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Estimation of the necessary order of Legendre-LMI conditions to assess stability of time-delay systems

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Abstract: This paper investigates the stability analysis of time-delay systems through Lyapunov arguments. Using the existence of a complete Lyapunov-Krasovskii functional and relying on polynomial approximation theory, our main goal is to approximate the complete Lyapunov functional and to take profit of a supergeometric convergence rate of the truncated error part. Necessary and sufficient conditions in the linear matrix inequality (LMI) framework for sufficiently large approximated orders are consequently proposed. Moreover, an estimation of the necessary order is provided analytically with respect to system parameters.

Keywords: Infinite dimensional systems; Stability conditions; Converse Lyapunov theorem; Pseudo-spectral methods.

1. INTRODUCTION

An extension of the Lyapunov necessary and sufficient theorem has been introduced in Datko (1970) to analyze the stability of linear infinite-dimensional systems. This contribution ensures the existence of a complete quadratic Lyapunov functional without giving more details. Focusing now on time-delay systems, such complete Lyapunov functional has been constructed in Kharitonov (2013). Nevertheless, the Lyapunov inequalities conditions stay theoretical and cannot be implemented. In parallel, infinite-dimensional systems are often modeled by approximated methods to be analyzed numerically. It is the case of Padé approximations in the frequency domain (see Golub and Van Loan (1989)) or pseudo-spectral methods in the time domain (see Gottlieb and Orszag (1977)). In this approximation context, most of the results release in the convergence of the solution towards the expected one. However, few results are able to make the link between stability properties provided by approximated model and the original ones. Indeed, not enough attention has been paid to error part.

In the wide variety of pseudo-spectral methods, two categories stand out. From one side, collocation methods amounts to interpolate the state. The idea is to create a discretization map such that the error is null at the interpolated points. In between these points, function approximation can be chosen to be constant (see Gu et al. (2003)), linear (see Gu (2013); Medvedeva and Zhabko (2015)), polynomial (see splines method) or exponential (see Egorov and Mondié (2014)). Applying an interpolation, it leads to necessary conditions of stability as in Egorov and Mondié (2014); Medvedeva and Zhabko (2019) or to sufficient conditions of stability based on

discretized Lyapunov functional as in Gu et al. (2003). Recently, these results have been proven to converge asymptotically using some convergence properties and deliver inner and outer estimation of the stability regions (see respectively Gu (2013) and Gomez et al. (2021)). From the other side, *tau* methods amounts to approximate the state. The idea is to take support of a complete Hilbert basis in order to proceed approximation through Galerkin-like methods. For instance, Fourier trigonometric basis or Legendre polynomials have been used. Simple Lyapunov functionals based on the first coefficient (i.e. the mean value of the state) of such basis has led to linear matrix inequality (LMI) conditions exposed in Fridman (2014), which are, however, quite conservative. To reduce this conservatism, hierarchic Lyapunov functionals taking support on the n first Legendre coefficients have been proposed in Seuret and Gouaisbaut (2013, 2015). It is only recently that the gap between these approximated functionals and the original complete functional has been closed. In Bajodek et al. (2022), it is proven that sufficient Legendre-LMI conditions converge and become necessary for sufficiently large orders.

This conference paper explains the methodology, which allows to obtain the necessary and sufficient stability conditions developed in Bajodek et al. (2022), and provides a better estimation of the necessary order n^* . First, the complete Lyapunov functional of time-delay systems is recalled. Then, an intermediate result quantifies the quality of the approximation of the complete Lyapunov functional so that the Lyapunov converse theorem holds. Based on the selection of a Legendre approximation which ensures exponentially fast uniform convergence, the necessary and sufficient condition of stability for time-delay systems resumes to an LMI of size n^* expressed with respect to

system parameters. Finally, the LMI test as well as the order n^* are computed on an academic example.

Notations: Throughout the paper, \mathbb{N} , $\mathbb{R}^{m \times p}$, \mathbb{S}^m and \mathbb{S}_+^m denote the sets of natural numbers, real matrices of size $m \times p$, symmetric matrices of size m and positive symmetric matrices of size m , respectively. For any square real matrix M , M^\top denotes the transpose of M and $\mathcal{H}(M)$ stands for $M + M^\top$. Furthermore, for any $M \in \mathbb{S}^m$, its minimal and maximal eigenvalues are denoted $\sigma(M)$ and $\bar{\sigma}(M)$. The 2-norm of matrix M in $\mathbb{R}^{m \times p}$ is $|M| = \sqrt{\bar{\sigma}(M^\top M)}$. The vector $u = \text{vec}(M)$ in $\mathbb{R}^{mp \times 1}$ collocates the columns of matrix M . Notation \otimes refers to the matrix Kronecker product. The set of square-integrable functions from $(-h, 0)$ to $\mathbb{R}^{m \times p}$ is noted $L^2(-h, 0; \mathbb{R}^{m \times p})$. Let also $H^1(-h, 0; \mathbb{R}^{m \times p})$ be the set of functions F , such that F and F' are in $L^2(-h, 0; \mathbb{R}^{m \times p})$. Lastly, denote x_t such that $x_t(\theta) = x(t + \theta)$, for any θ in $(-h, 0]$. The classical norm is given by $\|\varphi\| = \sqrt{|\varphi(0)|^2 + \int_{-h}^0 |\varphi(\theta)|^2 d\theta}$, for any φ in $\mathbb{R}^{n_x} \times L^2(0, 1; \mathbb{R}^{n_z})$. The sup-norm is given by $\|\varphi\|_\infty = \sup_{[-h, 0]} |\varphi(\theta)|$.

2. CONVERSE LYAPUNOV THEOREM FOR TIME-DELAY SYSTEMS

2.1 Presentation of the system

Consider a linear time-delay system given by

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t-h), & \forall t \geq 0, \\ x(\theta) = \varphi(\theta), & \forall \theta \in (-h, 0], \end{cases} \quad (1)$$

where the delay $h > 0$ and matrices A, A_d in $\mathbb{R}^{n_x \times n_x}$ are constant and known. Without loss of generality, decompose A_d into a product BC such that

$$A_d = BC \text{ with } |C| = 1, \quad (2)$$

with $B, C^\top \in \mathbb{R}^{n_x \times n_z}$ being full column rank matrices, with n_z the rank of matrix A_d . The initial condition $x_0 = \varphi$ belongs to the set \mathcal{D} given by

$$\mathcal{D} = \left\{ \begin{array}{l} \varphi(\theta) = \begin{cases} \varphi_0 & \text{if } \theta = 0, \\ \varphi_1(\theta) & \text{if } \theta \in (-h, 0), \end{cases} \text{ such that} \\ \left[\begin{array}{l} \varphi_0 \\ \varphi_1 \end{array} \right] \in \mathbb{R}^{n_x} \times H^1(-h, 0; \mathbb{R}^{n_x}), \varphi_1(0) = C\varphi_0 \end{array} \right\}.$$

