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# Lyapunov Functions for Singularly Perturbed Hybrid Systems with Frequent Jump Dynamics

Aneel Tanwani

Hyungbo Shim

**Abstract**—This article considers the stability analysis for a class of hybrid systems with the focus being on the frequently occurring jump dynamics. The system class is modelled as a singularly perturbed hybrid system where the singular perturbation parameter governs the frequency of jumps. Consequently, this results in a quasi steady-state system modeled by a differential equation without any jumps, and the boundary-layer system described by purely discrete dynamics. By imposing appropriate assumptions on the quasi steady-state system and the boundary-layer system, we derive results showing practical convergence to a compact attractor when the jumps occur frequently often. Our system class is motivated by the control design problem in a network of second-order continuous-time coupled oscillators, where each agent communicates the information about its position to the neighbors at discrete times. As a corollary to our main result, we show that if the information exchange between the agents and their neighbors is frequent enough, then the oscillators achieve practical consensus.

**Index Terms**—Hybrid system; Singular perturbation; Practical stability; Heterogeneous agents; Coupled nonlinear oscillators; Sampled-data system.

## I. INTRODUCTION

Hybrid systems, which comprise continuous and discrete dynamics, have found utility in modeling of a broad class of physical systems and algorithms. Conventional results on stability of hybrid systems, like [2, Theorem 3.18], require the continuous and discrete dynamics to be asymptotically stable. However, it is natural to develop stability conditions which provide an interplay between stability of continuous and discrete dynamics. Taking switched systems as an example of hybrid systems, it is seen that if the continuous dynamics are stable but discrete dynamics are not, then we can obtain stability conditions by constraining the frequency of switching times in the form of upper bound on *average dwell-time* (ADT) [8, Chapter 3]. On the other hand, if the continuous dynamics are not stable but the discrete dynamics compensate for the growth of the state during continuous-time evolution, then the overall system is asymptotically stable if the number of discrete jumps over each bounded interval is large enough [3]. This article considers an interconnected hybrid system where one subsystem has stable continuous dynamics, and the other subsystem has stable

discrete dynamics only. The particular structure that we consider in our work is analyzed using singular perturbation methods [4, Chapter 11].

Singular perturbation techniques have been conventionally studied for analysis of ordinary differential equations with slow and fast dynamics [4]. The basic idea is to see the system as a perturbation of a nominal system that comprises the boundary-layer system (describing the limit of fast dynamics) and the quasi steady-state system (described by slow system with limiting value of fast dynamics). More recently, we have seen the use of singular perturbation methods for a class of hybrid systems where fast evolution appears in continuous-time dynamics only [13], [16]. The solutions resulting from a continuous-time boundary-layer system generate an average vector field which approximates the slow subsystem. Using the stability of the average dynamics, with appropriate hypotheses on system data, the authors show semiglobal practical stability for original system.

Moreover, we see the use of singular perturbation methods for analysis in different applications. One such application of these methods, which is also studied in this article, is the synchronization problem in networked systems. For multi-agent systems with diffusive coupling and continuous-time heterogeneous agents [5], [12], it is observed that the agents achieve practical convergence if the coupling strength is large enough. One can view the coupling parameter as a singular perturbation parameter [7], [10]. More recently, we have also seen the use of discontinuous coupling for synchronization of networked system in [1]. Our work actually proposes singular perturbation methods for analysis of networked systems where the discontinuity in coupling arises due to time-sampled information exchange between neighbors.

Based on this last remark, another research direction which is particularly relevant for our paper is the stability of dynamical systems under fast switching or frequent impulses. A related application in this direction is studied in [11], where the authors propose the use of fast switching round-robin protocol for stability of a nonlinear controlled system. However, to the best of authors' knowledge, the use of singular perturbation methods for analysis of fast switching/impulses has not appeared in the literature, and this paper makes a contribution in this direction. An exception in this direction is the paper [17], where the authors take sampling period as the singular perturbation parameter but the resulting analysis is carried out entirely in discrete domain, which is completely different from the approach adopted in this paper.

This article proposes singular perturbation techniques for a class of hybrid systems, where frequency of jump dynamics

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is associated with the singular perturbation parameter. In particular, as the parameter gets smaller, the frequency of jumps increases. We construct an appropriate Lyapunov function for the interconnection between slow continuous dynamics and fast discrete dynamics, and show practical convergence under mild assumptions on system data. Such constructions are inspired from the analysis of sampled-data systems [15]. As an application of our main result, we study a heterogeneous multi-agent system where each node is described by a second order nonlinear oscillator. Each agent communicates one of its states (the output) to its neighbors at discrete times. We show that if the frequency of time instants, at which agents exchange information, is large enough then the oscillators achieve practical consensus.

## II. SINGULAR PERTURBATION THEOREMS FOR A CLASS OF HYBRID SYSTEMS

In this article, we are interested in studying a class of hybrid systems, parameterized by a scalar  $\epsilon \in (0, 1]$ ,

$$\begin{cases} \dot{x} = f_x(x, y, \epsilon), \\ \dot{y} = f_y(x, y, \epsilon), \\ \epsilon \dot{\tau} = \sigma, & \text{for } (x, y, \tau) \in C_x \times C_y \times [0, 1], \\ x^+ = x, \\ y^+ = g_y(x, y), \\ \tau^+ \in [0, \gamma], & \text{for } (x, y, \tau) \in C_x \times C_y \times \{1\}, \end{cases} \quad (1)$$

where  $x \in \mathbb{R}^n$ ,  $y \in \mathbb{R}^m$ , and  $\tau \in \mathbb{R}$ . The sets  $C_x \subset \mathbb{R}^n$  and  $C_y \subset \mathbb{R}^m$  are assumed to be closed, and contained in open and connected domains  $\mathcal{D}_x \subset \mathbb{R}^n$  and  $\mathcal{D}_y \subset \mathbb{R}^m$ , respectively. The vector fields  $f_x : \mathcal{D}_x \times \mathcal{D}_y \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$  and  $f_y : \mathcal{D}_x \times \mathcal{D}_y \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^m$ , and the function  $g_y : \mathcal{D}_x \times \mathcal{D}_y \rightarrow \mathbb{R}^m$  are assumed to be continuously differentiable in their arguments. The constant  $\gamma$  is assumed to belong to  $[0, 1)$ , and  $\sigma$  is a positive constant. In the terminology of hybrid systems used in [2], it is seen that the flow set is  $C_x \times C_y \times [0, 1]$  and the jump set is  $C_x \times C_y \times \{1\}$ .

