26)$$

holds by definition of the Lambert function. Consequently, inequalities (25) and (26) lead to $|\tilde{\varphi}_n(\tau)| \leq \varepsilon$ for all τ in $[-h, 0]$. ■

IV. A NEW NECESSARY AND SUFFICIENT STABILITY CONDITION FOR TIME-DELAY SYSTEMS

A. Approximated Lyapunov-Krasovskii functional

In this section, the Lyapunov-Krasovskii functional given by (2) is regarded for particular functions φ , taken from subsets of $\mathcal{C}_{pw}(-h, 0; \mathbb{R}^m)$. For instance, discrete values of φ equally distributed on $[-h, 0]$ have been considered in [11], [12]. Here, as presented in the preliminaries section, we proceed differently by taking support on the n first Legendre coefficients of φ denoted Φ_n and expressed in (18).

Let the approximated Lyapunov-Krasovskii functional at order n

$$V_n(\varphi) = \begin{bmatrix} \varphi(0) \\ \Phi_n \end{bmatrix}^\top \mathbf{P}_n \begin{bmatrix} \varphi(0) \\ \Phi_n \end{bmatrix}, \quad (27)$$

for any $\varphi \in \mathcal{C}_{pw}(-h, 0; \mathbb{R}^m)$, with matrix

$$\mathbf{P}_n = \begin{bmatrix} U(0) & \mathbf{Q}_n \\ * & \mathbf{T}_n + \mathbf{I}_n^{-1} \end{bmatrix}. \quad (28)$$

In the previous expression, we have

$$\begin{aligned} \mathbf{Q}_n &= \int_{-h}^0 U^\top(h+\tau) A_d \ell_n^\top(\tau) d\tau, \\ \mathbf{T}_n &= \int_{-h}^0 \int_{-h}^0 \ell_n(\tau_1) A_d^\top U(\tau_1 - \tau_2) A_d \ell_n^\top(\tau_2) d\tau_1 d\tau_2. \end{aligned} \quad (29)$$

Remark 2: Note that V_n does not take into consideration the Legendre remainder $\tilde{\varphi}_n$ and can then be seen as an approximation of the Lyapunov-Krasovskii functional V defined by (2).

Based on the previous section on polynomial approximation, the convergence of this approximated functional towards the complete Lyapunov-Krasovskii functional given by (2) will be established in the next section.

B. Convergence of the approximated Lyapunov-Krasovskii functional

Define the Lyapunov-Krasovskii functional remainder as

$$\tilde{V}_n(\varphi) = V(\varphi) - V_n(\varphi), \quad \forall \varphi \in \mathcal{C}_{pw}(-h, 0; \mathbb{R}^m). \quad (30)$$

Applying expansion (17), this remainder is rewritten as

$$\tilde{V}_n(\varphi) = \int_{-h}^0 \int_{-h}^0 \begin{bmatrix} \varphi(0) \\ \varphi_n(\tau_1) \\ \tilde{\varphi}_n(\tau_1) \\ \tilde{\varphi}_n(\tau_2) \end{bmatrix}^\top \Delta_n(\tau_1, \tau_2) \tilde{\varphi}_n(\tau_2) d\tau_1 d\tau_2, \quad (31)$$

where

$$\Delta_n(\tau_1, \tau_2) = \begin{bmatrix} \frac{2}{h} U^\top(h+\tau_2) A_d \\ 2 A_d^\top U(\tau_1 - \tau_2) A_d \\ A_d^\top U(\tau_1 - \tau_2) A_d \\ \frac{1}{h} I_m \end{bmatrix}. \quad (32)$$

The main idea is now to prove, at least in the compact subset \mathcal{S} of $\mathcal{C}_{pw}(-h, 0; \mathbb{R}^m)$ given by (10), that the approximated Lyapunov-Krasovskii functional V_n given by (27) converges towards the complete Lyapunov-Krasovskii functional V given by (2). On \mathcal{S} , its convergence rate is also quantified. This boils down to the study of the convergence of \tilde{V}_n towards zero, presented in the next lemma.

