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Abstract

We consider the so-called periodically aggregated resource-constrained project scheduling problem. This problem, introduced by Morin et al. 2017, is a variant of the well-known resource-constrained project scheduling problem that allows for a more flexible usage of the resource constraints. While the start and completion times of the activities can be arbitrary moments in time, the limitations on the resource usage are considered on average over aggregated periods of parameterized length. This paper presents new theoretical and experimental results for this problem. First, we settle the complexity status of the problem by proving NP-hardness of a number of special cases of the problem. Second, we propose a new mixed-integer programming formulation of the problem by disaggregating the precedence constraints over the periods. A theoretical comparison shows that the new formulation dominates the previously proposed one in terms of relaxation strength. Finally, we carry out computational experiments on instances from the literature to compare the merits of the different formulations.

keywords: Project Scheduling; Periodic Aggregation of resource-constraints; Computational complexity; Mixed-integer linear programming

1 Introduction

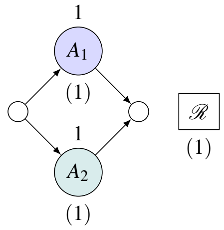
In the extensively studied standard resource-constrained project scheduling problem (RCPSP), at any time, the sum of the requirements of the activities that are currently processed must not exceed the resource capacity. This scheme permits to generalize a wide range of scheduling problems. However, in some practical applications, the time horizon is divided uniformly into consecutive intervals, and only the *average* activity requirements on each interval is considered. This aggregated form of resource constraints appears notably in employee scheduling where the load generated by the different activities and its compatibility with the number of present employees is evaluated on average in each shift (Paul and Knust 2015). However, the schedule of the activities can, and often should be determined on a more precise time scale for specific reasons such as the necessary anticipation for the usage of scarce resources or the contractual relationships with suppliers and customers (Artigues et al. 2009). Another example can be found in manufacturing or smart building applications, where the electricity consumption of jobs is only computed in intervals fixed by the electricity provider while the schedule of the jobs can be more detailed (Haït and Artigues 2011). In the literature, averaging the resource demand of activities inside fixed length periods has been proposed for problems with variable-intensity activities: the rough cut capacity planning (RCCP) (Hans 2001) and the resource-constrained project scheduling problem with variable intensity activities (RCPSVP) (Kis 2005). An extension of the RCPSP with partially renewable resources, entitled RCPSP/Π, has been introduced by Böttcher et al. (1999), that allows to define intervals with specific rules for resource consumption. However, the formulations proposed by Hans for the RCCP and by Kis for the RCPSVP do not involve variables representing start times; moreover, there is no assumption (a priori) nor algorithm (a posteriori) that provides values for start times. However, the average "energy" (duration $\times$ demand) is an explicit variable. For a given solution (average energy of each activity on each resource in each period), resource constraints induce bounds on start times for compatible schedules (i.e. schedules whose energy profile matches the solution). In some cases, any such schedule is precedence-infeasible. For the RCCP, a workaround has been proposed, but might fail. For the RCPSVP, in order to avoid this phenomenon, for each predecessor/successor pair, the standard end-to-start precedence constraint is replaced with the following constraint: if the predecessor completes in period ℓ, then the successor may start only in period $\ell + 1$ or later, which leads to overconstrained precedence constraints, compared to the standard ones. In both cases, no start times are involved. Apart from being undesirable for the above-mentioned specific reasons, generally, this enforcement also has a strong negative impact on the scheduling objectives.

For the RCPSP/Π, start times must coincide with sub-interval bounds and so can be seen as discrete variables: because of this, not only optimal but also feasible solutions, possibly even all of them, may be excluded.

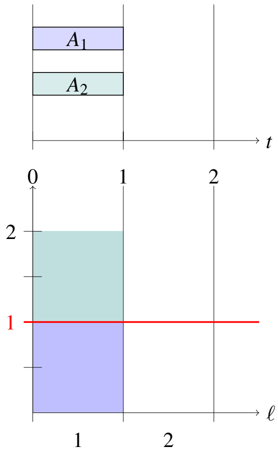
This paper focuses on the Periodically Aggregated Resource-Constrained Project Scheduling Problem (PARCPSP), introduced in Morin et al. (2017), that permits to address this modification of resource constraints, while considering the activity start times as continuous variables. In organizations, this model allows to consider decisions at an intermediate level between planning at the tactical level for the resource limitation constraints and scheduling at the operational level for time windows and precedence constraints.

Let us consider the following example (cf. figure 1), with a project composed of two activities and one resource (cf. figure 1a). Each activity has one unit processing time and a unit resource consumption, while the resource has one unit capacity. We suppose that the activities are not subject to a precedence relation. In the case of the RCPSP, the resource is disjunctive: the standard cumulative resource constraint forbids that the activity execution windows overlap, even partially. However, if these resource constraints are aggregated over time periods, e.g. of unit length, such that the total average request should not exceed the resource capacity, then there exists feasible schedules such that the two activities overlap, and even start and complete at the same moments in time. More precisely, among such schedules, some remain infeasible (cf. figure 1b), while others become feasible (cf. figure 1c). This example will be further commented in section 2.

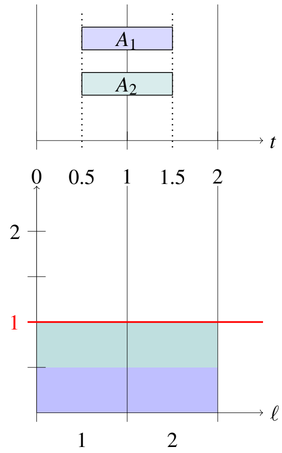
In Morin et al. (2017), a mixed-integer linear programming (MILP) formulation and heuristics are discussed, while the problem itself is conjectured to be NP-complete. The aim of the current paper is, first, to establish



(a) Project instance



(b) Infeasible solution



(c) Feasible solution

Figure 1: Example #1

the NP-completeness of the PARCPSP, with various restrictions on the input, to highlight non-standard structural properties of this problem and, second, to propose alternative mixed-integer linear programming formulations with tighter relaxations.

The paper is structured as follows. In section 2, the PARCPSP is defined formally and compared to traditional resource-constrained project scheduling problems. In section 3, the PARCPSP is proved to be strongly NP-hard, by focusing on the computational complexity characterization of three particular cases. In section 4, the MILP formulation proposed in Morin et al. (2017) is recalled and a new formulation is proposed. The new formulation is shown to dominate the previous formulation in terms of LP relaxation. Computational experiments to compare the new formulation with the previous one are given in section 5. Finally, in section 6, some concluding remarks are drawn and possible extensions of the problem are discussed.

2 PARCPSP – Problem statement

In this section, we formally introduce the problem studied. It is a variant of the extensively studied Resource Constrained Project Scheduling Problem (RCPSP), based on a temporal aggregation of resource constraints over periods defining a uniform subdivision of the time horizon, hence the name Periodically Aggregated Resource Constrained Project Scheduling Problem (PARCPSP).

2.1 Input and notations

The input of the problem can be split into two independent parts.

- On the one hand, a project instance X is considered. An activity set and a resource set are given. Activities require a given amount of capacity on some or all resources throughout their execution. They cannot be interrupted: preemption is not allowed. Precedence relations possibly exist between activities.

  The notations related to the project instance are listed hereafter.

$\mathscr{A}$	Finite set of n activities
$\mathscr{R}$	Finite set of m renewable resources
p_i	Processing time of activity $i \in \mathscr{A}$
b_k	Capacity of resource $k \in \mathscr{R}$
$r_{i,k}$	Request (demand) of activity $i \in \mathscr{A}$ on resource $k \in \mathscr{R}$
E	$\subseteq \mathscr{A} \times \mathscr{A}$; precedence relations (arc list)

  Let $\mathscr{X}$ the set of project instances.

- On the other hand, the time horizon is divided uniformly into L periods of parameterized length $\Delta \in \mathbb{R}_{>0}$. The convention chosen for period numbering is represented in figure 2.

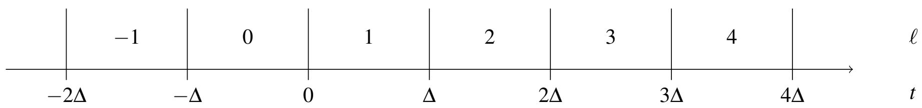


Figure 2: Uniform subdivision of the time horizon

A solution is a vector $S = (S_i)_{1 \leq i \leq n} \in \mathbb{R}^n$, where S_i is the start date of activity $i \in \mathscr{A}$. The start date and the completion date of the project are denoted by $S_0 = \min_{i \in \mathscr{A}}(S_i)$ and $S_{n+1} = \max_{i \in \mathscr{A}}(S_i + p_i)$, respectively.

2.2 A formulation of the PARCPSP

Let $S \in \mathbb{R}^n$ a solution of the PARCPSP. We consider two alternative objective functions linked to the temporal execution of the project.

$$C_{max} = S_{n+1} \qquad \text{(project makespan)}$$
$$dur(S) = S_{n+1} - S_0 \qquad \text{(project duration)}$$

Notice that, for the PARCPSP, unlike the RCPSP, these two objectives are not equivalent, because there is no guarantee that, for any of these objectives, at least one activity starts at $t = 0$ (hence $S_0 > 0$). This is further discussed in section 2.5. Moreover, if we set $S_0 \geq 0$, the problem defined with objective duration is indeed a relaxation of the one defined with objective makespan, since $S_{n+1} = C_{max}$.

Two families of constraints are taken into account.

1. Precedence constraints

   For each arc $(i_1, i_2) \in E$, activity i_1 has to complete before activity i_2 starts.

2. Resource constraints (periodically aggregated)

   For each resource $k \in \mathscr{R}$, in each period $\ell \in \mathbb{Z}$, the sum of the average requests of the activities cannot exceed the capacity of the resource.

Let $d_{i,\ell}(S) \in [0,\Delta]$ denote the execution duration of activity $i \in \mathscr{A}$ in period $\ell \in \mathbb{Z}$ depending on solution S, i.e., $d_{i,\ell}(S)$ is the length of the intersection of two intervals: the execution interval of activity i, and period ℓ (figure 3).

$$\begin{aligned} d_{i,\ell}(S) &= |[S_i, S_i+p_i] \cap [(\ell-1)\Delta, \ell\Delta]| \\ &= \max\Big(0\ ,\ \min\left(S_i+p_i\,,\,\ell\Delta\right) - \max\left(S_i\,,\,(\ell-1)\Delta\right)\Big) \end{aligned}$$

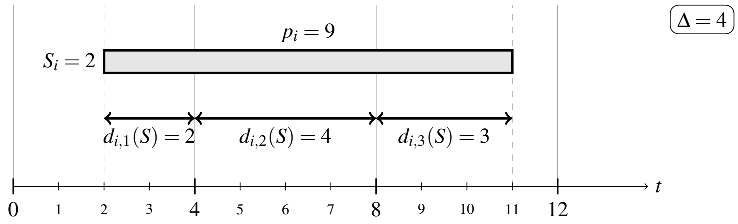


Figure 3: Evaluation of the execution duration in aggregated periods

Notice that, given a solution S, the expression of the average request of activity $i \in \mathscr{A}$ on resource $k \in \mathscr{R}$ over period $\ell \in \mathbb{Z}$ is $r_{i,k}\frac{d_{i,\ell}(S)}{\Delta}$. Therefore, the PARCPSP can be formulated as follows:

$$\begin{aligned} &\text{Minimize} && dur(S) && && (1)\\ &\text{subject to} && S_{i_2} - S_{i_1} \geq p_{i_1} && \forall (i_1,i_2) \in E && (2)\\ & && \sum_{i \in \mathscr{A}} r_{i,k}\frac{d_{i,\ell}(S)}{\Delta} \leq b_k && \forall k \in \mathscr{R}\ ,\ \forall \ell \in \mathbb{Z} && (3) \end{aligned}$$

Remark. Activities may start at any time within a period. In other words, the PARCPSP permits to tackle start and completion events in a precise way, as well as precedence constraints, while the resource consumption is evaluated on average over (aggregated) periods.

Remark. Notice that the formulation is easily adjusted when capacities depend on the period, in which case b_k is replaced by $b_{k,\ell}$. Non uniform period length can similarly be obtained by replacing Δ by Δ_ℓ.

In the following, the notation PARCPSP$[X,\Delta]$ is used to identify the problem instance considered, composed of a project $X \in \mathscr{X}$ and a period length $\Delta \in \mathbb{R}_{>0}$. Similarly, the notation RCPSP$[X]$ is used.

2.3 Conditions for the existence of feasible schedules

Let $X \in \mathscr{X}$ be a project instance and $\Delta \in \mathbb{R}_{>0}$ a (fixed) period length.

- The precedence constraints are satisfiable *iff* the precedence graph is acyclic.
- Let $i \in \mathscr{A}$. Let $k \in \mathscr{R}$. Let S denote a feasible solution of PARCPSP$[X,\Delta]$. Let $\ell_i = 1 + \left\lfloor \frac{S_i}{\Delta} \right\rfloor$ denote the period in which activity i starts (i.e. such that $(\ell_i - 1)\Delta \leq S_i < \ell_i\Delta$).
  - If $p_i \geq 2\Delta$

    The execution window $[S_i, S_i + p_i]$ fully includes period $\ell_i + 1$. So, in this period: $r_{i,k}d_{i,\ell_i+1}(S) = r_{i,k}\Delta \leq b_k\Delta$. Hence: $r_{i,k} \leq b_k$.

– If $p_i < 2\Delta$

The least restrictive configuration is such that the middle of the execution window is a period bound (otherwise by shifting the activity in any direction, the maximum overlapping with the left or the right period increases, which increases the maximum resource requirement of the activity among all periods) i.e.: $S_i + \frac{p_i}{2} = \ell_i\Delta$. In this case, $d_{i,\ell_i}(S) = d_{i,\ell_i+1}(S) = \frac{p_i}{2}$ while $d_{i,\ell}(S) = 0$ in all other periods $\ell \in \mathbb{Z} \smallsetminus \{\ell_i, \ell_i + 1\}$. In other words, the demand of activity i is split equally over two consecutive periods. So, the resource constraints in periods ℓ_i and $\ell_i + 1$ result in the same inequality: $r_{i,k}\frac{p_i}{2} \leq b_k\Delta$.

Hence: $r_{i,k} \leq b_k \frac{2\Delta}{p_i}$

Therefore, the (aggregated) resource constraints are satisfiable *iff* :

$$\forall i \in \mathscr{A} \quad \forall k \in \mathscr{R} \qquad r_{i,k} \leq b_k \max\left\{1, \frac{2\Delta}{p_i}\right\}$$

Remark If the project instance X satisfies the following conditions, then, whatever the value of Δ (period length), there exist feasible schedules.