For the problems studied in this paper, the parameter  $\epsilon$  appearing in (1) is taken to be small, which implies that the jumps occur frequently, and thus, the evolution of  $y$  relies highly on the jump dynamics. Since the jumps correspond to rapid evolution, we call  $x$  the *slow variable* and  $y$  the *fast variable*. The evolution of the timer  $\tau$ , determined by the parameters  $\epsilon$ ,  $\gamma$ , and  $\sigma$ , describes the interplay between the flow and the jump. The evolution of system (1) guarantees that, over an interval  $[0, T + \tau(0)]$ , there are at least  $T\sigma/\epsilon$  jumps, and after every jump, the system flows for at least  $(1-\gamma)\epsilon/\sigma$  units of time. Therefore, there is no accumulation time of jumps (i.e., there are finitely many jumps in any finite continuous-time interval) and the time domain of every complete solution has unbounded continuous-time element.

If  $\epsilon$  is sufficiently small, the behavior of the solution<sup>1</sup>

<sup>1</sup>According to the framework of [2], one writes the solution at a given time instant as  $(x(t, j), y(t, j))$  using the hybrid time index  $(t, j)$ . In our case, we find it convenient to write  $(x(t), y(t))$ , which implicitly refers to  $(x(t, j), y(t, j))$  for some  $j \in \mathbb{N}_{>0}$ . The choice is motivated by the fact that, in this paper, the complete solutions have unbounded  $t$ . Whenever  $t$  is a jump instant, we indicate the value of  $y$  after the jump with  $y(t^+)$ .

$(x(t), y(t))$  can be approximated by two different dynamic systems. The first system is called the *boundary-layer* system given by, for  $y \in C_y$ ,

$$y^+ = g_y(x, y) \quad \text{for a fixed value of } x \in C_x. \quad (2)$$

When  $\epsilon$  is sufficiently small, there are many jumps within a small continuous-time interval. This means that, from an initial condition  $(x(0), y(0))$ , while the solution  $x(t)$  remains close to  $x(0)$ , the solution  $y(t)$  obeys  $y^+ = g_y(x, y)$  which is approximated by (2). If (2) has a unique equilibrium for each fixed  $x \in C_x$ , then we may consider a continuously differentiable function  $h : \mathbb{R}^n \rightarrow \mathbb{R}^m$  such that

$$g_y(x, h(x)) = h(x), \quad \forall x \in C_x. \quad (3)$$

If, in addition, the equilibrium  $h(x)$  is asymptotically stable, then we can imagine that the solution  $y(t)$  rapidly approaches  $h(x(0))$  after some jumps, and remains near  $h(x(t))$  as time goes on. In order to approximate the behavior of  $x(t)$ , we define the *quasi steady-state* system given by

$$\dot{x} = f_x(x, h(x), 0), \quad x \in C_x \quad (4)$$

which basically approximates the evolution of  $x(t)$  on the *slow manifold*  $\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : y = h(x)\}$ . In the following, let us clarify the intuition discussed so far.

### A. Main Results

The basic problem studied in this paper relates to the stability of system (1). Toward this end, we work under the assumption that the quasi steady-state system (4) and the boundary-layer system (2) are both stable. As a principal result, we show that by choosing  $\epsilon$  small enough, system (1) achieves stability in an appropriate sense under aforementioned assumption.

**Assumption 1:** The boundary-layer system (2) is uniformly (in the fixed variable  $x$ ) exponentially stable at the equilibrium  $h(x) \in C_y$ ; that is, there exists a continuously differentiable function  $V_y : \mathcal{D}_x \times \mathcal{D}_y \rightarrow \mathbb{R}$  such that

$$\begin{aligned} \underline{b}|y - h(x)|^2 &\leq V_y(x, y) \leq \bar{b}|y - h(x)|^2, \\ V_y(x, g_y(x, y)) - V_y(x, y) &\leq -b|y - h(x)|^2, \\ |\nabla_y V_y(x, y)| &\leq l_y|y - h(x)|, \quad |\nabla_x V_y(x, y)| \leq l_x|y - h(x)|^2 \end{aligned}$$

for all  $(x, y) \in C_x \times C_y$  where  $\underline{b}$ ,  $\bar{b}$ ,  $b$ ,  $l_y$ , and  $l_x$  are positive constants.  $\triangleleft$

**Theorem 1:** Suppose that Assumption 1 holds and the quasi steady-state system (4) admits an asymptotically stable compact attractor  $\mathcal{A} \subset C_x$ , in the sense that there exists a continuously differentiable function  $V_x : \mathcal{D}_x \rightarrow \mathbb{R}$  such that, for all  $x \in C_x$ ,

$$\underline{\alpha}(|x|_{\mathcal{A}}) \leq V_x(x) \leq \bar{\alpha}(|x|_{\mathcal{A}}), \quad (5)$$

$$\nabla_x V_x(x) f_x(x, h(x), 0) \leq -\alpha(|x|_{\mathcal{A}}), \quad (6)$$

where  $\underline{\alpha}$  and  $\bar{\alpha}$  are class  $\mathcal{K}_\infty$  functions, and  $\alpha$  is a class  $\mathcal{K}$  function. Then, for any compact sets  $K_x \subset C_x$  and  $K_y \subset C_y$  and for every  $\rho > 0$ , there exists  $\epsilon^* > 0$  such that, for any  $0 < \epsilon \leq \epsilon^*$  and for any  $x(0) \in K_x$ ,  $y(0) \in K_y$ , and

