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CONVEXLY INDEPENDENT SUBSETS OF MINKOWSKI SUMS OF CONVEX POLYGONS

MATEUSZ SKOMRA$^1$ AND STÉPHAN THOMASSÉ$^{2,3}$

Abstract. We show that there exist convex $n$-gons $P$ and $Q$ such that the largest convex polygon in the Minkowski sum $P + Q$ has size $\Theta(n \log n)$. This matches an upper bound of Tiwary.

1. Introduction

Let $X$ be a finite set of points in the plane. A subset $C$ of $X$ is convexly independent if $C$ forms a convex polygon. We denote by $\text{ci}(X)$ the largest size of a convexly independent subset of $X$. The celebrated Happy Ending Theorem, from Erdős and Szekeres [5], asserts that $\text{ci}(X)$ goes to infinity when $|X|$ goes to infinity and $X$ does not have three points on the same line. More precisely, the minimum value one can achieve for $\text{ci}(X)$ is logarithmic in $|X|$. To the opposite, one can try to maximize $\text{ci}(X)$ when the set $X$ satisfies some geometrical constraint. A well-studied case is when $X$ is the Minkowski sum $P + Q := \{p + q : p \in P, q \in Q\}$ of two sets of points $P$ and $Q$.

Eisenbrand, Pach, Rothvoß, and Sopher [3] proved that $\text{ci}(P + Q) = O(n^{4/3})$ when $|P| = |Q| = n$. This result was complemented by a construction of Bílka, Buchin, Fulek, Kiyomi, Okamoto, Tanigawa, and Tóth [2] showing the existence of such $P$ and $Q$ satisfying $\text{ci}(P + Q) = \Theta(n^{4/3})$. Surprisingly, the set $Q$ they use in the extremal constructions can be chosen convex. A natural question is then to ask for the maximum possible value of $\text{ci}(P + Q)$ when both $P$ and $Q$ are convex polygons. In 2014, Tiwary [9] proposed an upper bound by showing that $\text{ci}(P + Q) = O((n + m) \log(n + m))$ when $P$ and $Q$ are respectively a convex $n$-gon and a convex $m$-gon. He concluded his paper by mentioning that his upper bound seemed very generous, and left as an open problem the existence of a matching lower bound.

Our main result in this paper is that Tiwary’s proof indeed provides a sharp bound by exhibiting a matching construction.

Theorem 1.1. There exist two families $(P_k)_{k \geq 1}, (Q_k)_{k \geq 1} \subset \mathbb{R}^2$ of convexly independent sets such that $|P_k| = |Q_k| = 2^k$ and $\text{ci}(P_k + Q_k) \geq (k + 2)2^{k-1}$ for all $k \geq 1$.

An equivalent point of view of Minkowski sums is to consider $(P + Q)/2$ instead of $P + Q$, and therefore $\text{ci}(P + Q)$ represents the maximum size of a set $M$ of midpoints of $P, Q$-segments in convex position. If we actually draw

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all segments corresponding to these midpoints, we get a bipartite graph with vertex set $P \cup Q$, whose edges are all pairs $(p, q) \in P \times Q$ such that $(p+q)/2 \in M$. In our construction, the number of edges of this graph is $cn \log n$, all vertices of $P$ and $Q$ are in convex position, and all midpoints of edges are also in convex position. This kind of drawing was introduced by Halman, Onn, and Rothblum [8] where they define a strong convex embedding of a graph $G = (V, E)$ as a function $f : V \to \mathbb{R}^2$ such that $f(V)$ is convex and $\{(f(x) + f(y))/2 : (x, y) \in E\}$ is also convex. They showed that if $G$ admits a strong convex embedding, then $|E| \leq 5n - 8$ when $n \geq 3$ is the number of vertices. Recently, García-Marco and Knauer [7] reduced this bound to $2n - 3$. Equivalently their result shows that $\ci(P + P) \leq 2n - 3$ when $P$ is a convex $n$-gon. Perhaps surprisingly, our construction shows that when slightly relaxing strong convex embedding to only ask convex positions for the two partite sets of a bipartite graph, the bound goes from linear to $n \log n$.

