Volumes of sublevel sets of nonnegative forms and complete monotonicity

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NON-NEGATIVE FORMS, VOLUMES OF SUBLEVEL SETS, COMPLETE MONOTONICITY AND MOMENT MATRICES

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ABSTRACT. Let $\mathcal{C}_{d,n}$ be the convex cone consisting of real n-variate degree d forms that are strictly positive on $\mathbb{R}^n\setminus\{\mathbf{0}\}$. We prove that the Lebesgue volume of the sublevel set $\{g\leq 1\}$ of $g\in\mathcal{C}_{d,n}$ is a completely monotone function on $\mathcal{C}_{d,n}$ and investigate the related properties. Furthermore, we provide (partial) characterization of forms, whose sublevel sets have finite Lebesgue volume. Finally, we discover an interesting property of a centered Gaussian distribution, establishing a connection between the matrix of its degree d moments and the quadratic form given by the inverse of its covariance matrix.

1. Introduction and main results

We bring together various constructions and studies associated with real homogeneous polynomials, see [1, 7, 8, 11, 12, 14, 15, 16, 22]. Our discoveries in Subsection 1.1 enrich the existing connection [8, 14, 22] between *real algebraic geometry* and *the theory of completely monotone functions*. Theorem 1.5 establishes a property of a *centered Gaussian distribution* that we also interpret in the context of *polynomial optimization* [16]. In Subsection 1.3 we solve a problem that contributes to a general study of *extremal properties of homogeneous polynomials* (see [1, 7, 9, 12]). Finally, the results stated in Subsection 1.4 naturally complement investigations from [7, 12] about *volumes of sublevel sets of non-negative homogeneous polynomials*.

Let $\mathcal{H}_{d,n}$ denote the space of real n-variate forms (homogeneous polynomials) of degree d. The space $\mathcal{H}_{d,n}$ is endowed with *the Bombieri inner product*

$$\langle g,h\rangle \;=\; \sum_{|\alpha|=d}\binom{d}{\alpha}^{-1} g_\alpha h_\alpha, \qquad \left(\text{with } \tbinom{d}{\alpha}=\tfrac{d!}{\alpha_1!\cdots\alpha_n!}\right), \tag{1.1}$$

where $g(\mathbf{x})=\sum_{|\alpha|=d} g_\alpha\mathbf{x}^\alpha$ and $h(\mathbf{x})=\sum_{|\alpha|=d} h_\alpha\mathbf{x}^\alpha$ are two forms written in the basis of monomials of degree d. For a form $g\in\mathcal{H}_{d,n}$ we consider its sublevel set

$$\mathbf{G}=\{\mathbf{x}\in\mathbb{R}^n : g(\mathbf{x})\leq 1\}.$$

We are interested in *the volume function* f that to a given $g\in\mathcal{H}_{d,n}$ associates the Lebesgue volume $f(g)=\mathrm{vol}\,(\mathbf{G})$ of its sublevel set. The volume of $\mathbf{G}$ is infinite if g takes negative values. Thus, it is natural to apply f only to polynomials that are non-negative on $\mathbb{R}^n$. We call a form g *positive definite* (PD for short), if it is positive on $\mathbb{R}^n\setminus\{\mathbf{0}\}$. The sublevel set $\mathbf{G}\subset\mathbb{R}^n$ of a PD form is compact and hence it has a finite Lebesgue volume. In this regard, we are concerned with the open convex cone $\mathcal{C}_{d,n}\subset\mathcal{H}_{d,n}$ of PD forms. We implicitly assume that the degree d is

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even, as only in this case $\mathcal{C}_{d,n}$ is non-empty. The closure $\overline{\mathcal{C}_{d,n}}$ of $\mathcal{C}_{d,n}$ with respect to the norm topology on $(\mathcal{H}_{d,n}, \langle\cdot,\cdot\rangle)$ consists of forms that are non-negative on $\mathbb{R}^n$.

1.1. **Complete monotonicity of the volume function.** The cone $\mathcal{C}_{d,n}$ is a natural domain of definition of the volume function

$$\begin{aligned} f : \mathcal{C}_{d,n} &\to \mathbb{R}, \\ g &\mapsto \operatorname{vol}(\mathbf{G}) = \int_{\mathbf{G}} \mathrm{d}\mathbf{x}, \end{aligned} \tag{1.2}$$

which is strictly convex and admits the following integral representation

$$f(g) = \operatorname{vol}(\mathbf{G}) = \frac{1}{\Gamma(1+n/d)} \int_{\mathbb{R}^n} \exp(-g(\mathbf{x}))\, \mathrm{d}\mathbf{x}, \tag{1.3}$$

see [12, Thm. 2.2]. In our first result we show that (1.2) is *completely monotone*, that is, it is *the Laplace transform* of some Borel measure on the closed dual cone $\mathcal{C}^*_{d,n} = \{L \in \mathcal{H}^*_{d,n} : L(g) \geq 0 \ \forall g \in \mathcal{C}_{d,n}\}$ to $\mathcal{C}_{d,n}$. To state it, let us recall (from, e.g., [5, Thm. 19]) that $\mathcal{C}^*_{d,n}$ is the conic hull of the image of the *Veronese map*

$$\begin{aligned} \boldsymbol{\Theta}_{d,n} : \mathbb{R}^n &\to \mathcal{C}^*_{d,n} \subset \mathcal{H}^*_{d,n}, \\ \boldsymbol{\ell} &\mapsto [g \mapsto g(\boldsymbol{\ell})]. \end{aligned} \tag{1.4}$$

By [21, Thm. 17.10], $\mathcal{C}^*_{d,n}$, also known as *the moment cone*, consists of *(truncated) moment functionals*, that is, such $L \in \mathcal{H}^*_{d,n}$ with $L(g) = \int_{\mathbb{S}^{n-1}} g(\mathbf{z})\, \mathrm{d}\nu(\mathbf{z})$, $g \in \mathcal{H}_{d,n}$, for some measure ν supported on *the unit sphere* $\mathbb{S}^{n-1} = \{\mathbf{z} \in \mathbb{R}^n : \mathbf{z}^\mathsf{T}\mathbf{z} = 1\}$. Then, an element $\boldsymbol{\Theta}_{d,n}(\ell)$, $\ell \in \mathbb{S}^{n-1}$, corresponds to *the Dirac measure* at ℓ. An alternative perspective on $\mathcal{C}^*_{d,n}$ comes with the identification of $\mathcal{H}^*_{d,n}$ and $\mathcal{H}_{d,n}$ via the inner product (1.1). Under this identification, one has $g(\boldsymbol{\ell}) = \left\langle \left(\boldsymbol{\ell}^\mathsf{T} \cdot\right)^d, g \right\rangle$ (see, e.g., [21, (19.6)]) and, in particular, (1.4) sends a vector $\boldsymbol{\ell} \in \mathbb{R}^n$ to the dth power of a linear form, $\theta_{\boldsymbol{\ell}} = \left(\boldsymbol{\ell}^\mathsf{T} \cdot\right)^d \in \mathcal{C}^*_{d,n} \subset \mathcal{H}^*_{d,n} \simeq \mathcal{H}_{d,n}$.

Theorem 1.1. *For $n \geq 2$ and any even $d \geq 2$ the volume function* (1.2) *admits an integral representation*

$$f(g) = \int_{\mathcal{C}^*_{d,n}} \exp(-\langle \theta, g\rangle)\, \frac{\mathrm{d}\mu(\theta)}{\Gamma(1+n/d)}, \tag{1.5}$$

where μ is the push-forward measure of the Lebesgue measure on $\mathbb{R}^n$ under the Veronese map (1.4). *In particular, $f : \mathcal{C}_{d,n} \to \mathbb{R}$ is completely monotone and for all $k \in \mathbb{N}$ and $v_1, \ldots, v_k \in \mathcal{C}_{d,n}$ we have that*

$$(-1)^k D_{v_1} \ldots D_{v_k} f(g) \geq 0, \quad g \in \mathcal{C}_{d,n}, \tag{1.6}$$

where D_v denotes the directional derivative along $v \in \mathcal{C}_{d,n}$.

Remark 1.2. *Complete monotonicity of a differentiable function $f : C \to \mathbb{R}$ on an open cone $C \subset \mathbb{R}^N$ is normally defined in terms of conditions on k-fold directional derivatives of f as in* (1.6). *Thanks to Bernstein-Hausdorff-Widder-Choquet theorem, recalled in Section 2, this is equivalent to the possibility of representing f as the Laplace transform of some Borel measure on the dual cone C^*.*

Scott and Sokal [22] investigated complete monotonicity of negative powers of some combinatorially defined polynomials . In [22, Theorem 1.3] they characterized values of s for which the function $G \mapsto (\det(G))^{-s}$, defined on the cone of positive definite matrices, is completely monotone. The volume function $f(g)$ associated to a

quadratic form $g(\mathbf{x}) = \mathbf{x}^\mathsf{T} G\mathbf{x}$ is proportional to $(\det(G))^{-1/2}$, see Proposition 2.10. Thus, in the case $d = 2$ complete monotonicity of the volume function (1.2) also follows from the mentioned characterization from [22]. The results from [22] along with exponential families in algebraic statistics [13] motivated Michałek, Sturmfels and the first author of the present work to investigate [8] complete monotonicity of negative powers g^{-s} of *hyperbolic polynomials*. A particular focus in their study was given to elementary symmetric polynomials and to products of linear forms.

In physics one often considers exponential probability density functions of the form $\frac{1}{Z_g}\exp(-g(\mathbf{x}))$, where $Z_g = \int_{\mathbb{R}^n}\exp(-g(\mathbf{x}))\,\mathrm{d}\mathbf{x}$ is the normalization constant, also known as *the partition function*. It is in general difficult to find closed-form expressions for Z_g (respectively, for (1.3)). As we see in Proposition 2.10, it is possible to write $f(g)$ in terms of the discriminant of a quadratic form $g \in \mathcal{C}_{2,n}$, which is also equal to the determinant of the real symmetric matrix associated to g. In [15] Morozov and Shakirov develop this line further, calling integrals in (1.3) *integral discriminants* and writing down their expressions in terms of $\mathrm{SL}(n,\mathbb{R})$-invariants of $g \in \mathcal{H}_{d,n}$. Indeed, the standard action of *the special linear group* on the space $\mathcal{H}_{d,n}$ preserves the integral (1.3) and as a consequence the integral discriminant depends on g only through the invariants of the action. For example, for $n = 2$ and $d = 4$ an expression for (1.3) in terms of the $\mathrm{SL}(2,\mathbb{R})$-invariants is given in [15, Section 5.1].

One can also generalize the above setting as explained in the following remark.

Remark 1.3. *For a non-negative form $h \in \overline{\mathcal{C}_{e,n}}$ of degree $e \geq 0$ consider a function*

$$\begin{aligned} f_h : \mathcal{C}_{d,n} &\to \mathbb{R}, \\ g &\mapsto \int_{\mathbf{G}} h(\mathbf{x})\, d\mathbf{x}, \end{aligned} \tag{1.7}$$

from which we retrieve the volume function (1.2) *by setting $h = 1$. In Section 2 we prove a generalization of Theorem 1.1, showing that* (1.7) *is completely monotone.*

Along with proving complete monotonicity of the volume function (1.2) (and its generalization defined in the above remark), in Section 2 we also discuss related properties as well as prove a version of Theorem 1.1 for *sum of squares forms*.

1.2. **Moment matrices of central Gaussian vectors and sums of squares.** A form $g \in \mathcal{H}_{d,n}$ is called *a sum of squares* (or SOS), if $g(\mathbf{x}) = \sum_{j=1}^r h_j(\mathbf{x})$ for some forms $h_1, \ldots, h_r \in \mathcal{H}_{d/2,n}$ of degree $d/2$. Equivalently, one can write g as

$$g(\mathbf{x}) \;=\; \mathbf{m}_{d/2}(\mathbf{x})^\mathsf{T}\, G\, \mathbf{m}_{d/2}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n, \tag{1.8}$$

where $\mathbf{m}_{d/2}(\mathbf{x}) = (\mathbf{x}^{\boldsymbol{\alpha}})_{|\boldsymbol{\alpha}|=d/2}$ is the column-vector of monomials[1] of degree $d/2$ and $G \in \mathrm{PD}_M$ is a positive semi-definite matrix of size $M \times M$, $M = \binom{d/2+n-1}{n-1}$ (called *a Gram matrix* of g). Below by PD_M and PSD_M we denote the convex cone of positive definite, respectively positive semi-definite, $M \times M$ real symmetric matrices. By definition, sums of squares are non-negative, and, if g is given by (1.8) with a positive definite matrix G, then the form $g \in \mathcal{C}_{d,n}$ is also positive definite.

Remark 1.4. *It is well-known that not every non-negative form is a sum of squares, see* [19]. *However, by a celebrated result of Artin* [2], *after multiplying a non-negative form by a suitable sum of squares form, one obtains a sum of squares.*

[1]From now on we fix an order on the set of monomials of a given degree.

Moreover, by a result [18, Thm. 3.12] *of Reznick, if a form* $g \in \mathcal{C}_{d,n}$ *is positive definite, there exists* $s \in \mathbb{N}$ *so that* $\|\mathbf{x}\|^{2s} g \in \mathcal{C}_{d+2s,n}$ *is a sum of squares, where* $\|\mathbf{x}\|^2 = \mathbf{x}^\mathsf{T}\mathbf{x} = \sum_{i=1}^n x_i^2$ *is the Euclidean norm of* $\mathbf{x} \in \mathbb{R}^n$*. In fact, by the same result,* $\|\mathbf{x}\|^{2s} g$ *is even a sum of* $(d+2s)$*-th powers of linear forms and hence* $\|\mathbf{x}\|^{2s} g \in \mathbf{\Theta}_{d+2s,n}(\mathbb{R}^n)$ *is identified with a point of the moment cone* $\mathcal{C}^*_{d+2s,n}$*. Thus, sums of squares constitute an important subclass of the class of non-negative forms. Furthermore, with the representation* (1.8) *for sums of squares, one can approach* polynomial optimization problems *via* semi-definite programming*, see* [10].