$\tau(0) \in [0, \gamma]$ , the solution  $(x(t), y(t))$  to (1) does not have finite escape time and satisfies<sup>2</sup>

$$\limsup_{t \rightarrow \infty} |x(t)|_{\mathcal{A}} \leq \rho, \quad (7)$$

$$\limsup_{t \rightarrow \infty} |y(t) - h(x(t))| \leq \rho. \quad (8)$$

### B. Comparisons with Existing Literature

Let us draw some comparisons between the system class (1) and some related frameworks from the literature.

1) *Singularly perturbed hybrid systems*: We observe that, by letting  $\tilde{\tau} = \epsilon\tau$ ,  $\tau$ -dynamics can be similarly written as

$$\begin{aligned} \dot{\tilde{\tau}} &= \sigma, & \tilde{\tau} &\in [0, \epsilon] \\ \tilde{\tau}^+ &\in [0, \gamma\epsilon], & \tilde{\tau} &\in \{\epsilon\}. \end{aligned}$$

Because we work with very small value of the parameter  $\epsilon$ , it is natural to think of (1) as a singularly perturbed hybrid system, as studied in [16]. The quasi steady-state system is a hybrid system, described as,

$$\begin{aligned} \dot{x} &= f_x(x, h(x), 0), & x &\in C_x \\ x^+ &= x, & x &\in C_x. \end{aligned}$$

The boundary-layer system is also a hybrid system, with  $x \in C_x$  fixed, and described as

$$\begin{cases} \dot{y} = f_y(x, y, 0), & y \in C_y \\ \dot{\tilde{\tau}} = \sigma, & \tilde{\tau} \in \{0\} \\ y^+ = g_y(x, y), & y \in C_y \\ \tilde{\tau}^+ = 0, & \tilde{\tau} \in \{0\}. \end{cases}$$

In other words, this reformulation is consistent with our earlier observation that the quasi steady-state system is purely continuous-time, and the boundary-layer system is purely discrete-time. This particular observation differentiates our system class from the one studied in [16].

2) *Impulsive systems with reverse average dwell-time*: One can also think of (1) as an impulsive system as studied in [3]. We find stability conditions for impulsive systems in terms of *average dwell-time* and *reverse average dwell-time* associated with the sequence of impulse times. For every solution of system (1), if  $N(t, s)$  denotes the number of jumps on the interval  $(s, t]$ , then

$$N(t, s) \geq \frac{\sigma(t-s)}{\epsilon}$$

which essentially associates a reverse average dwell-time property to our system. However, rapid switching does not stabilize  $x$ -dynamics, and the slow switching may lead to instability of  $y$ -dynamics. Due to this reason, the tools from [3] cannot be directly applied here.

In other words, our system (1) is an interconnection of a hybrid system with state  $x \in \mathbb{R}^n$  having asymptotically stable continuous-time dynamics but not necessarily stable

discrete dynamics, and another hybrid system with state  $y \in \mathbb{R}^m$  having stable jump dynamics but possibly unstable continuous dynamics. From this viewpoint, small gain theorems for hybrid systems from [9] are more suited for stability analysis of (1). However, for the applications studied in this article, imposing *input-to-state stability* assumptions on  $x$  and  $y$  subsystems is restrictive, and for this reason we propose analysis based on singular perturbation methods which results in practical stability rather than asymptotic stability.

### III. NETWORK OF HETEROGENEOUS OSCILLATORS

As a by-product of our theoretical result in Theorem 1, we study the problem of synchronization of coupled nonlinear oscillators. We consider  $N$  continuous-time controlled oscillators, which are connected to each other by an undirected and connected graph. The control action is chosen as a function of the states of their neighbors at discrete times. If this control is updated sufficiently often, then we show that all the oscillators converge to each other in practical sense.

Each of the  $N$  oscillators (the agent) has a two dimensional state  $(\omega_i, \xi_i) \in \mathbb{R}^2$ , for  $i = 1, \dots, N$ , whose dynamic equation is written as

$$\begin{aligned} \dot{\omega}_i &= -\omega_i + \xi_i \\ \dot{\xi}_i &= \phi_i(\omega_i, \xi_i) := (1 - f_i(\omega_i))(-\omega_i + \xi_i) - g_i(\omega_i) \end{aligned} \quad (9a)$$

where the functions  $f_i, g_i$  are assumed to be smooth. For example, if the agent  $i$  has  $f_i(\omega_i) = \mu_i(\omega_i^2 - 1)$  and  $g_i(\omega_i) = \omega_i$  with  $\mu_i > 0$ , then the agent  $i$  is a well-known Van der Pol oscillator [4].

Now we suppose that the agents are connected through an undirected and connected graph, and the state  $\xi_i$  is updated from time to time as a functions of its neighbors' state  $\xi_j$ ,  $j \in \mathcal{N}_i$ , where  $\mathcal{N}_i$  is the index set of neighboring agents of the agent  $i$ . In particular, we describe the update model as

$$\begin{aligned} \omega_i^+ &= \omega_i \\ \xi_i^+ &= \sum_{j \in \mathcal{N}_i} w_{ij} \xi_j \end{aligned} \quad (9b)$$

where  $w_{ij}$  is the  $(i, j)$ -th entry of a doubly stochastic matrix  $W \in \mathbb{R}^{N \times N}$ . If the network is modeled by a symmetric adjacency matrix  $A$ , and  $D$  is the diagonal matrix that describes the degree of each node, then  $W$  is given by  $I - \varphi D + \varphi A$ , for which every row and column sums to 1 by construction and the weight  $\varphi > 0$  is chosen small enough such that the matrix  $W$  becomes non-negative.