Lemma 4: Consider the Lyapunov-Krasovskii remainder defined by (31). For any φ in \mathcal{S} and $\eta > 0$, we have

$$\left| \tilde{V}_n(\varphi) \right| \leq \eta, \quad \forall n \geq \mathcal{N}(\mathcal{E}(\eta)), \quad (33)$$

where the order $\mathcal{N}(\varepsilon)$ is described in (21) and where

$$\begin{aligned} \mathcal{E}(\eta) &= \frac{-(\kappa_1 + \kappa_2) + \sqrt{(\kappa_1 + \kappa_2)^2 + (h + \kappa_2)\eta}}{h + \kappa_2}, \\ &= \frac{-(\kappa_1 + \kappa_2) + \sqrt{(\kappa_1 + \kappa_2)^2 + (h + \kappa_2)\eta}}{\eta}, \end{aligned} \quad (34)$$

with maximal values on bounded intervals

$$\kappa_1 = h \max_{[0, h]} |U(\tau)A_d|, \quad \kappa_2 = h^2 \max_{[-h, h]} |A_d^\top U(\tau)A_d|.$$

Proof: Roughly bounding the norm of \tilde{V}_n , we obtain

$$\begin{aligned} |\tilde{V}_n(\varphi)| &\leq \frac{2\kappa_1}{h} |\varphi(0)| \int_{-h}^0 |\tilde{\varphi}_n(\tau)| d\tau \\ &\quad + \frac{2\kappa_2}{h^2} \left(\int_{-h}^0 |\varphi_n(\tau)| d\tau \right) \left(\int_{-h}^0 |\tilde{\varphi}_n(\tau)| d\tau \right) \\ &\quad + \frac{\kappa_2}{h^2} \left(\int_{-h}^0 |\tilde{\varphi}_n(\tau)| d\tau \right)^2 + \left(\int_{-h}^0 |\tilde{\varphi}_n(\tau)|^2 d\tau \right). \end{aligned}$$

Hence having φ in \mathcal{S} leads to

$$|\tilde{V}_n(\varphi)| \leq 2(\kappa_1 + \kappa_2) \|\tilde{\varphi}_n\| + (h + \kappa_2) \|\tilde{\varphi}_n\|^2. \quad (35)$$

Similarly having $\|\tilde{\varphi}_n\| \leq \mathcal{E}(\eta)$ implies $|\tilde{V}_n(\varphi)| \leq \eta$. The conclusion is finally drawn thanks to Lemma 3, which states that $\|\tilde{\varphi}_n\| \leq \mathcal{E}(\eta)$ holds for any orders n greater than $\mathcal{N}(\mathcal{E}(\eta))$. ■

The proposed convergence property of the remainder will be therefore the key to design a stability test for time-delay systems.

C. Necessary and sufficient stability test

Lemmas 1 and 2 on the approximated Lyapunov-Krasovskii functional V_n defined by (27) provide then milestones of a new necessary and sufficient condition of stability for system (1) presented below.

Theorem 1: System (1) is exponentially stable if and only if matrix $\mathbf{P}_{\mathcal{N}(\mathcal{E}(\eta_0))}$ in (28) is positive definite where \mathcal{N}, \mathcal{E} are defined in (21), (34), respectively.

Proof: The necessity and sufficient sides of the proof are split.

Necessity: Assume that system (1) is exponentially stable. Let choose, for any vector $[\tilde{\Phi}_n^x]$ in $\mathbb{R}^{(n+1)m}$ and any n in \mathbb{N} , function φ as

$$\varphi(\tau) = \begin{cases} \ell_n^\top(\tau)\Phi_n, & \forall \tau \in [-h, 0), \\ x, & \text{if } \tau = 0. \end{cases}$$

Thanks to Lemma 1 with φ given above, there exists $\eta > 0$ such that

$$V(\varphi) = V_n(\varphi) = [\tilde{\Phi}_n^x]^\top \mathbf{P}_n [\tilde{\Phi}_n^x] \geq \eta \|[\tilde{\Phi}_n^x]\|^2,$$

which yields $\mathbf{P}_n \succ 0$, for all $n \in \mathbb{N}$, since x and Φ_n are any independent vectors.

