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CONVEXLY INDEPENDENT SUBSETS OF MINKOWSKI SUMS OF CONVEX POLYGONS

MATEUSZ SKOMRA\textsuperscript{1} AND STÉPHAN THOMASSÉ\textsuperscript{2,3}

Abstract. We show that there exist convex \( n \)-gons \( P \) and \( Q \) such that the largest convex polygon in the Minkowski sum \( P + Q \) has size \( \Theta(n \log n) \). This matches an upper bound of Tiwary.

1. Introduction

Let \( X \) be a finite set of points in the plane. A subset \( C \) of \( X \) is \textit{convexly independent} if \( C \) forms a convex polygon. We denote by \( \text{ci}(X) \) the largest size of a convexly independent subset of \( X \). The celebrated Happy Ending Theorem, from Erdős and Szekeres \cite{es}, asserts that \( \text{ci}(X) \) goes to infinity when \( |X| \) goes to infinity and \( X \) does not have three points on the same line. More precisely, the minimum value one can achieve for \( \text{ci}(X) \) is logarithmic in \( |X| \). To the opposite, one can try to maximize \( \text{ci}(X) \) when the set \( X \) satisfies some geometrical constraint. A well-studied case is when \( X \) is the Minkowski sum \( P + Q := \{ p + q : p \in P, q \in Q \} \) of two sets of points \( P \) and \( Q \).

Eisenbrand, Pach, Rothvoß, and Sopher \cite{epr} proved that \( \text{ci}(P + Q) = O(n^{4/3}) \) when \( |P| = |Q| = n \). This result was complemented by a construction of Bílka, Buchin, Fulek, Kiyomi, Okamoto, Tanigawa, and Tóth \cite{bfbkot} showing the existence of such \( P \) and \( Q \) satisfying \( \text{ci}(P + Q) = \Theta(n^{4/3}) \). Surprisingly, the set \( Q \) they use in the extremal constructions can be chosen convex. A natural question is then to ask for the maximum possible value of \( \text{ci}(P + Q) \) when both \( P \) and \( Q \) are convex polygons. In 2014, Tiwary \cite{ti} proposed an upper bound by showing that \( \text{ci}(P + Q) = O((n + m) \log(n + m)) \) when \( P \) and \( Q \) are respectively a convex \( n \)-gon and a convex \( m \)-gon. He concluded his paper by mentioning that his upper bound seemed very generous, and left as an open problem the existence of a matching lower bound. Our main result in this paper is that Tiwary’s proof indeed provides a sharp bound by exhibiting a matching construction.

\textbf{Theorem 1.1.} There exist two families \( (P_k)_{k \geq 1}, (Q_k)_{k \geq 1} \subset \mathbb{R}^2 \) of convexly independent sets such that \( |P_k| = |Q_k| = 2^k \) and \( \text{ci}(P_k + Q_k) \geq (k + 2)2^{k-1} \) for all \( k \geq 1 \).

An equivalent point of view of Minkowski sums is to consider \( (P + Q)/2 \) instead of \( P + Q \), and therefore \( \text{ci}(P + Q) \) represents the maximum size of a set \( M \) of midpoints of \( P, Q \)-segments in convex position. If we actually draw

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all segments corresponding to these midpoints, we get a bipartite graph with vertex set $P \cup Q$, whose edges are all pairs $(p, q) \in P \times Q$ such that $(p+q)/2 \in M$. In our construction, the number of edges of this graph is $cn \log n$, all vertices of $P$ and $Q$ are in convex position, and all midpoints of edges are also in convex position. This kind of drawing was introduced by Halman, Om, and Rothblum [8] where they define a strong convex embedding of a graph $G = (V, E)$ as a function $f: V \to \mathbb{R}^2$ such that $f(V)$ is convex and $\{(f(x) + f(y))/2: (x, y) \in E\}$ is also convex. They showed that if $G$ admits a strong convex embedding, then $|E| \leq 5n - 8$ when $n \geq 3$ is the number of vertices. Recently, García-Marco and Knauer [7] reduced this bound to $2n - 3$. Equivalently their result shows that $ci(P + P) \leq 2n - 3$ when $P$ is a convex $n$-gon. Perhaps surprisingly, our construction shows that when slightly relaxing strong convex embedding to only ask convex positions for the two partite sets of a bipartite graph, the bound goes from linear to $n \log n$.