- The precedence graph is acyclic.
- $\forall i \in \mathscr{A} \quad \forall k \in \mathscr{R} \qquad r_{i,k} \leq b_k$

2.4 Comparison with the RCPSP

A possible formulation for the RCPSP is:

$$\text{Minimize} \quad dur(S) \tag{4}$$

$$\text{subject to} \quad S_{i_2} - S_{i_1} \geq p_{i_1} \qquad \forall (i_1, i_2) \in E \tag{5}$$

$$\sum_{i \in \mathscr{A}_t(S)} r_{i,k} \leq b_k \qquad \forall k \in \mathscr{R}, \ \forall t \in \mathbb{R} \tag{6}$$

In this formulation, $\mathscr{A}_t(S)$ denote the set of activities in progress at $t \in \mathbb{R}$ depending on solution S.

$$\mathscr{A}_t(S) = \left\{ i \in \mathscr{A} \;\middle|\; t \in [S_i, S_i + p_i) \right\}$$

Notice that the only difference between the RCPSP and the PARCPSP lies in the definition of the resource constraints, which are evaluated either exactly at each instant $t \in \mathbb{R}$ or on average in each (aggregated) period $\ell \in \mathbb{Z}$ (see figure 4), which makes the PARCPSP a relaxation of the RCPSP.

Another way of viewing this is to consider the impact of very small Δ. Indeed, if Δ becomes small, then, given the definition of $d_{i,\ell}(S)$, intervals ℓ in which the activity is processed will be completely occupied by the activity, and hence feature $d_{i,\ell}(S) = 1$. This means that, as Δ becomes smaller, formulation (1)-(3) converges to formulation (4)-(6).

2.5 Impact of aggregation on resource feasibility

Finally, let us consider two simple examples, respectively in figure 1 page 3 (example 1, already investigated in the introduction), and in figure 5 page 8 (example 2).

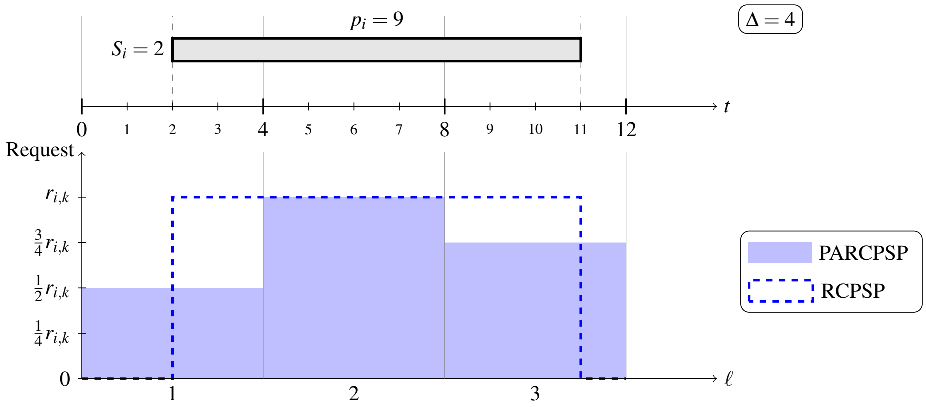


Figure 4: Evaluation of activity demands on resources

Example 1 We recall that, in this example, the project instance $X_1 \in \mathscr{X}$ (see figure 1a) is composed of a single resource of capacity 1 and two identical activities (same processing time equal to 1, same request on the resource equal to 1) with no precedence relations. We consider solutions such that the two activities are executed simultaneously, i.e. $S_1 = S_2$ (so, the project duration is equal to 1).

For the RCPSP$[X_1]$, such solutions are not feasible, since they violate resource constraints (evaluated at each instant). An optimal solution is obtained by executing one activity at a time, with no idle time; hence $\mathrm{Opt}(\mathrm{RCPSP}[X_1]) = 1 + 1 = 2$.

Let us consider a uniform subdivison of the temporal horizon, with periods of length $\Delta = 1$. What about the feasibility of such solutions for the PARCPSP$[X_1, 1]$?

- If $S_1 = S_2 = 0$, then both activities are completed at $t = 1$, i.e. both execution windows match exactly the first period ($\ell = 1$). In this period: $d_{1,1}(S) = d_{2,1}(S) = \Delta$. Therefore, the average request of each activity is equal to $\frac{\Delta}{\Delta} = 1$. So, as shown in figure 1b, the sum of the average requests in this period $(= 2)$ exceeds the capacity of the resource $(= 1)$. Hence this solution is not feasible.

- If $S_1 = S_2 = 0.5$, then both activities are completed at $t = 1.5$, i.e. both execution windows are split equally over two consecutive periods ($\ell \in \{1, 2\}$). In these periods: $d_{1,\ell}(S) = d_{2,\ell}(S) = \frac{\Delta}{2}$. Therefore, the average request of each activity is equal to $\frac{\Delta/2}{\Delta} = \frac{1}{2}$. So, as shown in figure 1c, the sum of the average requests in these periods $(= 1)$ does not exceed the capacity of the resource $(= 1)$. Hence this solution is feasible.

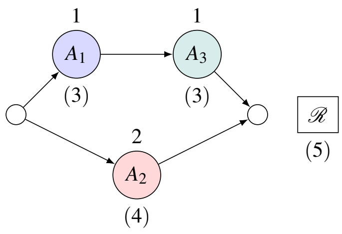
  Indeed, this solution is optimal (since the two activities run in parallel, no other configuration can lead to a shorter project duration).

This first example enhances the following points.

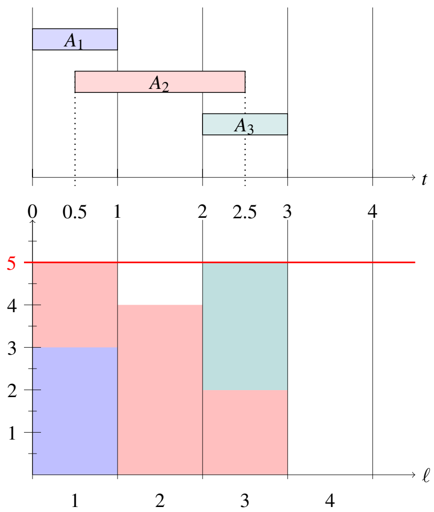
- Even when resources have a constant capacity over time, in the case of the PARCPSP, unlike the RCPSP, shifting a schedule can affect its feasibility.

- The gap between the optimum of the RCPSP and the PARCPSP can be large (here 50%) even with unit periods ($\Delta = 1$). In fact the example shows that the standard resource capacity lower bound equal to $\max_{k \in \mathscr{R}} \sum_{i \in \mathscr{A}} r_{i,k} p_i / b_k$ is not a valid lower bound for the PARCPSP.

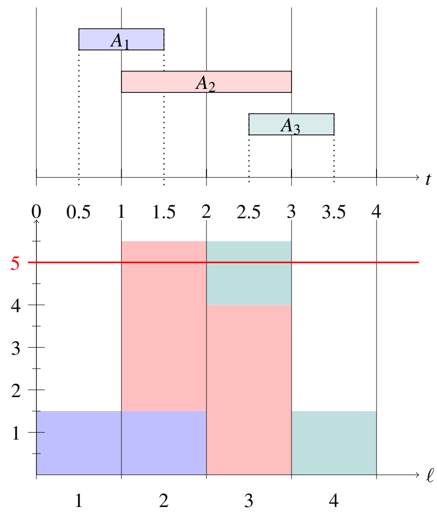
For this particular instance, the project duration is reduced by dispatching the average requests equally over two consecutive periods. However, the rule “the more periods used, the shorter the project duration” does not apply to all instances, as shown in the next example.



(a) Project instance



(b) Feasible solution



(c) Infeasible solution

Figure 5: Example #2

Example 2 The project instance $X_2 \in \mathscr{X}$ is composed of a single resource with capacity 5 and three activities with one precedence relation (see figure 5a for the numerical parameter values). We still consider unit periods ($\Delta = 1$).

- As shown in figure 5b, the solution $(0, 0.5, 2)$ is feasible. It is not optimal: one can shorten the project duration by shifting activity 1 to the right for an amount of 1/6 and shifting activity 3 to the left for an amount of 1/6, leading to a duration of 2.5.

  Notice that 3 periods are intersected by at least one activity execution window.

- As shown in figure 5c, the solution $(0.5, 1, 2.5)$, obtained by shifting the previous solution by $+0.5$, is not feasible. It is possible to repair it, by shifting activities 1 and 3 by $-\frac{1}{6}$ and $+\frac{1}{6}$, respectively, thus enlarging the project duration by $\frac{1}{3}$ but now using 4 periods.

Therefore, we showed that the feasibility of a schedule depends not only on the relative positions of activity execution windows as in the RCPSP, but also on their absolute positions, which determines the average resource usage in aggregated periods. This problem has consequently fundamental differences with the related RCPSP.

3 Complexity

The complexity of the problem was left open in Morin et al. 2017. This section first establishes that the problem is in NP, even if the time horizon is not part of the input. Then, the computational complexity of three particular cases are considered, which yields the complexity result for the PARCPSP.

3.1 Inclusion in NP

To check whether a start time solution vector is feasible w.r.t. a fixed makespan requires checking the resource and the precedence constraints. Since the number of time periods over a time horizon which covers both S_0 and S_{n+1} may not be bounded by a polynomial in the size of the input, computing the average resource consumption in every time period is not a viable approach to check the feasibility of a schedule in polytime. Testing resource constraints only in periods where the resource usage increases is sufficient. Recall that activities may start at any time in a period, and that preemption is not allowed. Therefore, each activity $i \in \mathscr{A}$ may increase the resource usage only in two periods: the period when it starts $\left(\ell_i = 1 + \lfloor \frac{S_i}{\Delta} \rfloor\right)$, and possibly the next one. So, for each activity, at most two periods have to be checked. A single test in a given period on a given resource consists in verifying that the sum of the mean demands of the activities is not greater than the capacity of the resource.

This yields an algorithm in $\mathscr{O}(1 + |E| + m \times 2n \times n)$, thus polynomial in the input size.

3.2 One resource, constant capacity

Theorem 1. *The PARCPSP with makespan objective and a single resource of fixed capacity $b \geq 2$ is weakly NP-hard.*

Proof. We show that PARTITION $\leq_P$ PARCPSP. In the PARTITION problem (Karp 1972), we have n items, and each item i has a size $a_i \geq 1$. All the data is integral, and $\sum_i a_i$ is an even integer. Is there a partitioning of the items into two subsets S_1 and S_2, such that $S_1 \cap S_2 = \emptyset$, $S_1 \cup S_2 = \{1, \ldots, n\}$, and $\sum_{i \in S_1} a_i = \sum_{i \in S_2} a_i$?

For any instance of the PARTITION problem, we define an instance of PARCPSP as follows. There is a single resource of capacity 2. There are n activities, activity i corresponds to item i in the PARTITION problem instance, and it has processing time $p_i := 2a_i$. The resource requirement of each activity is 1 from the single resource during its execution. We let $\Delta = 1$. We claim that the PARTITION problem instance has a YES answer if and only if the corresponding instance of PARCPSP admits a feasible schedule of length $(\sum_{i=1}^n p_i)/2$.

First suppose that the PARTITION problem instance has a YES answer. Then it must be the case that $\sum_{i \in S_1} p_i = \sum_{i \in S_2} p_i = (\sum_{i=1}^n p_i)/2$. We define the following schedule: the activities corresponding to the items in S_1 are scheduled in a single sequence from time 0 onwards. Notice that each activity starts and ends at integral time points. This sequence occupies one unit of the resource from time 0 to time $(\sum_{i=1}^n p_i)/2$. Now schedule all the activities corresponding to the items in S_2 in any sequence from time 0 onwards. Again, this sequence finishes at time $(\sum_{i=1}^n p_i)/2$. Since $\Delta = 1$, the total capacity of the resource is 2 in each interval $[t-1, t]$. Further on, in each interval $[t-1, t]$ with $t \leq (\sum_{i=1}^n p_i)/2$, the total resource usage is 2, because exactly two activities are processed in the intervals, each requiring one unit from the resource. Therefore, the schedule is feasible, and all jobs are completed by time $(\sum_{i=1}^n p_i)/2$.

Conversely, suppose there is a feasible schedule of length $(\sum_{i=1}^{n} p_i)/2$.

Claim 1a. *In any feasible schedule of length $(\sum_{i=1}^{n} p_i)/2$, exactly two units of resource are used in each interval.*

Proof. The total resource requirement of the activities is $\sum_{i=1}^{n} p_i$. Since the total capacity of the resource from time 0 to time $(\sum_{i=1}^{n} p_i)/2$ is equal to $(\sum_{i=1}^{n} p_i)$, the claim follows. ■

Claim 1b. *Exactly two activities start at time 0.*

Proof. Suppose it is not the case. Observe that there can be at most two activities processed in the interval $[0,1]$, because if there were 3 or more activities starting in the interval $[0,1]$, then all these 3 or more activities should be processed throughout the interval $[1,2]$, as each activity is of length 2 or more ($p_i = 2a_i$, and $a_i \geq 1$). But this is impossible, because the resource has capacity 2, and the activities would require 3 or more units of the resource. Now suppose that less than 2 activities start at time 0. Then the resource usage of the activities in interval $[0,1]$ must be less than 2, which contradicts claim 1a. ■

So far we have shown that exactly two activities start at time 0 in the feasible schedule. Since the processing times are integral, these two activities finish at integral time points, at t_1 and t_2, say. If $t_1 = t_2$, then we can repeat the same argument to show that there are exactly two activities starting right at time $t_1 = t_2$. If $t_1 \neq t_2$, then without loss of generality, $t_1 < t_2$. Then in the interval $[t_1, t_1+1]$, one unit of the resource is used by the activity which is still in progress. Since both t_1 and t_2 are divisible by 2 (as each p_i is divisible by 2), $t_2 \geq t_1 + 2$, and again, at most one activity may start in the interval $[t_1, t_1+1]$, otherwise in the interval $[t_1+1, t_1+2]$ the total resource usage would be more than 2. It follows that a new activity must be started at time t_1, otherwise in the interval $[t_1, t_1+1]$, less than 2 units of the resource would be used by the feasible schedule, which would contradict claim 1a. Proceeding in this way, we prove that all the activities start at integral time points, and at any time, at most two activities are processed. Hence, the schedule can be decomposed into two sequences of activities, S_1 and S_2, each of the same total length $(\sum_{i=1}^{n} p_i)/2$. Then S_1 and S_2 give rise to a partitioning of the items such that $\sum_{i \in S_1} a_i = \sum_{i \in S_2} a_i$. Hence, the instance of the PARTITION problem has answer YES. □

The reduction does not hold for the duration $(S_{n+1} - S_0)$ objective. Consider the simple multiset of 5 elements $\{1,1,1,1,1\}$. Obviously, there is no partition of such set. If we now consider the PARCPSP obtained by the reduction, we obtain a set of 5 tasks of duration 2, each having a unit resource requirement on a resource of capacity 2 and we also have $\Delta = 1$. Fig. 6 displays a feasible schedule of duration $S_{n+1} - S_0 = 5 = \sum_{i=1}^{5} p_i/2$.