Sufficiency: By contradiction, assume that system (1) is not exponentially stable, and that $\mathbf{P}_{\mathcal{N}(\mathcal{E}(\eta_0))} \succ 0$. This means that there exists an eigenvalue of (1) with a positive real part. Consequently, by application of Lemma 2, there necessarily exists φ in \mathcal{S} such that

$$V_n(\varphi) = V(\varphi) - \tilde{V}_n(\varphi) \leq -\eta_0 + |\tilde{V}_n(\varphi)|, \quad (36)$$

with η_0 given by (9). Finally, the convergence presented in Lemma 4 with $\eta = \eta_0$ leads to

$$V_n(\varphi) = [\tilde{\Phi}_n^x]^\top \mathbf{P}_n [\tilde{\Phi}_n^x] \leq 0, \quad \forall n \geq \mathcal{N}(\mathcal{E}(\eta_0)), \quad (37)$$

which contradicts $\mathbf{P}_{\mathcal{N}(\mathcal{E}(\eta_0))} \succ 0$. Therefore, if $\mathbf{P}_{\mathcal{N}(\mathcal{E}(\eta_0))}$ is positive, then the system is exponentially stable. ■

The proposed theorem provides a numerical test to guarantee stability or instability of time-delay systems, which follows the following sequence.

- 1) Calculate $n^* = \mathcal{N}(\mathcal{E}(\eta_0))$ with η_0 given by (9).
- 2) Evaluate each element of matrix \mathbf{P}_{n^*} .

- 3) Test the positive definiteness of matrix \mathbf{P}_{n^*} and conclude on the stability properties.

Notice that this necessary and sufficient condition of stability is of the same nature as the one in [11]. Indeed, the test is based on the positivity of matrix \mathbf{P}_n for a certain order n which can be given analytically.

As a background result, a hierarchical sufficient condition for instability of system (1) is also formulated below.

Corollary 1: If there exists $n \in \mathbb{N}$ such that matrix \mathbf{P}_n given by (28) is not definite positive then system (1) is not exponentially stable. Moreover, if this statement holds at order n_0 , then it also holds at any order $n \geq n_0$.

Proof: Relying on the necessity part of the proof of Theorem 1, the sufficient condition for instability is directly obtained. The hierarchy can then be proven because matrix \mathbf{P}_n at order $n \geq n_0$ can be written as

$$\mathbf{P}_n = \begin{bmatrix} \mathbf{P}_{n_0} & \mathbf{Q}_{n_0:n} \\ * & \bar{\mathbf{T}}_{n_0:n} + \bar{\mathbf{I}}_{n_0:n}^{-1} \end{bmatrix}, \quad (38)$$

with

$$\begin{aligned} \bar{\mathbf{Q}}_{n_0:n} &= \frac{1}{h} \int_{-h}^0 U^\top(h + \tau) A_d \bar{\ell}_{n_0:n}(\tau) d\tau, \\ \bar{\mathbf{T}}_{n_0:n} &= \frac{1}{h^2} \int_{-h}^0 \int_{-h}^0 \bar{\ell}_{n_0:n}(\tau_1) A_d^\top U(\tau_1 - \tau_2) A_d \bar{\ell}_{n_0:n}^\top(\tau_2) d\tau_1 d\tau_2, \\ \bar{\mathbf{I}}_{n_0:n} &= \begin{bmatrix} \frac{2n_0+1}{h} I_m & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \frac{2n-1}{h} I_m \end{bmatrix}, \quad \bar{\ell}_{n_0:n} = [l_{n_0} I_m \dots l_{n-1} I_m]^\top. \end{aligned}$$

Consequently, if \mathbf{P}_{n_0} is not positive definite then \mathbf{P}_n is also not positive definite at orders $n \geq n_0$. ■

The algorithm presented in Corollary 1 consists in testing $\mathbf{P}_n \succ 0$ from $n = 1$ to $n = n^*$. If \mathbf{P}_n is not definite positive, then the system is unstable. From Theorem 1, once order $n^* = \mathcal{N}(\mathcal{E}(\eta_0))$ is reached, one concludes that the system is stable. Compared to Theorem 1, unstable systems can then be detected faster, thanks to smaller orders.

It remains to solve the important problem of the numerical computation of \mathbf{P}_n , which is necessarily to implement the algorithm. This is detailed in the next section.