Another very attractive reason motivating the study of $\ci(P + Q)$ for convex $n$-gons is the famous unit distance problem of Erdős and Moser [4] asking for the maximum number of pairs of points in a convex $n$-gon $P$ with distance exactly one. The trick is to observe that if two points $p, p'$ of $P$ have distance 1, then $p - p'$ lies on the unit circle. A careful counting shows, in particular, that $\ci(P + (-P))$ is an upper bound on the number of unit distance pairs. Unfortunately, our construction shows that $\ci(P + Q)$ cannot be directly used to improve the already known $O(n \log n)$ upper bound on the unit distance problem (see Füredi [6] and Aggarwal [1] for a sharper estimate). We do not know however if the graphs $G_k$ arising from our construction can be realized as convex unit distance graphs. If not, it would be interesting to find a forbidden pattern in $G_k$. This would extend the list of forbidden matrices provided in [1], as none of them appears in $G_k$.

2. Proof of the main theorem

The proof of Tiwary [9] is based on the fact that any collection of convexly independent points in $\mathbb{R}^2$ can be split into at most four monotone chains as in Figure 1. We use the name “south-east chain” for the chain that is situated in the bottom-right part of this figure. Our construction is based on this type of chains. More formally, we make the following definition and point out the subsequent lemma.

**Definition 2.1.** A south-east chain is a sequence $(a^{(1)}, \ldots, a^{(n)}) \subset \mathbb{R}^2$ of $n \geq 2$ points in the plane that satisfies the following two conditions. First, the sequence is strictly increasing on both coordinates, i.e., we have $a^{(1)}_i < a^{(2)}_i < \cdots < a^{(n)}_i$ and $a^{(1)}_i < a^{(2)}_i < \cdots < a^{(n)}_i$ (where $a^{(k)}_i$ denotes the $i$th coordinate of the point $a^{(k)}$). Second, the corresponding sequence of consecutive slopes is strictly increasing, i.e., we have

$$
\frac{a^{(2)}_2 - a^{(1)}_2}{a^{(2)}_1 - a^{(1)}_1} < \frac{a^{(k+1)}_2 - a^{(k)}_2}{a^{(k+1)}_1 - a^{(k)}_1} < \frac{a^{(n)}_2 - a^{(n-1)}_2}{a^{(n)}_1 - a^{(n-1)}_1}.
$$

We say that $n$ is the length of a south-east chain $(a^{(1)}, \ldots, a^{(n)})$. 

Figure 1. Splitting convexly independent points into monotone chains.

Figure 2. Rotation of a flat south-east chain.

Lemma 2.2. If the points $(a^{(1)}, \ldots, a^{(n)})$ form a south-east chain, then they are convexly independent, i.e., the convex hull of $\{a^{(1)}, \ldots, a^{(n)}\}$ has $n$ vertices.

Before giving a formal proof of Theorem 1.1 let us explain it intuitively. Our proof is based on some elementary properties of rotations. Let $R: \mathbb{R}^2 \to \mathbb{R}^2$ denote the counterclockwise rotation by 60 degrees centered at zero. Let $a, b, c \in \mathbb{R}^2$ be three points on a plane such that $(a, b, c)$ is a south-east chain. Moreover, suppose that this chain is sufficiently flat (more precisely, that the line defined by extending the segment $[b, c]$ forms an angle smaller than 30 degrees with the horizontal axis). Then, the images $(R(a), R(b), R(c))$ also form a south-east chain. Moreover, if all the slopes between consecutive points of $(a, b, c)$ are close to 0, then the corresponding slopes of $(R(a), R(b), R(c))$ are close to $\sqrt{3}$ (in other words, the corresponding segments form angles close to 60 degrees with the horizontal axis). Even more, under these conditions, the triple $\frac{1}{2}(a + R(a), b + R(b), c + R(c))$ also forms a south-east chain, and its slopes are close to $1/\sqrt{3}$ (which corresponds to the angle of 30 degrees). We refer to Figure 2 for an illustration. The same applies to south-east chains formed by more than three points. In
this way, given a sufficiently flat south-east chain, we can construct two new chains, one with slopes close to $\sqrt{3}$ and one with slopes close to $\frac{1}{\sqrt{3}}$.