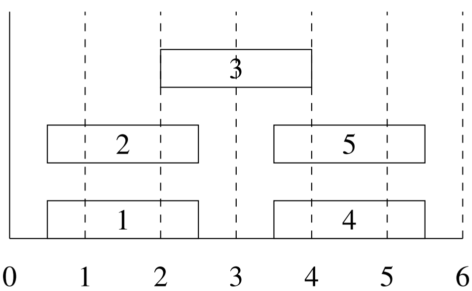


Figure 6: Feasible schedule, no partition

3.3 One resource, arbitrary capacity

The previous reduction (from the PARTITION problem) can be transformed slightly to derive the strong NP-hardness of the PARCPSP with objective makespan from the 3-PARTITION problem (Garey and Johnson 1979), when considering a single resource with arbitrary capacity.

Theorem 2. *The PARCPSP with makespan objective and a single resource is strongly NP-hard when capacity b is part of the input.*

Proof. Let us show that 3-PARTITION $\leq_P$ PARCPSP. Given $3n$ items of integral size a_i such that $\sum_i a_i = nD$, and $\frac{D}{4} < a_i < \frac{D}{2}$ for all $i \in \{1,\ldots,3n\}$, the 3-PARTITION problem consists in determining whether a partitioning of the items into n pairwise disjoint triples $T_1,\ldots,T_n$ of equal sum, i.e. $\sum_{i\in T_j} a_i = D$ for all $j \in \{1,\ldots,n\}$, exist.

The reduction from the 3-PARTITION problem to PARCPSP is almost the same as the previous reduction from PARTITION. Each item i is converted into an activity with processing time $p_i := 2a_i$, and a resource requirement of 1; we only change the capacity of the single resource, setting it to n (previously set to 2).

Let us show that the 3-PARTITION problem instance has a YES answer if and only if the corresponding instance of PARCPSP admits a feasible schedule of length $2D = (\sum_{i=1}^{n} p_i)/n$.

First suppose that the 3-PARTITION problem instance has a YES answer. A similar reasoning as the one presented in the previous proof entails that scheduling the activities corresponding to a triple T_j in any order from time 0 on in a single sequence yields a feasible schedule such that all jobs complete by time $2D$.

Conversely, suppose there is a feasible schedule of length $2D$. Claims 1a and 1b can be adapted seamlessly as follows.

Claim 2a (generalization of claim 1a)**.** *In each interval $[t-1,t]$ for $t \in \{1,\ldots,2D\}$, exactly n units of the resource is used.*

Claim 2b (generalization of claim 1b)**.** *Exactly n activities start at time 0.*

Moreover, there is no interval $[t-1,t]$, with $t \in \{1,...,2D\}$, containing a moment during which less than n activities are active (an activity being active at moment t if $S_j \leq t < S_j + p_j$). This can be seen using a contradiction argument; suppose there is an interval containing a moment with less than n activities being active. Since, in this interval, n units of resource must be used (after claim 2a), there must also be a moment in this interval in which more than n activities are active. But that implies that a neighboring interval must feature more than n activities active during that whole interval (since $p_j \geq 2$ and $\Delta = 1$), thereby exceeding the available capacity, which contradicts claim 2a.

Therefore, at each instant in $[0,2D]$, exactly n activities are active. Hence, the instance of the 3-PARTITION problem has answer YES. □

3.4 Multiple resources, constant capacities

The third reduction, inspired from Blazewicz et al. (1983), establishes the strong NP-hardness of the PARCPSP with objective duration or makespan for instances with an unlimited number of resources with constant capacities.

The proof presented hereafter considers the objective duration; notice that the proof for the objective makespan is very similar, because claim 3a holds regardless of the actual objective.

Theorem 3. *The PARCPSP with duration or makespan objective and unlimited number of resources with constant capacities is strongly NP-hard.*

Proof. We establish that Chromatic Number $\leq_P$ PARCPSP. Given a non-oriented graph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$, the Chromatic Number problem (Karp 1972) consists in coloring the vertices of $\mathscr{G}$ using a minimum number of colors $(c_j)_{j \in \mathscr{V}}$ so that no two adjacent vertices are assigned the same color. Let $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ a non-oriented graph. Let $\Delta \in \mathbb{R}_{>0}$ (e.g. $\Delta = 1$). Let $X(\mathscr{G}) \in \mathscr{X}$ the project instance defined by:

- $\mathscr{A} = \mathscr{V}$ (activity = vertex)
- $\mathscr{R} = \mathscr{E}$ (resource = edge)
- $\forall i \in \mathscr{A} \quad p_i = 2\Delta$
- $\forall k \in \mathscr{R} \quad b_k = 1$
- $\forall i \in \mathscr{A} \quad \forall k \in \mathscr{R} \quad r_{i,k} = 1$ if vertex i is one of the two extremities of edge k, 0 otherwise
- $E = \emptyset$ (no precedence relations)

Clearly, this is a polynomial time reduction; so, the theorem holds if the following assertions are equivalent.

1. $\mathscr{G}$ admits a feasible coloring c such that:
   $\max(c_j)_{1 \leq j \leq n} \leq \gamma$
2. There exists a feasible schedule S such that:
   $dur(S) \leq 2\gamma\Delta$

Suppose $\mathscr{G}$ admits a feasible coloring c such that $\max(c_j)_{1 \leq j \leq n} \leq \gamma$. Let S be the schedule defined by:

$$\forall i \in \mathscr{A} \quad S_i = 2(c_i - 1)\Delta$$

Given an edge (resource), its extremities (the two activities that require it) are colored differently (are not executed simultaneously, since processing times are all equal to 2Δ). So, S is feasible for the PARCPSP (indeed, it is even feasible for the RCPSP). Moreover:

$$dur(S) = S_{n+1} - S_0 \leq 2\gamma\Delta - 0 = 2\gamma\Delta$$

Hence, the direct implication holds.

Conversely, suppose there exists a feasible schedule S such that $dur(S) \leq 2\gamma\Delta$. Without loss of generality, the project execution starts in period $\ell = 1$, i.e., $0 \leq S_0 < \Delta$.

Claim 3a. *The execution windows of the two activities that share a common resource are disjoint.*

$$\forall (i_1, i_2) \in \mathscr{R} \quad (S_{i_1} + p_{i_1} \leq S_{i_2}) \vee (S_{i_2} + p_{i_2} \leq S_{i_1})$$

Proof. Let $k = (i_1, i_2) \in \mathscr{R}$. Suppose that $S_{i_1} \leq S_{i_2}$. For $i \in \mathscr{A}$, let $\ell_i = 1 + \left\lfloor \frac{S_i}{\Delta} \right\rfloor$ denote the period in which activity i starts. Notice that $\ell_{i_1} \leq \ell_{i_2}$.

For any activity $i \in \mathscr{A}$, including i_1 and i_2, since $p_i = 2\Delta$, one can determine bounds on $d_{i,\ell}(S)$ (see also figure 7):

$$\forall \ell \in \mathbb{Z} \quad d_{i,\ell}(S) \begin{cases} \in (0, \Delta] & \text{if } \ell = \ell_i \\ = \Delta & \text{if } \ell = \ell_i + 1 \\ \in [0, \Delta) & \text{if } \ell = \ell_i + 2 \\ = 0 & \text{otherwise} \end{cases}$$

Indeed: $d_{i,\ell_i+2}(S) = \Delta - d_{i,\ell_i}(S)$

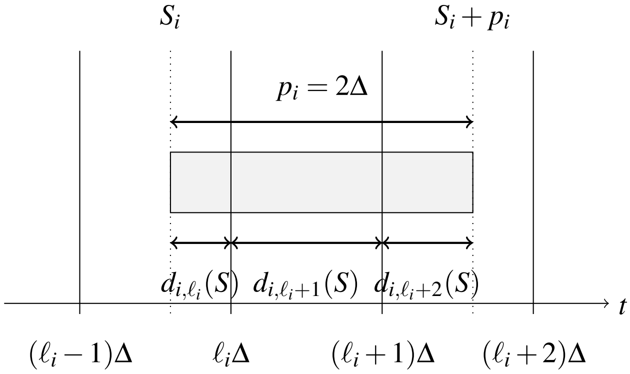


Figure 7: Execution interval (PARCPSP complexity proof)

Moreover, S is feasible; the resource constraints state that, in any period $\ell \in \mathbb{Z}$:

$$d_{i_1,\ell}(S) + d_{i_2,\ell}(S) \leq \Delta$$

- Suppose that: $\ell_{i_2} = \ell_{i_1}$

  Then, in period $\ell = \ell_{i_1} + 1 = \ell_{i_2} + 1$:

  $$d_{i_1,\ell}(S) + d_{i_2,\ell}(S) = 2\Delta > \Delta$$

  Therefore, this configuration cannot occur.

- Suppose that: $\ell_{i_2} = \ell_{i_1} + 1$

  Then, in period $\ell = \ell_{i_1} + 1 = \ell_{i_2}$:

  $$d_{i_1,\ell}(S) + d_{i_2,\ell}(S) = \Delta + d_{i_2,\ell_{i_2}}(S) > \Delta$$

  Therefore, this configuration cannot occur.

- Suppose that: $\ell_{i_2} = \ell_{i_1} + 2$

  Then, in period $\ell = \ell_{i_1} + 2 = \ell_{i_2}$:

  $$\begin{aligned} & d_{i_1,\ell}(S) \leq \Delta - d_{i_2,\ell}(S) \\ \Leftrightarrow \quad & (\ell - 1)\Delta + d_{i_1,\ell}(S) \leq \ell\Delta - d_{i_2,\ell}(S) \\ \Leftrightarrow \quad & (\ell_{i_1} + 1)\Delta + d_{i_1,\ell_{i_1}+2}(S) \leq \ell_{i_2}\Delta - d_{i_2,\ell_{i_2}}(S) \\ \Leftrightarrow \quad & S_{i_1} + p_{i_1} \leq S_{i_2} \end{aligned}$$

- Suppose that: $\ell_{i_2} \geq \ell_{i_1} + 3$

  Then:

$$S_{i_1} + p_{i_1} < (\ell_{i_1} + 2)\Delta \leq (\ell_{i_2} - 1)\Delta \leq S_{i_2}$$

It follows that $S_{i_1} \leq S_{i_2} \Rightarrow S_{i_1} + p_{i_1} \leq S_{i_2}$. Hence, the claim holds. ■

Let c the coloring defined by:

$$\forall j \in \mathscr{V} \quad c_j = 1 + \left\lfloor \frac{S_j}{2\Delta} \right\rfloor$$

Let $(j_1, j_2) \in \mathscr{E}$. Recall that processing times are all equal to 2Δ ; so, after claim 3a, $\left|S_{j_2} - S_{j_1}\right| \geq 2\Delta$. By construction, $\left|c_{j_2} - c_{j_1}\right| \geq 1$ i.e. $c_{j_1} \neq c_{j_2}$. Therefore, c is feasible.

Since $dur(S) \leq 2\gamma\Delta$:

$$\forall i \in \mathscr{A} \quad S_0 \leq S_i \leq S_{n+1} - p_i \leq (S_0 + 2\gamma\Delta) - 2\Delta$$

Consequently, $1 \leq c_j \leq \gamma$ for all $j \in \mathscr{V}$, and $\max(c_j)_{1 \leq j \leq n} \leq \gamma$.

Hence, the reciprocal implication also holds. □

3.5 General case

The general result comes from the reductions provided for the three particular cases. In table 1, an arrow points to a more general/less restricted context for makespan minimization. Hence, the destination problem is at least as difficult as the origin problem. The grey boxes correspond to complexity results holding also for duration minimization.

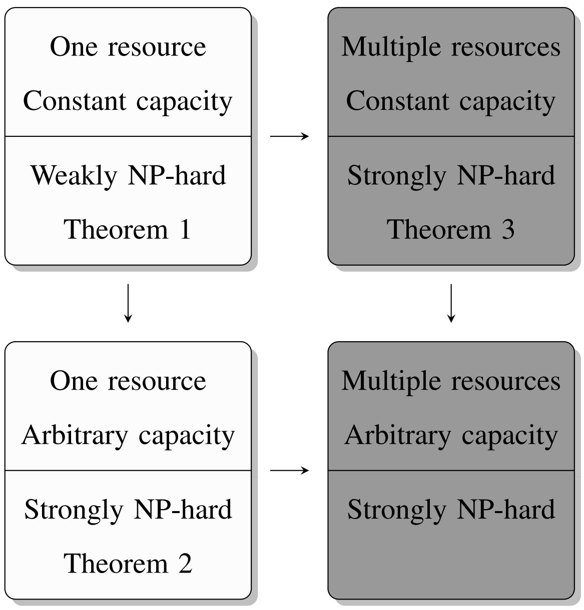


Table 1: Summary of the results on the NP-hardness of the PARCPSP for makespan minimization

It follows that the PARCPSP is strongly NP-hard in the general case. In the remaining of the paper, solution approaches are investigated.

4 A new mixed-integer linear programming formulation

In this Section, we consider mixed-integer linear programming formulations for the problem. Continuous variables are used to represent activity starting times while period-indexed variables allow to model the aggregated resource constraints. We consider two formulations and their strenghtened variants. The first one was proposed by Morin et al. (2017) and the second one is a new formulation based on the decomposition of a period relatively to the execution of an activity. Both formulations can be strengthened by using bounds on the number of periods possibly intersected by an activity. In addition, the new formulation allows to use disaggregated precedence constraints. We show that the disaggregated second formulation is stronger than the first one in terms of LP relaxation.

4.1 First formulation

4.1.1 Variables

The decision variables used in the model proposed by Morin et al. (2017) are summarized in table 2. A continuous start time variable S_i gives the start time of each activity $i \in \mathscr{A}$ while a continuous variable $d_{i,\ell}$ gives the length of the intersection of the time window of activity $i \in \mathscr{A}$ with period $\ell \in \mathscr{L}$. Two period-indexed binary step variables $zs_{i,\ell}$ and $zf_{i,\ell}$ are used to mark the first and last periods of an activity. An illustration of the link between these variables is given in figure 8.