The time when the update (9b) occurs is governed by a timer modeled as

$$\dot{\tau} = \sigma/\epsilon, \quad \tau \in [0, 1], \quad (9c)$$

$$\tau^+ \in [0, \gamma], \quad \tau \in \{1\}, \quad (9d)$$

where  $\epsilon \in (0, 1]$ ,  $\gamma \in [0, 1)$ , and  $\sigma > 0$ . This model implies that all the states  $\omega_i, \xi_i$ , and  $\tau$  are updated by (9b) and (9d) whenever  $\tau(t)$  becomes 1, and after the update, the timer state  $\tau(t^+)$  is chosen arbitrarily in  $[0, \gamma]$ . Such a framework

<sup>2</sup>In equations (7) and (8), we provide bounds on forward complete solutions  $(x(t), y(t))$  for  $t$  going to  $+\infty$ . It may happen that, for certain initial conditions, the resulting solution is defined only over a finite time interval, and the aforementioned bounds do not apply for such solutions.

for updating the state of agents at discrete times has been proposed in [14].

Now, let  $R \in \mathbb{R}^{N \times (N-1)}$  be a matrix such that  $R^\top R = I$  and  $1_N^\top R = 0$ . Then, it is seen that  $\Lambda := R^\top W R \in \mathbb{R}^{(N-1) \times (N-1)}$  is a positive definite Schur stable matrix. With the matrix  $R$ , we can write (9) in the form of (1). Let us introduce the new coordinates  $(s, \zeta)$  for  $\xi := \text{col}(\xi_1, \dots, \xi_N) \in \mathbb{R}^N$  as

$$s = \frac{1}{N} \sum_{i=1}^N \xi_i \in \mathbb{R}, \quad \zeta = R^\top \xi \in \mathbb{R}^{N-1}.$$

Then, the system (9) can be written as

$$\begin{aligned} \dot{\omega}_i &= -\omega_i + s + R_i \zeta, \quad i = 1, \dots, N, \\ \dot{s} &= \frac{1}{N} \sum_{i=1}^N (1 - f_i(\omega_i))(-\omega_i + s + R_i \zeta) - g_i(\omega_i) \\ \dot{\zeta} &= R^\top \phi(\omega, 1_N s + R \zeta) \\ \epsilon \dot{\tau} &= \tau \end{aligned} \quad (10a)$$

with the flow set  $\mathbb{R}^{2N} \times [0, 1]$ , and

$$\begin{aligned} \omega_i^+ &= \omega_i, \quad \text{for } i = 1, \dots, N, \\ s^+ &= s, \quad \zeta^+ = \Lambda \zeta, \quad \tau^+ \in [0, \gamma] \end{aligned} \quad (10b) \quad (10c)$$

with the jump set  $\mathbb{R}^{2N} \times \{1\}$ . In (10),  $R_i$  denotes the  $i$ -th row of  $R$ , and with  $\omega := \text{col}(\omega_1, \dots, \omega_N)$ ,

$$\phi(\omega, 1_N s + R \zeta) = \begin{bmatrix} \phi_1(\omega_1, s + R_1 \zeta) \\ \vdots \\ \phi_N(\omega_N, s + R_N \zeta) \end{bmatrix}.$$

Here, if we treat  $(\omega_1, \dots, \omega_N, s) = (\omega, s)$  as the slow variable and  $\zeta$  as the fast variable with both the flow set and the jump set for  $(\omega, s, \zeta)$  are the same as  $\mathbb{R}^{2N}$ , then the system is now in the form of (1).

On the other hand, the quasi steady-state (QSS) system is

$$\begin{aligned} \dot{\omega}_i &= -\omega_i + s, \quad i = 1, \dots, N, \\ \dot{s} &= \frac{1}{N} \sum_{i=1}^N (1 - f_i(\omega_i))(-\omega_i + s) - g_i(\omega_i), \end{aligned} \quad (11)$$

which corresponds to (4). The following assumption and lemma assures that this QSS system<sup>3</sup> satisfies the assumption of Theorem 1.

**Assumption 2:** The second order dynamical system

$$\begin{aligned} \dot{\chi} &= -\chi + s \\ \dot{s} &= (1 - f(\chi))(-\chi + s) - g(\chi) \end{aligned} \quad (12)$$

where

$$f(\chi) := \frac{1}{N} \sum_{i=1}^N f_i(\chi), \quad g(\chi) := \frac{1}{N} \sum_{i=1}^N g_i(\chi) \quad (13)$$

has a locally exponentially stable attractive compact set  $\mathcal{A}_0 \subset \mathbb{R}^2$  (i.e., a stable limit cycle).  $\triangleleft$

<sup>3</sup>The QSS system (11) is called ‘blended dynamics’ in the context of heterogeneous multi-agent systems [7].

**Lemma 1:** Under Assumption 2, the QSS system (11) has a locally asymptotically attractive compact set  $\mathcal{A} \subset \mathbb{R}^{N+1}$ , which is a stable limit cycle for (11), given by

$$\mathcal{A} = \{(\chi, \dots, \chi, s) \in \mathbb{R}^{N+1} : (\chi, s) \in \mathcal{A}_0\}. \quad (14)$$

Moreover, there exists  $V$  that satisfies (5) and (6) on a neighborhood  $\Omega$  of  $\mathcal{A}$ .