Consider now the linear map $L_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2$, $L_\varepsilon(x, y) := (\varepsilon x, \varepsilon^2 y)$, where $\varepsilon > 0$. It is immediate to see that this map preserves south-east chains (i.e., an image of a south-east chain under $L_\varepsilon$ is again a south-east chain). Moreover, this map flattens the chains. In other words, if $A$ is a south-east chain and $\varepsilon > 0$ is small enough, then the image of $A$ under $L_\varepsilon$ is a south-east chain that is sufficiently flat to apply the previous observations. Moreover, for small $\varepsilon$, the image of $A$ under $L_\varepsilon$ is contained in a small neighborhood of $0$.

Suppose now that we are given three south-east chains $A, B, C$ such that $C$ is included in the set of points $(A + B)/2$. By applying $L_\varepsilon$ to all three chains, we can suppose that they are arbitrarily flat, and contained in a small neighborhood of $0$. Then, we can apply the rotation $R$ to $A, B, C$. In this way, we obtain three chains $A', B', C'$ that are again contained in a neighborhood of $0$, but whose slopes are close to $\sqrt{3}$. We now translate the chains as follows. We do not apply any translation to $A$, we translate $B$ by the vector $(0, 2)$, and $C$ by the vector $(0, 1)$. Then, we translate $A'$ by the vector $(1, \frac{5}{2})$, $B'$ by the vector $(1, 1)$, and $C'$ by the vector $(1, \frac{7}{4})$. This gives the situation depicted in Figure 3. In this picture, the chain $A$ is contained in the small neighborhood of the point marked by $A$, and the slopes of this chain are close to the slope of the solid line passing through $A$. The same is true for the chains $B, C, A', B', C'$. In particular, for sufficiently
small ε, the concatenation \( A := (A, B') \) of chains \( A \) and \( B' \) forms a south-east chain (because the dashed line from \( A \) to \( B' \) has slope greater than the slope of the solid line passing through \( A \) but smaller than the slope of the solid line passing through \( B' \)). By the same reasoning, the concatenation \( B := (B, A') \) forms a south-east chain. Denote \( A = (a^{(1)}, \ldots, a^{(n)}) \) and \( A' = (a'^{(1)}, \ldots, a'^{(n)}) \). By the observation about rotation made above, the sequence \( D := (\frac{a^{(1)} + a'^{(1)}}{2}, \ldots, \frac{a^{(n)} + a'^{(n)}}{2}) \) is a south-east chain, and all the slopes of this chain are close to \( \sqrt{3} \). Moreover, this chain is contained in a small neighborhood of the point \((1/2, 5/4)\). We marked this chain in Figure 3 using the same conventions as for the remaining chains. By applying the same reasoning as above, the concatenation \( C := (C, D, C') \) is a south-east chain. To summarize, our construction shows the following statement. Given three south-east chains \( A, B, C \) such that \( C \) is included in the set of points \((A + B)/2\), we can construct three south-east chains \( A, B, C \) such that \( C \) is included in \((A + B)/2\) and \(|A| = |B| = |A| + |B|\), \(|C| = 2|C| + |A|\). Thus, if we suppose that \(|A| = |B| = n\), then we have \(|A| = |B| = 2n\) and \(|C| = 2|C| + n\). By iterating this reasoning, we obtain the claimed bound \( O(n \log n)\).