$S_i \geq 0$	Start time of Activity $i \in \mathscr{A}$ S_0 (resp S_{n+1}) represents the start (resp the end) of the project.
$d_{i,\ell} \in [0, \Delta]$	intersection length of intervals $[S_i, S_i + p_i]$ and $[(\ell-1)\Delta, \ell\Delta]$
$zs_{i,\ell} \in \{0,1\}$	Binary *step* variables: $zs_{i,\ell-1} \leq zs_{i,\ell}$ $zs_{i,\ell} = 1$ if S_i is in period ℓ, i.e. $S_i \in [(\ell-1)\Delta, \ell\Delta]$
$zf_{i,\ell} \in \{0,1\}$	Binary *step* variables: $zf_{i,\ell-1} \leq zf_{i,\ell}$ $zf_{i,\ell} = 1$ if $S_i + p_i$ is in period ℓ, i.e. $S_i + p_i \in [(\ell-1)\Delta, \ell\Delta]$

Table 2: Variables of the first period-indexed formulation

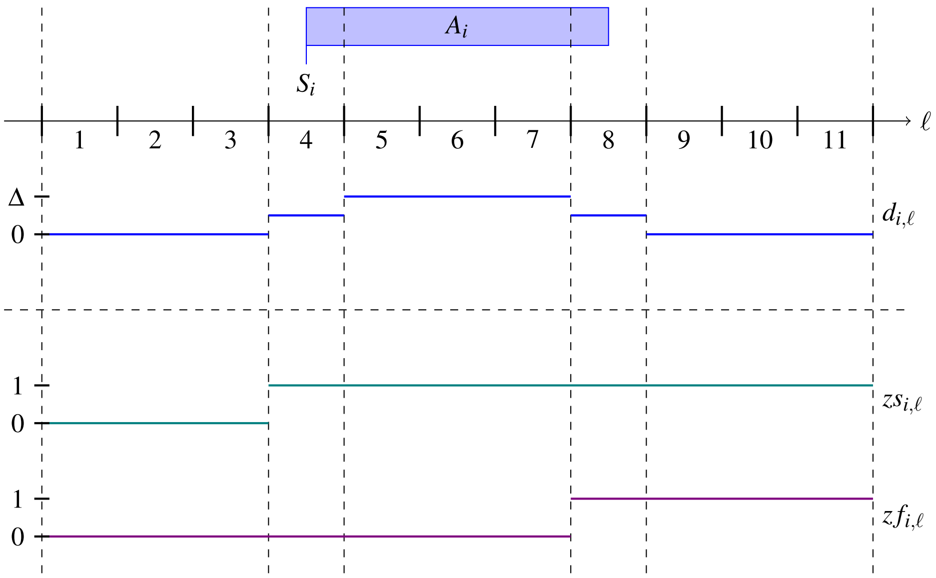


Figure 8: Representation of an execution time window with the variables of the first period-indexed formulation

4.1.2 Initial formulation

We recall below the main constraints of the formulation proposed by Morin et al. (2017), the domains of the decision variables being those of table 2.

$$(F1) \quad \text{Minimize} \quad S_{n+1} - S_0 \tag{7}$$

$$S_{i_2} - S_{i_1} \geq p_{i_1} \qquad \forall (i_1, i_2) \in E \tag{8}$$

$$\sum_{i \in \mathscr{A}} r_{i,k} d_{i,\ell} \leq b_k \Delta \qquad \forall k \in \mathscr{R}, \forall \ell \in \mathscr{L} \tag{9}$$

$$\ell\Delta(1 - zs_{i,\ell}) \leq S_i \leq L\Delta - (L-\ell)\Delta zs_{i,\ell} \qquad \forall i \in \mathscr{A}, \forall \ell \in \mathscr{L} \tag{10}$$

$$\ell\Delta(1 - zf_{i,\ell}) \leq S_i + p_i \leq L\Delta - (L-\ell)\Delta zf_{i,\ell} \qquad \forall i \in \mathscr{A}, \forall \ell \in \mathscr{L} \tag{11}$$

$$\Delta(zs_{i,\ell-1} - zf_{i,\ell}) \leq d_{i,\ell} \leq \Delta(zs_{i,\ell} - zf_{i,\ell-1}) \qquad \forall i \in \mathscr{A}, \forall \ell \in \mathscr{L} \tag{12}$$

$$d_{i,\ell} \geq \ell\Delta - S_i - \Delta zf_{i,\ell} - \ell\Delta zs_{i,\ell-1} \qquad \forall i \in \mathscr{A}, \forall \ell \in \mathscr{L} \tag{13}$$

$$d_{i,\ell} \geq S_i + p_i - (\ell-1)\Delta - \Delta(1 - zs_{i,\ell-1}) - (L-\ell+1)\Delta(1 - zf_{i,\ell}) \qquad \forall i \in \mathscr{A}, \forall \ell \in \mathscr{L} \tag{14}$$

$$\sum_{\ell \in \mathscr{L}} d_{i,\ell} = p_i \qquad \forall i \in \mathscr{A} \tag{15}$$

Objective (7) minimizes the project duration, under precedence constraints (8) and aggregated resource constraints (9). Constraints (10) link start time variables S_i and variables $zs_{i,\ell}$, while constraints (11) link completion time variables $S_i + p_i$ to variables $zf_{i,\ell}$.

The remaining constraints allow to compute the intersection lengths $d_{i,\ell}$. Constraints (12) enforce $d_{i,\ell}$ to take value 0 when period ℓ is either before or after the execution interval of activity i, and value Δ when period ℓ is integrally included in the execution interval of i. Constraints (13) allow to compute $d_{i,\ell}$ when ℓ is the period that contains S_i, while $S_i + p_i$ belongs to a period $\ell' > \ell$. Constraints (14) allow to compute $d_{i,\ell}$ when ℓ is the period that contains $S_i + p_i$ while S_i belongs to a period $\ell' < \ell$. Constraints (15) state that the sum of the intersection lengths of activity i over all the periods must be equal to the processing time of i. These constraints are necessary to compute the correct d_{il} when the duration of an activity is lower than Δ and the activity is fully included in one period (see proof of Theorem 4 for further details).

Theorem 4. *Formulation $(F1)$ is a correct formulation for the PARCPSP*

Proof is given in Appendix A.

4.1.3 Strengthening the first formulation

Morin et al. 2017 proposed to strengthen the formulation as follows. Since all periods have the same duration Δ, starting the project in the first period is a dominant policy. Hence the following constraint is valid.

$$0 \leq S_0 \leq \Delta \tag{16}$$

Furthermore, since preemption is not allowed, the number of periods intersected by an activity is bounded as stated by the following theorem. As in the proof of Theorem 4, let us define the first period of an activity ℓs^i as the one that satisfies $(\ell s^i - 1)\Delta \leq S_i < \ell s^i \Delta$ and let last period of an activity ℓf^i be defined by $(\ell f^i - 1)\Delta \leq S_i + p_i < \ell f^i \Delta$.

Lemma 1. *The first and the last period of an activity are such that either $\ell f^i = \ell s^i + \lfloor \frac{p_i}{\Delta} \rfloor$ or $\ell f^i = \ell s^i + \lceil \frac{p_i}{\Delta} \rceil$.*

Proof. Since we have $(\ell s^i - 1)\Delta \leq S_i < \ell s^i \Delta$, it follows:

$$(\ell s^i - 1)\Delta + p_i \leq S_i + p_i < \ell s^i \Delta + p_i$$

$$\Leftrightarrow (\ell s^i - 1 + \frac{p_i}{\Delta})\Delta \leq S_i + p_i < (\ell s^i + \frac{p_i}{\Delta})\Delta$$

$$\Rightarrow (\ell s^i - 1 + \left\lfloor \frac{p_i}{\Delta} \right\rfloor)\Delta \leq S_i + p_i < (\ell s^i + \left\lceil \frac{p_i}{\Delta} \right\rceil)\Delta,$$

which yields the desired result. □

In the proof of Theorem 4 (Appendix A), we show that for any solution S_i, $i \in \mathscr{A}$, a compatible assignment of the other variables can be obtained by setting $zs_{i,\ell} = 0$ for each $\ell < \ell s^i$, $zs_{i,\ell} = 1$ for each $\ell \geq \ell s^i$, $zf_{i,\ell} = 0$ for each $\ell < \ell f^i$ and $zf_{i,\ell} = 1$ for each $\ell \geq \ell f^i$.

Hence, a consequence of Lemma 1 is that variables $zs_{i,\ell}$ and $zf_{i,\ell}$ can be linked via a binary variable π_i (only one binary variable per activity), such that:

$$\begin{aligned} \pi_i = 0 \quad &\Leftrightarrow \quad zs_{i,\ell} = zf_{i,\ell + \lfloor \frac{p_i}{\Delta} \rfloor} \\ \pi_i = 1 \quad &\Leftrightarrow \quad zs_{i,\ell} = zf_{i,\ell + \lceil \frac{p_i}{\Delta} \rceil} \end{aligned}$$

In this case the integrality constraint on variables $zf_{i,\ell}$ can be relaxed and the linking constraint can be easily linearized by the adjunction of the following constraints:

$$zf_{i,\ell} \in [0,1] \quad \forall i \in \mathscr{A}, \forall \ell \in \mathscr{L} \tag{17}$$

$$\pi_i \in \{0,1\} \quad \forall i \in \mathscr{A} \tag{18}$$

$$zs_{i,\ell} \geq zf_{i,\ell+\lfloor \frac{p_i}{\Delta} \rfloor} \quad \forall i \in \mathscr{A}, \forall \ell \in \mathscr{L} \tag{19}$$

$$zs_{i,\ell} \leq zf_{i,\ell+\lceil \frac{p_i}{\Delta} \rceil} \quad \forall i \in \mathscr{A}, \forall \ell \in \mathscr{L} \tag{20}$$

$$zs_{i,\ell} \leq zf_{i,\ell+\lfloor \frac{p_i}{\Delta} \rfloor} + \pi_i \quad \forall i \in \mathscr{A}, \forall \ell \in \mathscr{L} \tag{21}$$

$$zs_{i,\ell} \geq zf_{i,\ell+\lceil \frac{p_i}{\Delta} \rceil} + \pi_i - 1 \quad \forall i \in \mathscr{A}, \forall \ell \in \mathscr{L} \tag{22}$$

If the number of periods is large this reduces considerably the number of explicit binary variables of the problem.

Remark. If activity $i \in \mathscr{A}$ is such that $p_i \bmod \Delta = 0$, the first and the last period of this activity are such that $\ell f^i = \ell s^i + \frac{p_i}{\Delta}$. Then there is no need to introduce variable π_i and the above-defined constraints can be simply replaced by:

$$zs_{i,\ell} = zf_{i,\ell+\frac{p_i}{\Delta}} \quad \forall i \in \mathscr{A}, \forall \ell \in \mathscr{L} \tag{23}$$

We denote by $(F1s)$ the strengthened formulation of Morin et al. (2017).

4.2 An alternative formulation

4.2.1 Variables description

In time-indexed formulations of scheduling problems, precedence constraints expressed directly under the form of constraints (8) are called aggregated precedence constraints. There exists indeed a disaggregated form of these precedence constraints that strengthen the relaxation (see e.g. Artigues (2017)). We show in this section that a disaggregated form of the precedence constraints can be proposed for the PARCPSP despite the continuous nature of the start time variables. For each activity $i \in \mathscr{A}$ and each time period $\ell \in \mathscr{L}$, let us define new variables $\lambda_{i,\ell}$ and $\mu_{i,\ell}$ such that

$$\lambda_{i,\ell} = |[0,S_i] \cap [(\ell-1)\Delta, \ell\Delta]| \text{ and } \mu_{i,\ell} = |[S_i + p_i, L\Delta] \cap [(\ell-1)\Delta, \ell\Delta]|.$$

The other decision variables used in the new model are described in table 3.

S_i	Start time of activity $i \in \mathscr{A}$ S_0 (respectively S_{n+1}) represents the start (resp. the end) of the project.
$d_{i,\ell}$	intersection length of intervals $[S_i, S_i + p_i]$ and $[(\ell-1)\Delta, \ell\Delta]$
$\lambda_{i,\ell}$	intersection length of intervals $[0, S_i]$ and $[(\ell-1)\Delta, \ell\Delta]$
$\mu_{i,\ell}$	intersection length of intervals $[S_i + p_i, L\Delta]$ and $[(\ell-1)\Delta, \ell\Delta]$
$z^{\lambda}_{i,\ell}$	Binary variables ensuring a decreasing step behavior for variables $\lambda_{i,\ell}$
$z^{\mu}_{i,\ell}$	Binary variables ensuring an increasing step behavior for variables $\mu_{i,\ell}$

Table 3: Variables of the second period-indexed formulation

From this definition it immediately follows that $\lambda_{i,\ell}$ is a decreasing step function of ℓ, while, symmetrically, $\mu_{i,\ell}$ is an increasing step function of ℓ. In the case that $p_i \geq \Delta$, illustrated by Figure 9 for activity A_i, $\lambda_{i,\ell}$ is equal to Δ for each period $\ell < \ell s^i$, then equal to $\Delta - d_{i,\ell}$ for $\ell = \ell s^i$ and finally equal to 0 for $\ell > \ell s^i$. Under the same condition ($p_i \geq \Delta$), $\mu_{i,\ell}$ is equal to 0 for $\ell < \ell f^i$, then equal to $\Delta - d_{i,\ell}$ for $\ell = \ell f^i$ and finally equal to Δ for $\ell > \ell f^i$.

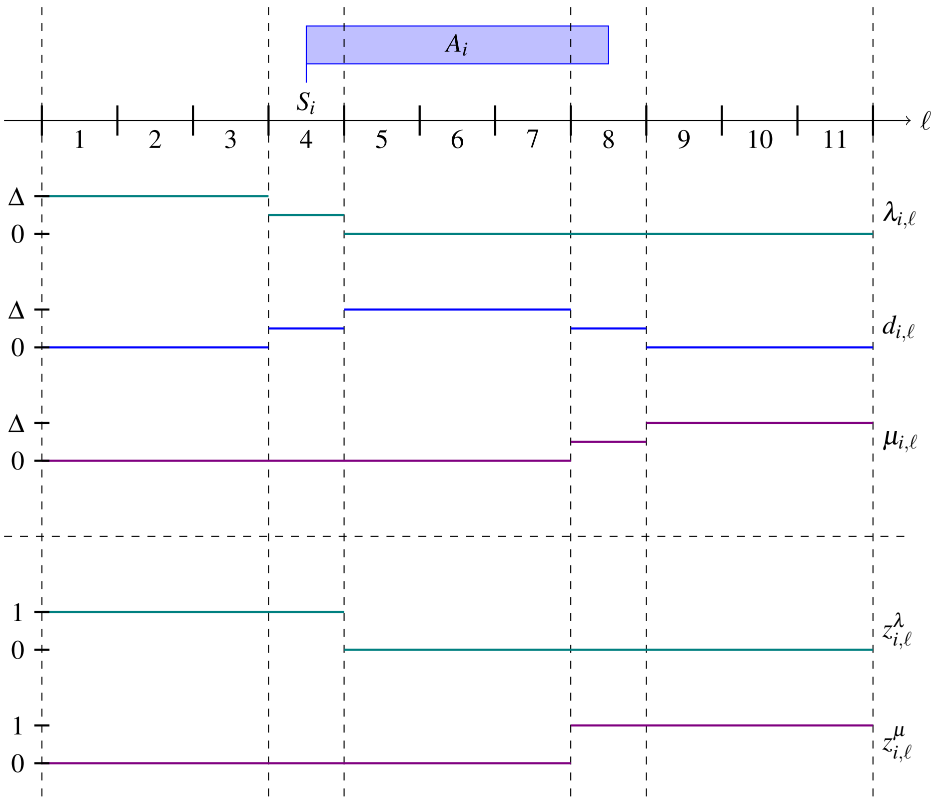


Figure 9: Scheduling variables of an activity for the second period-indexed formulation with $p_i \geq \Delta$

In the case where $p_i < \Delta$ and if the execution of activity i overlaps a period change (precisely $\ell s^i \Delta \in [S_i, S_i + p_i]$) the same behavior is observed.