A centered multivariate normal distribution is defined by its covariance matrix Q^{-1}, with the probability density function given by

$$p_Q(\mathbf{y}) \;=\; \frac{\sqrt{\det(Q)}}{\sqrt{(2\pi)^n}} \exp\left(-\frac{\mathbf{y}^\mathsf{T} Q \mathbf{y}}{2}\right), \quad \mathbf{y} \in \mathbb{R}^n. \tag{1.9}$$

The covariance matrix equals *the Hankel matrix* of moments of p_Q of degree 2, i.e.,

$$Q^{-1} \;=\; \mathbf{M}_2[Q] \;=\; \int_{\mathbb{R}^n} \mathbf{y}\mathbf{y}^\mathsf{T} p_Q(\mathbf{y})\,\mathrm{d}\mathbf{y} \;=\; \left(\int_{\mathbb{R}^n} y_i y_j p_Q(\mathbf{y})\,\mathrm{d}\mathbf{y}\right). \tag{1.10}$$

Motivated by this property of Gaussian distributions, in [11, 2.5] the second author of the present work considers *the Gaussian-like density*

$$P_G(\mathbf{y}) \;=\; \frac{\exp\left(-k\,\mathbf{m}_{d/2}(\mathbf{y})^\mathsf{T} G \mathbf{m}_{d/2}(\mathbf{y})\right)}{\int_{\mathbb{R}^n} \exp\left(-k\,\mathbf{m}_{d/2}(\mathbf{z})^\mathsf{T} G \mathbf{m}_{d/2}(\mathbf{z})\right)\mathrm{d}\mathbf{z}}, \quad \mathbf{y} \in \mathbb{R}^n, \tag{1.11}$$

associated to a positive definite SOS form $g(\mathbf{x}) = \mathbf{m}_{d/2}(\mathbf{x}) G \mathbf{m}_{d/2}(\mathbf{x})$, $G \in \mathrm{PD}_M$, and *the Hankel-like matrix*

$$\mathcal{M}_d[G] \;=\; \int_{\mathbb{R}^n} \mathbf{m}_{d/2}(\mathbf{y}) \mathbf{m}_{d/2}(\mathbf{y})^\mathsf{T} P_G(\mathbf{y})\,\mathrm{d}\mathbf{y}$$

of degree d moments of (1.11), where $(2k)^{-1} = \binom{d/2+n-1}{n}$. When $d = 2$, one has $k = 1/2$ and (1.11) is the Gaussian density (1.9), whose matrix $\mathcal{M}_2[G] = \mathbf{M}_2[Q]$ of degree 2 moments is the inverse of $G = Q \in \mathrm{PD}_n$, see (1.10). When $d > 2$ is higher, Lasserre shows in [11, Lemma 4] that $\mathcal{M}_d[G] = G^{-1}$ (for $G \in \mathrm{PD}_M$) if and only if G is a critical point of the function

$$G \in \mathrm{PD}_M \;\mapsto\; (\det(G))^k \int_{\mathbb{R}^n} \exp\left(-k \mathbf{m}_{d/2}(\mathbf{x})^\mathsf{T} G \mathbf{m}_{d/2}(\mathbf{x})\right)\mathrm{d}\mathbf{x}. \tag{1.12}$$

Theorem 1.5 below implies that forms $g(\mathbf{x}) = (\mathbf{x}^\mathsf{T} Q \mathbf{x})^{d/2}$, $Q \in \mathrm{PD}_n$, fulfil this condition. Before we state it, for $Q \in \mathrm{PD}_n$ and $d \geq 2$ consider the density

$$p_Q^{(d)}(\mathbf{y}) \;=\; \frac{\sqrt{\det(Q)}}{\sigma_d^n \sqrt{(2\pi)^n}} \exp\left(-\frac{\mathbf{y}^\mathsf{T} Q \mathbf{y}}{2\sigma_d^2}\right), \quad \sigma_d \;=\; \left(\frac{\binom{d/2+n-1}{n-1}}{\sqrt{2}^d \prod_{i=0}^{d/2-1}\left(\frac{n}{2}+i\right)}\right)^{1/d},$$

of a centered Gaussian distribution with covariance matrix $\sigma_d^2 Q^{-1}$, and let

$$\mathbf{M}_d[Q] \;=\; \int_{\mathbb{R}^n} \mathbf{m}_{d/2}(\mathbf{y}) \mathbf{m}_{d/2}(\mathbf{y})^\mathsf{T} p_Q^{(d)}(\mathbf{y})\,\mathrm{d}\mathbf{y} \;=\; \left(\int_{\mathbb{R}^n} \mathbf{y}^{\alpha+\beta} p_Q^{(d)}(\mathbf{y})\,\mathrm{d}\mathbf{y}\right) \tag{1.13}$$

be the Hankel-like matrix of moments of $p_Q^{(d)}$ of degree d. For $d = 2$ one has $\sigma_2 = 1$ and hence we recover $p_Q = p_Q^{(2)}$ given by (1.9).

Theorem 1.5. *Let $Q \in \mathrm{PD}_n$ be any positive definite matrix and let $d \geq 2$ be even. Then $\mathbf{M}_d[Q]^{-1}$ is a Gram matrix of the* SOS *form $(\mathbf{x}^\mathsf{T} Q \mathbf{x})^{d/2}$, that is,*

$$\mathbf{m}_{d/2}(\mathbf{x})^\mathsf{T} \mathbf{M}_d[Q]^{-1} \mathbf{m}_{d/2}(\mathbf{x}) \;=\; \left(\mathbf{x}^\mathsf{T} Q \mathbf{x}\right)^{d/2}. \tag{1.14}$$

Furthermore, we have $\mathbf{M}_d[Q] = \mathcal{M}_d[G]$, where $\mathcal{M}_d[G]$ is the matrix of degree d moments of the Gaussian-like density (1.11) *associated to $g(\mathbf{x}) = (\mathbf{x}^\mathsf{T} Q \mathbf{x})^{d/2}$.*

Remark 1.6. *By* [11, Lemma 4]*, for any $Q \in \mathrm{PD}_n$ the matrix $\mathcal{M}_d[Q]^{-1} \in \mathrm{PD}_M$ is a critical point of the function* (1.12)*. It is an interesting open question whether there exist other* SOS *forms with this property.*

Sums of squares of degree d form a closed convex subcone (called *the* SOS *cone*) $\Sigma_{d,n} \subseteq \overline{\mathcal{C}_{d,n}}$ of the cone of non-negative homogeneous polynomials. Its dual cone $\Sigma_{d,n}^* = \{L \in \mathcal{H}_{d,n}^* \,:\, L(g) \geq 0 \;\forall\, g \in \Sigma_{d,n}\}$ contains $\mathcal{C}_{d,n}^*$ and is known as *the pseudo-moment cone*. For any *(truncated) pseudo-moment functional* $L \in \Sigma_{d,n}^*$ let

$$\mathbf{M}_d(L) \;=\; L\left(\mathbf{m}_{d/2}(\mathbf{y})\mathbf{m}_{d/2}(\mathbf{y})^\mathsf{T}\right) \;=\; \left(L(\mathbf{y}^{\alpha+\beta})\right)_{|\alpha|=|\beta|=d/2} \tag{1.15}$$

be the $M \times M$ *Hankel-like matrix* of pseudo-moments of degree d associated to L. One has $\mathbf{M}_d(L) \in \mathrm{PSD}_M$, as $\mathbf{h}^\mathsf{T}\mathbf{M}_d(L)\mathbf{h} = L(\mathbf{h}^\mathsf{T}\mathbf{m}_{d/2}(\mathbf{y})\mathbf{m}_{d/2}(\mathbf{y})^\mathsf{T}\mathbf{h}) = L(h^2) \geq 0$ holds for any form $h \in \mathcal{H}_{d/2,n}$ whose coefficients in the basis of monomials form a column vector $\mathbf{h} = (h_\alpha)_{|\alpha|=d/2}$. Conversely, any matrix $\mathbf{M}_d \in \mathrm{PSD}_M$ that satisfies $(\mathbf{M}_d)_{\alpha\beta} = (\mathbf{M}_d)_{\alpha'\beta'}$ whenever $\alpha+\beta = \alpha'+\beta'$, $|\alpha| = |\beta| = |\alpha'| = |\beta'| = d/2$, is of the form (1.15) for $L \in \mathcal{H}_{d,n}^*$ defined by $L(\mathbf{y}^{\alpha+\beta}) = (\mathbf{M}_d)_{\alpha\beta}$, $|\alpha| = |\beta| = d/2$. Furthermore, for a linear functional $L \in \Sigma_{d,n}^*$ that lies in the interior of the pseudo-moment cone, the associated pseudo-moment matrix $\mathbf{M}_d(L) \in \mathrm{PD}_M$ is positive definite and hence invertible. If $L \in \mathcal{C}_{d,n}^*$ is even a moment functional, that is,

$$L(g) \;=\; \int_{\mathbb{S}^{n-1}} g(\mathbf{z})\,\mathrm{d}\nu(\mathbf{z}), \quad g \in \mathcal{H}_{d,n}, \tag{1.16}$$

for some measure ν supported on $\mathbb{S}^{n-1}$, then the entries of the matrix $\mathbf{M}_d(L)$ are moments of ν of degree d.

By a result of Nesterov (see [16, Thm. 2]), a SOS form $g \in \Sigma_{d,n}$ belongs to the interior of the cone $\Sigma_{d,n}$ if and only if there is some linear functional $L \in \Sigma_{d,n}^*$ in the interior of the pseudo-moment cone such that $g(\mathbf{x}) = \mathbf{m}_{d/2}(\mathbf{x})^\mathsf{T}\,\mathbf{M}_d(L)^{-1}\mathbf{m}_{d/2}(\mathbf{x})$. Moreover, $L \in \Sigma_{d,n}^*$ satisfying this condition is unique.

Remark 1.7. *Theorem 1.5 delivers Nesterov's characterization for a power of a quadratic form $g(\mathbf{x}) = (\mathbf{x}^\mathsf{T} Q \mathbf{x})^{d/2}$, associated to a positive definite matrix $Q \in \mathrm{PD}_n$. It implies that the unique $\mathbf{M}_d(L) \in \mathrm{PD}_M$ is the matrix of degree d moments of the centered Gaussian vector in $\mathbb{R}^n$, whose covariance matrix is proportional to Q^{-1}.*

By integrating out the radial part of the Gaussian measure from Theorem 1.5, one also obtains a measure on $\mathbb{S}^{n-1}$, whose degree d moments comprise $\mathbf{M}_d(L)$.

Corollary 1.8. *Let $Q \in \mathrm{PD}_n$ be a positive definite matrix and let L be the moment functional* (1.16)*, whose associated measure ν on $\mathbb{S}^{n-1}$ is defined by*

$$\mathrm{d}\nu(\mathbf{z}) \;=\; \binom{d/2+n-1}{n-1} \frac{\sqrt{\det(Q)}}{\sqrt{\mathbf{z}^\mathsf{T} Q \mathbf{z}}^{\,d+n}} \frac{\mathrm{d}\mathbb{S}^{n-1}(\mathbf{z})}{\mathrm{vol}\,(\mathbb{S}^{n-1})}, \quad \mathbf{z} \in \mathbb{S}^{n-1},$$

where $\mathrm{d}\mathbb{S}^{n-1}$ *is the standard Riemannian measure on the Euclidean sphere with* $\mathrm{vol}\,(\mathbb{S}^{n-1}) = 2\sqrt{\pi}^n/\,\Gamma\,(n/2)$ *being its total volume. Then,* $\mathbf{M}_d(L)^{-1}$ *is a Gram matrix of the* SOS *form* $(\mathbf{x}^\mathsf{T}Q\mathbf{x})^{d/2}$*, that is,*

$$\mathbf{m}_{d/2}(\mathbf{x})^\mathsf{T}\mathbf{M}_d(L)^{-1}\mathbf{m}_{d/2}(\mathbf{x}) \;=\; \left(\mathbf{x}^\mathsf{T}Q\mathbf{x}\right)^{d/2}.$$

Theorem 1.5 together with Corollary 1.8 are proven in Section 3.

1.3. **Comparison of L^1- and L^2-norms associated to a PD form.** Given a positive definite form $g \in \mathcal{C}_{d,n}$, let us denote by μ_g the Lebesgue measure restricted to the sublevel set $\mathbf{G} \subset \mathbb{R}^n$ of g. Integrating μ_g against any form $h \in \mathcal{H}_{e,n}$ of degree e, one obtains by [12, (2.1)] that

$$\mu_g(h) \;=\; \int_{\mathbf{G}} h(\mathbf{x})\,\mathrm{d}\mathbf{x} \;=\; \frac{1}{\Gamma\,(1+(n+e)/d)}\int_{\mathbb{R}^n} h(\mathbf{x})\exp(-g(\mathbf{x}))\,\mathrm{d}\mathbf{x}.$$

In this perspective, μ_g can be identified with a measure with exponential density $\exp(-g(\mathbf{x}))$ with respect to the Lebesgue measure on $\mathbb{R}^n$. We consider L^1- and L^2-norms associated to the measure μ_g,

$$\|h\|_{L^1(\mu_g)} \;=\; \int_{\mathbf{G}} |h(\mathbf{x})|\,\mathrm{d}\mathbf{x}, \quad \|h\|_{L^2(\mu_g)} \;=\; \left(\int_{\mathbf{G}} |h(\mathbf{x})|^2\,\mathrm{d}\mathbf{x}\right)^{1/2}, \tag{1.17}$$

where h is any Lebesgue measurable function on $\mathbf{G}$. We show that g minimizes the ratio of L^2- and L^1-norms over all non-zero $h \in \mathcal{H}_{d,n}$.