Definition 1. (GES). The trivial solution to (1) is said globally exponentially stable (GES), if there exist $\kappa \geq 1$ and $\mu > 0$ such that x_t , solution to (1) generated by any initial condition φ in \mathcal{D} , verifies

$$\|x_t\| \leq \kappa e^{-\mu t} \|\varphi\|, \quad \forall t \geq 0. \quad (3)$$

Remark 2. It is worth noticing that the regularity properties of the solutions of time-delay systems ensures that such a definition in the sense of $\|\cdot\|$ -norm implies that there also exist $\tilde{\kappa} \geq 1$ and $\tilde{\mu} > 0$ such that

$$|x(t)| \leq \tilde{\kappa} e^{-\tilde{\mu} t} \sup_{[-h, 0]} |\varphi(\theta)|, \quad \forall t \geq 0. \quad (4)$$

2.2 Existence of a complete Lyapunov functional

From Kharitonov and Zhabko (2003); Kharitonov (2013), it is well known that a necessary and sufficient stability condition of system (1) is the existence of a complete Lyapunov functional, which is recalled here.

For any $(W_1, W_2, W_3) \in \mathbb{S}_+^{n_x} \times \mathbb{S}_+^{n_z} \times \mathbb{S}_+^{n_z}$ and for any x_t in \mathcal{D} , define the following Lyapunov functional

$$\begin{aligned} \mathcal{V}(x_t) = & x_t^\top(0)U(0)x_t(0) + 2x_t^\top(0) \int_{-h}^0 U(\theta+h)A_d x_t(\theta) d\theta \\ & + \int_{-h}^0 \int_{-h}^0 x_t^\top(\theta_1)A_d^\top U(\theta_2 - \theta_1)A_d x_t(\theta_2) d\theta_1 d\theta_2 \\ & + \int_{-h}^0 x_t^\top(\theta)C^\top \left(\frac{\theta+h}{h}W_2 + W_3 \right) C x_t(\theta) d\theta, \end{aligned} \quad (5)$$

where U is a matrix function from $[-h, h]$ to $\mathbb{R}^{n_x \times n_x}$ and is called the Lyapunov matrix. Interestingly, referring to (Kharitonov, 2013, Section 2.10), a method is provided to build analytically this matrix. Denoting by K the fundamental matrix, its expression is given by

$$\begin{aligned} U(\theta) = & \int_0^\infty K^\top(t)WK(t+\theta)dt, \\ = & \begin{cases} \text{vec}^{-1} \left([I_{n_x} \ 0] e^{\theta M} N^{-1} [-\text{vec}_0^{\text{vec}(W)}] \right) & \text{if } \theta \leq 0, \\ U^\top(-\theta) & \text{if } \theta > 0, \end{cases} \end{aligned} \quad (6)$$

where matrices

$$\begin{aligned} W = & W_1 + C^\top(W_2 + W_3)C, \\ M = & \begin{bmatrix} -A^\top \otimes I_{n_x} & -A_d^\top \otimes I_{n_x} \\ I_{n_x} \otimes A_d^\top & I_{n_x} \otimes A^\top \end{bmatrix}, \\ N = & \begin{bmatrix} A^\top \otimes I_{n_x} + I_{n_x} \otimes A^\top & A_d^\top \otimes I_{n_x} \\ I_{n_x} \otimes A_d^\top & 0 \end{bmatrix} + \begin{bmatrix} I_{n_x} \otimes A_d^\top & 0 \\ 0 & -I_{n_x} \end{bmatrix} e^{-hM}, \end{aligned} \quad (7)$$

and where ‘vec’ is the operator that transforms a matrix in $\mathbb{R}^{n_x \times n_x}$ to a vector in $\mathbb{R}^{n_x^2}$ that collects all the columns of this matrix and ‘vec⁻¹’ denotes the inverse operator.

Such a complete Lyapunov functional \mathcal{V} has been built so that its time-derivative along the trajectories of the system yields

$$\begin{aligned} \dot{\mathcal{V}}(x_t) = & -x_t^\top(0)W_1 x_t(0) - x_t^\top(-h)C^\top W_3 C x_t(-h) \\ & - \frac{1}{h} \int_{-h}^0 x_t^\top(\theta)C^\top W_2 C x_t(\theta) d\theta. \end{aligned} \quad (8)$$

A necessary and sufficient condition of stability for system (1) based on this functional is formulated below.

2.3 Necessary and sufficient condition of stability

The following theorem is obtained by application of the Lyapunov theorem. The proposed formulation is issued from (Gu et al., 2003, Theorem 5.19).

Theorem 1. System (1) is GES if and only if there exist positive scalars $\alpha_1, \alpha_2, \alpha_3$ such that \mathcal{V} defined by (5) satisfies, for any x_t in \mathcal{D} , inequalities

$$\alpha_1 \|x_t\|^2 \leq \mathcal{V}(x_t) \leq \alpha_2 \|x_t\|^2, \quad (9a)$$

$$\dot{\mathcal{V}}(x_t) \leq -\alpha_3 \|x_t\|^2, \quad (9b)$$

where $\dot{\mathcal{V}}$ denotes the time derivative of \mathcal{V} along the trajectories of system (1).

Remark 1. It is worth noticing that in the sense of the stability definition given by (4), the Lyapunov-Krasovskii version of this theorem also holds replacing (9) by

$$\alpha_1 |x_t(0)|^2 \leq \mathcal{V}(x_t) \leq \alpha_2 \sup_{[-h, 0]} |x_t(\theta)|^2, \quad (10a)$$

$$\dot{\mathcal{V}}(x_t) \leq -\alpha_3 |x_t(0)|^2. \quad (10b)$$

Proof : For the sake of simplicity, the time argument will be discarded in the following proof. To begin with, the sufficiency refers to the application of the Lyapunov Theorem. The proof of necessity starts by noting that (9b) already holds by construction of the Lyapunov matrix (8). Indeed, expression (8) leads to $\alpha_3 = \max(\underline{\sigma}(W_1), \underline{\sigma}(W_2), \underline{\sigma}(W_3))$. The right-hand side of (9a) is also easy to prove since \mathcal{V} is quadratic with respect to x_t . Let focus now on the left-hand side of (9a) and introduce a functional \mathcal{W} as follows

$$\mathcal{W}(x_t) = \mathcal{V}(x_t) - \alpha_1 \|x_t\|^2.$$

According to (8), differentiating \mathcal{W} along the trajectories of (1) leads to

$$\begin{aligned} \dot{\mathcal{W}}(x_t) = & - \begin{bmatrix} x_t(0) \\ x_t(-h) \end{bmatrix}^\top \begin{bmatrix} W_1 & 0 \\ 0 & C^\top W_3 C \end{bmatrix} \begin{bmatrix} x_t(0) \\ x_t(-h) \end{bmatrix} \\ & - \frac{1}{h} \int_{-h}^0 x_t^\top(\theta) C^\top W_2 C x_t(\theta) d\theta \\ & - \alpha_1 \begin{bmatrix} x_t(0) \\ x_t(-h) \end{bmatrix}^\top \begin{bmatrix} \mathcal{H}(A) + C^\top C & B \\ B^\top & -I_{n_z} \end{bmatrix} \begin{bmatrix} x_t(0) \\ C x_t(-h) \end{bmatrix}. \end{aligned}$$