*Proof:* The basic idea for the proof comes from [6, Theorem 1]. Let

$$\chi := \frac{1}{N} 1_N^\top \omega, \quad \tilde{\chi} := R^\top \omega$$

so that  $\omega_i = \chi + R_i \tilde{\chi}$ . Then, we obtain

$$\begin{aligned} \dot{\chi} &= -\chi + s, \quad \dot{\tilde{\chi}} = -\tilde{\chi}, \\ \dot{s} &= \frac{1}{N} \sum_{i=1}^N (1 - f_i(\chi + R_i \tilde{\chi}))(s - \chi - R_i \tilde{\chi}) - g_i(\chi + R_i \tilde{\chi}) \\ &= (1 - f(\chi))(-\chi + s) - g(\chi) + \frac{1}{N} \sum_{i=1}^N \psi_i(s, \chi, \tilde{\chi}) \end{aligned}$$

where  $\psi_i$  vanishes when  $\tilde{\chi} = 0$  because

$$\begin{aligned} \psi_i &= f_i(\chi + R_i \tilde{\chi}) R_i \tilde{\chi} + (f_i(\chi + R_i \tilde{\chi}) - f_i(\chi))(\chi - s) \\ &\quad + (g_i(\chi) - g_i(\chi + R_i \tilde{\chi})). \end{aligned}$$

By continuous differentiability of  $\phi_i$ , there is a continuous function  $\Psi(s, \chi, \tilde{\chi}) \in \mathbb{R}^{1 \times (N-1)}$  such that  $(1/N) \sum_{i=1}^N \psi_i(s, \chi, \tilde{\chi}) = \Psi(s, \chi, \tilde{\chi}) \tilde{\chi}$ .

By Assumption 2, there is a Lyapunov function  $W(\chi, s)$  and a constant  $c > 0$  such that  $c_1 |(\chi, s)|_{\mathcal{A}_0} \leq W(\chi, s) \leq c_2 |(\chi, s)|_{\mathcal{A}_0}$ ,  $\dot{W}|_{\text{along (12)}} \leq -c_3 |(\chi, s)|_{\mathcal{A}_0}$ , and  $|\nabla W(\chi, s)| \leq c_4 |(\chi, s)|_{\mathcal{A}_0}$  on the set  $\{(\chi, s) : W(\chi, s) \leq c\}$ . With any  $C > 0$ , let  $M := \max_{W(\chi, s) \leq c, |\tilde{\chi}| \leq C} |\Psi(s, \chi, \tilde{\chi})|$ , and let

$$V(s, \chi, \tilde{\chi}) = W(\chi, s) + k |\tilde{\chi}|^2 \quad (15)$$

where  $k := c_4^2 M^2 / (2c_3)$ . Then,

$$\begin{aligned} \dot{V} &\leq -c_3 |(\chi, s)|_{\mathcal{A}_0}^2 + c_4 |(\chi, s)|_{\mathcal{A}_0} M |\tilde{\chi}| - 2k |\tilde{\chi}|^2 \\ &\leq -\frac{c_3}{2} |(\chi, s)|_{\mathcal{A}_0}^2 - k |\tilde{\chi}|^2 \end{aligned}$$

on the set

$$\Omega := \{(s, \chi, \tilde{\chi}) : V(s, \chi, \tilde{\chi}) < \min\{c, kC^2\}\}. \quad (16)$$

This completes the proof.  $\blacksquare$

We can now establish the following result about the networked system (9).

**Theorem 2:** Under Assumption 2, suppose that the initial condition of the multi-agent system (9) is such that  $(\omega_1, \omega_2, \dots, \omega_N, (1/N) \sum_{i=1}^N \xi_i) \in \Omega$  given in (16). Then, for each  $\eta > 0$ , there is  $\epsilon^* > 0$  such that, for each  $0 < \epsilon \leq \epsilon^*$ ,

$$\limsup_{t \rightarrow \infty} |(\omega_i(t), \xi_i(t)) - (\omega_j(t), \xi_j(t))| \leq \eta,$$

$$\limsup_{t \rightarrow \infty} |(\omega_i(t), \xi_i(t))|_{\mathcal{A}_0} \leq \eta, \quad \forall i, j = 1, \dots, N.$$

The proof follows from Theorem 1 and the fact that, if  $\zeta \equiv 0$  then  $\xi_i = s$  and  $\lim_{t \rightarrow \infty} |\omega_i(t) - \omega_j(t)| = 0$  by

(10a) for all  $i, j = 1, \dots, N$ . Indeed, the boundary-layer system is  $\zeta^+ = \Lambda\zeta$ , which corresponds to (2) and satisfies Assumption 1 (with the function  $h$  being the zero function). The stability of QSS system (11) follows from Lemma 1.

**Example 1:** We have simulated four Van der Pol oscillators (9a) with  $f_i(\omega_i) = \mu_i(\omega_i^2 - 1)$  where  $\mu_1 = 1$ ,  $\mu_2 = 1.5$ ,  $\mu_3 = 2$ , and  $\mu_4 = 2.5$ , and  $g_i(\omega_i) = \omega_i$  for  $i = 1, \dots, 4$ . The oscillators are connected through an undirected and connected graph, for which the adjacency matrix  $A$  is  $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$  and the matrix  $W$  in (9b) is  $W = 0.6I + 0.2A$ . For the timer, we work with,  $\sigma = 1$ ,  $\gamma = 0.9$ , and study different cases for  $\epsilon \in \{0.05, 0.2, 1\}$ . At every jump time,  $\tau^+$  is randomly chosen according to a uniform distribution on  $[0, \gamma]$ . The state trajectories of all the agents are plotted in Figure 1 for two different values of  $\epsilon$ .