Theorem 1.9. *Let* $g \in \mathcal{C}_{d,n}$*. Then*

$$g \;=\; \arg\min_{h\in\mathcal{H}_{d,n}} \left\{\|h\|_{L^2(\mu_g)} \,:\, \|h\|_{L^1(\mu_g)} = \|g\|_{L^1(\mu_g)}\right\}$$

or, equivalently,

$$\frac{\|g\|_{L^2(\mu_g)}}{\|g\|_{L^1(\mu_g)}} \;=\; \min\left\{\frac{\|h\|_{L^2(\mu_g)}}{\|h\|_{L^1(\mu_g)}} \,:\, h \in \mathcal{H}_{d,n}\setminus\{0\}\right\}.$$

Moreover, up to a factor, g *is the unique minimizer of these optimization problems.*

In particular, the positive definite form $g \in \mathcal{C}_{d,n}$ has the smallest $L^2(\mu_g)$-norm among all degree d forms $h \in \mathcal{H}_{d,n}$ that have the same $L^1(\mu_g)$-norm as g.

Remark 1.10. *The above result also holds when* g *and* h *are positively homogeneous functions of degree* d *with* $g(\mathbf{x})$ *being positive for* $\mathbf{x} \neq \mathbf{0}$*, see* [11, Lemma 3] *and our proof of Theorem 1.9 in Section 4.*

1.4. **Non-negative forms with sublevel sets of finite volume.** It turns out that, for some non-negative forms $g \in \partial\mathcal{C}_{d,n} = \overline{\mathcal{C}_{d,n}}\setminus\mathcal{C}_{d,n}$ in the boundary of $\mathcal{C}_{d,n}$, the sublevel set $\mathbf{G}$ has finite Lebesgue volume $f(g) = \mathrm{vol}\,(\mathbf{G})$, while it is not the case for all $g \in \partial\mathcal{C}_{d,n}$ for general d and n. Motivated by this observation, we consider the set $\mathcal{V}_{d,n}$ of forms $g \in \mathcal{H}_{d,n}$ with $f(g) < \infty$. One naturally has $\mathcal{C}_{d,n} \subseteq \mathcal{V}_{d,n} \subseteq \overline{\mathcal{C}_{d,n}}$ and $\mathcal{V}_{d,n} \subset \mathcal{H}_{d,n}$ is a convex cone by [12, Thm. 2.1]. It is straightforward to see that $\mathcal{V}_{2,n} = \mathcal{C}_{2,n}$, that is, only positive definite quadratic forms $g \in \overline{\mathcal{C}_{2,n}}$ have finite $f(g)$. We give a complete characterization of binary forms in $\mathcal{V}_{d,2}$ in terms of multiplicities of their real zeros.

Theorem 1.11. *A non-negative binary form* $g \in \overline{\mathcal{C}_{d,2}}$ *is in* $\mathcal{V}_{d,2}$ *if and only if* g *has zeros of order at most* $d/2-1$*. In particular,* $\mathcal{V}_{4,2} = \mathcal{C}_{4,2}$*.*

We also provide some sufficient conditions for membership in $\mathcal{V}_{d,n}$ for $n > 2$. For this let us call a non-negative form $g \in \overline{C}_{d,n}$ *generic*, if it is *round at every its real zero* $\mathbf{x} \in \mathbb{R}^n \setminus \{0\}$ (see [6, p. 47]), that is, the Hessian matrix $\mathrm{Hess}_{\mathbf{x}} g = \left(\frac{\partial^2 g}{\partial x_i \partial x_j}(\mathbf{x})\right)$ is positive definite when restricted to the orthogonal complement of $\mathbf{x}$ (equivalently, $\mathrm{Hess}_{\mathbf{x}} g$ is of corank one). Thus, a non-negative quadratic form $g(\mathbf{x}) = \mathbf{x}^{\mathsf{T}} G \mathbf{x}$ is generic if and only if the associated positive semi-definite matrix G is of corank one. Also, for a non-negative binary form $g \in \overline{C_{d,2}}$ the condition of being generic means that all real zeros of g are of order two.

Theorem 1.12. *For $d \geq 4$, $n \geq 3$ a generic non-negative form $g \in \overline{\mathcal{C}_{d,n}}$ is in $\mathcal{V}_{d,n}$.*

Theorems 1.11 and 1.12 are proven in Section 5. Now, we consider some examples of generic non-negative forms that lie on the boundary of $\mathcal{C}_{d,n}$.

Example 1. *The* Motzkin form

$$g(\mathbf{x}) \;=\; x_1^4 x_2^2 + x_1^2 x_2^4 + x_3^6 - 3x_1^2 x_2^2 x_3^2, \quad \mathbf{x} = (x_1, x_2, x_3), \tag{1.18}$$

was historically the first explicit example of a non-negative form that is not a sum of squares, see [19]. *A point $\mathbf{x} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$ is a zero of g if and only if $|x_1| = |x_2| = |x_3|$. Setting $x_3 = 1$ in g gives* the Motzkin polynomial $\tilde{g}(\mathbf{y}) = y_1^4 y_2^2 + y_1^2 y_2^4 + 1 - 3y_1^2 y_2^2$, *whose real zeros are* $(1,1)$, $(1,-1)$, $(-1,1)$, $(-1,-1)$. *At any real zero $\mathbf{x} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$ (say, $\mathbf{x} = (1,1,1)$) of g the Hessian $\mathrm{Hess}_{\mathbf{x}} g$ is positive semi-definite and has rank* 2 *and hence the Motzkin form $g \in \partial\mathcal{C}_{6,3}$ is generic in the above sense.*

As the following example shows, it is not difficult to find generic forms in $\overline{\mathcal{C}_{d,n}}$ also for higher $n \geq 3$ and even $d \geq 4$.

Example 2. *Consider a sum of squares form*

$$g(\mathbf{x}) \;=\; x_n^{d-2} \sum_{i=1}^{n-1} x_i^2 + \frac{2}{d} \sum_{i=1}^{n-1} x_i^d, \quad \mathbf{x} = (x_1, \ldots, x_{n-1}, x_n).$$

Then a real zero $\mathbf{x} \in \mathbb{R}^n$ of g must satisfy $x_1 = \cdots = x_{n-1} = 0$. At $\mathbf{x} = (0, \ldots, 0, 1)$ the Hessian of g has rank $n-1$ and hence $g \in \partial\mathcal{C}_{d,n}$ is generic.

By Theorem 1.12, the sublevel sets of the Motzkin form (1.18) and of forms constructed in Example 2 are of finite Lebegue volume.

One might wonder, whether the term "generic" is in place in the above context. The following remark motivates our choice for the terminology.

Remark 1.13. *The boundary $\partial\mathcal{C}_{d,n} \subset \mathcal{H}_{d,n}$ of the cone of non-negative forms is a semialgebraic subset of codimension one. In Proposition 5.4 we show that the set of non-generic forms is a semialgebraic subset of $\mathcal{H}_{d,n}$ of codimension at least* 2 *and that it is nowhere dense in the boundary $\partial\mathcal{C}_{d,n} \subset \mathcal{H}_{d,n}$ of $\mathcal{C}_{d,n}$, endowed with the topology induced from $(\mathcal{H}_{d,n}, \langle \cdot, \cdot \rangle)$.*

Structure of the paper

There are 4 sections, each corresponding to a subsection of the above introductory section. Thus, in the next section we prove complete monotonicity of the volume function (1.2) and discuss a generalization of this result as well as treat the case of sums of squares. In Section 3 we discuss moment matrices of centered Gaussian vectors and prove Theorem 1.5 and Corollary 1.8. In Section 4 a proof of

the extremal property from Theorem 1.9 is presented. And in the last section we characterize non-negative forms, whose sublevel sets have finite Lebesgue volume.

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2. Complete monotonicity of the volume function

Let $C \subset \mathbb{R}^N$ be an open convex cone and denote by C^* *the dual cone* to C,

$$C^* \;=\; \left\{ \mathbf{L} \in \left(\mathbb{R}^N\right)^* \,:\, \mathbf{L}(\mathbf{g}) \geq 0 \;\forall\; \mathbf{g} \in C \right\}.$$

Definition 2.1. *A function $f : C \to \mathbb{R}$ is* completely monotone*, if it is C^∞-differentiable and for all $k \in \mathbb{N}$ and all vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k \in C$*

$$(-1)^k D_{\mathbf{v}_1} \cdots D_{\mathbf{v}_k} f(\mathbf{g}) \;\geq\; 0, \quad \mathbf{g} \in C, \tag{2.1}$$

where $D_{\mathbf{v}}$ denotes the directional derivative along the vector $\mathbf{v}$.

The Bernstein-Hausdorff-Widder theorem [25, Thm. 12a] gives a characterization of completely monotone functions in one variable, these are exactly Laplace transforms of Borel measures on the positive reals. Choquet [4, Thm. 10] found a generalization of this result to convex cones in higher dimensional spaces.

Theorem 2.2 (Bernstein-Hausdorff-Widder-Choquet theorem)**.** *A smooth function $f : C \to \mathbb{R}$ on an open convex cone $C \subset \mathbb{R}^N$ is completely monotone if and only if it is* the Laplace transform *of a unique Borel measure μ supported on C^*, that is,*

$$f(\mathbf{g}) \;=\; \int_{C^*} \exp(-\mathbf{L}(\mathbf{g}))\, \mathrm{d}\mu(\mathbf{g}). \tag{2.2}$$

For a non-negative form $h \in \overline{\mathcal{C}_{e,n}}$ of degree $e \geq 0$ the function f_h defined in (1.7) admits an integral representation (see [11, Thm. 1])

$$f_h(g) \;=\; \int_{\mathbf{G}} h(\mathbf{x})\, \mathrm{d}\mathbf{x} \;=\; \frac{1}{\Gamma(1+(n+e)/d)} \int_{\mathbb{R}^n} h(\mathbf{x}) \exp(-g)\, \mathrm{d}\mathbf{x}, \quad g \in \mathcal{C}_{d,n}. \tag{2.3}$$

In particular, the Lebesgue volume of the sublevel set $\mathbf{G} = \{\mathbf{x} \in \mathbb{R}^n : g(\mathbf{x}) \leq 1\}$ of g is obtained with $h = 1$ as

$$f(g) \;=\; \int_{\mathbf{G}} \mathrm{d}\mathbf{x} \;=\; \frac{1}{\Gamma(1+n/d)} \int_{\mathbb{R}^n} \exp(-g(\mathbf{x}))\, \mathrm{d}\mathbf{x}, \quad g \in \mathcal{C}_{d,n}. \tag{2.4}$$

Moreover, f_h is C^∞-differentiable and its derivatives are expressed as follows.

Proposition 2.3. *Let $h \in \overline{\mathcal{C}_{e,n}}$. Then for any $k \in \mathbb{N}$ and any $v_1, \ldots, v_k \in \mathcal{H}_{d,n}$ we have for $g \in \mathcal{C}_{d,n}$*

$$\begin{aligned}(-1)^k D_{v_1} \ldots D_{v_k} f_h(g) \;&=\; \frac{\Gamma\left(1+k+\frac{n+e}{d}\right)}{\Gamma\left(1+\frac{n+e}{d}\right)} \int_{\mathbf{G}} h(\mathbf{x}) v_1(\mathbf{x}) \cdots v_k(\mathbf{x})\, \mathrm{d}\mathbf{x} \\ &=\; \frac{1}{\Gamma\left(1+\frac{n+e}{d}\right)} \int_{\mathbb{R}^n} h(\mathbf{x}) v_1(\mathbf{x}) \cdots v_k(\mathbf{x}) \exp(-g(\mathbf{x}))\, \mathrm{d}\mathbf{x}.\end{aligned} \tag{2.5}$$

Proof. Since $g(\mathbf{x})$ is positive for $\mathbf{x} \neq \mathbf{0}$, its minimum $g_{\min} = \min_{\|\mathbf{x}\|=1} g(\mathbf{x}) > 0$ over the unit sphere in $\mathbb{R}^n$ is positive. By homogeneity, $g(\mathbf{x}) \geq g_{\min} \|\mathbf{x}\|^d$, $\mathbf{x} \in \mathbb{R}^n$. Therefore, derivatives of the integrand in (2.3) read

$$(-1)^k D_{v_1} \ldots D_{v_k} \left(h(\mathbf{x}) \exp(-g(\mathbf{x}))\right) \;=\; h(\mathbf{x}) v_1(\mathbf{x}) \cdots v_k(\mathbf{x}) \exp(-g(\mathbf{x})), \quad \mathbf{x} \in \mathbb{R}^n.$$

Since this is majorized by the function $h(\mathbf{x})|v_1(\mathbf{x})\cdots v_k(\mathbf{x})|\exp(-g_{\min}\|\mathbf{x}\|^d)$, which is clearly integrable, dominated converge theorem implies that

(2.6)
$$(-1)^k D_{v_1}\dots D_{v_k} f_h(g) \;=\; \frac{1}{\Gamma\left(1+\frac{n+e}{d}\right)}\int_{\mathbb{R}^n} h(\mathbf{x})v_1(\mathbf{x})\cdots v_k(\mathbf{x})\exp(-g(\mathbf{x}))\,\mathrm{d}\mathbf{x}.$$

For any multiindex $\alpha\in\mathbb{N}^n$ formula [11, (20)] yields

$$\int_{\mathbb{R}^n} h(\mathbf{x})\,\mathbf{x}^\alpha\exp(-g(\mathbf{x}))\,\mathrm{d}\mathbf{x} \;=\; \Gamma(1+(n+e+|\alpha|)/d)\int_{\mathbf{G}} h(\mathbf{x})\,\mathbf{x}^\alpha\,\mathrm{d}\mathbf{x}.$$

From multilinearity of (2.6) in $v_1,\dots,v_k$ the remaining equality in (2.5) follows. □

Remark 2.4. *Proposition 2.3 implies that $f_h:\mathcal{C}_{d,n}\to\mathbb{R}$ and hence the volume function* (1.2) *are completely monotone. Indeed, for any $k\in\mathbb{N}$ and positive definite forms $v_1,\dots,v_k\in\mathcal{C}_{d,n}$ the derivative $(-1)^kD_{v_1}\dots D_{v_k}f_h(g)$ is non-negative for all $g\in\mathcal{C}_{d,n}$ by* (2.5)*. In fact, it is even strictly positive as one can see directly.*

Next, we give an alternative proof of complete monotonicity of the volume function, that also delivers an integral representation predicted by Theorem 2.2.