Another very attractive reason motivating the study of $ci(P + Q)$ for convex $n$-gons is the famous unit distance problem of Erdős and Moser [4] asking for the maximum number of pairs of points in a convex $n$-gon $P$ with distance exactly one. The trick is to observe that if two points $p, p'$ of $P$ have distance 1, then $p - p'$ lies on the unit circle. A careful counting shows, in particular, that $ci(P + (-P))$ is an upper bound on the number of unit distance pairs. Unfortunately, our construction shows that $ci(P + Q)$ cannot be directly used to improve the already known $O(n \log n)$ upper bound on the unit distance problem (see Füredi [6] and Aggarwal [1] for a sharper estimate). We do not know however if the graphs $G_k$ arising from our construction can be realized as convex unit distance graphs. If not, it would be interesting to find a forbidden pattern in $G_k$. This would extend the list of forbidden matrices provided in [1], as none of them appears in $G_k$.

2. Proof of the main theorem

The proof of Tiwary [9] is based on the fact that any collection of convexly independent points in $\mathbb{R}^2$ can be split into at most four monotone chains as in Figure 1. We use the name “south-east chain” for the chain that is situated in the bottom-right part of this figure. Our construction is based on this type of chains. More formally, we make the following definition and point out the subsequent lemma.

Definition 2.1. A south-east chain is a sequence $(a^{(1)}, \ldots, a^{(n)}) \subset \mathbb{R}^2$ of $n \geq 2$ points in the plane that satisfies the following two conditions. First, the sequence is strictly increasing on both coordinates, i.e., we have $a_1^{(1)} < a_2^{(2)} < \cdots < a_n^{(n)}$ and $a_1^{(1)} < a_2^{(2)} < \cdots < a_2^{(n)}$ (where $a_i^{(k)}$ denotes the $i$th coordinate of the point $a^{(k)}$). Second, the corresponding sequence of consecutive slopes is strictly increasing, i.e., we have

$$\frac{a_2^{(2)} - a_1^{(1)}}{a_2^{(2)} - a_1^{(1)}} < \cdots < \frac{a_2^{(k+1)} - a_2^{(k)}}{a_1^{(k+1)} - a_1^{(k)}} < \cdots < \frac{a_2^{(n)} - a_2^{(n-1)}}{a_1^{(n)} - a_1^{(n-1)}}.$$

We say that $n$ is the length of a south-east chain $(a^{(1)}, \ldots, a^{(n)})$. 
Lemma 2.2. If the points \((a^{(1)}, \ldots, a^{(n)})\) form a south-east chain, then they are convexly independent, i.e., the convex hull of \(\{a^{(1)}, \ldots, a^{(n)}\}\) has \(n\) vertices.

Before giving a formal proof of Theorem 1.1 let us explain it intuitively. Our proof is based on some elementary properties of rotations. Let \(R: \mathbb{R}^2 \to \mathbb{R}^2\) denote the counterclockwise rotation by 60 degrees centered at zero. Let \(a, b, c \in \mathbb{R}^2\) be three points on a plane such that \((a, b, c)\) is a south-east chain. Moreover, suppose that this chain is sufficiently flat (more precisely, that the line defined by extending the segment \([b, c]\) forms an angle smaller than 30 degrees with the horizontal axis). Then, the images \((R(a), R(b), R(c))\) also form a south-east chain. Moreover, if all the slopes between consecutive points of \((a, b, c)\) are close to 0, then the corresponding slopes of \((R(a), R(b), R(c))\) are close to \(\sqrt{3}\) (in other words, the corresponding segments form angles close to 60 degrees with the horizontal axis). Even more, under these conditions, the triple \(\frac{1}{2}(a + R(a), b + R(b), c + R(c))\) also forms a south-east chain, and its slopes are close to \(\frac{1}{\sqrt{3}}\) (which corresponds to the angle of 30 degrees). We refer to Figure 2 for an illustration. The same applies to south-east chains formed by more than three points. In
this way, given a sufficiently flat south-east chain, we can construct two new chains, one with slopes close to $\sqrt{3}$ and one with slopes close to $1/\sqrt{3}$.