In the case where $p_i < \Delta$ and there is no period ℓ such that $\ell\Delta \in [S_i, S_i + p_i]$ (such as for Activity A_i, fully included in period 2 in Figure 10), then a slightly different behavior is observed. The difference in this case is that for $\ell = \ell s^i = \ell c^i$, the period that fully includes the activity, we have $\lambda_{i,\ell} + d_{i,\ell} + \mu_{i,\ell} = \Delta$ with $\lambda_{i,\ell} > 0$ and $\mu_{i,\ell} > 0$. In Figure 10, we have $\lambda_{i,2} = 0.5\Delta$ and $\mu_{i,2} = d_{i,2} = 0.25\Delta$.

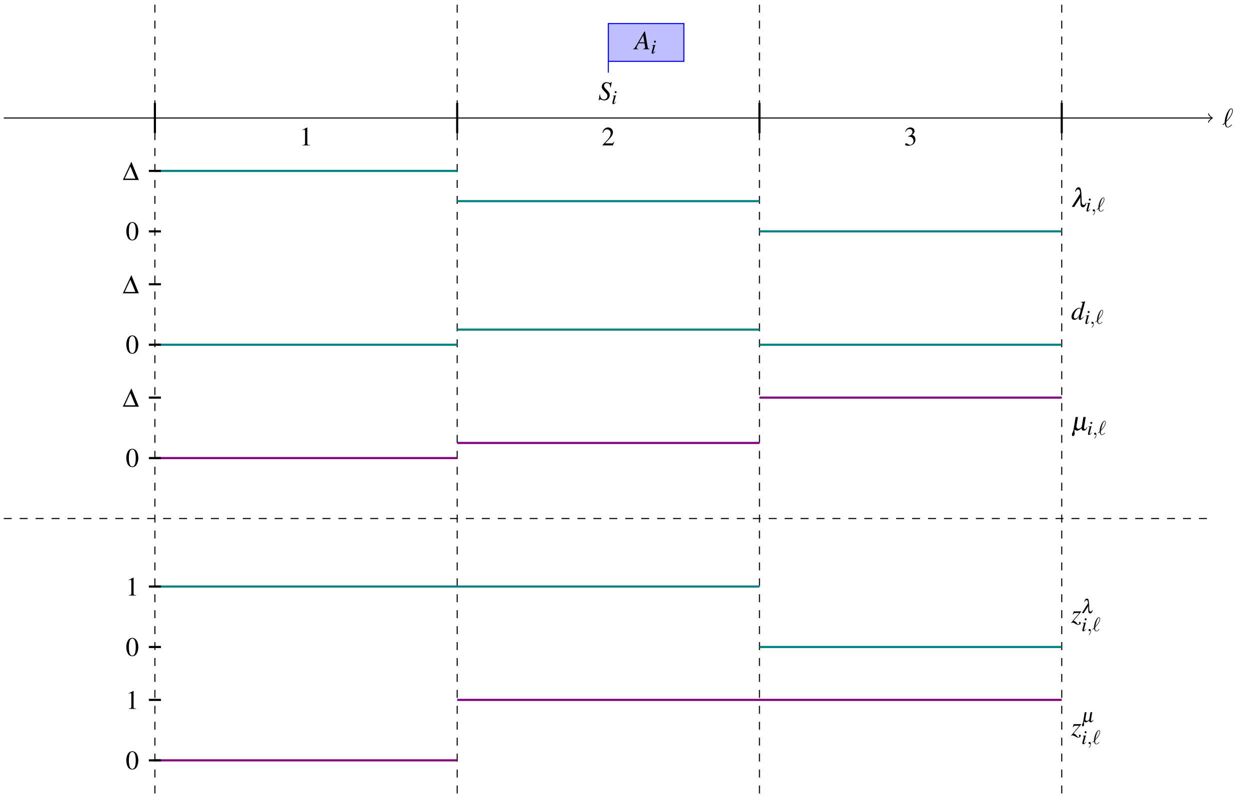


Figure 10: Scheduling variables of an activity for the second period-indexed formulation with $p_i < \Delta$

The monotonicity of the new variables allow a simpler linearization and furthermore, by definition, $\lambda_{i,\ell}$, $\mu_{i,\ell}$ and $d_{i,\ell}$ define a partition of period ℓ. More precisely, we always have $\lambda_{i,\ell} + d_{i,\ell} + \mu_{i,\ell} = \Delta$.

4.2.2 Initial formulation

Given the proposed variables, the new formulation can be written as follows.

$$(F2) \quad \text{Minimize} \quad S_{n+1} - S_0 \tag{24}$$

$$S_{i_2} - S_{i_1} \geq p_{i_1} \qquad \forall (i_1, i_2) \in E \tag{25}$$

$$\sum_{i \in \mathscr{A}} r_{i,k} d_{i,\ell} \leq b_k \Delta \qquad \forall k \in \mathscr{R}, \forall \ell \in \mathscr{L} \tag{26}$$

$$\lambda_{i,\ell} + d_{i,\ell} + \mu_{i,\ell} = \Delta \qquad \forall i \in \mathscr{A}, \forall \ell \in \mathscr{L} \tag{27}$$

$$S_i = \sum_{\ell \in \mathscr{L}} \lambda_{i,\ell} \qquad \forall i \in \mathscr{A} \tag{28}$$

$$\sum_{\ell \in \mathscr{L}} d_{i,\ell} = p_i \qquad \forall i \in \mathscr{A} \tag{29}$$

$$\lambda_{i,\ell} \leq \Delta z^{\lambda}_{i,\ell} \qquad \forall i \in \mathscr{A}, \forall \ell \in \mathscr{L} \tag{30}$$

$$\lambda_{i,\ell} \geq \Delta z^{\lambda}_{i,\ell+1} \qquad \forall i \in \mathscr{A}, \forall \ell \in \mathscr{L} \tag{31}$$

$$\mu_{i,\ell} \leq \Delta z^{\mu}_{i,\ell} \qquad \forall i \in \mathscr{A}, \forall \ell \in \mathscr{L} \tag{32}$$

$$\mu_{i,\ell} \geq \Delta z^{\mu}_{i,\ell-1} \qquad \forall i \in \mathscr{A}, \forall \ell \in \mathscr{L} \tag{33}$$

$$z^{\lambda}_{i,\ell} \in \{0,1\} \qquad \forall i \in \mathscr{A}, \forall \ell \in \mathscr{L} \tag{34}$$

$$z^{\mu}_{i,\ell} \in \{0,1\} \qquad \forall i \in \mathscr{A}, \forall \ell \in \mathscr{L} \tag{35}$$

$$d_{i,\ell}, \mu_{i,\ell}, \lambda_{i,\ell} \geq 0 \qquad \forall i \in \mathscr{A}, \forall \ell \in \mathscr{L} \tag{36}$$

Objective function (24) , precedence constraints (25) and resource constraints (26) are the same as in the first formulation. Constraints (27) define the partition of each period ℓ by variables $\lambda_{i,\ell}$, $\mu_{i,\ell}$ and $d_{i,\ell}$. Constraints (28) allow to express S_i from the $\lambda_{i,\ell}$ variables. Constraints (29) take the the activity processing times into account. Constraints (30) and (31) define the step behavior of variables $\lambda_{i,\ell}$ and $z^{\lambda}_{i,\ell}$, in such a way that a single variable $\lambda_{i,\ell}$ may vary between 0 and Δ, while the others take either value 0 or value Δ. Constraints (32) and (33) define the same process for variables $\mu_{i,\ell}$ and $z^{\mu}_{i,\ell}$ (cf figure 9). Finally $z^{\lambda}_{i,\ell}$ and $z^{\mu}_{i,\ell}$ are binary variables (constraints 34 and 35) while $d_{i,\ell}$, $\mu_{i,\ell}$ and $\lambda_{i,\ell}$ are non negative (constraints 36).

Theorem 5. *Formulation $(F2)$ is a correct formulation of the PARCPSP.*

Proof is given in Appendix B.

4.2.3 Formulation strengthening

As for the previous formulation the start time of the project can be assigned to the first period.

$$0 \leq S_0 \leq \Delta \tag{37}$$

As a consequence of Lemma 1, a binary variable π_i can also be defined for each activity to express the link between the start and the first period of an activity

$$\pi_i = 0 \quad \Leftrightarrow \quad z^{\lambda}_{i,\ell} + z^{\mu}_{i,\ell+\lfloor \frac{p_i}{\Delta} \rfloor -1} = 1$$
$$\pi_i = 1 \quad \Leftrightarrow \quad z^{\lambda}_{i,\ell} + z^{\mu}_{i,\ell+\lceil \frac{p_i}{\Delta} \rceil -1} = 1$$

The linearization of these constraints gives:

$$z^{\mu}_{i,\ell} \in [0,1] \qquad \forall i \in \mathscr{A}\ ,\ \forall \ell \in \mathscr{L} \tag{38}$$

$$\pi_i \in \{0,1\} \qquad \forall i \in \mathscr{A} \tag{39}$$

$$z^{\lambda}_{i,\ell} + z^{\mu}_{i,\ell+\lfloor \frac{p_i}{\Delta} \rfloor -1} \leq 1 \qquad \forall i \in \mathscr{A}\ ,\ \forall \ell \in \mathscr{L} \tag{40}$$

$$z^{\lambda}_{i,\ell} + z^{\mu}_{i,\ell+\lceil \frac{p_i}{\Delta} \rceil -1} \geq 1 \qquad \forall i \in \mathscr{A}\ ,\ \forall \ell \in \mathscr{L} \tag{41}$$

$$z^{\lambda}_{i,\ell} + z^{\mu}_{i,\ell+\lfloor \frac{p_i}{\Delta} \rfloor -1} \geq 1 - \pi_i \qquad \forall i \in \mathscr{A}\ ,\ \forall \ell \in \mathscr{L} \tag{42}$$

$$z^{\lambda}_{i,\ell} + z^{\mu}_{i,\ell+\lceil \frac{p_i}{\Delta} \rceil -1} \leq 2 - \pi_i \qquad \forall i \in \mathscr{A}\ ,\ \forall \ell \in \mathscr{L} \tag{43}$$

Remark. As for $(F1)$ if an activity $i \in \mathscr{A}$ is such that $p_i \bmod \Delta = 0$, there is no need to introduce π_i for this activity, as the last period of the activity can be obtained by a constant translation from the first period.

$$z^{\lambda}_{i,\ell} + z^{\mu}_{i,\ell+\frac{p_i}{\Delta}-1} = 1 \qquad \forall i \in \mathscr{A}\ ,\ \forall \ell \in \mathscr{L} \tag{44}$$

$$\lambda_{i,\ell} + \mu_{i,\ell+\frac{p_i}{\Delta}} = \Delta \qquad \forall i \in \mathscr{A}\ ,\ \forall \ell \in \mathscr{L} \tag{45}$$

We denote by $(F2s)$ the so-strengthened formulation.

4.2.4 Disaggregated precedence constraints

Consider two formulations A and B, and let $z_A(I)$ ($z_B(I)$) denote the value of the linear relaxation of model A (B) applied to instance I of the PARCPSP. Following standard terminology, we say that the linear relaxation of Formulation A is stronger than the linear relaxation of Formulation B when the two following conditions are fulfilled:
C1: for each instance I of PARCPSP, $z_A(I) \geq z_B(I)$, and
C2: there exists an instance I of PARCPSP for which $z_A(I) > z_B(I)$.

Thanks to the introduction of variables $\lambda_{i,\ell}$ and $\mu_{i,\ell}$, a further tightening of the formulation $(F2s)$ can be obtained through the definition of disaggregated precedence constraints. For that purpose, aggregated precedence constraints (25) can be replaced by:

$$\mu_{i_1,\ell} + \lambda_{i_2,\ell} \geq \Delta \qquad \forall (i_1,i_2) \in E\ ,\ \forall \ell \in \mathscr{L} \tag{46}$$

Theorem 6. *Replacing in formulation (F2s), aggregated precedence constraints* (25) *by disaggregated constraints* (46) *yields a correct formulation for the PARCPSP, which is stronger.*

Proof. It is easy to see that the aggregated precedence constraints (25) are implied by the conjunction of disaggregated constraints (46) and constraints (27–29). Indeed, summing up constraints (46) for all $l \in \mathscr{L}$ yields:

$$\sum_{l \in \mathscr{L}} \lambda_{i_2,\ell} \geq L\Delta - \sum_{l \in \mathscr{L}} \mu_{i_1,\ell}$$

This gives the aggregated precedence constraints since $S_i = \sum_{l\in\mathscr{L}} \lambda_{i_2,\ell}$ by (28) and $S_i + p_i = L\Delta - \sum_{l\in\mathscr{L}} \mu_{i,\ell}$ by (27–29). Hence we have shown that the LP relaxation of the new formulation with the disaggregated precedence constraints is not weaker than the new formulation with the aggregated precedence constraints. Consider now the problem instance with $L = 3$ periods of duration $\Delta = 1$ and $n = 3$ activities with durations $p_1 = p_2 = p_3 = 1$ and a single resource of capacity $b_1 = 3$ and activity requirements $b_1 = b_2 = 2$ and $b_3 = 3$. Furthermore there are two precedence constraints $E = \{(1,3),(2,3)\}$. Consider the following (optimal) fractional solution of (F2s), with objective value 2.

	S_i	$\lambda_{i,1}$	$\lambda_{i,2}$	$\lambda_{i,3}$	$z^{\lambda}_{i,1}$	$z^{\lambda}_{i,2}$	$z^{\lambda}_{i,3}$	$\mu_{i,1}$	$\mu_{i,2}$	$\mu_{i,3}$	$z^{\mu}_{i,1}$	$z^{\mu}_{i,2}$	$z^{\mu}_{i,3}$	$d_{i,1}$	$d_{i,2}$	$d_{i,3}$
1	¾	¾	0	0	1	0	0	0	¼	1	0	1	1	¼	¾	0
2	¾	¾	0	0	1	0	0	0	¼	1	0	1	1	¼	¾	0
3	7⁄4	⅞	⅞	0	1	⅞	0	0	⅛	⅛	0	⅛	1	⅛	0	⅞

This solution satisfies the LP relaxation of constraints (25–36) but violates the disagregated constraints. For period $\ell = 1$ and precedence $(1,3)$, we have $\mu_{1,1} + \lambda_{3,1} = \frac{7}{8} < \Delta$ although we have $S_3 = \frac{7}{4} \geq S_2 + p_2 = \frac{7}{4}$. Hence the new formulation augmented with the disaggregates precedence constraints is stronger. Furthermore solving the LP relaxation with the disaggregated constraint gives the following solution with optimal solution $\frac{25}{12} > 2$. Furthermore, since the $z^{\lambda}_{i,\ell}$ and $z^{\mu}_{i,\ell}$ variables are all integer-valued, the solution of the relaxation is feasible for the PARCPSP and consequently optimal, which illustrates the potential quality of the new valid inequalities.