Proof of Theorem 1.1. As already mentioned, complete monotonicity of the volume function (1.2) is guaranteed by Proposition 2.3.

Denote by μ the push-forward measure of the Lebesgue measure λ on $\mathbb{R}^n$ under the Veronese map (1.4), that is, $\mu(B) = \lambda(\mathbf{\Theta}_{d,n}^{-1}(B))$ for any Borel measurable subset $B\subseteq\mathbb{R}^n$. Then the definition (1.4) of $\mathbf{\Theta}_{d,n}$ and [3, Thm. 3.6.1] give

$$\int_{\mathbb{R}^n}\exp(-g(\boldsymbol{\ell}))\,\mathrm{d}\boldsymbol{\ell} \;=\; \int_{\mathbb{R}^n}\exp(-\langle\theta_{\boldsymbol{\ell}},g\rangle)\,\mathrm{d}\lambda(\boldsymbol{\ell}) \;=\; \int_{\mathcal{C}^*_{d,n}}\exp(-\langle\theta,g\rangle)\,\mathrm{d}\mu(\theta),$$

which combined with (1.3) yields (1.5). □

Remark 2.5. *Analogously, when $h\in\overline{\mathcal{C}_{d,n}}$ has degree d, one can show that*

$$f_h(g) \;=\; \int_{\mathcal{C}^*_{d,n}}\exp\left(-\langle\theta,g\rangle\right)\frac{\langle\theta,h\rangle\,\mathrm{d}\mu(\theta)}{\Gamma(1+(n+d)/d)},$$

where μ is the push-forward measure of the Lebesgue measure on $\mathbb{R}^n$ under the Veronese map (1.4)*. Thus, the unique Borel measure from Theorem 2.2, that is associated to the completely monotone function f_h is proportional to $\langle\cdot,h\rangle\mu$.*

By the Leibniz rule for derivatives, the product of two completely monotone functions is completely monotone. In particular, for any integer $s\in\mathbb{N}$ the s-th power f^s of the volume function (1.2) is completely monotone. Interestingly, in the setting of quadratic forms ($d=2$), f^s is completely monotone for all sufficiently large (not necessarily integer) powers $s>>1$.

Proposition 2.6. *The function $g\mapsto f(g)^s$ on $\mathcal{C}_{2,n}$ is completely monotone if and only if $s=0,1,2,\dots,n-2$ or $s\geq n-1$.*

Proof. By Proposition 2.10 below, we have that

$$f(g)^s \;=\; \left(\frac{\sqrt{\pi}^n}{\Gamma(1+n/2)}\right)^s\det(G)^{-s/2},\quad g(\mathbf{x})=\mathbf{x}^{\mathsf{T}}G\mathbf{x},$$

where G is the positive definite matrix representing the quadratic form $g\in\mathcal{C}_{2,n}$. The claim now directly follows from [22, Theorem 1.3]. □

By its construction, the representing measure μ from Theorem 1.1 is supported on the set $\mathbf{\Theta}_{d,n}(\mathbb{R}^n) \subset \mathcal{H}_{d,n}$ of d-th powers of linear forms, which is also known as (the real part of) *the Veronese cone*. By [8, Prop. 5.7], the representing measure of the power f^s of (1.2) is proportional to *the s-th convolution power μ^{*s} of μ*,

$$f^s \;=\; \prod_{j=1}^{s} \int_{\mathcal{C}^*_{d,n}} \exp(-\langle \theta, g\rangle)\, \frac{\mathrm{d}\mu(\theta)}{\Gamma\left(1+n/d\right)} \;=\; \int_{\mathcal{C}^*_{d,n}} \exp(-\langle \theta, g\rangle)\, \frac{\mathrm{d}\mu^{*s}(\theta)}{\Gamma\left(1+n/d\right)^s}.$$

The support of the measure μ^{*s} is *the s-th Minkowski power of $\mathbf{\Theta}_{d,n}(\mathbb{R}^n)$*,

$$\operatorname{supp}\left(\mu^{*s}\right) \;=\; \{\theta_{\boldsymbol{\ell}^1} + \cdots + \theta_{\boldsymbol{\ell}^s} \,:\, \boldsymbol{\ell}^1, \ldots, \boldsymbol{\ell}^s \in \mathbb{R}^n\}. \tag{2.7}$$

By Richter's theorem, for $s = \binom{n-1+d}{d}$ the support (2.7) of μ^{*s} fills the cone $\mathcal{C}^*_{d,n}$, see [20, Satz 4] and [5, Thm. 19]. Actually, by [5, Thm. 57], this already happens for some $s \leq \binom{n-1+d}{d} - n + 1$.

2.1. **Volume function of sums of squares.** Recall from (1.8) that a non-negative form $g \in \overline{\mathcal{C}_{d,n}}$ is SOS if and only if it can be written as $g(\mathbf{x}) = \mathbf{m}_{d/2}(\mathbf{x})^{\mathsf{T}} G \mathbf{m}_{d/2}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$, for some positive semi-definite (Gram) matrix G. Moreover, if $g \in \mathcal{C}_{d,n}$ is positive definite, it has a positive definite Gram matrix $G \in \mathrm{PD}_M$. Given a non-negative form $h \in \overline{\mathcal{C}_{e,n}}$, we consider a composed map

$$\begin{aligned} f_h^{\mathrm{sos}} : \mathrm{PD}_M \;&\to\; (0,\infty), \\ G \;&\mapsto\; f_h(\mathbf{m}_{d/2}(\mathbf{x})^{\mathsf{T}}\, G\, \mathbf{m}_{d/2}(\mathbf{x})), \end{aligned} \tag{2.8}$$

where f_h is defined in (1.7). When $h = 1$, we call

$$f^{\,\mathrm{sos}}(G) \;=\; f_1^{\mathrm{sos}}(G) \;=\; f(\mathbf{m}_{d/2}(\mathbf{x})^{\mathsf{T}}\, G\, \mathbf{m}_{d/2}(\mathbf{x})), \quad G \in \mathrm{PD}_M, \tag{2.9}$$

the SOS *volume function*. The map (1.8) is linear in G and therefore, by Proposition 2.3, (2.8) (and hence also (2.9)) must be completely monotone on its domain PD_M.

Proposition 2.7. *The function $f_h^{\mathrm{sos}} : \mathrm{PD}_M \to (0,\infty)$ is completely monotone.*

Proof. The claim follows directly from Proposition 2.3. Let $V_1, \ldots, V_k \in \mathrm{PD}_M$ be any positive definite matrices. Then by the definition of a directional derivative,

$$(-1)^k D_{V_1} \ldots D_{V_k} f_h^{\mathrm{sos}}(G) \;=\; (-1)^k D_{v_1} \ldots D_{v_k} f_h(g) \;\geq 0, \tag{2.10}$$

where g is given by (1.8) and $v_j(\mathbf{x}) = \mathbf{m}_{d/2}(\mathbf{x})^{\mathsf{T}}\, V_j\, \mathbf{m}_{d/2}(\mathbf{x}) \in \mathcal{C}_{d,n}$ is the positive definite form with Gram matrix V_j, $j = 1, \ldots, k$. □

Next, we describe an integral representation of (2.9), suggested by Theorem 2.2. For this we consider the evaluation of a sum of squares (1.8) at $\boldsymbol{\ell} \in \mathbb{R}^n$,

$$g(\boldsymbol{\ell}) \;=\; \mathbf{m}_{d/2}(\boldsymbol{\ell})^{\mathsf{T}}\, G\, \mathbf{m}_{d/2}(\boldsymbol{\ell}) \;=\; \operatorname{Tr}\left(\mathbf{m}_{d/2}(\boldsymbol{\ell}) \mathbf{m}_{d/2}(\boldsymbol{\ell})^{\mathsf{T}}\, G\right), \tag{2.11}$$

where $\Theta_{\boldsymbol{\ell}} = \mathbf{m}_{d/2}(\boldsymbol{\ell}) \mathbf{m}_{d/2}(\boldsymbol{\ell})^{\mathsf{T}}$ is a $M \times M$ rank-one positive semi-definite matrix. This motivates us to study "the matrix analogue" of the Veronese map (1.4),

$$\begin{aligned} \mathbf{\Theta}_{d,n}^{\mathrm{sos}} : \mathbb{R}^n \;&\to\; \mathrm{PD}_M^*, \\ \boldsymbol{\ell} \;&\mapsto\; \mathbf{m}_{d/2}(\boldsymbol{\ell}) \mathbf{m}_{d/2}(\boldsymbol{\ell})^{\mathsf{T}}, \end{aligned} \tag{2.12}$$

where $\mathrm{PSD}_M = \mathrm{PD}_M^* = \{\Theta \in \mathrm{Mat}(M,\mathbb{R}) : \operatorname{Tr}(\Theta\, G) \geq 0 \;\forall\; G \in \mathrm{PD}_M\}$ is the dual cone to PD_M, which coincides with the closure of $\mathrm{PD}_M \subset \mathrm{Mat}(M,\mathbb{R})$ and consists of positive semi-definite matrices. Here we identify the dual space $\mathrm{Mat}(M,\mathbb{R})^*$ and $\mathrm{Mat}(M,\mathbb{R})$ with the help of *the trace inner product* $\Theta, G \in \mathrm{Mat}(M,\mathbb{R}) \mapsto \operatorname{Tr}(\Theta\, G)$. The main result of this subsection is an analogue of (1.5) for SOS forms.

Theorem 2.8. *The* SOS *volume function* (2.9) *admits an integral representation*

$$f^{\,\mathrm{sos}}(G) \;=\; \int_{\mathrm{PSD}_M} \exp(-\mathrm{Tr}(\Theta\, G))\, \frac{\mathrm{d}\mu^{\mathrm{sos}}(\Theta)}{\Gamma\,(1+n/d)}, \tag{2.13}$$

where μ^{sos} *is the push-forward measure of the Lebesgue measure on* $\mathbb{R}^n$ *under* (2.12).

Proof. Denote by μ^{sos} the push-forward measure of the Lebesgue measure λ on $\mathbb{R}^n$ under the map (2.12). Then formula (2.11) and [3, Thm. 3.6.1] give

$$\begin{aligned} \int_{\mathbb{R}^n} \exp(-\mathbf{m}_{d/2}(\boldsymbol{\ell})^{\mathsf{T}}\, G\, \mathbf{m}_{d/2}(\boldsymbol{\ell}))\, \mathrm{d}\boldsymbol{\ell} \;&=\; \int_{\mathbb{R}^n} \exp(-\mathrm{Tr}\,(\Theta_{\boldsymbol{\ell}}\, G))\, \mathrm{d}\lambda(\boldsymbol{\ell}) \\ &=\; \int_{\mathrm{PSD}_M} \exp(-\mathrm{Tr}\,(\Theta\, G))\, \mathrm{d}\mu^{\mathrm{sos}}(\Theta), \end{aligned} \tag{2.14}$$

which combined with (1.3) and (2.9) yields (2.13). □

Remark 2.9. *Analogously, when* $h(\mathbf{x}) = \mathbf{m}_{d/2}(\mathbf{x})^{\mathsf{T}} H \mathbf{m}_{d/2}(\mathbf{x}) \in \Sigma_{d,n}$ *is a sum of squares of degree* d*, one can show that*

$$f_h^{\mathrm{sos}}(g) \;=\; \int_{\mathrm{PSD}_M} \exp\,(-\mathrm{Tr}\,(\Theta\, G))\, \frac{\mathrm{Tr}\,(\Theta\, H)\; \mathrm{d}\mu^{\mathrm{sos}}(\Theta)}{\Gamma(1+(n+d)/d)}, \tag{2.15}$$

where μ^{sos} *is the push-forward measure of the Lebesgue measure on* $\mathbb{R}^n$ *under* (2.12)*. Thus, the unique Borel measure from Theorem 2.2, that is associated to the completely monotone function* f_h^{sos} *is proportional to* $\mathrm{Tr}\,(\,\cdot\, H)\, \mu^{\mathrm{sos}}$.

The case of quadratic ($d = 2$) forms deserves a special attention. In this case any $g \in \mathcal{C}_{d,n}$ can be written as $g(\mathbf{x}) = \mathbf{x}^{\mathsf{T}} G \mathbf{x}$ for a unique positive definite matrix G (cf. (1.8)). By (1.3) and elementary Gaussian integration, one obtains a closed form expression for the volume function of g (see also [15, p. 38]).