Consider now the linear map $L_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2$, $L_\varepsilon(x, y) := (\varepsilon x, \varepsilon^2 y)$, where $\varepsilon > 0$. It is immediate to see that this map preserves south-east chains (i.e., an image of a south-east chain under $L_\varepsilon$ is again a south-east chain). Moreover, this map flattens the chains. In other words, if $A$ is a south-east chain and $\varepsilon > 0$ is small enough, then the image of $A$ under $L_\varepsilon$ is a south-east chain that is sufficiently flat to apply the previous observations. Moreover, for small $\varepsilon$, the image of $A$ under $L_\varepsilon$ is contained in a small neighborhood of 0.

Suppose now that we are given three south-east chains $A, B, C$ such that $C$ is included in the set of points $(A + B)/2$. By applying $L_\varepsilon$ to all three chains, we can suppose that they are arbitrarily flat, and contained in a small neighborhood of 0. Then, we can apply the rotation $R$ to $A, B, C$. In this way, we obtain three chains $A', B', C'$ that are again contained in a neighborhood of 0, but whose slopes are close to $\sqrt{3}$. We now translate the chains as follows. We do not apply any translation to $A$, we translate $B$ by the vector $(0, 2)$, and $C$ by the vector $(0, 1)$. Then, we translate $A'$ by the vector $(1, 5/2)$, $B'$ by the vector $(1, 1)$, and $C'$ by the vector $(1, 7/4)$. This gives the situation depicted in Figure 3. In this picture, the chain $A$ is contained in the small neighborhood of the point marked by $A$, and the slopes of this chain are close to the slope of the solid line passing through $A$. The same is true for the chains $B, C, A', B', C'$. In particular, for sufficiently
small \( \varepsilon \), the concatenation \( \mathcal{A} := (A, B') \) of chains \( A \) and \( B' \) forms a south-east chain (because the dashed line from \( A \) to \( B' \) has slope greater than the slope of the solid line passing through \( A \) but smaller than the slope of the solid line passing through \( B' \)). By the same reasoning, the concatenation \( \mathcal{B} := (B, A') \) forms a south-east chain. Denote \( A = (a^{(1)}, \ldots, a^{(n)}) \) and \( A' = (a'^{(1)}, \ldots, a'^{(n)}) \). By the observation about rotation made above, the sequence \( D := \left( \frac{a^{(1)} + a'^{(1)}}{2}, \ldots, \frac{a^{(n)} + a'^{(n)}}{2} \right) \) is a south-east chain, and all the slopes of this chain are close to \( 1/\sqrt{3} \). Moreover, this chain is contained in a small neighborhood of the point \( (1/2, 5/4) \). We marked this chain in Figure 3 using the same conventions as for the remaining chains. By applying the same reasoning as above, the concatenation \( \mathcal{C} := (C, D, C') \) is a south-east chain. To summarize, our construction shows the following statement. Given three south-east chains \( A, B, C \) such that \( C \) is included in the set of points \((A + B)/2\), we can construct three south-east chains \( \mathcal{A}, \mathcal{B}, \mathcal{C} \) such that \( \mathcal{C} \) is included in \((\mathcal{A} + \mathcal{B})/2\) and \(|\mathcal{A}| = |\mathcal{B}| = |A| + |B|\), \(|\mathcal{C}| = 2|C| + |A|\). Thus, if we suppose that \(|A| = |B| = n\), then we have \(|\mathcal{A}| = |\mathcal{B}| = 2n\) and \(|\mathcal{C}| = 2|C| + n\). By iterating this reasoning, we obtain the claimed bound \( \Theta(n \log n) \).