Then, there exists a positive number α_1 sufficiently small such that \mathcal{W} is negative. Integrating the above inequality from t to ∞ , we obtain

$$\lim_{T \rightarrow \infty} \mathcal{W}(x_T) - \mathcal{W}(x_t) \leq 0, \quad \forall t \geq 0.$$

As $x_T \xrightarrow{T \rightarrow \infty} 0$ because system (1) is assumed to be GES, inequality $\mathcal{W}(x_t) \geq 0$ holds, which concludes the proof. \square

In the following section, we will propose a modified version of Theorem 1.

3. MODIFIED CONVERSE LYAPUNOV THEOREM

3.1 Approximated complete Lyapunov functional

The Lyapunov matrix U is composed of a complex combination of exponential functions. In order to derive tractable numerical tests, the idea is to introduce an approximation \hat{U} of U , which has the nice properties to ease this purpose. Basically, define the function \hat{U} from $[-h, h]$ to $\mathbb{R}^{n_x \times n_x}$ satisfying the symmetric relation

$$\hat{U}(\theta) = \hat{U}^\top(-\theta), \quad \forall \theta \in [-h, 0]. \quad (11)$$

For any $(W_2, W_3) \in \mathbb{S}_+^{n_z} \times \mathbb{S}_+^{n_z}$ and x_t in \mathcal{D} , we introduce a new functional

$$\begin{aligned} \hat{\mathcal{V}}(x_t) = & x_t^\top(0) \hat{U}(0) x_t(0) + 2x_t^\top(0) \int_{-h}^0 \hat{U}(\theta+h) A_d x_t(\theta) d\theta \\ & + \int_{-h}^0 \int_{-h}^0 x_t^\top(\theta_1) A_d^\top \hat{U}(\theta_2 - \theta_1) A_d x_t(\theta_2) d\theta_1 d\theta_2 \\ & + \int_{-h}^0 x_t^\top(\theta) C^\top \left(\frac{\theta+h}{h} W_2 + W_3 \right) C x_t(\theta) d\theta. \end{aligned} \quad (12)$$

Such a functional will be considered as an approximation of the original \mathcal{V} . Denoting $\tilde{\mathcal{V}} = \mathcal{V} - \hat{\mathcal{V}}$, we obtain

$$\begin{aligned} \tilde{\mathcal{V}}(x_t) = & x_t^\top(0) \tilde{U}(0) x_t(0) + 2x_t^\top(0) \int_{-h}^0 \tilde{U}(\theta+h) A_d x_t(\theta) d\theta \\ & + \int_{-h}^0 \int_{-h}^0 x_t^\top(\theta_1) A_d^\top \tilde{U}(\theta_2 - \theta_1) A_d x_t(\theta_2) d\theta_1 d\theta_2, \end{aligned} \quad (13)$$

where $\tilde{U} = U - \hat{U}$ refers to the error done by comparing the exact Lyapunov matrix U and \hat{U} .

The use of approximations of the complete Lyapunov functional is worth of interest in two cases given below.

- The closed-form of the Lyapunov matrix U is unknown or cannot be calculated precisely enough.
- The necessary and sufficient inequalities conditions (9) need to be simplified to be tested.

Nevertheless, the price to pay is that the functional $\hat{\mathcal{V}}$ generated by this \hat{U} not necessarily verifies the condition (9) of the complete one. In this sequel, we provide a new theorem stating necessary and sufficient condition of stability even though the complete Lyapunov functional is not built by U but by \hat{U} . This theorem naturally relies on the distance between U and \hat{U} , which will be latter seen as an approximation error.

3.2 A new necessary and sufficient condition of stability

The main result of the paper is presented below.

Theorem 2. Let (W_1, W_2, W_3) in $\mathbb{S}_+^{n_x} \times \mathbb{S}_+^{n_z} \times \mathbb{S}_+^{n_z}$ and assume that the following inequality

$$\Psi(\theta; \tilde{U}) = \begin{bmatrix} W_1 + \Psi_1(\tilde{U}) & \Psi_2(\theta; \tilde{U}) \\ * & W_2 \end{bmatrix} \succ 0, \quad (14)$$

holds, for all θ in $(-h, 0)$, with $\tilde{U} = U - \hat{U}$ and

$$\begin{aligned} \Psi_1(\tilde{U}) &= \mathcal{H}(\tilde{U}(0)A) + \mathcal{H}(\tilde{U}(h)A_d), \\ \Psi_2(\theta; \tilde{U}) &= \left(A^\top \tilde{U}(\theta+h) + A_d^\top \tilde{U}(\theta) - \tilde{U}'(\theta+h) \right) hB. \end{aligned} \quad (15)$$

System (1) is GES if and only if there exist positive scalars $\alpha_1, \alpha_2, \alpha_3$ such that $\hat{\mathcal{V}}$ defined by (12) satisfies, for any x_t in \mathcal{D} , inequalities

$$\alpha_1 \|x_t\|^2 \leq \hat{\mathcal{V}}(x_t) \leq \alpha_2 \|x_t\|^2, \quad (16a)$$

$$\dot{\hat{\mathcal{V}}}(x_t) \leq -\alpha_3 \|x_t\|^2, \quad (16b)$$

where $\dot{\hat{\mathcal{V}}}$ denotes the time derivative of $\hat{\mathcal{V}}$ along the trajectories of system (1).