	S_i	$\lambda_{i,1}$	$\lambda_{i,2}$	$\lambda_{i,3}$	$z^{\lambda}_{i,1}$	$z^{\lambda}_{i,2}$	$z^{\lambda}_{i,3}$	$\mu_{i,1}$	$\mu_{i,2}$	$\mu_{i,3}$	$z^{\mu}_{i,1}$	$z^{\mu}_{i,2}$	$z^{\mu}_{i,3}$	$d_{i,1}$	$d_{i,2}$	$d_{i,3}$
1	¼	¼	0	0	1	0	0	0	¾	1	0	1	1	¾	¼	0
2	¼	¼	0	0	1	0	0	0	¾	1	0	1	1	¾	¼	0
3	4⁄3	1	⅓	0	1	1	0	0	0	⅔	0	0	1	0	⅔	⅓

□

We denote by (F2s+) the new formulation with the disaggregated precedence constraints.

4.3 Theoretical comparison of formulations $(F1s)$ and $(F2s+)$

Theorem 7. *The linear relaxation of $(F2s+)$ is stronger than the linear relaxation of $(F1s)$.*

Proof. Let us first compare the relaxations of formulations (F1s) and (F2s), with aggregated precedence constraints only. We remark there exist linear non singular transformations between the binary variables ($zs_{i,\ell}$ and $zf_{i,\ell}$) of the first model and the one of the second model ($z^{\lambda}_{i,\ell}$ and $z^{\mu}_{i,\ell}$).

$$zs_{i,\ell} = 1 - z^{\lambda}_{i,\ell+1}$$

$$zf_{i,\ell} = z^{\mu}_{i,\ell}$$

Continuous variables (S_i and $d_{i,\ell}$) appear in both models with the same meaning, while variables ($\lambda_{i,\ell}$ and $\mu_{i,\ell}$) appear only in the second model.

The start time of an activity i is a linear expression of variables $\lambda_{i,\ell}$ (Constraints (28)).

$$S_i = \sum_{\ell=1}^{L} \lambda_{i,\ell}$$

Similarly, recall that the completion time of an activity i is a linear expression of variables $\mu_{i,\ell}$ (using constraints (27) to (29)).

$$S_i + p_i = L\Delta - \sum_{\ell=1}^{L} \mu_{i,\ell}$$

Aggregated precedence constraints have the same expression in both models (constraints (8) and (25)). We remark that rewriting the other constraints of formulation $(F1s)$ by substituting variables of the first model by the variables of the second model yields constraints that are implied by the constraints of $(F2s)$. Let us provide the proof for the lower bound part of Constraints (10). We first rewrite the constraint for activity i and period $\ell - 1$, by using the transformation $zs_{i,\ell-1} = 1 - z^{\lambda}_{i,\ell}$, we obtain the following equivalent constraint in variable $z^{\lambda}_{i,\ell}$.

$$S_i \geq (\ell-1)\Delta - (\ell-1)\Delta\left(1 - z^{\lambda}_{i,\ell}\right) \tag{$10'_{LB}$}$$

Now we evaluate expression $S_i - (\ell-1)\Delta + (\ell-1)\Delta\left(1 - z^{\lambda}_{i,\ell}\right)$ by using $S_i = \sum_{\ell=1}^{L} \lambda_{i,\ell}$. We obtain:

$$\begin{aligned}
&S_i - (\ell-1)\Delta + (\ell-1)\Delta\left(1 - z^{\lambda}_{i,\ell}\right) \\
&= -(\ell-1)\Delta + \left(\textstyle\sum_{\ell'=1}^{L} \lambda_{i,\ell'}\right) + (\ell-1)\Delta - \left(\textstyle\sum_{\ell'=1}^{\ell-1} \Delta\right) z^{\lambda}_{i,\ell} \\
&\geq \textstyle\sum_{\ell'=1}^{L} \lambda_{i,\ell'} - \sum_{\ell'=1}^{\ell-1} \Delta z^{\lambda}_{i,\ell'+1} \\
&\geq \textstyle\sum_{\ell'=1}^{L} \lambda_{i,\ell'} - \sum_{\ell'=1}^{\ell-1} \lambda_{i,\ell'} \\
&= \textstyle\sum_{\ell'=\ell}^{L} \lambda_{i,\ell'} \\
&\geq 0
\end{aligned}$$

The proof for the upper bound part of Constraints (10) and Constraints (11) (link between S_i, $zs_{i,\ell}$ and $zf_{i,\ell}$) and the proof for Constraints (12) to (14) (expression of $d_{i,\ell}$) are given in the Appendix. Constraints (15) of the first model are also present in the second model (Constraints (36) and (29)).

Lastly, Constraints (19) to (23) that link variables $zs_{i,\ell}$ and $zf_{i,\ell}$ of the first model via binary variable π_i are equivalent to constraints (40) to (44) that link variables $z^{\lambda}_{i,\ell}$ and $z^{\mu}_{i,\ell}$ of the second model via the same binary variable π_i: The above described linear transformations can be used to switch from one formulation to the other. From what precedes, we conclude that formulation $(F2s)$ cannot be weaker than formulation $(F1s)$ in terms of linear programming relaxation. As Theorem 6 states that formulation $(F2s+)$ is stronger than formulation $(F2s)$, the result follows. $\square$

5 Computational experiments

In this section, we compare the different MILP formulations on a set of benchmark instances from the literature. As in Morin et al. (2017), we select standard resource-constrained-project scheduling instances, to which we associate a period Δ with $\Delta = 1, 2, 3, 4$ and 5. We use IBM ILOG CPLEX 20.1 for solving the (mixed-integer) linear programs with default parameters. All experiments were run with 2 threads on 8 cluster nodes, each with 36 Intel Xeon CPU E5-2695 v3 2.10GHz cores running Linux Ubuntu 16.04.4.

We first compare the LP relaxations of the Morin et al. (2017) strengthened formulation $(F1s)$ with the new ones ($F2s$ and $F2s+$) on the 30, 60, 90 and 120 activity RCPSP instances from the PSPLIB library, named KSD30, KSD60, KSD90 and KSD120 (Kolisch and Sprecher 1996), as well as on the Pack instances (Carlier and Néron

2003). With the different values of Δ, we obtain a set of 2400 KSD30 instances, 2400 KSD60 instances, 2400 KSD90 instances, 3000 KSD120 instances and 280 Pack instances. For each instance, an upper bound of the number of periods is obtained by selecting the best solution in terms of project duration returned by the randomized multi-start priority-rule based heuristic presented in Morin et al. (2017) with 1000 iterations.

Table 4 reports the results of the LP relaxations compared to the trivial critical path lower bound (CPM) given by the precedence constraints only. Each row of the table correspond to the instances of a specific set for a given Δ, except the last row "all" of each instance set that regroups all the Δ values and the last "all" row that regroups the statistics over all instances and Δ values. Column `#UB>CPM` displays the number of instances for which the upper bound is not equal to the CPM lower bound. Indeed, the LP relaxations have the potential of increasing the CPM lower bound only on these instances. A first remark is that this number is a decreasing function of Δ, which illustrates the fact that decreasing the period lengths globally tightens the resource constraints, yielding larger project durations. There are two columns of results for each formulation $F1s$, $F2s$ and $F2s+$. The first column (`gap CPM`) gives the average improvement upon the CPM bound, only on the instances for which the CPM bound is strictly lower than UB (number given in column `#UB>CPM`). The second column (cpu) gives the average cpu time in seconds. The best results in terms of gap and CPU time are highlighted in bold.

The ranking $F1s < F2s < F2s+$ from the weakest to the strongest upper bound is well illustrated by the results. Globally, for the larger values of Δ, all bounds are rather weak and each bound gets tighter as Δ decreases. The improvement brought by the new formulation with aggregated precedence constraints ($F2s$) on the previous formulation is modest, except on the Pack set. The new formulation with disaggregated precedence constraints ($F2s+$) significantly improves the previous formulation upon the CPM-based lower bound on all instances with small Δ. The large gaps observed for the Pack instance set are explained by the small number of precedence constraints in this set and the predominance of resource constraints. This allows to remark that the improvement brought by $F2s$ and $F2s+$ on the previous formulation $F1s$ can be drastic and indicates that the new formulation better captures resource conflicts. About the computational times, the fastest bounds are obtained either by $F1s$ or $F2s$, the latter offering the best compromise quality/speed. The computational times become very large for the KSD120 set and illustrates the limits of time-indexed MILP approaches for large scheduling horizons, even with aggregated resource constraints.

The $F2s+$ formulation is superior to the other ones in terms of LP relaxations. We now switch to the comparison of the quality of the integer solutions found by CPLEX under a limited time.

We limit the CPU time to 1 hour for the KSD30 instances and to 2 hours for the remaining instances. The randomized multi-start priority-rule based heuristic of Morin et al. (2017) with 1000 iterations is used to obtained an initial feasible solution provided as a "MILP start".

Table 5 reports the obtained results on the KSD30 and KSD60 sets for the three formulations. For each formulation and value of Δ, the table displays the number of optimal solutions found and certified within the alloted time (column `#opt`), the average gap between the lower and the upper bound returned by the solver, and the average CPU time. The last column (`av. gap LB RCPSP`) gives the average gap for each value of Δ of the optimal solution (or the best found lower bound when optimailty is not verified) for the PARCPSP to the optimal solution (of the best known lower bound when the optimum is unknown) of the RCPSP. The number in this column for row `all` is the

set	Δ	#UB>CPM	$F1s$ gap CPM	$F1s$ cpu (s)	$F2s$ gap CPM	$F2s$ cpu (s)	$F2s+$ gap CPM	$F2s+$ cpu (s)
KSD30	5	146	0.00%	**0.11**	0.00%	0.23	**0.23%**	0.30
	4	171	0.00%	**0.15**	0.00%	0.27	**0.21%**	0.38
	3	198	0.00%	**0.20**	0.02%	0.33	**0.85%**	0.41
	2	234	0.00%	**0.31**	0.20%	0.47	**1.89%**	0.41
	1	264	0.02%	0.54	1.00%	0.62	**4.00%**	**0.33**
	all	1013	0.01%	**0.29**	0.31%	0.41	**1.71%**	0.37
KSD60	5	158	0.00%	**0.51**	0.00%	0.76	**0.33%**	2.02
	4	167	0.00%	**0.91**	0.00%	1.01	**0.42%**	2.93
	3	181	0.00%	**1.40**	0.04%	1.52	**1.51%**	3.29
	2	202	0.00%	3.32	0.49%	**2.56**	**3.16%**	4.57
	1	233	0.00%	6.97	2.05%	**3.19**	**5.73%**	7.76
	all	941	0.00%	2.96	0.62%	**1.94**	**2.52%**	4.39
KSD90	5	164	0.00%	**1.68**	0.00%	2.88	**0.27%**	5.14
	4	174	0.00%	3.82	0.00%	**2.96**	**0.34%**	6.64
	3	179	0.00%	8.11	0.01%	**5.16**	**1.39%**	19.06
	2	198	0.00%	28.82	0.52%	**13.86**	**3.60%**	37.46
	1	214	0.00%	57.28	2.43%	**22.01**	**7.12%**	46.93
	all	929	0.00%	21.91	0.67%	**10.08**	**2.79%**	24.62
KSD120	5	485	0.00%	10.71	0.07%	**5.03**	**1.48%**	14.43
	4	496	0.00%	27.98	0.22%	**8.45**	**1.84%**	31.09
	3	508	0.00%	72.78	1.07%	**17.56**	**3.75%**	66.96
	2	521	0.00%	145.90	2.90%	**48.35**	**6.83%**	138.41
	1	550	0.01%	282.09	6.22%	**73.84**	**11.52%**	247.16
	all	2560	0.00%	112.19	2.19%	**31.78**	**5.25%**	103.31
Pack	5	54	0.00%	**0.09**	1.74%	0.13	**15.72%**	0.19
	4	55	0.00%	**0.13**	3.97%	0.19	**20.37%**	0.22
	3	55	0.00%	**0.26**	14.05%	**0.26**	**35.64%**	0.30
	2	55	0.31%	0.51	32.26%	**0.37**	**51.32%**	0.41
	1	55	4.45%	1.28	64.17%	**0.46**	**73.89%**	0.62
	all	274	0.96%	0.46	23.32%	**0.28**	**39.48%**	0.35
all	all	5717	0.00%	54.34	1.25%	**16.26**	**3.52%**	51.05

Table 4: Comparisons of $F1s$, $F2s$ and $F2s+$ LP relaxations on various instance sets

best gap over all Δ values. Note that for KSD30 instances the optimal makespan for the RCPSP are known while for KSD60 we use for comparison the best current LB[1].

The `av. gap LB RCPSP` gaps is increasing on average in function of Δ and is of significant magnitude. This confirms that aggregating the resource constraints without restricting the start time values is highly beneficial for reducing the makespan, as mentioned in the introduction. For each instance set, the largest obtained bound for the different values of Δ gives a gap to the best known RCPSP LB of less than 3%. No lower bound is improved on the KSD60 set. The results in table 5, compared to the best results obtained by MILP for the standard RCPSP in Koné et al. (2011), suggest that the PARCPSP is not much easier to solve than the RCPSP. So it is still unclear whether the PARCPSP can be used as an efficient bounding scheme of the RCPSP.

Turning now to the comparison of formulations, the best results are displayed in bold. The aggregated variant of the new formulation ($F2s$) appears dominated on all criteria, including the CPU time[2]. The previous formulation ($F1s$) obtains the best results for solving the KSD30 instances with $\Delta = 1$ and $\Delta = 2$ as well as the KSD60 instances with $\Delta = 3$ in terms of optimal solutions found with a faster or equivalent CPU time. This indicates that the quality of the LP relaxation of $F2s+$ does not always compensate the search slowdown it incurs. However, the $F2s+$ dominates on all criteria for the remaining instances and is always the best one in terms of average gap for all Δ values. Averaging all instances and all Δ values, the $F2s+$ formulation outperforms the other ones for all criteria.