Before presenting a formal proof, let us discuss the types of graph drawings that we obtain in this way. Here, we are interested in a drawing \( f: (U \uplus V) \to \mathbb{R}^2 \) of a bipartite graph \( G = (U \uplus V, E) \) such that \( f(U) \) is a south-east chain, \( f(V) \) is a south-east chain, and the midpoints \( \{f(u) + f(v))/2: (u, v) \in E\} \) also form a south-east chain. The construction described above implies that if \( G \) is drawable in this way and \( G' := (U' \uplus V', E') \) is a copy of \( G \), then the graph \( \mathcal{G} := (U \uplus V, E) \) defined as \( U := U \uplus V', V := V \uplus U' \), and

\[
E := E \uplus E' \uplus \{(u, u'): u \in U\}
\]

is also drawable in this fashion. By starting from a graph

\[
G_1 = (\{u_1, u_2\} \uplus \{v_1, v_2\}, \{(u_1, v_1), (u_2, v_1), (u_2, v_2)\})
\]

and iterating the procedure, we obtain a family \( \{G_k\}_{k \geq 1} \) of drawable graphs, each having \( 2k+1 \) vertices and \( (k+2)2^{k-1} \) edges. Figure 4 depicts this family of graphs for \( k \leq 3 \) and Figure 5 depicts a drawing of \( G_3 \) using south-east chains.

In the remaining part of this section, we give a formal proof of the argument described above. Let \( R: \mathbb{R}^2 \to \mathbb{R}^2 \) denote the counterclockwise rotation by 60 degrees centered at zero, i.e., the linear transformation given
by the matrix
\[ R := \frac{1}{2} \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}. \]

Furthermore, if \( a := (a_1, a_2), b := (b_1, b_2) \in \mathbb{R}^2 \) are two points such that \( a_1 < b_1 \) and \( a_2 < b_2 \), then we denote by
\[ \text{sl}(a, b) := \frac{b_2 - a_2}{b_1 - a_1} \]
the corresponding slope of the segment \([a, b]\). The following lemma, which can be proven using elementary trigonometric identities, gathers the properties of rotation that were mentioned above.

**Lemma 2.3.** Suppose that two points \( a := (a_1, a_2), b := (b_1, b_2) \in \mathbb{R}^2 \) are such that \( a_1 < b_1, a_2 < b_2, \) and \( \text{sl}(a, b) < \frac{1}{\sqrt{3}} = \tan\left(\frac{\pi}{6}\right) \). Let \( \theta := \arctan(\text{sl}(a, b)) < \frac{\pi}{6} \) and denote \( \ddot{a} = R(a), \ddot{b} = R(b) \). Then, we have \( \ddot{a}_1 < \ddot{b}_1, \ddot{a}_2 < \ddot{b}_2 \),
\[ \text{sl}(\ddot{a}, \ddot{b}) = \tan\left(\frac{\pi}{3} + \theta\right), \quad \text{and} \quad \text{sl}\left(\frac{a + \ddot{a}}{2}, \frac{b + \ddot{b}}{2}\right) = \tan\left(\frac{\pi}{6} + \theta\right). \]

**Proof.** We have \( \ddot{a} = \frac{1}{\sqrt{3}}(a_1 - \sqrt{3}a_2, \sqrt{3}a_1 + a_2) \) and \( \ddot{b} = \frac{1}{\sqrt{3}}(b_1 - \sqrt{3}b_2, \sqrt{3}b_1 + b_2) \).
The inequality \( \ddot{a}_2 < \ddot{b}_2 \) is trivial. Moreover, \( \ddot{b}_1 > \ddot{a}_1 \iff b_1 - a_1 > \sqrt{3}(b_2 - a_2), \) which is true by our assumptions. Furthermore, we have
\[
\tan\left(\frac{\pi}{3} + \theta\right) = \frac{\tan\left(\frac{\pi}{3}\right) + \tan(\theta)}{1 - \tan\left(\frac{\pi}{3}\right) \tan(\theta)} = \frac{\sqrt{3}(b_1 - a_1) + b_2 - a_2}{b_1 - a_1 - \sqrt{3}(b_2 - a_2)} = \frac{\ddot{b}_2 - \ddot{a}_2}{\ddot{b}_1 - \ddot{a}_1} = \text{sl}(\ddot{a}, \ddot{b}).
\]
Similarly,
\[
\tan\left(\frac{\pi}{6} + \theta\right) = \frac{\tan\left(\frac{\pi}{6}\right) + \tan(\theta)}{1 - \tan\left(\frac{\pi}{6}\right)\tan(\theta)} = \frac{b_1 - a_1 + \sqrt{3}(b_2 - a_2)}{\sqrt{3}(b_1 - a_1) - (b_2 - a_2)} = \frac{b_2 + \tilde{b} - a_2 - \tilde{a}_2}{b_1 + b_1 - a_1 - \tilde{a}_1} = \text{sl}\left(\frac{a + \tilde{a}}{2}, \frac{b + \tilde{b}}{2}\right).
\]
\[
\square
\]