Proof : Trivially, the sufficiency of the statement is verified. For the necessity of the proposed statement, let us first note that the right-hand part of (16a) is directly obtained because the Lyapunov functional is quadratic with respect to x_t . Let now focus on the two remained inequalities. To begin with, the derivative of $\hat{\mathcal{V}}$ given by (12) along the trajectories of system (1) leads, after performing some integration by parts and thanks to symmetric relation (11) to the following expression

$$\begin{aligned} \dot{\hat{\mathcal{V}}}(x_t) = & 2x_t^\top(0) \tilde{U}(0) A x_t(0) + \frac{2x_t^\top(0) \tilde{U}(0) A_d x_t(-h)}{\cancel{2}} \\ & + 2 \left(x_t^\top(0) \tilde{U}(\theta+h) A_d x_t(\theta) \right) \Big|_{\theta=-h}^0 - x_t^\top(0) \int_{-h}^0 \tilde{U}'(\theta+h) A_d x_t(\theta) d\theta \\ & + 2(Ax_t(0) + \cancel{A_d x_t(-h)})^\top \int_{-h}^0 \tilde{U}(\theta+h) A_d x_t(\theta) d\theta \\ & + 2 \int_{-h}^0 \left[x_t^\top(\theta_1) A_d^\top \tilde{U}(\theta_2 - \theta_1) A_d x_t(\theta_2) \right] \Big|_{\theta_1=-h}^{\theta_1=0} d\theta_2 \\ & - \int_{-h}^0 \int_{-h}^0 x_t^\top(\theta_1) A_d^\top \underbrace{(\partial_{\theta_1} + \partial_{\theta_2}) \tilde{U}(\theta_2 - \theta_1)}_{=0} A_d x_t(\theta_2) d\theta_1 d\theta_2. \end{aligned}$$

The last term of the previous expression vanishes and simplifications are made. Then, using the previous expression and (8), the time derivative of $\hat{\mathcal{V}} = \mathcal{V} - \tilde{\mathcal{V}}$ along the trajectories of system (1) boils down to

$$\dot{\hat{V}}(x_t) = -\frac{1}{h} \int_{-h}^0 \begin{bmatrix} x_t(0) \\ Cx_t(\theta) \end{bmatrix}^\top \Psi(\theta; \tilde{U}) \begin{bmatrix} x_t(0) \\ Cx_t(\theta) \end{bmatrix} - x_t^\top(-h) C^\top W_3 C x_t(-h). \quad (17)$$

Condition $\Psi(\theta; \tilde{U}) \succ 0$ for all $\theta \in (-h, 0)$ ensures the existence of a sufficiently small scalar $\alpha_3 > 0$ such that inequality (16b) holds. On the other part, introduce the functional \hat{W} given by

$$\hat{W}(x_t) = \hat{V}(x_t) - \alpha_1 \|x_t\|^2.$$

According to the previous calculations (17), differentiating \hat{W} along the trajectories of (1) leads to

$$\begin{aligned} \dot{\hat{W}}(x_t) = & -\frac{1}{h} \int_{-h}^0 \begin{bmatrix} x_t(0) \\ Cx_t(\theta) \end{bmatrix}^\top \Psi(\theta; \tilde{U}) \begin{bmatrix} x_t(0) \\ Cx_t(\theta) \end{bmatrix} \\ & - x_t^\top(-h) C^\top W_3 C x_t(-h) \\ & - \alpha_1 \begin{bmatrix} x_t(0) \\ x_t(-h) \end{bmatrix}^\top \begin{bmatrix} \mathcal{H}(A) + C^\top C & B \\ B^\top & -I_{n_z} \end{bmatrix} \begin{bmatrix} x_t(0) \\ Cx_t(-h) \end{bmatrix}. \end{aligned}$$

Then, there exists a sufficiently small scalar $\alpha_1 > 0$ such that $\dot{\hat{W}}$ is negative. Integrating the above from t to ∞ , we obtain

$$\lim_{T \rightarrow \infty} \hat{W}(x_T) - \hat{W}(x_t) \leq 0, \quad \forall t \geq 0.$$

Since system (1) is GES, we have $x_T \xrightarrow{T \rightarrow \infty} 0$ so that inequality $\hat{W}(x_t) \geq 0$ holds. That corresponds to the left-hand part of (16a) and concludes the proof. \square

Remark 2. In Bajodek et al. (2022), a similar theorem was provided but the main difference relies on the fact that functional \hat{V} has been defined $U(0)$ instead of $\hat{U}(0)$, which leads to a different and more complex condition.

It is worth mentioning that, taking support on a judicious choice of \hat{U} , inequalities (16) are easier to implement numerically compared to inequalities (9). In practice, the approximated function \hat{U} can come from interpolation or approximation techniques. In particular, the Legendre polynomial approximation will lead to the efficient LMI developed in the next section.

4. APPLICATION TO LEGENDRE POLYNOMIALS

4.1 Legendre approximation

Consider a polynomial approximation \hat{U}_n of U , which is defined from $[-h, h]$ to $\mathbb{R}^{n_x \times n_x}$ as

$$\hat{U}_n(\theta) = \begin{cases} \ell_n^{n_x \top}(\theta) \mathcal{U}_n & \text{if } \theta \in [-h, 0], \\ U_n^\top(-\theta) & \text{if } \theta \in [0, h], \end{cases} \quad (18)$$

where matrix $\ell_n^{n_x}$ in $\mathbb{R}^{n_x \times n_x}$ collocates the n first Legendre polynomials scaled on $[-h, 0]$

$$\ell_n^{n_x \top}(\theta) = [l_0(\theta) \dots l_{n-1}(\theta)] \otimes I_{n_x}, \quad \forall \theta \in [-h, 0], \quad (19)$$

where the matrix \mathcal{U}_n in $\mathbb{R}^{n_x \times n_x}$ collocates the n first Legendre coefficients of the Lyapunov matrix U

$$\mathcal{U}_n = \mathcal{I}_n^{n_x} \left(\int_{-h}^0 \ell_n^{n_x}(\theta) U(\theta) d\theta \right), \quad (20)$$

and where the normalization matrix of Legendre polynomials is equal to

$$\mathcal{I}_n^{n_x} = \left(\int_{-h}^0 \ell_n^{n_x}(\theta) \ell_n^{n_x \top}(\theta) d\theta \right)^{-1}. \quad (21)$$

4.2 Convergence properties

In the light of Dunkl and Xu (2001), the approximation is selected in order to have uniform convergence on the closed intervals $[-h, 0]$ and $[0, h]$ of U_n and U_n' towards functions U and U' , respectively.

Property 1. The Lyapunov matrix U associated to W and defined in (6) satisfies

$$\|U^{(d)}\|_\infty \leq \left(\frac{2\mu}{h} \right)^d \rho, \quad \forall d \in \mathbb{N}, \quad (22)$$

where $U^{(d)}$ is the d^{th} derivative of U and where parameter μ and ρ are given by

$$\mu = \frac{h|M|}{2}, \quad \rho = \sqrt{n_x} e^{h|M|} |N^{-1}| |W|, \quad (23)$$

and where the matrices W, M, N are given by (7).