Proposition 2.10. *For any* $n \geq 2$ *we have*

$$f^{\,\mathrm{sos}}(G) \;=\; f(g) \;=\; \frac{\sqrt{\pi}^{\,n}}{\Gamma(1+n/2)}\, \frac{1}{\sqrt{\det(G)}}.$$

Remark 2.11. *Complete monotonicity of the function* $G \mapsto (\det(G))^{-1/2}$ *on the cone of positive definite matrices is a quite known fact, see* [22, pp. 355-356].

We conclude this chapter with an explicit formula for (2.8) in the case $d = 2$.

Theorem 2.12. *For any* $n \geq 2$ *and* $h(\mathbf{x}) = \mathbf{x}^{\mathsf{T}} H \mathbf{x}$*,* $H \in \mathrm{PSD}_n$*, we have*

$$f_h^{\mathrm{sos}}(G) \;=\; \frac{\sqrt{\pi}^{\,n}}{2\,\Gamma\,(1+(n+2)/2)}\, \frac{\mathrm{Tr}(G^{-1}H)}{\sqrt{\det(G)}}, \quad G \in \mathrm{PD}_n. \tag{2.16}$$

Proof. Spectral theorem applied to real symmetric matrices G and H yields decompositions $G = \sum_{i=1}^n \lambda_i \mathbf{c}_i \mathbf{c}_i^{\mathsf{T}}$ and $H = \sum_{j=1}^n \eta_j\, \mathbf{d}_j \mathbf{d}_j^{\mathsf{T}}$. Here $\{\mathbf{c}_1, \ldots, \mathbf{c}_d\} \subset \mathbb{R}^n$ and $\{\mathbf{d}_1, \ldots, \mathbf{d}_n\} \subset \mathbb{R}^n$ are orthonormal bases consisting of eigenvectors of G and, respectively, H, whose associated eigenvalues are $\lambda_1, \ldots, \lambda_n > 0$ and, respectively, $\eta_1, \ldots, \eta_n \geq 0$. Performing an orthogonal change of variables $\mathbf{x} = \sum_{k=1}^n y_k \mathbf{c}_k$ in

the last integral of (2.3), we write

$$\begin{aligned}\int_{\mathbb{R}^n} \mathbf{x}^\mathsf{T} H\mathbf{x} \exp(-\mathbf{x}^\mathsf{T} G\mathbf{x})\,\mathrm{d}\mathbf{x} &= \int_{\mathbb{R}^n} \sum_{j=1}^n \eta_j \left(\sum_{k=1}^n y_k \mathbf{c}_k^\mathsf{T}\mathbf{d}_j\right)^2 \exp\left(-\sum_{i=1}^n \lambda_i y_i^2\right)\,\mathrm{d}\mathbf{y}\\ &= \int_{\mathbb{R}^n} \sum_{j=1}^n \eta_j \left(\sum_{k,\ell=1}^n y_k y_\ell \mathbf{c}_k^\mathsf{T}\mathbf{d}_j \mathbf{c}_\ell^\mathsf{T}\mathbf{d}_j\right) \exp\left(-\sum_{i=1}^n \lambda_i y_i^2\right)\,\mathrm{d}\mathbf{y}\\ &= \int_{\mathbb{R}^n} \sum_{j=1}^n \eta_j \sum_{k=1}^n y_k^2 \left(\mathbf{c}_k^\mathsf{T}\mathbf{d}_j\right)^2 \exp\left(-\sum_{i=1}^n \lambda_i y_i^2\right)\,\mathrm{d}\mathbf{y},\end{aligned}$$

where in the last step we use the fact that the integral $\int_{\mathbb{R}^n} y_k y_\ell \exp(-\sum_{i=1}^n \lambda_i y_i^2)\,\mathrm{d}\mathbf{y}$ equals zero whenever $k \neq \ell$. After one more change of variables $y_i = z_i/\sqrt{\lambda_i}$, $i = 1, \ldots, n$, and elementary Gaussian integration, the above integral reads

$$\sum_{i,j=1}^n \frac{\eta_j}{\lambda_i}\left(\mathbf{c}_i^\mathsf{T}\mathbf{d}_j\right)^2 \int_{\mathbb{R}^n} z_i^2 \exp\left(-\mathbf{z}^\mathsf{T}\mathbf{z}\right) \frac{\mathrm{d}\mathbf{z}}{\sqrt{\lambda_1 \cdots \lambda_n}} = \frac{\sqrt{\pi}^n}{2\sqrt{\det(G)}} \sum_{i,j=1}^n \frac{\eta_j}{\lambda_i}\left(\mathbf{c}_i^\mathsf{T}\mathbf{d}_j\right)^2.$$

The inverse matrix of $G \in \mathrm{PD}_n$ is given by $G^{-1} = \sum_{i=1}^n \lambda_i^{-1}\mathbf{c}_i\mathbf{c}_i^\mathsf{T}$. Using properties of the trace, we can rewrite the double sum in the last expression above as

$$\begin{aligned}\sum_{i,j=1}^n \frac{\eta_j}{\lambda_i}\left(\mathbf{c}_i^\mathsf{T}\mathbf{d}_j\right)^2 &= \sum_{i,j=1}^n \frac{\eta_j}{\lambda_i}\mathrm{Tr}\left((\mathbf{c}_i^\mathsf{T}\mathbf{d}_j)(\mathbf{d}_j^\mathsf{T}\mathbf{c}_i)\right) = \sum_{i,j=1}^n \lambda_i^{-1}\eta_j \mathrm{Tr}\left(\mathbf{c}_i\mathbf{c}_i^\mathsf{T}\mathbf{d}_j\mathbf{d}_j^\mathsf{T}\right)\\ &= \mathrm{Tr}\left(\left(\sum_{i=1}^n \lambda_i^{-1}\mathbf{c}_i\mathbf{c}_i^\mathsf{T}\right)\left(\sum_{j=1}^n \eta_j\,\mathbf{d}_j\mathbf{d}_j^\mathsf{T}\right)\right) = \mathrm{Tr}\left(G^{-1}H\right).\end{aligned}$$

By gathering all together and combining with (2.3) we obtain the claim (2.16). □

3. Moment matrices of central Gaussian vectors and sums of squares

Our goal in this section is to prove Theorem 1.5 and then derive Corollary 1.8.

Proof of Theorem 1.5. Let us consider the vector $\hat{\mathbf{m}}_{d/2}(\mathbf{x}) = \left(\sqrt{\binom{d/2}{\alpha}}\,\mathbf{x}^\alpha\right)_{|\alpha|=d/2}$ of normalized monomials of degree $d/2$. The advantage of working with the normalized monomials consists in the fact that they (unlike $\mathbf{x}^\alpha$, $|\alpha| = d/2$) form an orthonormal basis of $\mathcal{H}_{d/2,n}$ with respect to the Bombieri inner product (1.1). It is a well-known fact (see, e.g., [9, p. 17]) that (1.1) is invariant under the standard action of $O(n)$ on $\mathcal{H}_{d/2,n}$ by changes of variables. Thus, an orthogonal change of variables $\mathbf{x} \mapsto a\mathbf{x}$, $a \in O(n)$, induces a change of basis in $\mathcal{H}_{d/2,n}$ given by some "big" orthogonal matrix $A \in O(M)$, $M = \binom{d/2+n-1}{n-1}$, that is,

$$\hat{\mathbf{m}}_{d/2}(a\mathbf{x}) = A\hat{\mathbf{m}}_{d/2}(\mathbf{x}). \tag{3.1}$$

The entries of the "big" matrix $A \in O(M)$ are some forms of degree $d/2$ in the entries of the "small" matrix $a \in O(n)$. The matrix

$$\hat{\mathbf{M}}_d[Q] = \int_{\mathbb{R}^n} \hat{\mathbf{m}}_{d/2}(\mathbf{y})\hat{\mathbf{m}}_{d/2}(\mathbf{y})^\mathsf{T} p_Q^{(d)}(\mathbf{y})\,\mathrm{d}\mathbf{y} = \left(\int_{\mathbb{R}^n} \sqrt{\binom{d/2}{\alpha}}\sqrt{\binom{d/2}{\beta}}\mathbf{y}^{\alpha+\beta} p_Q^{(d)}(\mathbf{y})\,\mathrm{d}\mathbf{y}\right)$$

of normalized moments of degree d of $p_Q^{(d)}$ is related to $\mathbf{M}_d[Q]$ via the formula

$$\hat{\mathbf{M}}_d[Q] \;=\; S\,\mathbf{M}_d[Q]\,S, \tag{3.2}$$

where

$$S \;=\; \begin{pmatrix} \ddots & \cdots & 0 \\ \vdots & \sqrt{\binom{d/2}{\alpha}} & \vdots \\ 0 & \cdots & \ddots \end{pmatrix}$$

is the diagonal matrix of square roots of multinomial coefficients. We now observe that forms $\mathbf{m}_{d/2}(\mathbf{x})^{\mathsf{T}}\mathbf{M}_d[Q]^{-1}\mathbf{m}_{d/2}(\mathbf{x})$ and $\hat{\mathbf{m}}_{d/2}(\mathbf{x})^{\mathsf{T}}\hat{\mathbf{M}}_d[Q]^{-1}\hat{\mathbf{m}}_{d/2}(\mathbf{x})$ coincide. Indeed, the relation (3.2) together with $\hat{\mathbf{m}}_{d/2}(\mathbf{x}) = S\mathbf{m}_{d/2}(\mathbf{x})$ imply

$$\begin{aligned} \mathbf{m}_{d/2}(\mathbf{x})^{\mathsf{T}}\mathbf{M}_d[Q]^{-1}\mathbf{m}_{d/2}(\mathbf{x}) \;&=\; \hat{\mathbf{m}}_{d/2}(\mathbf{x})S^{-1}\mathbf{M}_d[Q]^{-1}S^{-1}\hat{\mathbf{m}}_{d/2}(\mathbf{x}) \\ &=\; \hat{\mathbf{m}}_{d/2}(\mathbf{x})\hat{\mathbf{M}}_d[Q]^{-1}\hat{\mathbf{m}}_{d/2}(\mathbf{x}). \end{aligned} \tag{3.3}$$

Next, we show that it is enough to prove the claim (1.14) in the case when Q is diagonal. For this, let us choose an orthogonal $a \in O(n)$ so that the conjugate to Q matrix $a^{\mathsf{T}}Qa = \Lambda$ is the diagonal matrix of its eigenvalues $\lambda_1, \ldots, \lambda_n > 0$. Using formula (3.1), we derive

(3.4)

$$\begin{aligned} &\hat{\mathbf{m}}_{d/2}(\mathbf{x})^{\mathsf{T}}\hat{\mathbf{M}}_d^{-1}[Q]\hat{\mathbf{m}}_{d/2}(\mathbf{x}) \\ &=\; \hat{\mathbf{m}}_{d/2}(\mathbf{x})^{\mathsf{T}}\left(\int_{\mathbb{R}^n} \hat{\mathbf{m}}_{d/2}(\mathbf{y})\hat{\mathbf{m}}_{d/2}(\mathbf{y})^{\mathsf{T}}\frac{\sqrt{\det(Q)}}{\sigma_d^n\sqrt{(2\pi)^n}}\exp\left(-\frac{\mathbf{y}^{\mathsf{T}}a\Lambda a^{\mathsf{T}}\mathbf{y}}{2\sigma_d^2}\right)\mathrm{d}\mathbf{y}\right)^{-1}\hat{\mathbf{m}}_{d/2}(\mathbf{x}) \\ &=\; \hat{\mathbf{m}}_{d/2}(\mathbf{x})^{\mathsf{T}}\left(\int_{\mathbb{R}^n} \hat{\mathbf{m}}_{d/2}(a\mathbf{z})\hat{\mathbf{m}}_{d/2}(a\mathbf{z})^{\mathsf{T}}\frac{\sqrt{\det(\Lambda)}}{\sigma_d^n\sqrt{(2\pi)^n}}\exp\left(-\frac{\mathbf{z}^{\mathsf{T}}\Lambda\mathbf{z}}{2\sigma_d^2}\right)\mathrm{d}\mathbf{z}\right)^{-1}\hat{\mathbf{m}}_{d/2}(\mathbf{x}) \\ &=\; \hat{\mathbf{m}}_{d/2}(\mathbf{x})^{\mathsf{T}}\left(\int_{\mathbb{R}^n} A\hat{\mathbf{m}}_{d/2}(\mathbf{z})\hat{\mathbf{m}}_{d/2}(\mathbf{z})^{\mathsf{T}}A^{\mathsf{T}}\frac{\sqrt{\det(\Lambda)}}{\sigma_d^n\sqrt{(2\pi)^n}}\exp\left(-\frac{\mathbf{z}^{\mathsf{T}}\Lambda\mathbf{z}}{2\sigma_d^2}\right)\mathrm{d}\mathbf{z}\right)^{-1}\hat{\mathbf{m}}_{d/2}(\mathbf{x}) \\ &=\; \hat{\mathbf{m}}_{d/2}(\mathbf{x})^{\mathsf{T}}A\,\hat{\mathbf{M}}_d[\Lambda]^{-1}A^{\mathsf{T}}\hat{\mathbf{m}}_{d/2}(\mathbf{x}) \;=\; \hat{\mathbf{m}}_{d/2}(a^{\mathsf{T}}\mathbf{x})^{\mathsf{T}}\hat{\mathbf{M}}_d[\Lambda]^{-1}\hat{\mathbf{m}}_{d/2}(a^{\mathsf{T}}\mathbf{x}), \end{aligned}$$