For any \(\varepsilon > 0\) we denote by \(L_\varepsilon: \mathbb{R}^2 \rightarrow \mathbb{R}^2\) the linear transformation \(L_\varepsilon(x, y) := (\varepsilon x, \varepsilon^2 y)\). As a corollary of Lemma 2.3 we may now prove the properties of the three transformations of south-east chains discussed above.

To improve readability, we use the following notation: if \(A\) is a sequence, then we denote by \(A(k)\) its \(k\)th element, so that \(A = (A(1), \ldots, A(n))\). If \(A\) is a south-east chain of length \(n\) and \(\varepsilon > 0\), then we consider the following three sequences:
\[
\begin{align*}
A_\varepsilon & := \left(L_\varepsilon(A(1)), \ldots, L_\varepsilon(A(n))\right), \\
A_\varepsilon' & := \left(R(A_\varepsilon(1)), \ldots, R(A_\varepsilon(n))\right), \\
A_\varepsilon'' & := \left(\frac{A_\varepsilon(1) + A_\varepsilon'(1)}{2}, \ldots, \frac{A_\varepsilon(n) + A_\varepsilon'(n)}{2}\right).
\end{align*}
\]

Using this notation, \(A_\varepsilon\) is a chain obtained by flattening \(A\), \(A_\varepsilon'\) is the rotated version of this flattened chain, and \(A_\varepsilon''\) is the chain formed by taking the midpoints of the two previous chains. The next result follows from Lemma 2.3.

**Lemma 2.4.** Suppose that \(A := (A(1), \ldots, A(n))\) is a south-east chain. Then, for sufficiently small \(\varepsilon > 0\), the sequences \(A_\varepsilon, A_\varepsilon', A_\varepsilon''\) are south-east chains. Moreover, for every \(k \in [n - 1]\) we have the equalities
\[
\begin{align*}
\lim_{\varepsilon \to 0^+} \text{sl}(A_\varepsilon(k), A_\varepsilon(k + 1)) &= 0, \\
\lim_{\varepsilon \to 0^+} \text{sl}(A_\varepsilon'(k), A_\varepsilon'(k + 1)) &= \tan\left(\frac{\pi}{3}\right) = \sqrt{3}, \\
\lim_{\varepsilon \to 0^+} \text{sl}(A_\varepsilon''(k), A_\varepsilon''(k + 1)) &= \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}.
\end{align*}
\]