Proof : Noticing that $|W| \leq |\text{vec}(W)| \leq \sqrt{n_x} |W|$, the following inequalities hold

$$\begin{aligned} |U^{(d)}(\theta)| & \leq \left| \begin{bmatrix} I_{n_x} & 0 \\ 0 & 0 \end{bmatrix} e^{\theta M} M^k N^{-1} \begin{bmatrix} -\text{vec}(W) \\ 0 \end{bmatrix} \right|, \\ & \leq \sqrt{n_x} |e^{\theta M}| |M|^k |N^{-1}| |W|, \quad \forall \theta \in [-h, 0], \end{aligned}$$

Moreover, recalling $e^{\theta M} = \sum_{k=0}^{\infty} \frac{(\theta M)^k}{k!}$, the following upper bound is obtained

$$|e^{\theta M}| \leq \sum_{k=0}^{\infty} \frac{(|\theta M|)^k}{k!} \leq \sum_{k=0}^{\infty} \frac{|\theta|^k |M|^k}{k!} \leq \sum_{k=0}^{\infty} \frac{h^k |M|^k}{k!} = e^{h|M|},$$

and yields the result. \square

Based on Property 1, the Legendre approximation errors \tilde{U}_n and its derivative \tilde{U}_n' are proven to converge uniformly and exponentially fast to zero on the closed interval $[-h, 0]$.

Property 2. For all $d \geq 4$ and $n \geq d + 2$, the Legendre truncated error $\tilde{U}_n = U - \hat{U}_n$ satisfies the inequalities

$$\|\tilde{U}_n\|_\infty \leq \sqrt{\frac{\pi^3}{2(n-1-d)(n-\frac{3}{2}) \dots (n-d+\frac{1}{2})}} \frac{\mu^{d+1} \rho}{}, \quad (24a)$$

$$h \|\tilde{U}_n'\|_\infty \leq \sqrt{\frac{2\pi^3}{(n-1-d)(n-\frac{7}{2}) \dots (n-d+\frac{1}{2})}} \frac{\mu^{d+1} \rho}{}, \quad (24b)$$

where parameter μ and ρ are given by (23).

Proof : The proof can be found in (Wang and Xiang, 2012, Theorem 5.12) using Legendre polynomials properties and d successive integrations by parts. \square

When function U is smooth on $[-h, 0]$, the convergence is supergeometric since both errors $\|\tilde{U}_n\|_\infty, \|\tilde{U}_n'\|_\infty$ are in the range of $\frac{1}{n!} \sim \frac{1}{n^\infty} = e^{-n \log(n)}$ from Stirling formula.

These properties unlock the following results.

4.3 Necessary and sufficient LMI condition of stability

For any $(\mathbf{P}_n, R, S) \in \mathbb{S}^{n_x + n n_x} \times \mathbb{S}_+^{n_x} \times \mathbb{S}_+^{n_x}$, introduce the Lyapunov candidate functional considered in Seuret and Gouaisbaut (2014, 2015)

$$\begin{aligned} \mathcal{V}_n(x_t) = & \left[\int_{-h}^0 \ell_n^{n_z}(\theta) C x_t(\theta) d\theta \right]^\top \mathbf{P}_n \left[\int_{-h}^0 \ell_n^{n_z}(\theta) C x_t(\theta) d\theta \right] \\ & + \int_{-h}^0 x_t^\top(\theta) C^\top \left(\frac{\theta+h}{h} R + S \right) C x_t(\theta) d\theta. \end{aligned} \quad (25)$$

The time derivative of \mathcal{V}_n along the trajectories of system (1) are calculated using integrations by parts as follows

$$\begin{aligned} \dot{\mathcal{V}}_n(x_t) = & 2 \left[\int_{-h}^0 \ell_n^{n_z}(\theta) C x_t(\theta) d\theta \right]^\top \mathbf{P}_n \left[-\ell_n^{n_z}(-h) \right] C x_t(-h) \\ & + 2 \left[\int_{-h}^0 \ell_n^{n_z}(\theta) C x_t(\theta) d\theta \right]^\top \mathbf{P}_n \left[\ell_n^{n_z}(0) C \mathcal{L}_n \right] \left[\int_{-h}^0 \ell_n^{n_z}(\theta) C x_t(\theta) d\theta \right] \\ & + \left[\begin{array}{c} x_t(0) \\ C x_t(-h) \end{array} \right]^\top \left[\begin{array}{cc} C^\top(hR+S)C & 0 \\ 0 & -S \end{array} \right] \left[\begin{array}{c} x_t(0) \\ C x_t(-h) \end{array} \right] \\ & - \frac{1}{h} \int_{-h}^0 x_t^\top(\theta) C^\top R C x_t(\theta) d\theta, \end{aligned} \quad (26)$$

where the derivation matrix of Legendre polynomials is equal to

$$\mathcal{L}_n = -\text{tril}(\ell_n^{n_z}(0) \ell_n^{n_z \top}(0) - \ell_n^{n_z}(-h) \ell_n^{n_z \top}(-h)) \mathcal{I}_n^{n_z}. \quad (27)$$

In this context, the following theorem can be established.

Theorem 3. System (1) is GES if and only if there exists an order n and matrices $(\mathbf{P}_n, S, R) \in \mathbb{S}^{n_x + nn_z} \times \mathbb{S}_+^{n_z} \times \mathbb{S}_+^{n_z}$ the following linear matrix inequalities hold

$$\begin{aligned} \Xi_n^+ &= \mathbf{P}_n + (\mathbf{I}_n^S + \mathbf{J}_n^R) \succ 0, \\ \Xi_n^- &= \left[\begin{array}{c} \mathcal{H}(\mathbf{P}_n \mathbf{A}_n) + \mathbf{C}_n(hR + S) \mathbf{C}_n - \frac{1}{h} \mathbf{I}_n^R \mathbf{P}_n \mathbf{B}_n \\ * \\ -S \end{array} \right] \prec 0. \end{aligned} \quad (28)$$

where matrices are defined by

$$\begin{aligned} \mathbf{I}_n^S &= \left[\begin{array}{c} 0 \\ 0 \quad \mathcal{I}_n^{n_z} \int_{-h}^0 \ell_n^{n_z}(\theta) S \ell_n^{n_z \top}(\theta) d\theta \mathcal{I}_n^{n_z} \end{array} \right], \quad \forall S \in \mathbb{S}_+^{n_z}, \\ \mathbf{J}_n^S &= \left[\begin{array}{c} 0 \\ 0 \quad \mathcal{I}_n^{n_z} \int_{-h}^0 \ell_n^{n_z}(\theta) \left(\frac{\theta+h}{h} \right) S \ell_n^{n_z \top}(\theta) d\theta \mathcal{I}_n^{n_z} \end{array} \right], \\ \mathbf{A}_n &= \left[\begin{array}{cc} A & 0 \\ \ell_n^{n_z}(0) C & \mathcal{L}_n \end{array} \right], \\ \mathbf{B}_n &= \left[\begin{array}{c} B \\ -\ell_n^{n_z}(-h) \end{array} \right], \quad \mathbf{C}_n = [C \ 0]. \end{aligned} \quad (29)$$