Before presenting a formal proof, let us discuss the types of graph drawings that we obtain in this way. Here, we are interested in a drawing \( f : (U \uplus V) \rightarrow \mathbb{R}^2 \) of a bipartite graph \( G = (U \uplus V, E) \) such that \( f(U) \) is a south-east chain, \( f(V) \) is a south-east chain, and the midpoints \( \{f(u) + f(v))/2\} : (u, v) \in E \) also form a south-east chain. The construction described above implies that if \( G \) is drawable in this way and \( G' := (U \uplus V', E') \) is a copy of \( G \), then the graph \( G := (U \uplus V, E) \) defined as \( U := U \uplus V', \quad V := V \uplus U', \quad E := E \uplus E' \uplus \{(u, u') : u \in U\} \) is also drawable in this fashion. By starting from a graph \( G_1 = (\{u_1, u_2\} \uplus \{v_1, v_2\}, \{(u_1, v_1), (u_2, v_1), (u_2, v_2)\}) \) and iterating the procedure, we obtain a family \( (G_k)_{k \geq 1} \) of drawable graphs, each having \( 2k^2 + 1 \) vertices and \((k + 2)2^{k-1}\) edges. Figure 4 depicts this family of graphs for \( k \leq 3 \) and Figure 5 depicts a drawing of \( G_3 \) using south-east chains.

In the remaining part of this section, we give a formal proof of the argument described above. Let \( R : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) denote the counterclockwise rotation by 60 degrees centered at zero, i.e., the linear transformation given.
by the matrix
\[
R := \frac{1}{2} \begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}.
\]
Furthermore, if \( a := (a_1, a_2), b := (b_1, b_2) \in \mathbb{R}^2 \) are two points such that \( a_1 < b_1 \) and \( a_2 < b_2 \), then we denote by
\[
\text{sl}(a, b) := \frac{b_2 - a_2}{b_1 - a_1}
\]
the corresponding slope of the segment \([a, b] \). The following lemma, which can be proven using elementary trigonometric identities, gathers the properties of rotation that were mentioned above.

**Lemma 2.3.** Suppose that two points \( a := (a_1, a_2), b := (b_1, b_2) \in \mathbb{R}^2 \) are such that \( a_1 < b_1, a_2 < b_2, \) and \( \text{sl}(a, b) < \frac{1}{\sqrt{3}} = \tan\left(\frac{\pi}{6}\right) \). Let \( \theta := \arctan(\text{sl}(a, b)) < \frac{\pi}{6} \) and denote \( \tilde{a} = R(a), \tilde{b} = R(b) \). Then, we have \( \tilde{a}_1 < \tilde{b}_1, \) \( \tilde{a}_2 < \tilde{b}_2, \)
\[
\text{sl}(\tilde{a}, \tilde{b}) = \tan\left(\frac{\pi}{3} + \theta\right), \quad \text{and} \quad \text{sl}\left(\frac{\tilde{a} + \tilde{b}}{2}, \frac{\tilde{a} + \tilde{b}}{2}\right) = \tan\left(\frac{\pi}{6} + \theta\right).
\]

**Proof.** We have \( \tilde{a} = \frac{1}{2}(a_1 - \sqrt{3}a_2, \sqrt{3}a_1 + a_2) \) and \( \tilde{b} = \frac{1}{2}(b_1 - \sqrt{3}b_2, \sqrt{3}b_1 + b_2) \).

The inequality \( \tilde{a}_2 < \tilde{b}_2 \) is trivial. Moreover, \( \tilde{b}_1 > \tilde{a}_1 \iff b_1 - a_1 > \sqrt{3}(b_2 - a_2) \), which is true by our assumptions. Furthermore, we have
\[
\tan\left(\frac{\pi}{3} + \theta\right) = \frac{\tan\left(\frac{\pi}{3}\right) + \tan(\theta)}{1 - \tan\left(\frac{\pi}{3}\right) \tan(\theta)} = \frac{\sqrt{3}(b_1 - a_1) + b_2 - a_2}{b_1 - a_1 - \sqrt{3}(b_2 - a_2)} = \frac{\tilde{b}_2 - \tilde{a}_2}{\tilde{b}_1 - \tilde{a}_1} = \text{sl}(\tilde{a}, \tilde{b}).
\]
Similarly,
\[
\tan(\frac{\pi}{6} + \theta) = \frac{\tan(\frac{\pi}{6}) + \tan(\theta)}{1 - \tan(\frac{\pi}{6}) \tan(\theta)} = \frac{b_1 - a_1 + \sqrt{3}(b_2 - a_2)}{\sqrt{3}(b_1 - a_1) - (b_2 - a_2)}
\]
\[
= \frac{b_2 + \tilde{b}_2 - a_2 - \tilde{a}_2}{b_1 + b_1 - a_1 - a_1} = \text{sl}\left(\frac{a + \tilde{a}}{2}, \frac{b + \tilde{b}}{2}\right).
\]
\[\square\]