**Proof.** It is obvious that the sequence \(A_\varepsilon\) is strictly increasing on both coordinates. Moreover, for every \(k \in [n - 1]\) we have \(\text{sl}(A_\varepsilon(k), A_\varepsilon(k + 1)) = \varepsilon \text{sl}(A(k), A(k + 1))\). Hence, \(A_\varepsilon\) is a south-east chain and we have \(\lim_{\varepsilon \to 0^+} \text{sl}(A_\varepsilon(k), A_\varepsilon(k + 1)) = 0\). To prove the claim for the remaining two sequences, note that for sufficiently small \(\varepsilon > 0\), the inequality \(\text{sl}(A_\varepsilon(k), A_\varepsilon(k + 1)) = \varepsilon \text{sl}(A(k), A(k + 1)) < 1/\sqrt{3}\) is satisfied for all \(k \in [n - 1]\). Hence, by Lemma 2.3, the sequence \(A_\varepsilon' = \left(R(A_\varepsilon(1)), \ldots, R(A_\varepsilon(n))\right)\) is strictly increasing on both coordinates and the same is true for the sequence \(A_\varepsilon'' = \left(\frac{A_\varepsilon(1) + A_\varepsilon'(1)}{2}, \ldots, \frac{A_\varepsilon(n) + A_\varepsilon'(n)}{2}\right)\). Moreover, if we denote \(\theta^{(k, \varepsilon)} := \arctan\left(\text{sl}(A_\varepsilon(k), A_\varepsilon(k + 1))\right) < \frac{\pi}{6}\), then the sequence \(\theta^{(1, \varepsilon)}, \ldots, \theta^{(n - 1, \varepsilon)}\) is strictly increasing and Lemma 2.3 shows that
\[
\text{sl}(A_\varepsilon'(k), A_\varepsilon'(k + 1)) = \tan\left(\frac{\pi}{3} + \theta^{(k, \varepsilon)}\right).
\]
and
\[ \mathrm{sl}(A'_\varepsilon(k), A''_\varepsilon(k + 1)) = \tan\left(\frac{\pi}{6} + \theta(k, \varepsilon)\right). \]
In particular, the sequences \(A'_\varepsilon, A''_\varepsilon\) are south-east chains. Moreover, we have \(\lim_{\varepsilon \to 0^+} \theta(k, \varepsilon) = 0\) for all \(k\), which gives the equalities
\[ \lim_{\varepsilon \to 0^+} \mathrm{sl}(A'_\varepsilon(k), A'_\varepsilon(k + 1)) = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}, \]
\[ \lim_{\varepsilon \to 0^+} \mathrm{sl}(A''_\varepsilon(k), A''_\varepsilon(k + 1)) = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}. \]

We now show how the transformations given in Lemma 2.4 can be used to prove Theorem 1. To do so, we take three south-east chains \(A := (A(1), \ldots, A(n)), B := (B(1), \ldots, B(n)), \) and \(C := (C(1), \ldots, C(m))\) such that \(C \subset (A + B)/2\). As discussed before, we let \(u := (1, 5/2), v := (0, 2), w := (1, 1)\) and we consider the chains \(A_\varepsilon, A'_\varepsilon, B_\varepsilon, B'_\varepsilon\) defined as in [1].

**Lemma 2.5.** If \(\varepsilon > 0\) is sufficiently small, then the sequences \(A_\varepsilon := (A_\varepsilon, w + B'_\varepsilon), B_\varepsilon := (v + B_\varepsilon, u + A'_\varepsilon)\) are south-east chains. Moreover, the set \((A_\varepsilon + B_\varepsilon)/2\) contains a south-east chain of length at least \(2m + n\).

In the statement above, \(w + B'_\varepsilon\) denotes the sequence obtained by translating every element of \(B'_\varepsilon\) by the vector \(w\) and \((A_\varepsilon, w + B'_\varepsilon)\) denotes the concatenation of two sequences. The same applies to \((v + B_\varepsilon, u + A'_\varepsilon)\).

**Proof of Lemma 2.5.** We start by proving that \(A_\varepsilon\) is a south-east chain. By Lemma 2.4 the sequences \(A_\varepsilon\) and \(B'_\varepsilon\) are south-east chains for sufficiently small \(\varepsilon\). Hence, the sequence \(w + B'_\varepsilon\) is also a south-east chain. Furthermore, we have \(\lim_{\varepsilon \to 0^+} (A_\varepsilon(n)) = (0, 0)\) and \(\lim_{\varepsilon \to 0^+} (w + B'_\varepsilon(1)) = w = (1, 1)\). In particular, for sufficiently small \(\varepsilon\), the sequence \(A_\varepsilon\) is strictly increasing on both coordinates. Moreover, Lemma 2.4 shows the equalities \(\lim_{\varepsilon \to 0^+} \mathrm{sl}(A_\varepsilon(n - 1), A_\varepsilon(n)) = 0\) and \(\lim_{\varepsilon \to 0^+} \mathrm{sl}(w + B'_\varepsilon(1), w + B'_\varepsilon(2)) = \sqrt{3}\).