V. COMPUTATIONAL ISSUES

A. Problem identification

To perform the numerical test presented above, each coefficient of matrix \mathbf{P}_n given by (28) needs to be evaluated numerically. This amounts to calculate the integral terms \mathbf{Q}_n and \mathbf{T}_n given by (29). Such computation can be done analytically but may turn out to be a tough task, especially for large n or m . Technically speaking, for $m = 4$ and $n = 100$, the exact calculation of \mathbf{P}_n can take days on a basic computer.

It is worth noticing that this problem is not encountered in [11] to the matter of fact that the matrix under process contains pointwise evaluations of the Lyapunov matrix U . Here, the situation is more complicated since matrix $\mathbf{Q}_n = [Q_0^\top A_d \dots Q_{n-1}^\top A_d]$ contains $\{Q_k\}_{k \in \{0, \dots, n-1\}}$ the first Legendre coefficients of U given by

$$Q_k = \int_{-h}^0 U(h + \tau) l_k(\tau) d\tau, \quad \forall k \in \{0, \dots, n-1\}, \quad (39)$$

and since \mathbf{T}_n is composed of coefficients $A_d^\top (T_{jk} + (T_{jk}^b)^\top) A_d$

where

$$\begin{aligned} T_{jk} &= \int_{-h}^0 \left(\int_{-h}^{\tau_1} U(\tau_1 - \tau_2) l_k(\tau_2) d\tau_2 \right) l_j(\tau_1) d\tau_1, \\ T_{jk}^b &= \int_{-h}^0 \left(\int_{\tau_1}^0 U(\tau_2 - \tau_1) l_k(\tau_2) d\tau_2 \right) l_j(\tau_1) d\tau_1, \end{aligned} \quad (40)$$

$$\forall (j, k) \in \{0, \dots, n-1\}^2.$$

The question of the numerical implementation of these integral terms in a reasonable time is then raised.

To face the problem and make our results tractable numerically, an alternative for the computation of \mathbf{Q}_n and \mathbf{T}_n is proposed.

B. Iterative calculation of Legendre coefficients of exponential functions

To begin with, by the use of (4), the Legendre coefficients Q_k, T_{jk} and T_{jk}^b of the Lyapunov matrix U given in (39),(40), can be rewritten in the vector form as follows

$$\begin{cases} \mathbf{Q}_k = \text{vec}(Q_k) = [I_{m^2} \ 0] \Gamma_k N^{-1} \begin{bmatrix} -I_m \\ 0 \end{bmatrix}, \\ \mathbf{T}_{jk} = \text{vec}(T_{jk}) = [I_{m^2} \ 0] \bar{\Gamma}_{jk} N^{-1} \begin{bmatrix} -I_m \\ 0 \end{bmatrix}, \\ \mathbf{T}_{jk}^b = \text{vec}(T_{jk}^b) = [I_{m^2} \ 0] \bar{\Gamma}_{jk}^b N^{-1} \begin{bmatrix} -I_m \\ 0 \end{bmatrix}, \end{cases} \quad (41)$$

with $\Gamma_k, \bar{\Gamma}_{jk}$ and $\bar{\Gamma}_{jk}^b$ defined by

$$\Gamma_k = \int_{-h}^0 e^{(h+\tau)M} l_k(\tau) d\tau, \quad \forall k \in \{0, \dots, n-1\}, \quad (42)$$

$$\bar{\Gamma}_{jk} = \int_{-h}^0 \left(\int_{-h}^{\tau_1} e^{(\tau_1-\tau_2)M} l_k(\tau_2) d\tau_2 \right) l_j(\tau_1) d\tau_1, \quad (43)$$

$$\bar{\Gamma}_{jk}^b = \int_{-h}^0 \left(\int_{\tau_1}^0 e^{(\tau_2-\tau_1)M} l_k(\tau_2) d\tau_2 \right) l_j(\tau_1) d\tau_1, \quad (44)$$

$$\forall (j, k) \in \{0, \dots, n-1\}^2.$$

The numerical issue can then be resumed to the calculation of Legendre polynomials coefficients of exponential matrices. To perform this calculation in a fast manner, the following relations are used.