For any \(\varepsilon > 0\) we denote by \(L_\varepsilon : \mathbb{R}^2 \rightarrow \mathbb{R}^2\) the linear transformation \(L_\varepsilon(x, y) := (\varepsilon x, \varepsilon^2 y)\). As a corollary of Lemma 2.3 we may now prove the properties of the three transformations of south-east chains discussed above.

To improve readability, we use the following notation: if \(A\) is a sequence, then we denote by \(A(k)\) its \(k\)th element, so that \(A = (A(1), \ldots, A(n))\). If \(A\) is a south-east chain of length \(n\) and \(\varepsilon > 0\), then we consider the following three sequences:

\[
A_\varepsilon := \left(\left.L_\varepsilon(A(1)), \ldots, L_\varepsilon(A(n))\right\}\right),
\]
\[
(1)
\]
\[
A'_\varepsilon := \left(\left.R(A_\varepsilon(1)), \ldots, R(A_\varepsilon(n))\right\}\right),
\]
\[
A''_\varepsilon := \left(\frac{A_\varepsilon(1) + A'_\varepsilon(1)}{2}, \frac{A_\varepsilon(n) + A'_\varepsilon(n)}{2}\right).
\]

Using this notation, \(A_\varepsilon\) is a chain obtained by flattening \(A\), \(A'_\varepsilon\) is the rotated version of this flattened chain, and \(A''_\varepsilon\) is the chain formed by taking the midpoints of the two previous chains. The next result follows from Lemma 2.3.

**Lemma 2.4.** Suppose that \(A := (A(1), \ldots, A(n))\) is a south-east chain. Then, for sufficiently small \(\varepsilon > 0\), the sequences \(A_\varepsilon, A'_\varepsilon, A''_\varepsilon\) are south-east chains. Moreover, for every \(k \in [n - 1]\) we have the equalities

\[
\lim_{\varepsilon \to 0^+} \text{sl}(A_\varepsilon(k), A_\varepsilon(k + 1)) = 0,
\]
\[
\lim_{\varepsilon \to 0^+} \text{sl}(A'_\varepsilon(k), A'_\varepsilon(k + 1)) = \tan\left(\frac{\pi}{3}\right) = \sqrt{3},
\]
\[
\lim_{\varepsilon \to 0^+} \text{sl}(A''_\varepsilon(k), A''_\varepsilon(k + 1)) = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}.
\]

**Proof.** It is obvious that the sequence \(A_\varepsilon\) is strictly increasing on both coordinates. Moreover, for every \(k \in [n - 1]\) we have \(\text{sl}(A_\varepsilon(k), A_\varepsilon(k + 1)) = \varepsilon \text{sl}(A(k), A(k + 1))\). Hence, \(A_\varepsilon\) is a south-east chain and we have \(\lim_{\varepsilon \to 0^+} \text{sl}(A_\varepsilon(k), A_\varepsilon(k + 1)) = 0\). To prove the claim for the remaining two sequences, note that for sufficiently small \(\varepsilon > 0\), the inequality \(\text{sl}(A_\varepsilon(k), A_\varepsilon(k + 1)) = \varepsilon \text{sl}(A(k), A(k + 1)) < 1/\sqrt{3}\) is satisfied for all \(k \in [n - 1]\).