		$F1s$			$F2s$			$F2s+$			
set	Δ	#opt	gap	time	#opt	gap	time	#opt	gap	time	gap LB RCPSP
KSD30	5	457	0.13%	226.40	447	0.24%	323.09	**474**	**0.02%**	**110.41**	6.70%
	4	441	0.39%	342.46	432	0.52%	411.52	**459**	**0.14%**	**234.62**	6.25%
	3	438	0.62%	386.32	423	0.89%	465.93	**437**	**0.42%**	**380.21**	5.69%
	2	**436**	0.97%	**386.46**	425	1.41%	477.13	433	**0.83%**	430.54	4.72%
	1	**447**	1.05%	**336.47**	422	2.04%	492.52	437	**1.03%**	406.05	2.82%
	all	2219	0.63%	335.62	2149	1.02%	434.04	**2240**	**0.49%**	**312.36**	2.82%
KSD60	5	402	0.97%	1235.91	400	1.22%	1290.65	**426**	**0.32%**	**974.67**	3.57%
	4	397	1.36%	1278.81	394	1.89%	1337.82	**406**	**0.79%**	**1224.08**	3.48%
	3	**395**	2.08%	**1335.99**	390	3.64%	1388.57	394	**1.57%**	1354.49	3.35%
	2	**383**	3.37%	1504.17	381	5.86%	1534.21	**383**	**2.77%**	**1498.22**	3.17%
	1	377	6.98%	1604.12	372	6.91%	1664.11	**380**	**3.59%**	**1535.03**	2.50%
	all	1954	2.95%	1391.80	1937	3.90%	1443.07	**1989**	**1.81%**	**1317.30**	2.45%

Table 5: Comparisons of integer solutions for instances KSD30 and KSD60

We now switch to the KSD90, KSD120 and Pack benchmarks, which are much harder to solve in the RCPSP setting. Here, only the non dominated formulations ($F1s$ and $F2s+$) are compared. As seen in Table 6, except for three exceptions (average CPU time criterion for KSD90-$\Delta = 3$ instances and number of optima found for KSD90-$\Delta = 2$ instances), the new formulation outperforms the previous one on all instances and all criteria. Two additional observations are worth mentioning. First while the optimality gaps moderately increase for KSD30, KSD60 and

[1] Recorded at `http://solutionsupdate.ugent.be/`, last visit Novembre 9, 2021

[2] Recall that the time limit is 1 hour for KSD30 and 2 hours for KSD60

KSD90 sets, the limit of the time-indexed MILP approach seems to be reached for the KSD120 set since large gaps are observed as Δ decreases. This is inline with the large needed CPU time for solving the LP relaxation. A second remark is the relative quality of the RCPSP bbound on the Pack instances. These instances seem as challenging in the PARCPSP setting as they are in the RCPSP setting, even for $\Delta = 5$ instances since only 38 instances out of 55 are solved to optimality. However 4 of the RCPSP lower bounds reported in Schutt et al. (2013) were improved, while no lower bound was improved for KSD60, KSD90 and KSD120 instances. The main notorious difference between the Pack and the KSD sets is that the Pack instances are "highly cumulative" in the sense that many activities can be scheduled in parallel and have very few precedence constraints (which explains the name Pack with reference to the 2D packing problem). In this case, the resource aggregation seems to pay off although all improvements were obtained for $\Delta = 1$. The improved lower bounds are reported in Table 7.

		$F1s$			$F2s+$			
set	Δ	#opt	gap	time	#opt	gap	time	gap LB RCPSP
KSD90	5	390	1.55%	1366.66	**411**	**0.86%**	**1208.99**	1.88%
	4	388	2.25%	1401.56	**393**	**1.85%**	**1334.47**	2.01%
	3	385	3.36%	**1439.20**	**387**	**2.89%**	1441.82	2.14%
	2	**384**	6.79%	**1490.47**	378	**4.16%**	1575.43	2.33%
	1	376	10.69%	1655.45	**378**	**4.64%**	**1561.48**	1.78%
	all	1923	4.93%	1470.67	**1947**	**2.88%**	**1424.44**	1.42%
KSD120	5	302	8.82%	3738.88	**353**	**7.66%**	**3270.11**	5.81%
	4	283	13.92%	3946.06	**314**	**10.34%**	**3731.16**	6.83%
	3	260	20.30%	4232.44	**272**	**12.74%**	**4137.12**	7.25%
	2	232	28.12%	4555.50	**238**	**16.79%**	**4420.86**	7.76%
	1	203	35.02%	4958.74	**223**	**18.05%**	**4601.85**	6.47%
	all	1280	21.24%	4286.32	**1400**	**13.12%**	**4032.22**	4.82%
Pack	5	27	1.12%	4937.18	**38**	**2.02%**	**2959.33**	11.77%
	4	19	2.04%	5577.67	**37**	**0.47%**	**3292.10**	9.24%
	3	16	3.27%	5787.41	**27**	**3.10%**	**4179.22**	7.44%
	2	13	5.08%	6185.74	**26**	**2.24%**	**4632.17**	5.01%
	1	6	8.26%	7085.33	**32**	**1.89%**	**3857.74**	1.85%
	all	63	3.95%	5914.67	**160**	**1.94%**	**3784.11**	1.83%

Table 6: Comparisons of integer solutions for instances KSD90, KSD120, Pack

name	LB (Schutt et al. 2013)	LB $F2s+$
Pack037	116	125
Pack046	110	118
Pack050	94	100
Pack053	97	105

Table 7: Improved RCPSP lower bounds on the Pack instance set compared to Schutt et al. (2013)

6 Conclusion and perspectives

In this paper, an original variant of the RCPSP, namely the PARCPSP, has been studied from a theoretical point of view. This problem is indeed a relaxation of the RCPSP, that permits to model periodically aggregated resource constraints arising from practical applications, where the resource usage is limited only on average over periods of parameterized length. Contrarily to the RCPSP, the feasibility of a solution (with respect to the resource constraints) is no more invariant by shifting. We proposed three reductions to establish the computational complexity of particular cases of the problem, which is strongly NP-hard in the general case. We designed a new period-indexed mixed-integer linear programming formulation of the problem, defining the precedence constraint in a disaggregated form. We carried out a polyhedral study that established that the new formulation is stronger than the previously proposed formulation in terms of linear programming relaxation. A computational experiment on the set of PSPLIB project scheduling instances with five different period lengths, showed that the practical improvement of the lower bound is significant. When using the formulations for exact solution approaches in a commercial MILP solver, the new formulation is still globally better in terms of optimal solution found and optimality gaps, except for a few exceptions. The PARCPSP appears as a challenging NP-hard problem. Although it provides a bounding scheme for the widely studied RCPSP, it is still unclear whether efficient approaches can be designed to this aim. However this research direction is worth pursuing as a few lower bounds were improved for the difficult RCPSP instance set Pack. For a global improvement of mixed-integer linear programming approches, the disaggregated precedence constraints could be added on-the-fly to obtain a better compromise between the formulation size and the relaxation quality. The question whether an extended formulation based on a Dantzig-Wolfe decomposition of the resource constraint, as successfully done for the RCPSP (Mingozzi et al. 1998; Brucker and Knust 2000; Baptiste and Demassey 2004), would yield a competitive relaxation is open, as the aggregated resource constraints are less tight than the standard ones. In order to fit practical applications, various extensions can be considered. For instance, the definition of a consumption rate, either fixed (data) or variable (decision to make), on resources for each activity would allow to model a wider range of resource usage profiles. Also, one could take into account additional limitations, in a similar way as in Okubo et al. 2015, where a RCPSP/Π original formulation is enriched with specific constraints. More flexible activities with variables intensities should also be considered such as in Hans 2001; Kis 2005. A promising research direction consists in considering varying period lengths. Indeed, models with time buckets of non homogeneous lengths were successfully applied to a scheduling problem issued from particle therapy for cancer treatment (Riedler et al. 2020). The latter work reveals that this approach has a double potential: to better model practical situations where resource scarceness is time-dependent, and to improve primal and dual bound for the RCPSP.

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Appendix

A Proof of Theorem 4: formal correctness of formulation $(F1)$

Proof. We first show that a feasible solution for formulation $(F1)$ is a feasible solution for the PARCPSP with the same objective function value. Consider a solution S_i the MILP and suppose it is unfeasible for the PARCPSP. Since constraints (8) translate directly the precedence constraints, the solution must be resource-unfeasible, which means that constraints (3) of the conceptual model is violated. This can only be the case if there exists $i \in \mathscr{A}$ and $\ell \in \mathscr{L}$ such $d_{i,\ell} < \max\Big(0\,,\, \min\big(S_i + p_i\,,\, \ell\Delta\big) - \max\big(S_i\,,\, (\ell-1)\Delta\big)\Big)$, i.e. $d_{i,\ell}$ is strictly smaller than the intersection length of intervals $[S_i, S_i + p_i]$ and $[(\ell-1)\Delta, \ell\Delta]$. Otherwise constraints (9) ensure that constraints (3) are satisfied. The lower bound to $d_{i,\ell}$ for each $i \in \mathscr{A}$ and $\ell \in \mathscr{L}$ is set by its non-negativity and by constraints (12–14). The latter constraints involve binary variables $zs_{i,\ell-1}$ and $zf_{i,\ell}$. We show that $d_{i,\ell}$ is not smaller than the intersection length for each of the possible values for pair $(zs_{i,\ell-1}, zf_{i,\ell})$.

- $zs_{i,\ell-1} = 0$ and $zf_{i,\ell} = 0$. According to (10), since $zs_{i,\ell-1} = 0$ we have $S_i \geq (\ell-1)\Delta$. Similarly, with $zf_{i,\ell} = 0$ constraint (11) yields $S_i + p_i \geq \ell\Delta$. In this case the intersection length is 0 if $S_i \geq \ell\Delta$ and $\ell\Delta - S_i$ otherwise. This is ensured by the non-negativity of $d_{i,\ell}$ in conjunction with constraints (13).

- $zs_{i,\ell-1} = 0$ and $zf_{i,\ell} = 1$. As for the previous case, inserting $zs_{i,\ell-1} = 0$ in (10) gives $S_i \geq (\ell-1)\Delta$. Setting $zf_{i,\ell} = 1$ in (11) yields $S_i + p_i \leq \ell\Delta$. This is the case where interval $[S_i, S_i + p_i]$ is included in $[(\ell-1)\Delta, \ell\Delta]$, so the intersection length is equal to p_i. Remark that $S_i \geq (\ell-1)\Delta \implies S_i + p_i > (\ell-2)\Delta$. Hence, we have $zf_{i,\ell-2} = 0$ since $zf_{i,\ell-2} = 1$ and (11) would imply that $S_i + p_i \leq (\ell-2)\Delta$. Then, since $zf_{i,\ell-2} = 0$ and $zs_{i,\ell-1} = 0$, (12) yields $d_{i,\ell-1} = 0$. We have also $d_{i,\ell'} = 0$ for all $\ell' \leq \ell - 1$ because $zf_{i,\ell''-1} = 0$ and $zs_{i,\ell'} = 0$ according to step constraints. Another remark is that $S_i + p_i \leq \ell\Delta \implies S_i < (\ell+1)\Delta$. With (10), this yields $zs_{i,\ell+1} = 1$. Since in addition $zf_{i,\ell} = 1$, constraint (12) yields $d_{i,\ell+1} = 0$. With step constraints, we have $zf_{i,\ell''-1} = 1$ and $zs_{i,\ell'} = 1$ and so with (12) $d_{i,\ell'} = 0$ for all $\ell' \geq \ell + 1$. It follows that $d_{i,\ell'} = 0$ for all $\ell' \in \mathscr{L} \setminus \{\ell\}$. According to constraint (15), we obtain $d_{i,\ell} = p_i$.

- $zs_{i,\ell-1} = 1$ and $zf_{i,\ell} = 0$. (10) and $zs_{i,\ell-1} = 1$ implies that $S_i \leq (\ell-1)\Delta$, while (11) and $zf_{i,\ell} = 0$ imply that $S_i + p_i \geq (\ell)\Delta$. In this case interval $[(\ell-1)\Delta, \ell\Delta]$ is included in interval $[S_i, S_i + p_i]$ and the intersection length is equal to Δ. Inserting $zs_{i,\ell-1} = 1$ and $zf_{i,\ell} = 0$ in (12) directly gives $d_{i,\ell} \geq \Delta$.

- $zs_{i,\ell-1} = 1$ and $zf_{i,\ell} = 1$. In this case, we obtain $S_i \leq (\ell-1)\Delta$ with (10) and $S_i + p_i \leq \ell\Delta$ with (11). It follows that the intersection length is $S_i + p_i - (\ell-1)\Delta$ if $S_i + p_i \geq (\ell-1)\Delta$ and 0 otherwise. Constraints (14) yields $d_{i,\ell} \geq S_i + p_i - (\ell-1)\Delta$.

As in all case $d_{i,\ell}$ is not smaller than the actual length of the intersection of intervals $[S_i, S_i + p_i]$ and $[(\ell-1)\Delta, \ell\Delta]$, any feasible solution of the MILP is also feasible for the PARCPSP. Furthermore the objective functions are exactly the same.

It remains to show that for any feasible solution of the PARCPSP, there is a compatible assignment of the other decision variables that satisfies all the constraints of the MILP. Let S_i, $i \in \mathscr{A}$ denote a feasible solution of the PARCPSP. Obviously the precedence constraints (8) are satisfied. Setting variables $d_{i,\ell}$ to $d_{i,\ell}(S)$ according to its definition in §2.2 allows to satisfy constraints (9) and (15). Consider the following assignment for variables $zs_{i,\ell}$. Let ℓs^i the period such that $(\ell s^i - 1)\Delta \leq S_i < \ell s^i \Delta$. For all $i \in \mathscr{A}$, let us set $zs_{i,\ell} = 0$ for each $\ell < \ell s^i$ and $zs_{i,\ell} = 1$ for each $\ell \geq \ell s^i$. Similarly, let ℓf^i the period verifying $(\ell f^i - 1)\Delta \leq S_i + p_i < \ell f^i \Delta$. For all $i \in \mathscr{A}$, let us set $zf_{i,\ell} = 0$ for each $\ell < \ell f^i$ and $zf_{i,\ell} = 1$ for each $\ell \geq \ell f^i$. This assignment obviously satisfies the step behavior constraints of variables $zs_{i,\ell}$ and $zf_{i,\ell}$. Start time lower bound constraints (10) are satisfied as they give $S_i \geq \ell\Delta$ for $\ell = 1, \cdots, \ell s^i - 1$ and $S_i \geq 0$ for $\ell = \ell s^i, \cdots, L$. Start time upper bound constraints (10) are also satisfied as they can be written $S_i \leq L\Delta$ for $\ell = 1, \cdots, \ell s^i - 1$ and $S_i \leq \ell\Delta$ for $\ell = \ell s^i, \cdots, L$. The same holds for completion time lower and upper bound constraints (11), since we obtain $\ell\Delta \leq S_i + p_i \leq L\Delta$ for $\ell = 1, \cdots, \ell f^i - 1$ and $0 \leq S_i + p_i \leq \ell\Delta$ for $\ell = \ell f^i, \cdots, L$. Now, let us consider the constraints (10-14) that link $d_{i,\ell}$, S_i, $zs_{i,\ell}$ and $zf_{i,\ell}$ variables. Recall that $d_{i,\ell}$ is set to $\max\Big(0\,,\, \min\big(S_i + p_i\,,\, \ell\Delta\big) - \max\big(S_i\,,\, (\ell-1)\Delta\big)\Big)$. For each task i, we consider the following sets $L1 = \{\ell \in \mathscr{L} | \ell < \ell s^i\}$, $L2 = \{\ell \in \mathscr{L} | \ell > \ell f^i\}$ and $L3 = \{\ell s^i + 1, \ldots, \ell f^i - 1\}$. Note that $\mathscr{L} = L1 \cup \{\ell s^i, \ell f^i\} \cup L2 \cup L3$. By definition of ℓs^i and ℓf^i, $d_{i,\ell s^i} = \ell s^i \Delta - S_i$ and $d_{i,\ell f^i} = S_i + p_i - (\ell f^i - 1)\Delta$ if $\ell f^i > \ell s^i$ and $d_{i,\ell f^i} = d_{i,\ell s^i} = p_i$ otherwise. For $\ell \in L1 \cup L2$, $d_{i,\ell} = 0$. For $\ell \in L3$, $d_{i,\ell} = \Delta$. We show below that constraints (13–14) are all compatible with these values.