where we use orthogonality of matrices $a \in O(n)$, $A \in O(N)$, linearity of the integral and the fact that $\hat{\mathbf{m}}_{d/2}(a^{\mathsf{T}}\mathbf{x}) = A^{\mathsf{T}}\hat{\mathbf{m}}_{d/2}(\mathbf{x})$. Therefore, (1.14) holds for Λ if and only if it holds for $Q = a\Lambda a^{\mathsf{T}}$, since by the last derivation and (3.3) we have

$$\begin{aligned} \big(\mathbf{x}^{\mathsf{T}}Q\mathbf{x}\big)^d \;&=\; \big(\mathbf{x}^{\mathsf{T}}a\Lambda a^{\mathsf{T}}\mathbf{x}\big)^d \;=\; \big((a^{\mathsf{T}}\mathbf{x})^{\mathsf{T}}\Lambda(a^{\mathsf{T}}\mathbf{x})\big)^d \quad \text{and} \\ \mathbf{m}_{d/2}(\mathbf{x})^{\mathsf{T}}\mathbf{M}_d[Q]^{-1}\mathbf{m}_{d/2}(\mathbf{x}) \;&=\; \hat{\mathbf{m}}_{d/2}(\mathbf{x})^{\mathsf{T}}\hat{\mathbf{M}}_d[Q]^{-1}\hat{\mathbf{m}}_{d/2}(\mathbf{x}) \\ &=\; \hat{\mathbf{m}}_{d/2}(a^{\mathsf{T}}\mathbf{x})^{\mathsf{T}}\hat{\mathbf{M}}_d[\Lambda]^{-1}\hat{\mathbf{m}}_{d/2}(a^{\mathsf{T}}\mathbf{x}) \\ &=\; \mathbf{m}_{d/2}(a^{\mathsf{T}}\mathbf{x})^{\mathsf{T}}\mathbf{M}_d[\Lambda]^{-1}\mathbf{m}_{d/2}(a^{\mathsf{T}}\mathbf{x}). \end{aligned}$$

Now, we reduce proving the claim (1.14) for a diagonal matrix $Q = \Lambda$ to proving it for the identity matrix $Q = \mathrm{Id}$. Since the eigenvalues of $Q = \Lambda$ are positive we can write them as $\lambda_i = \mu_i^2$ for some $\mu_i \in \mathbb{R}$, $i = 1, \ldots, n$. Let us observe that, under the coordinate-wise multiplication $\mathbf{x} = (x_1, \ldots, x_n) \mapsto \mu * \mathbf{x} = (\mu_1 x_1, \ldots, \mu_n x_n)$ by real numbers $\mu_1, \ldots, \mu_n \in \mathbb{R}$, the vector of monomials transforms as

$$\mathbf{m}_{d/2}(\mu * \mathbf{x}) \;=\; (\mu^\alpha \mathbf{x}^\alpha)_{|\alpha|=d} \;=\; \mathbf{m}_{d/2}(\mu) * \mathbf{m}_{d/2}(\mathbf{x}) \;=\; \mathcal{N}(\mu)\,\mathbf{m}_{d/2}(\mathbf{x}), \tag{3.5}$$

where $\mathcal{N}(\mu)$ is the diagonal matrix with numbers μ^α, $|\alpha| = d$, on its diagonal. We relate the forms $\mathbf{m}_{d/2}(\mathbf{x})^\mathsf{T}\mathbf{M}_d[Q]^{-1}\mathbf{m}_{d/2}(\mathbf{x})$ with $Q = \Lambda$ and $Q = \mathrm{Id}$ by

$$\begin{aligned}
&\mathbf{m}_{d/2}(\mathbf{x})^\mathsf{T}\mathbf{M}_d[\Lambda]^{-1}\mathbf{m}_{d/2}(\mathbf{x})\\
&= \mathbf{m}_{d/2}(\mathbf{x})^\mathsf{T}\left(\int_{\mathbb{R}^n}\mathbf{m}_{d/2}(\mathbf{y})\mathbf{m}_{d/2}(\mathbf{y})^\mathsf{T}\frac{\sqrt{\det(\Lambda)}}{\sigma_d^n\sqrt{(2\pi)^n}}\exp\left(-\frac{(\mu*\mathbf{y})^\mathsf{T}(\mu*\mathbf{y})}{2\sigma_d^2}\right)\mathrm{d}\mathbf{y}\right)^{-1}\mathbf{m}_{d/2}(\mathbf{x})\\
&= \mathbf{m}_{d/2}(\mathbf{x})^\mathsf{T}\left(\int_{\mathbb{R}^n}\mathbf{m}_{d/2}(\mu^{-1}*\mathbf{z})\mathbf{m}_{d/2}(\mu^{-1}*\mathbf{z})^\mathsf{T}\exp\left(-\frac{\mathbf{z}^\mathsf{T}\mathbf{z}}{2\sigma_d^2}\right)\frac{\mathrm{d}\mathbf{z}}{\sigma_d^n\sqrt{(2\pi)^n}}\right)^{-1}\mathbf{m}_{d/2}(\mathbf{x})\\
&= \mathbf{m}_{d/2}(\mathbf{x})^\mathsf{T}\left(\int_{\mathbb{R}^n}\mathcal{N}(\mu^{-1})\mathbf{m}_{d/2}(\mathbf{z})\mathbf{m}_{d/2}(\mathbf{z})^\mathsf{T}\mathcal{N}(\mu^{-1})\exp\left(-\frac{\mathbf{z}^\mathsf{T}\mathbf{z}}{2\sigma_d^2}\right)\frac{\mathrm{d}\mathbf{z}}{\sigma_d^n\sqrt{(2\pi)^n}}\right)^{-1}\mathbf{m}_{d/2}(\mathbf{x})\\
&= \mathbf{m}_{d/2}(\mathbf{x})^\mathsf{T}\mathcal{N}(\mu)^\mathsf{T}\mathbf{M}_d[\mathrm{Id}]^{-1}\mathcal{N}(\mu)\mathbf{m}_{d/2}(\mathbf{x}) = \mathbf{m}_{d/2}(\mu*\mathbf{x})^\mathsf{T}\mathbf{M}_d[\mathrm{Id}]^{-1}\mathbf{m}_{d/2}(\mu*\mathbf{x}),
\end{aligned}$$

where we perform a change of variables $\mathbf{z} = \mu * \mathbf{y}$, use linearity of the integral and the formula $\mathcal{N}(\mu^{-1}) = \mathcal{N}(\mu)^{-1}$ with $\mu^{-1} = (\mu_1^{-1}, \dots, \mu_n^{-1})$. This together with the formula $(\mathbf{x}^\mathsf{T}\Lambda\mathbf{x})^d = ((\mu*\mathbf{x})^\mathsf{T}(\mu*\mathbf{x}))^d$ imply that (1.14) holds for a diagonal matrix $Q = \Lambda$ if and only if it holds for the identity matrix $Q = \mathrm{Id}$.

To prove (1.14) for $Q = \mathrm{Id}$, we first note that the form $\mathbf{m}_{d/2}(\mathbf{x})^\mathsf{T}\mathbf{M}_d[\mathrm{Id}]^{-1}\mathbf{m}_{d/2}(\mathbf{x})$ is invariant under orthogonal changes of variables. Indeed, this property follows from formula (3.4) with $Q = \Lambda = \mathrm{Id}$ and (3.3), since (by definition) $a\,\mathrm{Id}\,a^\mathsf{T} = \mathrm{Id}$ holds for all orthogonal matrices $a \in O(n)$. As an $O(n)$-invariant form of degree d is proportional to $(\mathbf{x}^\mathsf{T}\mathbf{x})^{d/2}$ (see, e.g., [7, Lemma 2.1]), we obtain

$$\mathbf{m}_{d/2}(\mathbf{x})^\mathsf{T}\mathbf{M}_d[\mathrm{Id}]^{-1}\mathbf{m}_{d/2}(\mathbf{x}) \;=\; c\,(\mathbf{x}^\mathsf{T}\mathbf{x})^{d/2}$$

for some constant $c \in \mathbb{R}^n$. We now integrate this equality against the measure $p^{(d)}_{\mathrm{Id}}(\mathbf{x})\,\mathrm{d}\mathbf{x}$. Using linearity of the integral and properties of the trace and performing elementary integration, we compute

$$\begin{aligned}
\binom{d/2+n-1}{n-1} &= \mathrm{Tr}\left(\mathbf{M}_d[\mathrm{Id}]^{-1}\mathbf{M}_d[\mathrm{Id}]\right) = \sum_{|\alpha|=|\beta|=d/2}\mathbf{M}_d[\mathrm{Id}]^{-1}_{\alpha\beta}\mathbf{M}_d[\mathrm{Id}]_{\alpha\beta}\\
&= \int_{\mathbb{R}^n}\sum_{|\alpha|=|\beta|=d/2}\mathbf{M}_d[\mathrm{Id}]^{-1}_{\alpha\beta}\,\mathbf{x}^{\alpha+\beta}\,p^{(d)}_{\mathrm{Id}}(\mathbf{x})\,\mathrm{d}\mathbf{x}\\
&= \int_{\mathbb{R}^n}\mathbf{m}_{d/2}(\mathbf{x})^\mathsf{T}\mathbf{M}_d[\mathrm{Id}]^{-1}\mathbf{m}_{d/2}(\mathbf{x})\,p^{(d)}_{\mathrm{Id}}(\mathbf{x})\,\mathrm{d}\mathbf{x}\\
&= c\int_{\mathbb{R}^n}(\mathbf{x}^\mathsf{T}\mathbf{x})^{d/2}\,p^{(d)}_{\mathrm{Id}}(\mathbf{x})\,\mathrm{d}\mathbf{x}\\
&= c\,\sigma_d^d\frac{1}{\sqrt{(2\pi)^n}}\int_{\mathbb{R}^n}(\mathbf{x}^\mathsf{T}\mathbf{x})^{d/2}\exp\left(-\frac{\mathbf{x}^\mathsf{T}\mathbf{x}}{2}\right)\mathrm{d}\mathbf{x}\\
&= c\,\sigma_d^d\,\frac{\mathrm{vol}\,(\mathbb{S}^{n-1})}{\sqrt{(2\pi)^n}}\int_0^\infty r^{d+n-1}\exp\left(-\frac{r^2}{2}\right)\mathrm{d}r\\
&= c\,\frac{\binom{d/2+n-1}{n-1}}{\sqrt{2}^d\prod_{i=0}^{d/2-1}\left(\frac{n}{2}+i\right)}\sqrt{2}^d\frac{\Gamma\left(\frac{d+n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} = c\binom{d/2+n-1}{n-1},
\end{aligned}$$

where we use the formula $\mathrm{vol}\,(\mathbb{S}^{n-1}) = 2\sqrt{\pi}^n/\,\Gamma\,(n/2)$ for the volume of the unit sphere $\mathbb{S}^{n-1}$. This derivation yields $c = 1$ and hence completes our proof of (1.14).

It remains to prove that

$$\mathbf{M}_d[Q] \;=\; \frac{\int_{\mathbb{R}^n} \mathbf{m}_{d/2}(\mathbf{x})\mathbf{m}_{d/2}(\mathbf{x})^\mathsf{T} \exp\left(-k(\mathbf{x}^\mathsf{T}Q\mathbf{x})^{d/2}\right) \mathrm{d}\mathbf{x}}{\int_{\mathbb{R}^n} \exp\left(-k(\mathbf{y}^\mathsf{T}Q\mathbf{y})^{d/2}\right) \mathrm{d}\mathbf{y}} \tag{3.6}$$

is the matrix of degree d moments of the Gaussian-like density (1.11), associated to the SOS form $g(\mathbf{x}) = (\mathbf{x}^\mathsf{T}Q\mathbf{x})^{d/2}$. Applying a change of variables $\mathbf{y} = \sqrt{2}\sigma_d\,\mathbf{z}$ in the definition (1.13) of $\mathbf{M}_d[Q]$ and then using [11, (20)], we write