Hence, by Lemma 2.3, the sequence \(A'_\varepsilon = \left(R(A_\varepsilon(1)), \ldots, R(A_\varepsilon(n))\right)\) is strictly increasing on both coordinates and the same is true for the sequence \(A''_\varepsilon = \left(\frac{A_\varepsilon(1) + A'_\varepsilon(1)}{2}, \ldots, \frac{A_\varepsilon(n) + A'_\varepsilon(n)}{2}\right)\). Moreover, if we denote \(\theta^{(k, \varepsilon)} := \arctan\left(\text{sl}(A_\varepsilon(k), A_\varepsilon(k + 1))\right) < \frac{\pi}{6}\), then the sequence \(\theta^{(1, \varepsilon)}, \ldots, \theta^{(n - 1, \varepsilon)}\) is strictly increasing and Lemma 2.3 shows that

\[
\text{sl}(A'_\varepsilon(k), A'_\varepsilon(k + 1)) = \tan\left(\frac{\pi}{3} + \theta^{(k, \varepsilon)}\right).
\]
and
\[ \text{sl}(A'_\varepsilon(k), A''_\varepsilon(k+1)) = \tan(\frac{\pi}{6} + \theta(k,\varepsilon)). \]

In particular, the sequences $A'_\varepsilon, A''_\varepsilon$ are south-east chains. Moreover, we have $\lim_{\varepsilon \to 0^+} \theta(k,\varepsilon) = 0$ for all $k$, which gives the equalities
\[ \lim_{\varepsilon \to 0^+} \text{sl}(A'_\varepsilon(k), A'_\varepsilon(k+1)) = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}, \]
\[ \lim_{\varepsilon \to 0^+} \text{sl}(A''_\varepsilon(k), A''_\varepsilon(k+1)) = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}. \]

We now show how the transformations given in Lemma 2.4 can be used to prove Theorem 1. To do so, we take three south-east chains $A \subset (A(1), \ldots, A(n))$, $B \subset (B(1), \ldots, B(n))$, and $C \subset (C(1), \ldots, C(m))$ such that $C \subset (A + B)/2$. As discussed before, we let $u := (1,5/2)$, $v := (0,2)$, $w := (1,1)$ and we consider the chains $A_\varepsilon, A'_\varepsilon, B_\varepsilon, B'_\varepsilon$ defined as in (1).

**Lemma 2.5.** If $\varepsilon > 0$ is sufficiently small, then the sequences $\overline{A}_\varepsilon := (A_\varepsilon, w + B'_\varepsilon)$, $\overline{B}_\varepsilon := (v + B_\varepsilon, u + A'_\varepsilon)$ are south-east chains. Moreover, the set $(\overline{A}_\varepsilon + \overline{B}_\varepsilon)/2$ contains a south-east chain of length at least $2m+n$.

In the statement above, $w + B'_\varepsilon$ denotes the sequence obtained by translating every element of $B'_\varepsilon$ by the vector $w$ and $(A_\varepsilon, w + B'_\varepsilon)$ denotes the concatenation of two sequences. The same applies to $(v + B_\varepsilon, u + A'_\varepsilon)$.

**Proof of Lemma 2.5.** We start by proving that $\overline{A}_\varepsilon$ is a south-east chain. By Lemma 2.4, the sequences $A_\varepsilon$ and $B'_\varepsilon$ are south-east chains for sufficiently small $\varepsilon$. Hence, the sequence $w + B'_\varepsilon$ is also a south-east chain. Furthermore, we have $\lim_{\varepsilon \to 0^+} (A_\varepsilon(n)) = (0,0)$ and $\lim_{\varepsilon \to 0^+} (w + B'_\varepsilon(1)) = w = (1,1)$. In particular, for sufficiently small $\varepsilon$, the sequence $\overline{A}_\varepsilon$ is strictly increasing on both coordinates. Moreover, Lemma 2.4 shows the equalities $\lim_{\varepsilon \to 0^+} \text{sl}(A_\varepsilon(n-1), A_\varepsilon(n)) = 0$ and $\lim_{\varepsilon \to 0^+} \text{sl}(w + B'_\varepsilon(1), w + B'_\varepsilon(2)) = \sqrt{3}$.