We also have
\[ \lim_{\varepsilon \to 0^+} \mathrm{sl}(A_\varepsilon(n), w + B'_\varepsilon(1)) = \mathrm{sl}(0, w) = 1. \]
Since \(0 < 1 < \sqrt{3}\), the sequence \(A_\varepsilon\) is a south-east chain for sufficiently small \(\varepsilon\). The proof for \(B_\varepsilon\) is analogous—it is enough to observe that \(\lim_{\varepsilon \to 0^+} (v + B_\varepsilon(n)) = (0, 2)\) and \(\lim_{\varepsilon \to 0^+} (u + A'_\varepsilon(1)) = (1, 5/2)\) to show that \(B_\varepsilon\) is strictly increasing on both coordinates. Then, \(B_\varepsilon\) is a south-east chain by the equalities \(\lim_{\varepsilon \to 0^+} \mathrm{sl}(v + B_\varepsilon(n - 1), v + B_\varepsilon(n)) = 0\), \(\lim_{\varepsilon \to 0^+} \mathrm{sl}(u + A'_\varepsilon(1), u + A'_\varepsilon(2)) = \sqrt{3}\), and
\[ \lim_{\varepsilon \to 0^+} \mathrm{sl}(v + B_\varepsilon(n), u + A'_\varepsilon(1)) = \mathrm{sl}(v, u) = \frac{1}{2}. \]
It remains to show that \((A_\varepsilon + B_\varepsilon)/2\) contains a south-east chain of length at least \(2m + n\). To do so, consider the chain \(C \subset (A + B)/2, |C| = m,\) and let \(C_\varepsilon, C'_\varepsilon\) be defined as in [1]. Furthermore, let \(t := (0, 1) = v/2\) and \(z := (1, 7/4) = (u + w)/2\). Since the transformations \(L_\varepsilon\) and \(R\) are linear, we have \(t + C_\varepsilon \subset (A_\varepsilon + B_\varepsilon)/2 \subset (A_\varepsilon + B_\varepsilon)/2\) and \(z + C'_\varepsilon \subset (u + A'_\varepsilon + w + B'_\varepsilon)/2 \subset (A_\varepsilon + B_\varepsilon)/2\). Moreover, we define the chain \(A''_\varepsilon\) as in [1] we let \(s := (1/2, 5/4) = u/2\), and we note that \(s + A''_\varepsilon \subset (A_\varepsilon + u +
\(A_\varepsilon'/2 \subset (A_\varepsilon + B_\varepsilon)/2\). Hence, the sequence \(D_\varepsilon := (t + C_\varepsilon, s + A_\varepsilon', z + C_\varepsilon')\) is contained in \((A_\varepsilon + B_\varepsilon)/2\) and its length is equal to \(2m + n\). Therefore, it is enough to prove that \(D_\varepsilon\) is a south-east chain. The proof is similar to the proofs before. By Lemma 2.2, the sequences \(t + C_\varepsilon, s + A_\varepsilon\), and \(z + C_\varepsilon\) are south-east chains for sufficiently small \(\varepsilon\). Moreover, we have the equalities \(\lim_{\varepsilon \to 0^+} (t + C_\varepsilon(m)) = (0, 1)\), \(\lim_{\varepsilon \to 0^+} (s + A_\varepsilon(1)) = (1/2, 5/4)\), and \(\lim_{\varepsilon \to 0^+} (z + C_\varepsilon(1)) = (1, 7/4)\), which show that \(D_\varepsilon\) is strictly increasing on both coordinates for sufficiently small \(\varepsilon\). Furthermore, we use Lemma 2.4 to observe that \(\lim_{\varepsilon \to 0^+} \text{sl}(t + C_\varepsilon(m) - 1, t + C_\varepsilon(m)) = 0\), \(\lim_{\varepsilon \to 0^+} \text{sl}(s + A_\varepsilon(1), s + A_\varepsilon(2)) = \lim_{\varepsilon \to 0^+} \text{sl}(s + A_\varepsilon(n - 1), s + A_\varepsilon(n)) = 1/\sqrt{3}\), and \(\lim_{\varepsilon \to 0^+} \text{sl}(z + C_\varepsilon(1), z + C_\varepsilon(2)) = \sqrt{3}\). To finish, we have

\[
\lim_{\varepsilon \to 0^+} \text{sl}(t + C_\varepsilon(m), s + A_\varepsilon'(1)) = \text{sl}(t, s) = \frac{1}{2},
\]

\[
\lim_{\varepsilon \to 0^+} \text{sl}(s + A_\varepsilon'(n), z + C_\varepsilon'(1)) = \text{sl}(s, z) = 1,
\]

and \(0 < \frac{1}{2} < \frac{1}{\sqrt{3}} < 1 < \sqrt{3}\). Hence, for sufficiently small \(\varepsilon\), the sequence \(D_\varepsilon\) is a south-east chain. \(\square\)

As a corollary, we may prove our main theorem.