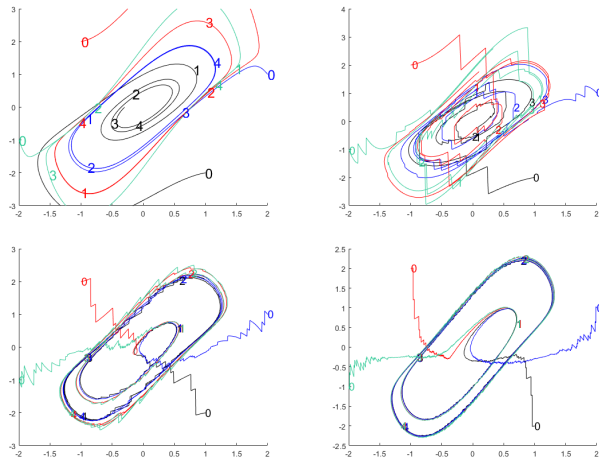


Fig. 1. Plot of  $(\omega_i(t), \xi_i(t))$  for four agents. Numbers are time stamps, and all trajectories circulate clock-wise. (Top left): Case of no coupling; Started from four different initial conditions (time stamps ‘0’), their motions are not synchronized, and the black agent does not have a stable limit cycle so that it converges to the origin. (Top right): Coupled with  $\epsilon = 1$ ; Jumps are observed and they approximately synchronize. (Bottom left): Coupled with  $\epsilon = 0.2$ . (Bottom right): Coupled with  $\epsilon = 0.05$ . The animated version can be seen at [https://youtu.be/f\\_TYWumAD-w](https://youtu.be/f_TYWumAD-w).

#### IV. ANALYSIS AND PROOFS

For convenience of the proof, let us define  $z := y - h(x)$  and consider the closed set

$$C_{xz} := \{(x, y - h(x)) : (x, y) \in C_x \times C_y\}.$$

With the state  $(x, z, \tau) \in C_{xz} \times [0, 1]$ , we can rewrite the flow dynamics as

$$\begin{aligned} \dot{x} &= f_x(x, h(x), 0) + [f_x(x, z + h(x), \epsilon) - f_x(x, h(x), 0)] \\ &=: f_x(x, h(x), 0) + F_x(x, z, \epsilon), \end{aligned}$$

$$\begin{aligned} \dot{z} &= f_y(x, z + h(x), \epsilon) - \frac{\partial h}{\partial x}(x) f_x(x, z + h(x), \epsilon) \\ &=: F_z(x, z, \epsilon) \end{aligned}$$

$$\epsilon \dot{\tau} = \sigma,$$

and the jump dynamics, for  $(x, z, \tau) \in C_{xz} \times \{1\}$ , as

$$x^+ = x,$$

$$z^+ = g_y(x, z + h(x)) - h(x) =: g_z(x, z),$$

$$\tau^+ \in [0, \gamma].$$

##### A. Proof of Theorem 1

For the stability analysis of the hybrid system in the translated coordinates, let us consider the Lyapunov function candidate  $V$ , in an open neighborhood of  $[0, 1] \times C_{xz}$ ,

$$V(\tau, x, z) := e^{c_x \epsilon \tau} V_x(x) + e^{-c_z \tau} V_z(x, z) \quad (17)$$

where  $V_z(x, z) := V_y(x, z + h(x))$ , and the positive scalars  $c_x$  and  $c_z$  are chosen to be sufficiently small (less than 1) and will be specified later. The maximum value  $\epsilon^*$  of the parameter  $\epsilon$  is also to be chosen less than or equal to 1. Since  $\tau \in [0, 1]$ , the function  $V$  satisfies the following inequalities:

$$\underline{\alpha}(|x|_{\mathcal{A}} + e^{-1}\bar{b}|z|)^2 \leq V(\tau, x, z) \leq e^{\bar{\alpha}}(|x|_{\mathcal{A}} + \bar{b}|z|)^2$$

and hence the lower and upper bounds on  $V$  are class  $\mathcal{K}_\infty$  functions with respect to distance from the set  $[0, 1] \times \mathcal{A} \times \{0\}$ . In particular, for any  $\eta > 0$ , define the following compact sets:

$$\begin{aligned} \underline{\Omega}_\eta &:= \left\{ (x, z) \in C_{xz} : |x|_{\mathcal{A}} \leq \bar{\alpha}^{-1} \left( \frac{\eta}{2e} \right), |z| \leq \sqrt{\frac{\eta}{2\bar{b}}} \right\} \\ \Omega_\eta(\tau) &:= \{(x, z) \in C_{xz} : V(\tau, x, z) \leq \eta\} \\ \bar{\Omega}_\eta &:= \left\{ (x, z) \in C_{xz} : |x|_{\mathcal{A}} \leq \underline{\alpha}^{-1}(\eta), |z| \leq \sqrt{\frac{\epsilon\eta}{\bar{b}}} \right\}. \end{aligned}$$

Then, an immediate consequence is that

$$\underline{\Omega}_\eta \subset \Omega_\eta(\tau) \subset \bar{\Omega}_\eta \subset C_{xz}, \quad \forall \tau \in [0, 1].$$

Now, pick  $\kappa$  large enough so that  $K_{xz} \subset \underline{\Omega}_\kappa$  where  $K_{xz} := \{(x, y - h(x)) : (x, y) \in K_x \times K_y\}$ . Then, we restrict our Lyapunov analysis to the compact set  $\bar{\Omega}_\kappa$ .

With the given  $\rho$ , take  $\eta = \min\{\kappa, \underline{\alpha}(\rho), \rho^2 \bar{b}/e\}$ . From now on, we show that  $\dot{V} < 0$  and  $\Delta V := V(\tau^+, x^+, z^+) - V(\tau, x, z) < 0$ , for every  $(x, z) \in \bar{\Omega}_\kappa \setminus \underline{\Omega}_\eta$ , and every  $\tau \in [0, 1]$ . This allows us to conclude that the state variables  $(x, z)$  converge to  $\bar{\Omega}_\eta$  as long as the maximal solution exists (see [2, Theorem 3.18]), and thus, proves Theorem 1.