We also have
\[ \lim_{\varepsilon \to 0^+} \text{sl}(A_\varepsilon(n), w + B'_\varepsilon(1)) = \text{sl}(0, w) = 1. \]

Since $0 < 1 < \sqrt{3}$, the sequence $\overline{A}_\varepsilon$ is a south-east chain for sufficiently small $\varepsilon$. The proof for $\overline{B}_\varepsilon$ is analogous—it is enough to observe that $\lim_{\varepsilon \to 0^+} (v + B_\varepsilon(n)) = (0,2)$ and $\lim_{\varepsilon \to 0^+} (u + A'_\varepsilon(1)) = (1,5/2)$ to show that $\overline{B}_\varepsilon$ is strictly increasing on both coordinates. Then, $\overline{B}_\varepsilon$ is a south-east chain by the equalities $\lim_{\varepsilon \to 0^+} \text{sl}(v + B_\varepsilon(n-1), v + B_\varepsilon(n)) = 0$, $\lim_{\varepsilon \to 0^+} \text{sl}(u + A'_\varepsilon(1), u + A'_\varepsilon(2)) = \sqrt{3}$, and
\[ \lim_{\varepsilon \to 0^+} \text{sl}(v + B_\varepsilon(n), u + A'_\varepsilon(1)) = \text{sl}(v, u) = \frac{1}{2}. \]

It remains to show that $(\overline{A}_\varepsilon + \overline{B}_\varepsilon)/2$ contains a south-east chain of length at least $2m+n$. To do so, consider the chain $C \subset (A + B)/2$, $|C| = m$, and let $C_\varepsilon, C'_\varepsilon$ be defined as in (1). Furthermore, let $t := (0,1) = v/2$ and $z := (1,7/4) = (u + w)/2$. Since the transformations $L_\varepsilon$ and $R$ are linear, we have $t + C_\varepsilon \subset (A_\varepsilon + v + B_\varepsilon)/2 \subset (\overline{A}_\varepsilon + \overline{B}_\varepsilon)/2$ and $z + C'_\varepsilon \subset (u + A'_\varepsilon + w + B'_\varepsilon)/2 \subset (\overline{A}_\varepsilon + \overline{B}_\varepsilon)/2$. Moreover, we define the chain $A''_\varepsilon$ as in (1) we let $s := (1/2,5/4) = u/2$, and we note that $s + A''_\varepsilon \subset (A_\varepsilon + u +
$A'_ε/2 \subset (\bar{A}_ε + \bar{B}_ε)/2$. Hence, the sequence $D_ε := (t + C_ε, s + A'_ε, z + C'_ε)$ is contained in $(\bar{A}_ε + \bar{B}_ε)/2$ and its length is equal to $2m + n$. Therefore, it is enough to prove that $D_ε$ is a south-east chain. The proof is similar to the proofs before. By Lemma 2.4, the sequences $t + C_ε, s + A'_ε,$ and $z + C'_ε$ are south-east chains for sufficiently small $ε$. Moreover, we have the equalities $\lim_{ε\to 0^+} (t + C_ε(m)) = (0, 1), \lim_{ε\to 0^+} (s + A'_ε(1)) = \lim_{ε\to 0^+} (s + A'_ε(n)) = (1/2, 5/4), \lim_{ε\to 0^+} (z + C'_ε(1)) = (1, 7/4)$, which show that $D_ε$ is strictly increasing on both coordinates for sufficiently small $ε$. Furthermore, we use Lemma 2.4 to observe that $\lim_{ε\to 0^+} \text{sl}(t + C_ε(m), t + C_ε(m)) = 0$, $\lim_{ε\to 0^+} \text{sl}(s + A'_ε(1), s + A'_ε(2)) = \lim_{ε\to 0^+} \text{sl}(s + A'_ε(n-1), s + A'_ε(n)) = \frac{1}{\sqrt{3}}$, $\lim_{ε\to 0^+} \text{sl}(z + C'_ε(1), z + C'_ε(2)) = \sqrt{3}$. To finish, we have

$$\lim_{ε\to 0^+} \text{sl}(t + C_ε(m), s + A'_ε(1)) = \text{sl}(t, s) = \frac{1}{2},$$

$$\lim_{ε\to 0^+} \text{sl}(s + A'_ε(n), z + C'_ε(1)) = \text{sl}(s, z) = 1,$$

and $0 < \frac{1}{2} < \frac{1}{\sqrt{3}} < 1 < \sqrt{3}$. Hence, for sufficiently small $ε$, the sequence $D_ε$ is a south-east chain.