Proof : For the sake of simplicity, the time argument has been discarded in the following proof. Firstly, if LMI condition (28) is true, then the Lyapunov candidate functional given by (25) and its derivatives given by (26) satisfy the positivity and negativity conditions, respectively. Applying the Lyapunov theorem, the sufficiency of the statement is verified. Secondly, concerning the necessity, we take support on Theorem 2. For all (W_1, W_2, W_3) in $\mathbb{S}_+^{n_x} \times \mathbb{S}_+^{n_z} \times \mathbb{S}_+^{n_z}$, let matrices

$$\mathbf{P}_n = \left[\begin{array}{cc} U_n(0) & \mathcal{Q}_n \\ * & \mathcal{T}_n \end{array} \right], \quad R = W_2, \quad S = W_3, \quad (30)$$

$$\mathcal{Q}_n = \int_{-h}^0 U^\top(\theta) B \ell_n^{n_z \top}(\theta) \mathcal{I}_n^{n_z} d\theta,$$

$$\mathcal{T}_n = \int_{-h}^0 \int_{-h}^0 \mathcal{I}_n^{n_z} \ell_n^{n_z}(\theta_1) B^\top U_n(\theta_2 - \theta_1) B \ell_n^{n_z \top}(\theta_2) \mathcal{I}_n^{n_z} d\theta_1 d\theta_2.$$

In that case, the Lyapunov candidate functional \mathcal{V}_n given by (25) is the Legendre approximation of the Lyapunov converse functional \mathcal{V} given by (5). Indeed, the functional $\hat{\mathcal{V}}$ given by (12) is recognized with $\hat{U} = \hat{U}_n$ in (18). More interestingly, Property 2 ensures that both $\|U_n\|_\infty$ and $\|\hat{U}'_n\|_\infty$ tend to zero as n goes to infinity. Focusing on matrix Ψ given by (14) with $\tilde{U} = \tilde{U}_n$, we have

$$\Psi(\theta; \tilde{U}_n) = \begin{bmatrix} W_1 & 0 \\ * & W_2 \end{bmatrix} + \begin{bmatrix} \Psi_1(\tilde{U}_n) & 0 \\ * & \Psi_2(\theta; \tilde{U}_n) \end{bmatrix}, \quad (31)$$

where matrices Ψ_1, Ψ_2 are given by

$$\begin{aligned} \Psi_1(\tilde{U}_n) &= \mathcal{H}(\tilde{U}_n(0)A) + \mathcal{H}(\tilde{U}_n(h)A_d), \\ \Psi_2(\theta; \tilde{U}_n) &= \left(A^\top \tilde{U}_n(\theta+h) + A_d^\top \tilde{U}_n(\theta) - \tilde{U}'_n(\theta+h) \right) hB. \end{aligned}$$

Then, the second term in (31) tends to zero as n goes to infinity. Therefore, there exists an order $n^* \in \mathbb{N}$ such that $\Psi(\theta; \tilde{U}_n) \succ 0$ holds for all $\theta \in [-h, 0]$. Assume now that system (1) is GES. According to Theorem 2, for all $n \geq n^*$, there exist positive scalars $\alpha_1, \alpha_2, \alpha_3$ such for the following inequalities hold

$$\alpha_1 \|x_t\|^2 \leq \mathcal{V}_n(x_t) \leq \alpha_2 \|x_t\|^2, \quad (32a)$$

$$\dot{\mathcal{V}}_n(x_t) \leq -\alpha_3 \|x_t\|^2. \quad (32b)$$

Then, these inequalities are regarded for x_t in $\mathcal{D}_n \subset \mathcal{D}$ given by

$$\mathcal{D}_n = \left\{ \varphi(\theta) = \begin{cases} \varphi_0 & \text{if } \theta = 0, \\ C^\top (CC^\top)^{-1} \ell_n^{n_z \top}(\theta) \zeta_n & \text{if } \theta \in (-h, 0), \\ \varphi_{-h} & \text{if } \theta = -h, \end{cases} \right\} \\ \text{such that } [\varphi_0 \ \zeta_n \ \varphi_{-h}]^\top \in \mathbb{R}^{2n_x + nn_z}.$$

Thus, inequalities (32a),(32b) lead to $\Xi_n^+ \succ 0$ and $\Xi_n^- \prec 0$, respectively, and conclude the proof. \square

Theorem 3 provides a necessary and sufficient LMI condition for the stability of system (1). Nonetheless, it only guarantees the existence of an order n such that the LMI must be satisfied. It does not provide however an estimation of this order. The calculation of the order is reserved to the next subsection.

4.4 Estimation of the necessary order n^*

The following property takes the advantages of the bounds provided by Property 2 to estimate the smallest order from which both $\|\tilde{U}_n\|_\infty$ and $h\|\tilde{U}'_n\|_\infty$ are smaller than a prescribed $\varepsilon > 0$.

Property 3. For any scalar $\varepsilon > 0$, the Legendre truncated error $\tilde{U}_n = U - \hat{U}_n$ verifies

$$\max \left(\|\tilde{U}_n\|_\infty, h\|\tilde{U}'_n\|_\infty \right) \leq \varepsilon, \quad \forall n \geq \mathcal{N}(\varepsilon), \quad (33)$$

where $\mathcal{N}(\varepsilon)$ is given by

$$\mathcal{N}(\varepsilon) = \max \left(6, \left\lceil \frac{7}{2} + \mu e^{1+\mathcal{W}\left(\frac{-\log(\lambda\varepsilon)}{\mu e}\right)} \right\rceil \right), \quad (34)$$

where

$$\lambda = \frac{1}{\rho \mu^4} \sqrt{\frac{|\mu|}{2\pi^3}} \left(\frac{\mu e}{|\mu| + \frac{1}{2}} \right)^{\lceil \mu \rceil + \frac{1}{2}}, \quad (35)$$

where scalars μ, ρ are given by (23) and where \mathcal{W} is the Lambert function defined from \mathbb{R}_+ to \mathbb{R}_+ by $\mathcal{W}(z) = y$ where y is uniquely defined by the relation $e^y y = z$.