$$\begin{aligned}
\mathbf{M}_d[Q] \;&=\; \frac{\sqrt{\det(Q)}}{\sigma_d^n\sqrt{(2\pi)^n}} \int_{\mathbb{R}^n} \mathbf{m}_{d/2}(\mathbf{y})\mathbf{m}_{d/2}(\mathbf{y})^\mathsf{T} \exp\left(-\frac{\mathbf{y}^\mathsf{T}Q\mathbf{y}}{2\sigma_d^2}\right) \mathrm{d}\mathbf{y} \\
&=\; \frac{\sqrt{2}^d\sigma_d^d\sqrt{\det(Q)}}{\sqrt{\pi^n}} \int_{\mathbb{R}^n} \mathbf{m}_{d/2}(\mathbf{z})\mathbf{m}_{d/2}(\mathbf{z})^\mathsf{T} \exp\left(-\mathbf{z}^\mathsf{T}Q\mathbf{z}\right) \mathrm{d}\mathbf{z} \\
&=\; \frac{\sqrt{2}^d\sigma_d^d\sqrt{\det(Q)}}{\sqrt{\pi^n}}\Gamma\left(1+\frac{n+d}{2}\right) \int_{\left\{\mathbf{z}^\mathsf{T}Q\mathbf{z}\le 1\right\}} \mathbf{m}_{d/2}(\mathbf{z})\mathbf{m}_{d/2}(\mathbf{z})^\mathsf{T}\,\mathrm{d}\mathbf{z} \\
&=\; \frac{\sqrt{2}^d\sigma_d^d\sqrt{\det(Q)}}{\sqrt{\pi^n}}\Gamma\left(1+\frac{n+d}{2}\right) \int_{\left\{(\mathbf{z}^\mathsf{T}Q\mathbf{z})^{d/2}\le 1\right\}} \mathbf{m}_{d/2}(\mathbf{z})\mathbf{m}_{d/2}(\mathbf{z})^\mathsf{T}\,\mathrm{d}\mathbf{z} \\
&=\; \frac{\sqrt{2}^d\sigma_d^d\sqrt{\det(Q)}}{\sqrt{\pi^n}}\frac{\Gamma\left(1+\frac{n+d}{2}\right)}{\Gamma\left(1+\frac{n+d}{d}\right)} \int_{\mathbb{R}^n} \mathbf{m}_{d/2}(\mathbf{z})\mathbf{m}_{d/2}(\mathbf{z})^\mathsf{T} \exp\left(-(\mathbf{z}^\mathsf{T}Q\mathbf{z})^{d/2}\right) \mathrm{d}\mathbf{z} \\
&=\; \frac{k^{1+\frac{n}{d}}\sqrt{2}^d\sigma_d^d\sqrt{\det(Q)}}{\sqrt{\pi^n}}\frac{d\,\Gamma\left(\frac{n+d}{2}\right)}{2\,\Gamma\left(\frac{n+d}{d}\right)} \int_{\mathbb{R}^n} \mathbf{m}_{d/2}(\mathbf{z})\mathbf{m}_{d/2}(\mathbf{z})^\mathsf{T} \exp\left(-k(\mathbf{y}^\mathsf{T}Q\mathbf{y})^{d/2}\right) \mathrm{d}\mathbf{y},
\end{aligned}$$

where in the last step we change variables $\mathbf{z} = k^{1/d}\mathbf{y}$. Now, using the definition of σ_d and formulas $k^{-1} = 2\binom{d/2+n-1}{n} = \binom{d/2+n-1}{n-1}d/n$, $\mathrm{vol}\,(\mathbb{S}^{n-1}) = 2\sqrt{\pi^n}/\Gamma(n/2)$, we simplify the constant in front of the last integral and write

$$\mathbf{M}_d[Q] \;=\; \frac{k^{\frac{n}{d}}n\sqrt{\det(Q)}}{\mathrm{vol}\,(\mathbb{S}^{n-1})\Gamma\left(\frac{n+d}{d}\right)} \int_{\mathbb{R}^n} \mathbf{m}_{d/2}(\mathbf{z})\mathbf{m}_{d/2}(\mathbf{z})^\mathsf{T} \exp(-k(\mathbf{y}^\mathsf{T}Q\mathbf{y})^{d/2})\,\mathrm{d}\mathbf{y}. \tag{3.7}$$

Performing a sequence of changes of variables and using basic properties of Gamma function, one can see that the mass of the measure $\exp(-k(\mathbf{y}^\mathsf{T}Q\mathbf{y})^{d/2})\,\mathrm{d}\mathbf{y}$ equals

$$\int_{\mathbb{R}^n} \exp\left(-k(\mathbf{y}^\mathsf{T}Q\mathbf{y})^{d/2}\right) \mathrm{d}\mathbf{y} \;=\; \frac{\mathrm{vol}\,(\mathbb{S}^{n-1})\Gamma\left(\frac{n+d}{d}\right)}{k^{\frac{n}{d}}n\sqrt{\det(Q)}},$$

which together with (3.7) yields the claimed representation (3.6) for $\mathbf{M}_d[Q]$. □

We conclude this section with a proof of Corollary 1.8.

Proof of Corollary 1.8. Since the matrix $Q \in \mathrm{PD}_n$ is positive definite, we have that $\mathbf{y}^\mathsf{T}Q\mathbf{y} > 0$ for non-zero $\mathbf{y} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$. We first perform spherical change of variables $\mathbf{y} = r\,\mathbf{z}$, $r \in (0,\infty)$, $\mathbf{z} \in \mathbb{S}^{n-1}$, in the integral (1.13) defining $\mathbf{M}_d[Q]$ and then, after

changing to $t := \frac{\mathbf{z}^\mathsf{T} Q \mathbf{z}}{2\sigma_d^2} r^2$ in the inner integral, we integrate out the t variable:

$$\begin{aligned}
\mathbf{M}_d[Q] &= \frac{\sqrt{\det(Q)}}{\sigma_d^n \sqrt{(2\pi)^n}} \int_{\mathbb{R}^n} \mathbf{m}_{d/2}(\mathbf{y}) \mathbf{m}_{d/2}(\mathbf{y})^\mathsf{T} \exp\left(-\frac{\mathbf{y}^\mathsf{T} Q \mathbf{y}}{2\sigma_d^2}\right) \mathrm{d}\mathbf{y} \\
&= \frac{\sqrt{\det(Q)}}{\sigma_d^n \sqrt{(2\pi)^n}} \int_{\mathbb{S}^{n-1}} \mathbf{m}_{d/2}(\mathbf{z}) \mathbf{m}_{d/2}(\mathbf{z})^\mathsf{T} \int_0^\infty r^{d+n-1} \exp\left(-\frac{\mathbf{z}^\mathsf{T} Q \mathbf{z}}{2\sigma_d^2} r^2\right) \mathrm{d}r \, \mathrm{d}\mathbb{S}^{n-1}(\mathbf{z}) \\
&= \sqrt{\det(Q)} \frac{\sqrt{2}^{d-2} \sigma_d^d}{\sqrt{\pi}^n} \Gamma\left(\frac{d+n}{2}\right) \int_{\mathbb{S}^{n-1}} \frac{\mathbf{m}_{d/2}(\mathbf{z}) \mathbf{m}_{d/2}(\mathbf{z})^\mathsf{T}}{\sqrt{\mathbf{z}^\mathsf{T} Q \mathbf{z}}^{d+n}} \, \mathrm{d}\mathbb{S}^{n-1}(\mathbf{z}) \\
&= \sqrt{\det(Q)} \binom{d/2+n-1}{n-1} \frac{\Gamma\left(\frac{n}{2}\right)}{2\sqrt{\pi}^n} \int_{\mathbb{S}^{n-1}} \frac{\mathbf{m}_{d/2}(\mathbf{z}) \mathbf{m}_{d/2}(\mathbf{z})^\mathsf{T}}{\sqrt{\mathbf{z}^\mathsf{T} Q \mathbf{z}}^{d+n}} \, \mathrm{d}\mathbb{S}^{n-1}(\mathbf{z}).
\end{aligned}$$

This, together with the formula $\mathrm{vol}\,(\mathbb{S}^{n-1}) = 2\sqrt{\pi}^n / \Gamma\,(n/2)$ for the volume of the sphere, implies that $\mathbf{M}_d[Q]$ is the matrix of degree d moments of the measure ν on $\mathbb{S}^{n-1}$ from the statement of Corollary 1.8. The rest follows from Theorem 1.5. □

4. Comparison of L^1- and L^2-norms associated to a PD form

In this section we prove an extremal property of positive definite forms, that is stated in Theorem 1.9.

Proof of Theorem 1.9. For a positive definite form $g \in \mathcal{C}_{d,n}$ and any $h \in \mathcal{H}_{d,n}$ one has by formula (56) in [11, Lemma 3] that

$$\int_{\mathbf{G}} |h(\mathbf{x})| g(\mathbf{x}) \, \mathrm{d}\mathbf{x} = \frac{n+d}{n+2d} \int_{\mathbf{G}} |h(\mathbf{x})| \, \mathrm{d}\mathbf{x} = \frac{n+d}{n+2d} \|h\|_{L^1(\mu_g)}$$

and in particular with $h = g$ one obtains $\|g\|^2_{L^2(\mu_g)} = \frac{n+d}{n+2d} \|g\|_{L^1(\mu_g)}$.

Let now $h \in \mathcal{H}_{d,n}$ be such that $\|h\|_{L^1(\mu_g)} = \|g\|_{L^1(\mu_g)}$. Then we derive

$$\begin{aligned}
0 \le \||h| - g\|^2_{L^2(\mu_g)} &= \|h\|^2_{L^2(\mu_g)} - 2 \int_{\mathbf{G}} |h(\mathbf{x})| g(\mathbf{x}) \, \mathrm{d}\mathbf{x} + \|g\|^2_{L^2(\mu_g)} \\
&= \|h\|^2_{L^2(\mu_g)} - 2 \frac{n+d}{n+2d} \|h\|_{L^1(\mu_g)} + \frac{n+d}{n+2d} \|g\|_{L^1(\mu_g)} \\
&= \|h\|^2_{L^2(\mu_g)} - \frac{n+d}{n+2d} \|g\|_{L^1(\mu_g)} = \|h\|^2_{L^2(\mu_g)} - \|g\|^2_{L^2(\mu_g)},
\end{aligned}$$

which yields the claim. Moreover, if $\|h\|_{L^2(\mu_g)} = \|g\|_{L^2(\mu_g)}$, we necessarily have that $\||h| - g\|_{L^2(\mu_g)} = 0$ and hence either $h = g$ or $h = -g$. □

5. Non-negative forms with sublevel sets of finite volume

In this section we prove Theorems 1.11 and 1.12 and start with some auxiliary discussion. Let $g \in \overline{\mathcal{C}_{d,n}}$ be a non-negative form and consider the dehomogenized non-negative polynomial $\tilde{g}(\mathbf{y}) = g(\mathbf{y}, 1)$, where $\mathbf{y} = (y_1, \ldots, y_{n-1}) \in \mathbb{R}^{n-1}$. We consider the following change of variables on $\{\mathbf{x} \in \mathbb{R}^n : x_n \neq 0\}$,

$$\mathbf{x} = (x_1, \ldots, x_{n-1}, x_n) = r(y_1, \ldots, y_{n-1}, 1), \quad (\mathbf{y}, r) \in \mathbb{R}^{n-1} \times (\mathbb{R} \setminus \{0\}),$$

which allows us to write $g(\mathbf{x}) = r^d \tilde{g}(\mathbf{y})$. Then the sublevel set of g is expressed as

$$\mathbf{G} = \{g(\mathbf{x}) \le 1\} = \left\{(\mathbf{y}, r) \in \mathbb{R}^{n-1} \times (\mathbb{R} \setminus \{0\}) : -\tilde{g}(\mathbf{y})^{-1/d} \le r \le \tilde{g}(\mathbf{y})^{-1/d}\right\}$$

and, as a consequence, its Lebesgue volume equals

$$f(g) \;=\; \int_{\mathbf{G}} \mathrm{d}\mathbf{x} \;=\; \int_{\mathbb{R}^{n-1}} \int_{-\tilde{g}(\mathbf{y})^{-1/d}}^{\tilde{g}(\mathbf{y})^{-1/d}} |r|^{n-1} \mathrm{d}r \, \mathrm{d}\mathbf{y} \;=\; \frac{2}{n} \int_{\mathbb{R}^{n-1}} \frac{1}{\tilde{g}(\mathbf{y})^{n/d}} \, \mathrm{d}\mathbf{y}. \tag{5.1}$$

As we show in the following lemma, the question about finiteness of $f(g)$ has a "local nature", at least for some class of non-negative forms $g \in \overline{\mathcal{C}_{d,n}}$.

Lemma 5.1. *Let $\tilde{g} \in \mathbb{R}[y_1, \dots, y_{n-1}]$ be a non-negative polynomial of degree d with only isolated real zeros $\mathbf{y}^{(1)}, \dots \mathbf{y}^{(m)} \in \mathbb{R}^{n-1}$ and whose degree d part is positive definite. Then for any $\alpha > \frac{n-1}{d}$ the integral $\int_{\mathbb{R}^{n-1}} \tilde{g}(\mathbf{y})^{-\alpha} \, \mathrm{d}\mathbf{y}$ is finite if and only if integrals $\int_{U_i} \tilde{g}(\mathbf{y})^{-\alpha} \, \mathrm{d}\mathbf{y}$ over some neighborhoods U_i of $\mathbf{y}^{(i)}$, $i = 1, \dots, m$, are finite.*

Proof. The only if direction is obvious, since $\int_U \tilde{g}(\mathbf{y})^{-\alpha} \, \mathrm{d}\mathbf{y} \le \int_{\mathbb{R}^{n-1}} \tilde{g}(\mathbf{y})^{-\alpha} \, \mathrm{d}\mathbf{y}$ for any open subset $U \subseteq \mathbb{R}^{n-1}$.