Non-negativity constraints are satisfied for all $\ell \in \mathscr{L}$. Constraints (12) are equivalent to $d_{i,\ell} \leq 0$ for $\ell \in L1 \cup L2$ and $d_{i,\ell} \leq \Delta$ for $\ell = \{\ell s^i, \ell f^i\} \cup L3$. Constraints (12) can be written $d_{i,\ell} \geq 0$ for $\ell \in L1 \cup L2 \cup \{\ell s^i, \ell f^i\}$ and $d_{i,\ell} \geq \Delta$ for $\ell \in L3$. Constraints (13) give $d_{i,\ell} \geq \ell\Delta - S_i < 0$ for $l \in L1$, $d_{i,\ell} \geq \ell s^i \Delta - S_i$ for $\ell = \ell s^i$ in the case where $zf_{i,\ell s^i} = 0$ (i.e. $\ell f^i > \ell s^i$) and $d_{i,\ell} \geq (\ell s^i - 1)\Delta - S_i < 0$ for $\ell = \ell s^i$ in the case where $zf_{i,\ell s^i} = 1$. For $\ell \in L3$, we obtain $d_{i,\ell} \geq -S_i$. For $\ell = \ell f^i > \ell s^i$ and for $\ell \in L2$, we have $d_{i,\ell} \geq -\Delta - S_i$. Last, constraints (14) give precisely $d_{i,\ell} \geq S_i + p_i - (\ell f^i - 1)\Delta$ for $\ell = \ell f^i$ and $\ell f^i > \ell s^i$. For $\ell = \ell f^i = \ell s^i$, we have $d_{i,\ell} \geq S_i + p_i - \ell f^i \Delta < 0$. For $\ell \in L1$ or $\ell = \ell s^i < \ell f^i$, the constraints is written $d_{i,\ell} \geq S_i + p_i - (L+1)\Delta < 0$. For $\ell \in L3$, we obtain $d_{i,\ell} \geq S_i + p_i - L\Delta < 0$. For $\ell \in L2$, the constraint yields $d_{i,\ell} \geq S_i + p_i - (\ell-1)\Delta < 0$.

□

B Proof of Theorem 5: formal correctness of formulation $(F2)$

Proof. Given the common structure with the first formulation and the definition of variables $d_{i,\ell}$, $\lambda_{i,\ell}$ and $\mu_{i,\ell}$, we just have to show that constraints (27–36) properly model the relationships $d_{i,\ell} = [S_i, S_i + p_i] \cap [(\ell-1)\Delta, \ell\Delta]$ for each activity i. Constraints (30-31) can be rewritten $z^\lambda_{i,\ell+1} \leq \frac{\lambda_{i,\ell}}{\Delta} \leq z^\lambda_{i,\ell}$, meaning that variables $z^\lambda_{i,\ell}$ have a decreasing step behavior. Suppose that a period ℓ is such that $\ell\Delta \geq S_i$ and $\lambda_{i,\ell} > 0$. Then we have $z^\lambda_{i,\ell} = 1$ and so $z^\lambda_{i,\ell'} = 1$ and $\lambda_{i,\ell'} = \Delta$, for all $\ell' < \ell$. In this case we would have $\sum_{\ell'=1}^{\ell} \lambda_{i,\ell'} > S_i$, a contradiction. It follows that any non zero $\lambda_{i,\ell}$ variable is such that $\ell\Delta < S_i$. It follows that if $S_i > 0$, the first $\ell \in \{1, \cdots, \lfloor \frac{S_i}{\Delta} \rfloor\}$ periods are such that $\lambda_{i,\ell} = \Delta$ and period $\ell = \lfloor \frac{S_i}{\Delta} \rfloor + 1$ is such that $\lambda_{i,\ell} = S_i \mod \Delta$. We have precisely $\lambda_{i,\ell} = [0, S_i] \cap [(\ell-1)\Delta, \ell\Delta]$ for all $\ell \in \mathscr{L}$. Similarly, constraints (32–33) yield $z^\mu_{i,\ell-1} \leq \frac{\mu_{i,\ell}}{\Delta} \leq z^\mu_{i,\ell}$ for all $\ell \in \mathscr{L}$. Hence variables $z^\mu_{i,\ell-1}$ have an increasing step behavior. Composition of constraints (27), (28) and (29) give $L\Delta - S_i - p_i = \sum_{\ell \in \mathscr{L}} \mu_{i,\ell}$. With the same reasoning it comes that $\mu_{i,\ell} = [S_i + p_i, L\Delta] \cap [(\ell-1)\Delta, \ell\Delta]$. From constraints (27), we obtain $d_{i,\ell} = \Delta - [S_i + p_i, L\Delta] \cap [(\ell-1)\Delta, \ell\Delta] - [0, S_i] \cap [(\ell-1)\Delta, \ell\Delta] = [S_i, S_i + p_i] \cap [(\ell-1)\Delta, \ell\Delta]$. □

C Proof details of Theorem 7

Proof. We provide below the full proof of the implication of model (F1s) by model (F2s) for Constraints (10) and (11) (link between S_i, $zs_{i,\ell}$ and $zf_{i,\ell}$) and Constraints (12) to (14) (expression of $d_{i,\ell}$). For each constraint, the variables of model (F1s) are substituted by the variable of model (F2s), which yields the constraint with a prime (') that are then shown to be always satisfied.

$$S_i \leq (\ell-1)\Delta + (L-\ell+1)\Delta z^\lambda_{i,\ell} \quad \text{using } 10_{UB} \text{ for } l-1 \text{ and } zs_{i,\ell-1} = 1 - z^\lambda_{i,\ell} \tag{10'$_{UB}$}$$

$$\begin{aligned} & S_i - (\ell-1)\Delta - (L-\ell+1)\Delta z^\lambda_{i,\ell} \\ &= -(\ell-1)\Delta + \left(\textstyle\sum_{\ell'=1}^{L} \lambda_{i,\ell'}\right) - \left(\textstyle\sum_{\ell'=\ell}^{L} \Delta\right) z^\lambda_{i,\ell} \\ &\leq -\textstyle\sum_{\ell'=\ell}^{L} \left(\Delta z^\lambda_{i,\ell'} - \lambda_{i,\ell'}\right) - \textstyle\sum_{\ell'=1}^{\ell-1} \left(\Delta - \lambda_{i,\ell'}\right) \\ &\leq 0 \end{aligned}$$

$$S_i + p_i \geq \ell\Delta - \ell\Delta z^\mu_{i,\ell} \tag{11'$_{LB}$}$$

$$\begin{aligned} & S_i + p_i - \ell\Delta + \ell\Delta z^\mu_{i,\ell} \\ &= -\ell\Delta + \left(L\Delta - \textstyle\sum_{\ell'=1}^{L} \mu_{i,\ell'}\right) + \left(\textstyle\sum_{\ell'=1}^{\ell} \Delta\right) z^\mu_{i,\ell} \\ &\geq \textstyle\sum_{\ell'=1}^{\ell} \left(\Delta z^\mu_{i,\ell'} - \mu_{i,\ell'}\right) + \textstyle\sum_{\ell'=\ell+1}^{L} \left(\Delta - \mu_{i,\ell'}\right) \\ &\geq 0 \end{aligned}$$

$$S_i + p_i \leq \ell\Delta + (L-\ell)\Delta\left(1 - z^\mu_{i,\ell}\right) \tag{11'$_{UB}$}$$

$$
\begin{aligned}
& S_i + p_i - \ell\Delta - (L-\ell)\Delta\left(1 - z^{\mu}_{i,\ell}\right) \\
&= -\ell\Delta + \left(L\Delta - \textstyle\sum_{\ell'=1}^{L} \mu_{i,\ell'}\right) - (L-\ell)\Delta + \left(\textstyle\sum_{\ell'=\ell+1}^{L} \Delta\right) z^{\mu}_{i,\ell} \\
&\leq -\textstyle\sum_{\ell'=1}^{L} \mu_{i,\ell'} + \textstyle\sum_{\ell'=\ell+1}^{L} \Delta z^{\mu}_{i,\ell'-1} \\
&\leq -\textstyle\sum_{\ell'=1}^{L} \mu_{i,\ell'} + \textstyle\sum_{\ell'=\ell+1}^{L} \mu_{i,\ell'} \\
&= -\textstyle\sum_{\ell'=1}^{\ell} \mu_{i,\ell'} \\
&\leq 0
\end{aligned}
$$

$$
d_{i,\ell} \geq \Delta\left(1 - z^{\lambda}_{i,\ell} - z^{\mu}_{i,\ell}\right) \tag{12'$_{LB}$}
$$

$$
\begin{aligned}
& d_{i,\ell} - \Delta\left(1 - z^{\lambda}_{i,\ell} - z^{\mu}_{i,\ell}\right) \\
&= \Delta - \lambda_{i,\ell} - \mu_{i,\ell} - \Delta + \Delta z^{\lambda}_{i,\ell} + \Delta z^{\mu}_{i,\ell} \\
&= \left(\Delta z^{\lambda}_{i,\ell} - \lambda_{i,\ell}\right) + \left(\Delta z^{\mu}_{i,\ell} - \mu_{i,\ell}\right) \\
&\geq 0
\end{aligned}
$$

$$
d_{i,\ell} \leq \Delta\left(1 - z^{\lambda}_{i,\ell+1} - z^{\mu}_{i,\ell-1}\right) \tag{12'$_{UB}$}
$$

$$
\begin{aligned}
& d_{i,\ell} - \Delta\left(1 - z^{\lambda}_{i,\ell+1} - z^{\mu}_{i,\ell-1}\right) \\
&= \Delta - \lambda_{i,\ell} - \mu_{i,\ell} - \Delta + \Delta z^{\lambda}_{i,\ell+1} + \Delta z^{\mu}_{i,\ell-1} \\
&= \left(\Delta z^{\lambda}_{i,\ell+1} - \lambda_{i,\ell}\right) + \left(\Delta z^{\mu}_{i,\ell-1} - \mu_{i,\ell}\right) \\
&\leq 0
\end{aligned}
$$

$$
d_{i,\ell} \geq \ell\Delta - S_i - \Delta z^{\mu}_{i,\ell} - \ell\Delta\left(1 - z^{\lambda}_{i,\ell}\right) \tag{13'}
$$

$$
\begin{aligned}
& d_{i,\ell} - \ell\Delta + S_i + \Delta z^{\mu}_{i,\ell} + \ell\Delta\left(1 - z^{\lambda}_{i,\ell}\right) \\
&= d_{i,\ell} - \ell\Delta + \left(\textstyle\sum_{\ell'=1}^{L} \lambda_{i,\ell'}\right) + \Delta z^{\mu}_{i,\ell} + \ell\Delta - \ell\Delta z^{\lambda}_{i,\ell} \\
&= \textstyle\sum_{\ell'=1}^{L} \lambda_{i,\ell'} + d_{i,\ell} + \Delta\left(z^{\mu}_{i,\ell} - z^{\lambda}_{i,\ell}\right) - \left(\textstyle\sum_{\ell'=1}^{\ell-1} \Delta\right) z^{\lambda}_{i,\ell} \\
&\geq \textstyle\sum_{\ell'=1}^{L} \lambda_{i,\ell'} + \left(\Delta - \lambda_{i,\ell} - \mu_{i,\ell}\right) + \Delta\left(z^{\mu}_{i,\ell} - z^{\lambda}_{i,\ell}\right) - \left(\textstyle\sum_{\ell'=1}^{\ell-1} \Delta z^{\lambda}_{i,\ell'+1}\right) \\
&\geq \left(\textstyle\sum_{\ell'=1}^{L} \lambda_{i,\ell'} - \lambda_{i,\ell} - \textstyle\sum_{\ell'=1}^{\ell-1} \lambda_{i,\ell'}\right) + \left(\Delta z^{\mu}_{i,\ell} - \mu_{i,\ell}\right) + \Delta\left(1 - z^{\lambda}_{i,\ell}\right) \\
&\geq \textstyle\sum_{\ell'=\ell+1}^{L} \lambda_{i,\ell'} \\
&\geq 0
\end{aligned}
$$

$$
d_{i,\ell} \geq S_i + p_i - (\ell-1)\Delta - \Delta z^{\lambda}_{i,\ell} - (L-\ell+1)\Delta\left(1 - z^{\mu}_{i,\ell}\right) \tag{14'}
$$

$$
\begin{aligned}
& d_{i,\ell} - S_i - p_i + (\ell-1)\Delta + \Delta z^{\lambda}_{i,\ell} + (L-\ell+1)\Delta \left(1 - z^{\mu}_{i,\ell}\right) \\
&= d_{i,\ell} + (\ell-1)\Delta - \left(L\Delta - \textstyle\sum_{\ell'=1}^{L} \mu_{i,\ell'}\right) + \Delta z^{\lambda}_{i,\ell} + (L-\ell+1)\Delta - (L-\ell+1)\Delta z^{\mu}_{i,\ell} \\
&= \textstyle\sum_{\ell'=1}^{L} \mu_{i,\ell'} + d_{i,\ell} + \Delta \left(z^{\lambda}_{i,\ell} - z^{\mu}_{i,\ell}\right) - \left(\textstyle\sum_{\ell'=\ell+1}^{L} \Delta\right) z^{\mu}_{i,\ell} \\
&\geq \textstyle\sum_{\ell'=1}^{L} \mu_{i,\ell'} + (\Delta - \lambda_{i,\ell} - \mu_{i,\ell}) + \Delta \left(z^{\lambda}_{i,\ell} - z^{\mu}_{i,\ell}\right) - \left(\textstyle\sum_{\ell'=\ell+1}^{L} \Delta\, z^{\mu}_{i,\ell'-1}\right) \\
&\geq \left(\textstyle\sum_{\ell'=1}^{L} \mu_{i,\ell'} - \mu_{i,\ell} - \textstyle\sum_{\ell'=\ell+1}^{L} \mu_{i,\ell'}\right) + \left(\Delta z^{\lambda}_{i,\ell} - \lambda_{i,\ell}\right) + \Delta \left(1 - z^{\mu}_{i,\ell}\right) \\
&\geq \textstyle\sum_{\ell'=1}^{\ell-1} \mu_{i,\ell'} \\
&\geq 0
\end{aligned}
$$

□