**Proof of Theorem 1.1** As noted in the introduction, it is enough to prove the claim for \((P_k + Q_k)/2\) instead of \(P_k + Q_k\). We show, by induction over \(k\), that we can find two south-east chains \(P_k, Q_k\), each of length \(2^k\), such that \((P_k + Q_k)/2\) contains a south-east chain of length \((k + 2)2^{k-1}\). Let \(P_1 := ((0, 0), (1, 1))\) and \(Q_1 := ((0, 1), (1, 2))\). The sequences \(P_1\) and \(Q_1\) are south-east chains and the set \((P_1 + Q_1)/2\) contains a south-east chain of length three. Moreover, given \(P_k, Q_k\) we may apply Lemma 2.5 to obtain two south-east chains \(P_{k+1}, Q_{k+1}\), each of length \(2^{k+1}\), such that \((P_{k+1} + Q_{k+1})/2\) contains a south-east chain of length at least \((k + 2)2^k + 2^k = (k + 3)2^k\). Therefore, the claim follows from Lemma 2.2. \(\square\)

### 3. Final remarks

Let us discuss some natural questions for further research. Firstly, we do not know how far from optimal is our construction. More precisely, consider the function \(f : \mathbb{N}^* \to \mathbb{N}^*\) defined as

\[
f(n) := \max\{c_i(P + Q) : P, Q \subset \mathbb{R}^2 \text{ are convexly independent and } |P| = |Q| = n\}.
\]

By joining our analysis with the result of Tiwary [9], we obtain the asymptotic bound \(f(n) = \Theta(n \log n)\). One can ask for an optimal constant \(C\) such that \(f(n) \leq C n \log n\). Secondly, we wonder if our family of graphs \(G_k = (U_k \cup V_k, E_k)\) can be still embedded in the plane if we impose a stronger condition on the midpoints of the edges. For instance, we may ask for an embedding such that \(U_k\) and \(V_k\) are south-east chains and the midpoints of \(E_k\) are contained in some convex curve of small degree, such as a circle or a parabola. Our interest in this question is motivated by the following observation: if \(G_k\) are realizable in the south-east quadrant of the plane in such a way that the midpoints of \(E_k\) are on the unit circle, then we obtain a
\(\Theta(n \log n)\) bound for the convex version of the unit distance problem, proposed by Erdős and Moser [4] (see [6, 1] for more information on lower and upper bounds for this variant of the unit distance problem). Indeed, suppose that \(h_k: (U_k \uplus V_k) \to \mathbb{R}^2\) is an embedding of \(G_k\) such that \(h_k(U_k)\) and \(h_k(V_k)\) are south-east chains, \(h_k(U_k), h_k(V_k) \subset \{x \in \mathbb{R}^2: x_1 \geq 0, x_2 \leq 0\}\), and \(\frac{1}{2} \|h_k(u)+h_k(v)\|_2 = 1\) for all \((u,v) \in E_k\). Then, the points \(-h_k(U_k)\) form a monotone concave chain in the north-west quadrant of the plane and \(h_k(V_k)\) form a monotone convex chain in the south-east quadrant of the plane, which implies that the collection \(\{-h_k(U_k), h_k(V_k)\}\) is convexly independent. Therefore, the polygon \(P_k := \text{conv}\{-h_k(U_k)/2, h_k(V_k)/2\}\) has \(2^{k+1}\) vertices and \(\Theta(k2^k)\) diagonals of length one. As noted in the introduction, the existence of such an embedding cannot be excluded using the current list of forbidden matrices [1]. We were neither able to construct these embeddings nor find a new forbidden pattern that occurs in \(G_k\).

\textbf{References}