Now, by the compactness of  $\bar{\Omega}_\kappa$ , there exist constants  $M_x$  and  $M_z$  such that

$$\begin{aligned} |f_x(x, z + h(x), \epsilon)| &\leq M_x, & |F_z(x, z, \epsilon)| &\leq M_z, \\ |\nabla_x V_x(x)| &\leq M_V \end{aligned}$$

for all  $(x, z, \epsilon) \in \bar{\Omega}_\kappa \times [0, 1]$ . Also, by invoking Lemma 2 in Appendix, there exist  $N_x$  and  $N_\epsilon$  such that

$$|F_x(x, z, \epsilon)| \leq N_x |z| + N_\epsilon \epsilon$$

for all  $(x, z, \epsilon) \in \bar{\Omega}_\kappa \times [0, 1]$ .

To analyze  $\Delta V$  along the jump dynamics, we observe that

$$\begin{aligned} \Delta V &= V(\tau^+, x^+, z^+) - V(\tau, x, z) \\ &\leq e^{c_x \epsilon \gamma} V_x(x) - e^{c_x \epsilon} V_x(x) + V_z(x, g_z(x, z)) - e^{-c_z} V_z(x, z) \\ &\leq -(e^{c_x \epsilon} - e^{c_x \epsilon \gamma}) V_x(x) + V_z(x, g_z(x, z)) - V_z(x, z) \\ &\quad + (1 - e^{-c_z}) V_z(x, z) \\ &\leq -(e^{c_x \epsilon} - e^{c_x \epsilon \gamma}) \underline{\alpha}(|x|_{\mathcal{A}} - \bar{b}|z|)^2 + (1 - e^{-c_z}) \bar{b}|z|^2. \end{aligned}$$

Let the positive scalar  $c_z$  be sufficiently small such that

$$(1 - e^{-c_z})\bar{b} < b.$$

Then, since  $\gamma < 1$ , for the chosen  $c_z$  and for any  $c_x > 0$  and  $\epsilon > 0$ ,

$$\Delta V < 0, \quad \forall(\tau, x, z) \in [0, 1] \times \bar{\Omega}_{\kappa}.$$

Next we compute the bound on the derivative of  $V$  along the flow dynamics as follows:

$$\begin{aligned} \dot{V} &= c_x \sigma e^{c_x \epsilon \tau} V_x(x) + e^{c_x \epsilon \tau} \nabla_x V_x \cdot f_x(x, h(x), 0) \\ &\quad + e^{c_x \epsilon \tau} \nabla_x V_x \cdot F_z(x, z, \epsilon) - \frac{c_z \sigma}{\epsilon} e^{-c_z \tau} V_z(x, z) \\ &\quad + e^{-c_z \tau} (\nabla_z V_z \cdot F_z(x, z, \epsilon) + \nabla_x V_z \cdot f_x(x, z + h(x), \epsilon)) \\ &\leq c_x \sigma e^{c_x \epsilon} \bar{\alpha}(|x|_{\mathcal{A}}) - \alpha(|x|_{\mathcal{A}}) + e^{c_x \epsilon} M_V (N_x |z| + N_\epsilon \epsilon) \\ &\quad - \frac{c_z \sigma}{\epsilon} e^{-c_z} \underline{b} |z|^2 + M_z l_y |z| + M_x l_x |z|^2. \end{aligned}$$

Using Lemma 3 in Appendix, with  $\bar{s} = \max_{(x,z) \in \bar{\Omega}_{\kappa}} |x|_{\mathcal{A}}$ , it follows that for every  $o_x > 0$ , there exists  $v_x > 0$  such that  $\bar{\alpha}(|x|_{\mathcal{A}}) \leq v_x \alpha(|x|_{\mathcal{A}}) + o_x$ . Using this bound and Young's inequality, it follows that

$$\begin{aligned} \dot{V} &\leq (c_x \sigma e^{c_x \epsilon} v_x - 1) \alpha(|x|_{\mathcal{A}}) + c_x \sigma e^{c_x \epsilon} o_x - \frac{c_z \sigma}{\epsilon} e^{-c_z} \underline{b} |z|^2 \\ &\quad + \frac{c_z \sigma}{2\epsilon} e^{-c_z} \underline{b} |z|^2 + \frac{\epsilon (e^{c_x \epsilon} M_V N_x + M_z l_y)^2}{2c_z \sigma e^{-c_z} \underline{b}} + M_x l_x |z|^2 \\ &\quad + e^{c_x \epsilon} M_V N_\epsilon \epsilon \\ &\leq -(1 - c_x \sigma e^{c_x \epsilon} v_x) \alpha(|x|_{\mathcal{A}}) - \left( \frac{\kappa_1}{\epsilon} - \kappa_0 \right) |z|^2 \\ &\quad + \epsilon \kappa_c + c_x \sigma e o_x \end{aligned}$$

where  $\kappa_0 := M_x l_x$ ,  $\kappa_1 := c_z \sigma e^{-c_z} \underline{b}_z / 2$ , and  $\kappa_c := (e M_V N_x + M_z l_y)^2 / (4\kappa_1) + e M_V N_\epsilon$ . With  $m := \max \left\{ \frac{1}{2} \alpha \left( \bar{\alpha}^{-1} \left( \frac{\eta}{2\epsilon} \right) \right), \frac{\eta}{4\bar{b}} \right\}$ , the choice of  $\epsilon^*$  and  $c_x$  such that  $\epsilon^* \leq \min \left\{ 1, \frac{\kappa_1}{\kappa_0 + 1/2}, \frac{m}{2\kappa_c} \right\}$ ,  $c_x \leq \min \left\{ 1, \frac{1}{2\sigma e v_x}, \frac{m}{2\sigma e o_x} \right\}$  makes

$$\dot{V} \leq -(1/2) \alpha(|x|_{\mathcal{A}}) - (1/2) |z|^2 + m < 0$$

if  $|x|_{\mathcal{A}} > \bar{\alpha}^{-1}(\eta/(2\epsilon))$  or  $|z| > \sqrt{\eta/(2\bar{b})}$ . This proves that

$$\dot{V} < 0, \quad \forall(\tau, x, z) \in [0, 1] \times \bar{\Omega}_{\kappa} \setminus \underline{\Omega}_{\eta}.$$