Proof: According to Property 2 with $d = n - 1 - \lceil \mu \rceil$, the Legendre truncated error function \tilde{U}_n and its derivatives are bounded by

$$\begin{aligned} \varepsilon_n &:= \max \left(\|\tilde{U}_n\|_\infty, h \|\tilde{U}'_n\|_\infty \right), \\ &\leq \sqrt{\frac{2\pi^3}{\lceil \mu \rceil} \frac{\mu^{n-\lceil \mu \rceil} \rho}{(n - \frac{7}{2}) \dots (1 + \lceil \mu \rceil + \frac{1}{2})}}, \quad \forall n \geq 6. \end{aligned} \quad (36)$$

Applying the logarithm to (36) leads to

$$\log(\varepsilon_n) \leq \log \left(\sqrt{\frac{2\pi^3}{\lceil \mu \rceil} \rho} \right) + (n - \lceil \mu \rceil) \log(\mu) - \sum_{k=1+\lceil \mu \rceil}^{n-4} \log \left(k + \frac{1}{2} \right). \quad (37)$$

By series-integral comparison, we obtain

$$\begin{aligned} \sum_{k=1+\lceil \mu \rceil}^{n-4} \log \left(k + \frac{1}{2} \right) &\geq \int_{x=\lceil \mu \rceil}^{n-4} \log \left(x + \frac{1}{2} \right) dx, \\ &= \left[\left(x + \frac{1}{2} \right) \log \left(\frac{x + \frac{1}{2}}{e} \right) \right]_{x=\lceil \mu \rceil}^{n-4}. \end{aligned}$$

Then, introducing λ in (35), inequality (37) yields

$$\log(\lambda \varepsilon_n) \leq - \left(n - \frac{7}{2} \right) \log \left(\frac{n - \frac{7}{2}}{\mu e} \right). \quad (38)$$

The orders n which allow to satisfy the following inequality

$$\log(\varepsilon_n) := \log \left(\max \left(\|\tilde{U}_n\|_\infty, h \|\tilde{U}'_n\|_\infty \right) \right) \leq \log(\varepsilon)$$

are solution to

$$- \left(\frac{n - \frac{7}{2}}{\mu e} \right) \log \left(\frac{n - \frac{7}{2}}{\mu e} \right) \leq \frac{\log(\lambda \varepsilon)}{\mu e}. \quad (39)$$

The definition of Lambert function implies that

$$y_n := \log \left(\frac{n - \frac{7}{2}}{\mu e} \right) \geq \mathcal{W} \left(- \frac{\log(\lambda \varepsilon)}{\mu e} \right). \quad (40)$$

Therefore, the corresponding orders satisfy

$$n \geq \frac{7}{2} + \mu e^{1 + \mathcal{W} \left(- \frac{\log(\lambda \varepsilon)}{\mu e} \right)}, \quad (41)$$

which leads to the expression of $\mathcal{N}(\varepsilon)$ together with the initial constraint $n \geq 6$. \square

The following theorem is the final piece to estimate the necessary order n^* for which Theorem 3 holds.

Theorem 4. Theorem 3 holds for orders $n \geq n^*$, where

$$n^* = \mathcal{N}(\mathcal{E}(\beta_1, \beta_2)), \quad (42)$$

where scalars β_1, β_2 are equal to

$$\begin{cases} \beta_1 = \frac{|A| + |A_d|}{3(1 + h(|A| + |A_d|))^2 |B|^2}, \\ \beta_2 = \frac{1}{9(1 + h(|A| + |A_d|))^2 |B|^2}, \end{cases} \quad (43)$$

where functions \mathcal{N} and \mathcal{E} are defined by

$$\begin{aligned} \mathcal{N}(\varepsilon) &= \max \left(6, \left\lceil \frac{7}{2} + \mu e^{1 + \mathcal{W} \left(- \frac{\log(\lambda \varepsilon)}{\mu e} \right)} \right\rceil \right), \\ \mathcal{E}(\beta_1, \beta_2) &= -\beta_1 + \sqrt{\beta_1^2 + \beta_2} = \frac{\beta_2}{\beta_1 + \sqrt{\beta_1^2 + \beta_2}}, \end{aligned} \quad (44)$$

and where parameters μ, ρ and λ are given by

$$\begin{aligned} \mu &= \frac{h|M|}{2}, \quad \rho = 3\sqrt{n_x} e^{h|M|} |N^{-1}|, \\ \lambda &= \frac{1}{\rho \mu^4} \sqrt{\frac{\lceil \mu \rceil}{2\pi^3}} \left(\frac{\mu e}{\lceil \mu \rceil + \frac{1}{2}} \right)^{\lceil \mu \rceil + \frac{1}{2}}. \end{aligned} \quad (45)$$

Proof: Fix matrices $W_1 = I_{n_x}$, $W_2 = I_{n_z}$ and $W_3 = I_{n_x}$. The estimation of the order n^* is made possible thanks to the convergence properties of Legendre approximation. According to Property 3, if $n \geq n^* = \mathcal{N}(\mathcal{E}(\beta_1, \beta_2))$, then the following inequality holds

$$\varepsilon_n := \max \left(\|\tilde{U}_n\|_\infty, h \|\tilde{U}'_n\|_\infty \right) \leq \mathcal{E}(\beta_1, \beta_2), \quad (46)$$

and implies that the following inequality holds

$$\beta_2 - 2\beta_1 \varepsilon_n - \varepsilon_n^2 \geq 0. \quad (47)$$

In addition, it is worth noticing that $\Psi_1(\tilde{U})$ and $\Psi_{n,2}$ given by (15) are upper bounded by

$$\begin{aligned} \left| \Psi_1(\tilde{U}_n) \right| &= \left| \mathcal{H}(\tilde{U}_n(0)A) + \mathcal{H}(\tilde{U}_n(h)A_d) \right|, \\ &\leq 6(|A| + |A_d|) \varepsilon_n = \frac{2\beta_1}{\beta_2} \varepsilon_n, \end{aligned}$$

and by

$$\begin{aligned} \|\Psi_2(\theta; \tilde{U}_n)\|_\infty &= \left| \left(A^\top \tilde{U}_n(\theta+h) + A_d^\top \tilde{U}_n^\top(\theta) - \tilde{U}'_n(\theta+h) \right) h B \right|, \\ &\leq 3(1 + h(|A| + |A_d|)) |B| \varepsilon_n = \sqrt{\frac{1}{\beta_2}} \varepsilon_n, \end{aligned}$$

because $|W_1 + C^\top(W_2 + W_3)C| \leq 3$, $\|U\|_\infty \leq \varepsilon_n$ and $h\|U'\|_\infty \leq \varepsilon_n$. Consequently, inequality (47) means that the following condition holds

$$I_{n_x} + \Psi_1(\tilde{U}) - \Psi_{n,2}^\top(\theta) \Psi_2(\theta; \tilde{U}) \succ 0, \quad \forall \theta \in [-h, 0]. \quad (48)$$

From Shur complement, inequality $\Psi(\theta; \tilde{U}) \succ 0$ is satisfied in the interval $(-h, 0)$ and the proof of Theorem 3 can be conducted for any orders $n \geq n^*$, where the estimated order n^* is given analytically by (42). \square

Thanks to the approximation of U by Legendre polynomials and leaning on its supergeometric convergence rate, we greatly reduce the estimate of the order n^* done in Bajodek et al. (2022). Here, the necessary order n^* grow extremely slow with respect to system parameters and make the necessary and sufficient LMI test provided in Theorem 3 tractable numerically as shown in the last example section.