Proposition 1: If M is a non singular matrix, then matrices Γ_k in (42) can be computed by the recursive relation

$$\Gamma_k = \Gamma_{k-2} - \frac{2(2k-1)}{h} M^{-1} \Gamma_{k-1}, \quad \forall k \geq 2, \quad (45)$$

and initialized with

$$\begin{bmatrix} \Gamma_0 \\ \Gamma_1 \end{bmatrix} = \begin{bmatrix} M^{-1}(e^{hM} - I_{2m^2}) \\ M^{-1}(e^{hM} + I_{2m^2}) - \frac{2}{h} M^{-1} \Gamma_0 \end{bmatrix}. \quad (46)$$

Proposition 2: For any matrix M and for matrices $\bar{\Gamma}_{jk}, \bar{\Gamma}_{jk}^b$ expressed in (43),(44), the following equality holds

$$\bar{\Gamma}_{jk} = (-1)^{j+k} \bar{\Gamma}_{jk}^b, \quad \forall (j, k) \in \{0, \dots, n-1\}^2. \quad (47)$$

Proposition 3: If M is a non singular matrix, then matrices $\bar{\Gamma}_{jk}$ in (43) can be computed by the following relations

$$\bar{\Gamma}_{jk} = \begin{cases} (-1)^{j+k} \bar{\Gamma}_{kj}, & \forall k < j, \\ \left(\bar{\Gamma}_{jk-2} + M^{-1} \frac{2(2k-1)}{h} \bar{\Gamma}_{jk-1} \right), & \forall k \geq \max(2, j), \\ -M^{-1} \frac{h}{2j+1} (\delta_{jk} - \delta_{jk-2}), & \end{cases} \quad (48)$$

and initialized with

$$\begin{bmatrix} \bar{\Gamma}_{00} \\ \bar{\Gamma}_{01} \\ \bar{\Gamma}_{11} \end{bmatrix} = \begin{bmatrix} M^{-1}(\Gamma_0 - h I_{2m^2}) \\ -M^{-1} \Gamma_1 \\ M^{-1} \left(\left(\frac{2}{h} M^{-1} - I_{2m^2} \right) \Gamma_1 - \frac{h}{3} I_{2m^2} \right) \end{bmatrix}. \quad (49)$$

Proof: The proofs of these propositions are respectively given in Appendix A, B and C. ■

TABLE I: Evaluation and comparison of necessary and sufficient conditions for stability for Example 1 with several delays.

Delay h	Result	Order n^*	ΔT	Order r^* [11]
0.1	Stable	4	0.2s	36
0.604	Stable	13	0.9s	$\approx 10^8$
0.605	Unstable	13	0.9s	$\approx 10^8$
2	Unstable	23	2.3s	$\approx 10^{12}$

TABLE II: Evaluation of necessary and sufficient conditions for stability for Example 1 with several control gains K and delays h .

Parameters	Result	Order n^*	ΔT
$K = 10, h = 0.552$	Stable	64	150s
$K = 10, h = 0.553$	Unstable	64	150s

C. Suggested solution

The problem of fast evaluation of elements of matrix \mathbf{P}_n is solved, as summarized in the following proposition.

Proposition 4: Assume matrix M non-singular. For any $n \in \mathbb{N}$,

$$\mathbf{P}_n = \begin{bmatrix} U(0) & Q_0^\top A_d & \dots & Q_{n-1}^\top A_d \\ * & A_d^\top \mathcal{H}(T_{00}) A_d + h I_m & \dots & A_d^\top (T_{0n-1} + (-1)^{n-1} T_{0n-1}^\top) A_d \\ * & * & \ddots & \vdots \\ * & * & * & A_d^\top \mathcal{H}(T_{n-1n-1}) A_d + \frac{h}{2n-1} I_m \end{bmatrix},$$

where Q_k and T_{jk} are taken from (41) with Γ_k and $\bar{\Gamma}_{jk}$ calculated recursively according to (45) and (48), respectively.

Proof: This result comes from matrix manipulations to construct integrals (29) with Q_k, T_{jk} and T_{jk}^b given by (39),(40). Then, Proposition 2 is traduced by $T_{jk} + (T_{jk}^b)^\top = T_{jk} + (-1)^{j+k} T_{jk}^\top$ and Propositions 1 and 3 complete the proof. ■

By computing matrix \mathbf{P}_n as proposed in this proposition, the necessary and sufficient condition of stability for time-delay systems evoked in this paper is tractable numerically. In the last section, the effectiveness of our numerical tests of stability are commented on several numerical examples.