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#### APPENDIX

**Lemma 2:** Consider a continuously differentiable function  $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^l$  such that  $f(x, 0) = 0$  for all  $x \in \mathbb{R}^n$ . For any compact set  $K \subset \mathbb{R}^n \times \mathbb{R}^m$ , there are nonnegative constants  $N_i$  such that

$$|f(x, y)| \leq N_1 |y_1| + \dots + N_m |y_m|, \quad \forall(x, y) \in K.$$

*Proof:* The result follows by applying the mean-value theorem and taking the maximum of the continuous function  $(\partial f / \partial y)(x, y)$  over the compact set  $K$ . ■

**Lemma 3:** Consider two class  $\mathcal{K}$  functions  $\alpha_1$  and  $\alpha_2$ . Fix  $\bar{s} > 0$ . For every  $\delta > 0$ , there exists  $v_{12} > 0$  such that

$$\alpha_1(s) \leq v_{12} \alpha_2(s) + \delta, \quad \forall s \in [0, \bar{s}].$$

*Proof:* For a given  $\delta > 0$ , pick a positive  $s^* \leq \bar{s}$  such that  $\alpha_1(s^*) \leq \delta$ , and take

$$v_{12} = \max_{s \in [s^*, \bar{s}]} \frac{\alpha_1(s)}{\alpha_2(s)}$$

which is well-defined. We immediately get

$$\alpha_1(s) \leq \begin{cases} \delta, & 0 \leq s < s^*, \\ v_{12} \alpha_2(s), & s^* \leq s \leq \bar{s} \end{cases}$$

which is the desired inequality. ■

#### REFERENCES

- [1] M. Coraggio, P. De Lellis, and M. di Bernardo. Distributed discontinuous coupling for convergence in heterogeneous networks. *IEEE Control Systems Letters*, 5(3):1037–1042, 2020.
- [2] R. Goebel, R. G. Sanfelice, and A. R. Teel. *Hybrid Dynamical Systems: Modeling, Stability, and Robustness*. Princeton University Press, 2012.
- [3] J. P. Hespanha, D. Liberzon, and A. Teel. Lyapunov conditions for input-to-state stability of impulsive systems. *Automatica*, 44:2735 – 2744, 2008.
- [4] H.K. Khalil. *Nonlinear Systems*. Prentice-Hall, Inc., 3rd edition, 2002.
- [5] J. Kim, J. Yang, H. Shim, J.-S. Kim, and J.H. Seo. Robustness of synchronization of heterogeneous agents by strong coupling and a large number of agents. *IEEE Transactions on Automatic Control*, 61(10):3096–3102, 2016.
- [6] J.G. Lee and H. Shim. Behavior of a network of heterogeneous Liénard systems under strong output coupling. *IFAC-PapersOnLine*, 52(16):256–261, 2019.
- [7] J.G. Lee and H. Shim. A tool for analysis and synthesis of heterogeneous multi-agent systems under rank-deficient coupling. *Automatica*, 117, 2020.
- [8] D. Liberzon. *Switching in Systems and Control*. Birkhäuser, Boston, 2003.
- [9] D. Liberzon, D. Nesić, and A.R. Teel. Lyapunov-based small-gain theorems for hybrid systems. *IEEE Transactions on Automatic Control*, 59(6):1395 – 1410, 2014.
- [10] M. Maghenem, E. Panteley, and A. Loria. Singular-perturbations-based analysis of synchronization in heterogeneous networks: A case-study. In *Proceedings of 55th IEEE Conference on Decision and Control*, page 2581–2586, 2016.
- [11] C. Maheshwari, S. Srikant, and D. Chatterjee. Stabilization under round robin scheduling of control inputs in nonlinear systems. Available online: <https://arxiv.org/abs/1905.09507>, 2019.
- [12] E. Panteley and A. Loria. Synchronization and dynamic consensus of heterogeneous networked systems. *IEEE Transactions on Automatic Control*, 62(8):3758–3773, 2017.
- [13] R.G. Sanfelice and A.R. Teel. On singular perturbations due to fast actuators in hybrid control systems. *Automatica*, 47:692–701, 2011.
- [14] A. Tanwani. Suboptimal filtering over sensor networks with random communication. *IEEE Transactions on Automatic Control*, 2022. DOI: 10.1109/TAC.2021.3116180.
- [15] A. Tanwani and A.R. Teel. Stabilization with event-driven controllers over a digital communication channel with random transmissions. In *Proc. 56th IEEE Conf. on Decision and Control*, pages 6063–6068, 2017.
- [16] W. Wang, A.R. Teel, and D. Nesić. Analysis for a class of singularly perturbed hybrid systems via averaging. *Automatica*, 48:1057–1068, 2012.
- [17] H. Yun, H. Shim, and H.J. Chang. Singular perturbation for sampled-data systems with fast subsystems. *IFAC-PapersOnLine*, 50(1):8145–8150, 2017.