5. NUMERICAL EXAMPLE

Consider the system (1) with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0 & 0 \\ -k_1 & -k_2 \end{bmatrix}, \quad h = 1, \quad (49)$$

for proportional gains $k_1 > 0$ and $k_2 \geq 0$.

Thanks to Theorems 3 and 4, the necessary and sufficient LMI conditions (28) can be tested for the orders n^* given by (42) and several gains (k_1, k_2) in $[0, 1] \times [0, 2]$. If the test is true, then system (49) is GES. If the test is false, then system (49) is unstable. Figure 1 shows the result of such an LMI test. One can see that stable and unstable regions with respect to parameters k_1 and k_2 can be easily detected (see green and red bars, respectively).

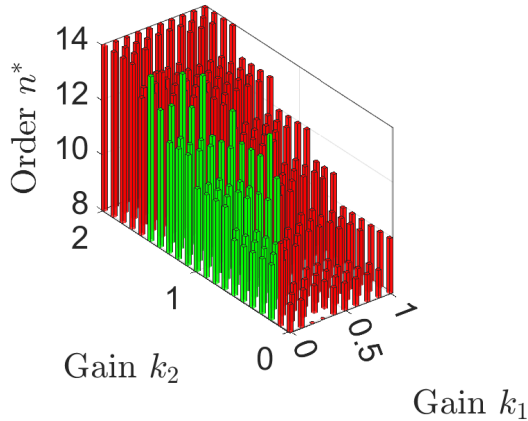


Fig. 1. Application of Theorem 3 for order n^* .

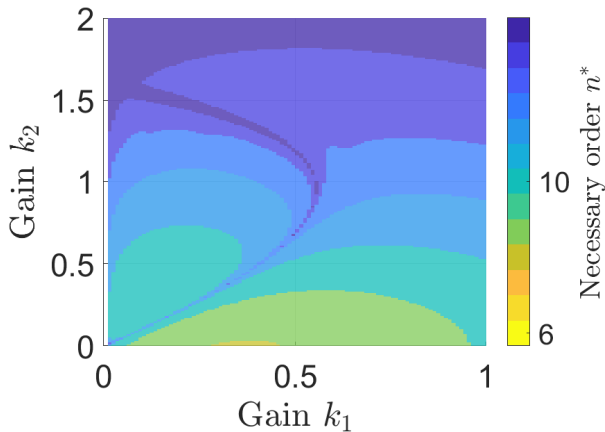


Fig. 2. Estimation of the necessary order n^* .

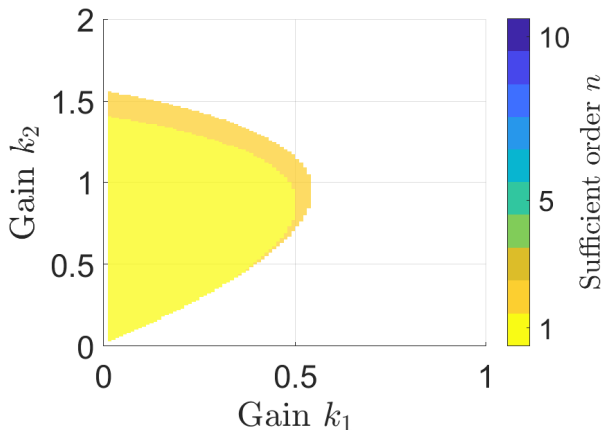


Fig. 3. Sufficient order n to satisfy LMI (28).

Furthermore, Figure 2 reports the necessary orders n^* given by (42) that corresponds to system (49). To begin with, the order n^* is lower than 12 for almost every (k_1, k_2) in $[0, 1]^2$. This is rewarded by a low computational load and a fast LMI test (less than 1 second). Indeed, it is worth noticing that the number of LMI variables is given by

$$N_{var} = (n^* + 1)(2(n^* + 1) + 1) + 6, \quad (50)$$

and that the feasibility of LMI (28) by the point interior method requires at least a spacial and time complexity which is polynomial with respect to order n^* . In addition, when gains k_1 and k_2 approach the limit of stability, the estimated order n^* increases and exceeds 15. Indeed, in the case that some characteristic roots of system (49) cross the imaginary axis, matrix N is a singular matrix as proved in (Kharitonov, 2013, Theorem 2.10) and scalar ρ_0 as well as order n^* tend to infinity.

As a complement, Figure 3 reports the sufficient order n such that our sufficient condition of stability holds. Increasing the order n from 1 to n^* , we are looking for the first order \tilde{n} from which the LMI condition (28) is true. One remarks that there are lower than the necessary order, which leaves a room for forthcoming improvements.

6. CONCLUSIONS

In this work, three steps leading to necessary and sufficient LMI conditions of stability are highlighted. Firstly, the complete Lyapunov functional is presented. Secondly, from Theorem 2, we are now able to quantify the degree of conservatism introduced by an approximation process. Thirdly, the necessary and sufficient Legendre-LMI test provided by Theorem 3 is established. The size of the LMI test can be calculated thanks to Theorem 4 and the estimation of the LMI size was minimized by taking advantage of the supergeometric convergence of Legendre approximation.

Following the same methodology, this work open the way to develop new LMI conditions for larger classes of systems. Future directions of research concern time-delay systems with multiple delays (see Alexandrova and Zhabko (2021)), distributed delays (see Egorov et al. (2017)) or non linearities (see Di Ferdinando et al. (2021)). More generally, an extension to linear ordinary-partial interconnected differential equations would be promising to regard.

Future works could also be dedicated to the reduction of the necessary order n^* from which the necessary and sufficient condition holds. The goal would be to get closer to the sufficient order \tilde{n} . To do so, we could consider other pair of matrices W_1 , W_2 and W_3 and treat directly the necessary condition $\Psi(\theta; U_n) \succ 0$ or investigate other pseudo-spectral methods.

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