VI. APPLICATION TO NUMERICAL EXAMPLES

A. Presentation of the examples

Example 1: Consider (1) with $A = 1$ and $A_d = -2$.

Example 2: Consider (1) with $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -10 & -K & 0 & 1 \\ 5 & -15 & 0 & -0.25 \end{bmatrix}$ and

$$A_d = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ K & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ for any } K > 0.$$

B. Numerical results on stability analysis

To evaluate the stability of these systems with respect to the delay, Theorem 1 is applied for point-wise values of h . The results are gathered in Table I and II for Example 1 and 2, respectively. For both systems, one can see that the maximal h , which guarantees the stability can be given with a precision 0.001.

The minimal order to obtain such results is also given on both Tables. More precisely, a map of orders $n^* = \mathcal{N}(\mathcal{E}(\eta_0))$ from which the necessary condition of stability becomes sufficient is depicted in Fig 1. By D-partition, the exact limit between the stable (on the right) and unstable (on the left) regions is superposed. Surprisingly, order n^* do not depends on the location of the stability areas but rather on parameters h and K . That means that the convergence towards the expected sets of stability is not asymptotic but reached at a finite order n^* .

The CPU time is obviously impacted by the size of the different matrices to be calculated. Actually, the time of processing does not

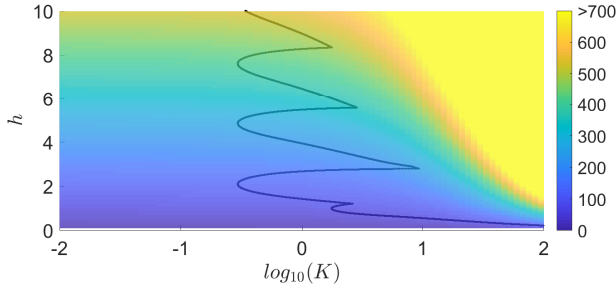


Fig. 1: Example 2: Required orders with respect to (K, h) .

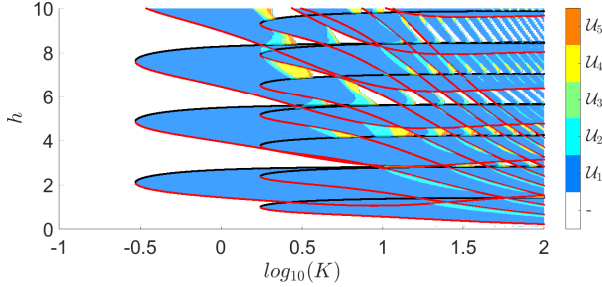


Fig. 2: Example 2: Unstable areas with respect to (K, h) .

come from the computation of the eigenvalues of matrix \mathbf{P}_n but from the evaluation of matrices $\mathbf{Q}_n, \mathbf{T}_n$. By using the recurrence relation given by Proposition 4, this time has been drastically reduced. Nonetheless, since errors are accumulated in the recurrence relations, the Matlab precision requirement has to be refined as n increases (10^{-32} for $n \simeq 20$ against 10^{-300} for $n \simeq 70$) and explain the large difference between CPU time in Table I and II.

In [11], a similar necessary and sufficient condition of stability has also been developed. It is based on a Dirac comb on $[-h, 0]$ whereas, in this paper, the Legendre approximation is used. In both cases, an approximation of the complete Lyapunov-Krasovskii functional is made and a positivity test of finite size is deployed to analyze the stability of time-delay systems. Orders n^* and r^* can then be compared. For Example 1, one remarks that our approach is more efficient since it requires smaller orders. This is probably due to our approximation technique, which allows to better fit with function φ and to obtain faster convergence rate enlightened in Lemma 4.