As a corollary, we may prove our main theorem.

**Proof of Theorem 1.1.** As noted in the introduction, it is enough to prove the claim for $(P_k + Q_k)/2$ instead of $P_k + Q_k$. We show, by induction over $k$, that we can find two south-east chains $P_k, Q_k$, each of length $2^k$, such that $(P_k + Q_k)/2$ contains a south-east chain of length $(k + 2)2^{k-1}$. Let $P_1 := ((0, 0), (2, 1))$ and $Q_1 := ((0, 2), (2, 4))$. The sequences $P_1$ and $Q_1$ are south-east chains and the set $(P_1 + Q_1)/2$ contains a south-east chain of length three. Moreover, given $P_k, Q_k$ we may apply Lemma 2.5 to obtain two south-east chains $P_{k+1}, Q_{k+1}$, each of length $2^{k+1}$, such that $(P_{k+1} + Q_{k+1})/2$ contains a south-east chain of length at least $(k + 2)2^k + 2^k = (k + 3)2^k$. Therefore, the claim follows from Lemma 2.2.

**3. Final remarks**

Let us discuss some natural questions for further research. Firstly, we do not know how far from optimal is our construction. More precisely, consider the function $f: \mathbb{N}^* \to \mathbb{N}^*$ defined as

$$f(n) := \max\{\text{ci}(P + Q) : P, Q \subset \mathbb{R}^2 \text{ are convexly independent and } |P| = |Q| = n\}.$$ 

By joining our analysis with the result of Tiwary [9], we obtain the asymptotic bound $f(n) = \Theta(n \log n)$. One can ask for an optimal constant $C$ such that $f(n) \leq C_n \log n$. Secondly, we wonder if our family of graphs $G_k = (U_k \cup V_k, E_k)$ can be still embedded in the plane if we impose a stronger condition on the midpoints of the edges. For instance, we may ask for an embedding such that $U_k$ and $V_k$ are south-east chains and the midpoints of $E_k$ are contained in some convex curve of small degree, such as a circle or a parabola. Our interest in this question is motivated by the following observation: if $G_k$ are realizable in the south-east quadrant of the plane in such a way that the midpoints of $E_k$ are on the unit circle, then we obtain a
\( \Theta(n \log n) \) bound for the convex version of the unit distance problem, proposed by Erdős and Moser \cite{erdos-moser-1959} (see \cite{furedi-1990} for more information on lower and upper bounds for this variant of the unit distance problem). Indeed, suppose that \( h_k : (U_k \uplus V_k) \to \mathbb{R}^2 \) is an embedding of \( G_k \) such that \( h_k(U_k) \) and \( h_k(V_k) \) are south-east chains, \( h_k(U_k), h_k(V_k) \subset \{ x \in \mathbb{R}^2 : x_1 \geq 0, x_2 \leq 0 \} \), and \( \| h_k(u) + h_k(v) \|_2 = 1 \) for all \( (u, v) \in E_k \). Then, the points \( -h_k(U_k) \) form a monotone concave chain in the north-west quadrant of the plane and \( h_k(V_k) \) form a monotone convex chain in the south-east quadrant of the plane, which implies that the collection \( \{ -h_k(U_k), h_k(V_k) \} \) is convexly independent. Therefore, the polygon \( P_k := \mathrm{conv}\{ -h_k(U_k)/2, h_k(V_k)/2 \} \) has \( 2^{k+1} \) vertices and \( \Theta(k2^k) \) diagonals of length one. As noted in the introduction, the existence of such an embedding cannot be excluded using the current list of forbidden matrices \cite{furedi-1990}. We were neither able to construct these embeddings nor find a new forbidden pattern that occurs in \( G_k \).

**References**