Lastly, outside of Theorem 1, a scalable sufficient criterion of instability is mentioned in Corollary 1. It permit to detect some unstable systems by finding a low order n such that $\mathbf{P}_n \succ 0$ does not hold. Indeed, for Example 2, to avoid a direct test $\mathbf{P}_n \succ 0$ at order $n = n^* \gg 100$ which is time consuming, alternative tests at low orders $n = \{1, \dots, 5\}$ are realized. Then, the corresponding unstable areas with respect to parameters h and K , denoted $\{\mathcal{U}_1, \dots, \mathcal{U}_5\}$, are drawn on Fig. 2. For instance, at order $n = 1$, if matrix

$$\mathbf{P}_0 = \begin{bmatrix} U(0) & Q_0 A_d \\ * & A_d^\top \mathcal{H}(T_{00}) A_d + h I_m \end{bmatrix}$$

is not definite positive, then the time-delay system is unstable and takes part of \mathcal{U}_1 . Even if no certificate on stability can be deduced, great information on the unstable regions of Example 2 is obtained. We also verified that $\mathcal{U}_1 \subset \dots \subset \mathcal{U}_5$ as proven in Corollary 1 but it is worth noticing that \mathcal{U}_1 already spans the main part of unstable systems. More interestingly, the hard-to-reach areas are located around the passage of eigenvalues on the right-half plane (see red lines).

VII. CONCLUSIONS AND PERSPECTIVES

This contribution is an extension to the necessary and sufficient condition for stability of time-delay systems developed in [11]. It derives from the positivity of the complete Lyapunov-Krasovskii functional, where an approximation of the delay Lyapunov matrix has been considered. Stability can then be linked with the positive definiteness of a certain matrix of finite size n^* , which depends on systems parameters. The originality of our work is that the approximation is made on polynomial coefficients instead of discretized elements. The properties of convergence satisfied by Legendre approximation involve that our condition requires smaller orders n^* than in [11]. Based on recurrence relations satisfied by Legendre polynomials, our stability criterion can finally be implemented numerically in an easy and fast way compared to linear matrix inequalities.

New tracks of research would be to cover other classes of delay systems and infinite-dimensional systems by this methodology.

APPENDIX

A. Proof of Proposition 1

Proof: The proof is based on the relation

$$l'_k - l'_{k-2} = \frac{2(2k-1)}{h} l_{k-1}, \quad \forall k \geq 2, \quad (50)$$

satisfied by Legendre polynomials [8]. To calculate Γ_k , an integration by parts leads to

$$\begin{aligned} \Gamma_k - \Gamma_{k-2} &= \int_{-h}^0 e^{(h+\tau)M} (l_k - l_{k-2})(\tau) d\tau, \\ &= -\frac{2(2k-1)}{h} M^{-1} \int_{-h}^0 e^{(h+\tau)M} l_{k-1}(\tau) d\tau \\ &\quad + M^{-1} \left[e^{(h+\tau)M} (l_k - l_{k-2})(\tau) \right]_{-h}^0, \end{aligned}$$

and knowing that $l_k(-h) = l_{k-2}(-h) = (-1)^k$ and $l_k(0) = l_{k-2}(0) = 1$, the last term vanishes. Moreover, for $k \in \{0, 1\}$, we directly have

$$\begin{aligned} \Gamma_0 &= \int_{-h}^0 e^{(h+\tau)M} d\tau = M^{-1} (e^{hM} - I_{2m^2}), \\ \Gamma_1 &= \int_{-h}^0 e^{(h+\tau)M} \left(\frac{2\tau+h}{h} \right) d\tau, \\ &= M^{-1} (e^{hM} + I_{2m^2}) - \frac{2}{h} M^{-1} \Gamma_0. \end{aligned}$$

B. Proof of Proposition 2

Proof: The successive changes of variables $\tau'_2 = -(\tau_2 + h)$ and $\tau'_1 = -(\tau_1 + h)$ lead directly to

$$\begin{aligned} \Gamma_{jk} &= \int_{-h}^0 \left(\int_0^{-(\tau_1+h)} e^{(\tau_1+\tau'_2+h)M} l_k(-\tau'_2-h) d\tau'_2 \right) l_j(\tau_1) d\tau_1, \\ &= \int_0^{-h} \left(\int_0^{\tau'_1} e^{(\tau'_2-\tau'_1)M} l_k(-\tau'_2-h) d\tau'_2 \right) l_j(-\tau'_1-h) d\tau'_1, \\ &= (-1)^{j+k} \Gamma_{jk}^b, \end{aligned}$$

following the parity properties of Legendre polynomials. ■

C. Proof of Proposition 3

Proof: As in Appendix A, an integration by parts and (50) ensure that $\bar{\Gamma}_{jk}^+$ satisfies the recursive relation

$$\begin{aligned} & \bar{\Gamma}_{jk} - \bar{\Gamma}_{jk-2} \\ &= \int_{-h}^0 \left(\int_{-h}^{\tau_1} e^{(\tau_1-\tau_2)M} (l_k - l_{k-2})(\tau_2) d\tau_2 \right) l_j(\tau_1) d\tau_1, \\ &= \frac{2(2k-1)}{h} M^{-1} \int_{-h}^0 \left(\int_{-h}^{\tau_1} e^{(\tau_1-\tau_2)M} l_{k-1}(\tau_2) d\tau_2 \right) l_j(\tau_1) d\tau_1 \\ &\quad - M^{-1} \int_{-h}^0 (l_k - l_{k-2})(\tau_1) l_j(\tau_1) d\tau_1, \\ &= \frac{2(2k-1)}{h} M^{-1} \bar{\Gamma}_{jk-1} - \frac{h}{2j+1} M^{-1} (\delta_{jk} - \delta_{jk-2}). \end{aligned}$$

It is also important to notice that

$$\begin{aligned} \mathcal{D} &= \{(\tau_1, \tau_2); \tau_1 \in [-h, 0], \tau_2 \in [-h, \tau_1]\}, \\ &= \{(\tau_1, \tau_2); \tau_2 \in [-h, 0], \tau_1 \in [\tau_2, 0]\}, \end{aligned}$$

which means that

$$\begin{aligned} \bar{\Gamma}_{jk} &= \int_0^{-h} \int_{-h}^{\tau_1} e^{(\tau_1-\tau_2)M} l_j(\tau_1) l_k(\tau_2) d\tau_2 d\tau_1, \\ &= \int_0^{-h} \int_{\tau_2}^0 e^{(\tau_1-\tau_2)M} l_j(\tau_1) l_k(\tau_2) d\tau_1 d\tau_2, \\ &= \int_0^{-h} \int_{\tau_1}^0 e^{(\tau_2-\tau_1)M} l_k(\tau_1) l_j(\tau_2) d\tau_2 d\tau_1, \\ &= \bar{\Gamma}_{kj}^b = (-1)^{j+k} \bar{\Gamma}_{kj}. \end{aligned}$$

The initial values are given by

$$\begin{aligned} \bar{\Gamma}_{00} &= \int_{-h}^0 \left(\int_{-h}^{\tau_1} e^{(\tau_1-\tau_2)M} d\tau_2 \right) d\tau_1, \\ &= \int_{-h}^0 \left(-M^{-1} (I_{2m^2} - e^{(\tau_1+h)M}) \right) d\tau_1, \\ &= -hM^{-1} + M^{-1}\Gamma_0, \end{aligned}$$

then by

$$\begin{aligned} \bar{\Gamma}_{01} &= -\bar{\Gamma}_{10}, \\ &= - \int_{-h}^0 \left(\int_{-h}^{\tau_1} e^{(\tau_1-\tau_2)M} d\tau_2 \right) \left(\frac{2\tau_1+h}{h} \right) d\tau_1, \\ &= M^{-1} \int_{-h}^0 \left(I_{2m^2} - e^{(\tau_1+h)M} \right) \left(\frac{2\tau_1+h}{h} \right) d\tau_1, \\ &= -M^{-1}\Gamma_1. \end{aligned}$$

and finally by

$$\begin{aligned} \bar{\Gamma}_{11} &= \int_{-h}^0 \left(\int_{-h}^{\tau_1} e^{(\tau_1-\tau_2)M} \left(\frac{2\tau_2+h}{h} \right) d\tau_2 \right) \left(\frac{2\tau_1+h}{h} \right) d\tau_1, \\ &= \frac{2}{h} M^{-1} \bar{\Gamma}_{10} \\ &\quad - M^{-1} \int_{-h}^0 \left[e^{(\tau_1-\tau_2)M} \left(\frac{2\tau_2+h}{h} \right) \right]_{-h}^{\tau_1} \left(\frac{2\tau_1+h}{h} \right) d\tau_1, \\ &= \frac{2}{h} M^{-2} \Gamma_1 - \frac{h}{3} M^{-1} - M^{-1} \Gamma_1. \end{aligned}$$

■

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