Let us now consider such open neighborhoods U_i of $\mathbf{y}^{(i)}$, $i = 1, \dots, m$, that satisfy $\int_{U_i} \tilde{g}(\mathbf{y})^{-\alpha} \, \mathrm{d}\mathbf{y} < \infty$ and take a ball $B_R = \left\{ \mathbf{y} \in \mathbb{R}^{n-1} : |\mathbf{y}|^2 = \sum_{i=1}^{n-1} y_i^2 < R^2 \right\}$ that contains $\mathbf{y}^{(1)}, \dots, \mathbf{y}^{(m)}$. First, we show that the integral of $\tilde{g}^{-\alpha}$ over the complement of B_R is finite for large $R > 0$. By our assumption, the degree d homogeneous part $\tilde{g}_d$ of $\tilde{g} = \sum_{j=0}^{d} \tilde{g}_j$ is a positive definite form and hence $\tilde{g}_d(\mathbf{y}) \ge \varepsilon_d |\mathbf{y}|^d$ holds for some $\varepsilon_d > 0$. For $j = 0, 1, \dots, d-1$ the degree j homogeneous part $\tilde{g}_j$ of $\tilde{g}$ satisfies $|\tilde{g}_j(\mathbf{y})| \le \varepsilon_j |\mathbf{y}|^j$ for some $\varepsilon_j > 0$.[2] We put all these inequalities together and obtain

$$\tilde{g}(\mathbf{y}) \;=\; \tilde{g}_d(\mathbf{y}) + \sum_{j=0}^{d-1} \tilde{g}_j(\mathbf{y}) \;\ge\; \tilde{g}_d(\mathbf{y}) - \sum_{j=0}^{d-1} |\tilde{g}_j(\mathbf{y})| \;\ge \varepsilon_d |\mathbf{y}|^d - \sum_{j=0}^{d-1} \varepsilon_j |\mathbf{y}|^j, \quad \mathbf{y} \in \mathbb{R}^{n-1}.$$

For sufficiently large $R > 0$ and all points $\mathbf{y} \in \mathbb{R}^{n-1}$ with $|\mathbf{y}| \ge R$ the rightmost expression in the last formula can be estimated as

$$\tilde{g}(\mathbf{y}) \;\ge\; \varepsilon_d |\mathbf{y}|^d - \sum_{j=0}^{d-1} \varepsilon_j |\mathbf{y}|^j \;\ge\; \varepsilon |\mathbf{y}|^d \tag{5.2}$$

for some $\varepsilon > 0$. Using (5.2) and writing $\mathbf{y} = r\,\mathbf{z} \in \mathbb{R}^{n-1} \setminus B_R$, $\mathbf{z} \in \mathbb{S}^{n-2}$, $r \ge R$, in spherical coordinates, we estimate the integral,

$$\int_{\mathbb{R}^{n-1} \setminus B_R} \frac{1}{\tilde{g}(\mathbf{y})^{\alpha}} \, \mathrm{d}\mathbf{y} \;\le \int_{\mathbb{R}^{n-1} \setminus B_R} \frac{1}{\varepsilon^{\alpha} |\mathbf{y}|^{d\alpha}} \, \mathrm{d}\mathbf{y} \;=\; \frac{\mathrm{vol}(\mathbb{S}^{n-2})}{\varepsilon^{\alpha}} \int_R^{\infty} \frac{1}{r^{d\alpha - n + 2}} \, \mathrm{d}r < \infty, \tag{5.3}$$

where convergence of the last integral follows from the condition $\alpha > (n-1)/d$. Since the function $\tilde{g}^{-\alpha}$ is bounded over the compact set $K = \overline{B_R \setminus (\bigcup_{i=1}^{m} U_i)}$, the integral $\int_K \tilde{g}(\mathbf{y})^{-\alpha} \, \mathrm{d}\mathbf{y}$ is finite. This, together with (5.3) and finiteness of integrals $\int_{U_i} \tilde{g}(\mathbf{y})^{-\alpha} \, \mathrm{d}\mathbf{y}$, $i = 1, \dots, m$, finally yield

$$\int_{\mathbb{R}^{n-1}} \frac{1}{\tilde{g}(\mathbf{y})^{\alpha}} \, \mathrm{d}\mathbf{y} \;\le\; \int_K \frac{1}{\tilde{g}(\mathbf{y})^{\alpha}} \, \mathrm{d}\mathbf{y} + \int_{\mathbb{R}^{n-1} \setminus B_R} \frac{1}{\tilde{g}(\mathbf{y})^{\alpha}} \, \mathrm{d}\mathbf{y} + \sum_{i=1}^{m} \int_{U_i} \frac{1}{\tilde{g}(\mathbf{y})^{\alpha}} \, \mathrm{d}\mathbf{y} \;<\; \infty.$$

□

[2] The constant ε_j, $j = 0, 1, \dots, d-1$, can be chosen to be the maximum value of the restriction of $|\tilde{g}_j|$ to the sphere $\mathbb{S}^{n-2} \subset \mathbb{R}^{n-1}$ and ε_d is a positive constant not greater than the minimum of $\tilde{g}_d$ over $\mathbb{S}^{n-2}$.

Remark 5.2. *For any $g \in \overline{\mathcal{C}_{d,n}}$ the form $g(\mathbf{y},0)$ is the degree d part of the (inhomogeneous) polynomial $\tilde{g} \in \mathbb{R}[y_1,\dots,y_{n-1}]$ defined by $\tilde{g}(\mathbf{y}) = g(\mathbf{y},1)$. Therefore, the degree d part of $\tilde{g}$ is positive definite if and only if a non-negative form $g \in \overline{\mathcal{C}_{d,n}}$ has no zeros in $\{(\mathbf{y},0) \in \mathbb{R}^n : \mathbf{y} \neq \mathbf{0}\}$.*

For a non-negative binary form $g \in \overline{\mathcal{C}_{d,2}}$, the non-negative univariate polynomial $\tilde{g} \in \mathbb{R}[y]$ has only isolated real zeros, each of which is of even order. Moreover, $\tilde{g}$ is of degree d if and only if g does not vanish at $(1,0)$. This can be achieved by composing g with a rotation of $\mathbb{R}^2$. Under such an operation the sublevel set $\mathbf{G} \subset \mathbb{R}^2$ of g gets rotated and, in particular, its Lebesgue volume remains preserved. Thus, by (5.1) and Lemma 5.1, finiteness of $f(g) = \mathrm{vol}\,(\mathbf{G})$ can be decided by looking at the integrals of $\tilde{g}^{-n/d}$ over small neighborhoods of real zeros of $\tilde{g}$.

Proof of Theorem 1.11. Let us consider a non-negative binary form $g \in \overline{\mathcal{C}_{d,2}}$. By Remark 5.2 and the above discussion, without loss of generality we can assume that $\tilde{g} \in \mathbb{R}[y]$ satisfies assumptions of Lemma 5.1.

By (5.1) the volume of $\mathbf{G} = \{g \leq 1\} \subset \mathbb{R}^2$ is proportional to the one-dimensional integral $\int_{\mathbb{R}} \tilde{g}(y)^{-2/d}\,\mathrm{d}y$, where $\tilde{g}(y) = g(y,1)$ is the dehomogenization of g. If $y' \in \mathbb{R}$ is a real zero of $\tilde{g} \in \mathbb{R}[y]$ of order k, it is possible to write $\tilde{g}(y) = (y-y')^k h(y)$, where $h \in \mathbb{R}[y]$ is a polynomial that does not vanish at y'. Performing an affine change of variables $y = z + y'$, $z \in \mathbb{R}$, we are left with the integral $\int_{\mathbb{R}} z^{-2k/d}\tilde{h}(z)^{-2/d}\,\mathrm{d}z$, where $\tilde{h}(z) = h(z+y')$. For a small $\delta > 0$, the function $\tilde{h}^{-2/d}$ is bounded (from below and from above by some positive constants) over the interval $(-\delta,\delta)$. Thus, $\int_{-\delta}^{\delta} z^{-2k/d}\tilde{h}(z)^{-2/d}\,\mathrm{d}z < \infty$ if and only if $\int_{-\delta}^{\delta} z^{-2k/d}\,\mathrm{d}z < \infty$. The latter condition is in turn equivalent to $2k/d < 1$, which means that the order k of the zero $y' \in \mathbb{R}$ is smaller than $d/2$. Since the argument applies to an arbitrary real zero of $\tilde{g}$, the main claim follows from Lemma 5.1.

For $d = 4$ the above condition on the order of real zeros reads $k < 2$. Since the order of a real zero of a non-negative binary form is always an even number, only positive definite forms g in $\overline{\mathcal{C}_{4,2}}$ have finite $f(g)$ and hence $\mathcal{V}_{4,2} = \mathcal{C}_{4,2}$. □

If $g \in \overline{\mathcal{C}_{d,n}}$ is a generic non-negative form, the polynomial $\tilde{g} \in \mathbb{R}[y_1,\dots,y_n]$ can have only isolated real zeros. In fact, a stronger result holds.

Lemma 5.3. *The restriction of a generic non-negative form $g \in \overline{\mathcal{C}_{d,n}}$ to the sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ has only finitely many zeros. In particular, $\tilde{g}$ has finitely many zeros in $\mathbb{R}^{n-1}$ and, after possibly an orthogonal change of variables, g has no zeros in $\{(\mathbf{y},0) \in \mathbb{R}^n : \mathbf{y} \neq 0\}$.*

Proof. If $g \in \overline{\mathcal{C}_{d,n}}$ is a generic form and $\mathbf{x} \in \mathbb{S}^{n-1}$ is a zero of g, then for all unit vectors $\mathbf{v} \in \mathbb{S}^{n-1}$ that are orthogonal to $\mathbf{x}$, the Hessian satisfies

$$\mathbf{v}^{\mathsf{T}}\mathrm{Hess}_{\mathbf{x}} g\,\mathbf{v} = \sum_{i,j=1}^{n} \frac{\partial^2 g}{\partial x_i \partial x_j}(\mathbf{x}) v_i v_j \geq \varepsilon$$

for some $\varepsilon > 0$. Since a real zero $\mathbf{x} \in \mathbb{S}^{n-1}$ of a non-negative form g must also be its singular point, that is, $\frac{\partial g}{\partial x_i}(\mathbf{x}) = 0$, $i = 1,\dots,n$, we have

$$g(\mathbf{x} + t\mathbf{v} + o(t)) \;=\; t^2 \sum_{i,j=1}^{n} \frac{\partial^2 g}{\partial x_i \partial x_j}(\mathbf{x})\, v_i v_j + o(t^3) \;\geq\; t^2(\varepsilon + o(t))$$

for points $\mathbf{x}+t\mathbf{v}+o(t)\in\mathbb{S}^{n-1}$ in a neighborhood of $\mathbf{x}\in\mathbb{S}^{n-1}$. In particular, for a sufficiently small $t>0$ and any $\mathbf{v}\in\mathbb{S}^{n-1}$ orthogonal to $\mathbf{x}$ we have $g(\mathbf{x}+t\mathbf{v}+o(t))>0$ and hence the zero $\mathbf{x}\in\mathbb{S}^{n-1}$ is isolated. By compactness of the sphere, g can have only finitely many isolated zeros in $\mathbb{S}^{n-1}$.

Injectivity of the mapping

$$\mathbf{y}\in\mathbb{R}^{n-1}\mapsto\frac{(\mathbf{y},1)}{\sqrt{|\mathbf{y}|^2+1}}\in\mathbb{S}^{n-1}$$

together with the homogeneity of g implies that $\tilde{g}(\mathbf{y})=g(\mathbf{y},1)=0$ holds for only finitely many $\mathbf{y}\in\mathbb{R}^{n-1}$. Finally, after possibly an orthogonal change of variables, the last coordinate of any real zero $\mathbf{x}\in\mathbb{S}^{n-1}$ of $g\in\overline{\mathcal{C}_{d,n}}$ is non-zero. □

We are now ready to prove Theorem 1.12.

Proof of Theorem 1.12. Let $g\in\overline{\mathcal{C}_{d,n}}$ be a generic non-negative form. This means in particular that for any real zero of g of the form $\mathbf{x}=(\mathbf{y},1)$ the kernel of the positive semi-definite $n\times n$ matrix $\mathrm{Hess}_{\mathbf{x}}g=\left(\frac{\partial^2 g}{\partial x_i\partial x_j}(\mathbf{x})\right)$ is one-dimensional and spanned by $\mathbf{x}$. Note that $\mathbf{y}\in\mathbb{R}^{n-1}$ is a zero of $\tilde{g}$ and the $(n-1)\times(n-1)$ Hessian matrix $\mathrm{Hess}_{\mathbf{y}}\tilde{g}=\left(\frac{\partial^2\tilde{g}}{\partial y_i\partial y_j}(\mathbf{y})\right)$ is positive definite. Indeed, because $\tilde{g}\in\mathbb{R}[y_1,\dots,y_{n-1}]$ is non-negative, the matrix $\mathrm{Hess}_{\mathbf{y}}\tilde{g}$ must be at least positive semi-definite. If $\mathbf{u}\in\mathbb{R}^{n-1}$ is a vector in the kernel of $\mathrm{Hess}_{\mathbf{y}}\tilde{g}$, then $\mathbf{w}=(\mathbf{u},0)\in\mathbb{R}^n$ must be in the kernel of $\mathrm{Hess}_{\mathbf{x}}g$, because

$$\mathbf{w}^{\mathsf{T}}\mathrm{Hess}_{\mathbf{x}}g\,\mathbf{w}\;=\;\mathbf{u}^{\mathsf{T}}\mathrm{Hess}_{\mathbf{y}}\tilde{g}\,\mathbf{u}\;=\;\mathbf{0}$$

and both Hessian matrices are positive semi-definite. But then $\mathbf{w}=(\mathbf{u},0)$ has to be proportional to $\mathbf{x}=(\mathbf{y},1)$ and hence $\mathbf{u}=\mathbf{0}$. Positive definitedness of $\mathrm{Hess}_{\mathbf{y}}\tilde{g}$ means in particular that

$$\mathbf{v}^{\mathsf{T}}\mathrm{Hess}_{\mathbf{y}}\tilde{g}\,\mathbf{v}\;=\;\sum_{i,j=1}^{n-1}\frac{\partial^2\tilde{g}}{\partial y_i\partial y_j}(\mathbf{y})v_iv_j\;\geq\varepsilon \tag{5.4}$$

for some $\varepsilon>0$ and all unit vectors $\mathbf{v}\in\mathbb{S}^{n-2}\subset\mathbb{R}^{n-1}$.

Now, by Lemma 5.3 and Remark 5.2, we can assume that the non-negative polynomial $\tilde{g}\in\mathbb{R}[y_1,\dots,y_n]$ satisfies assumptions of Lemma 5.1. Thus, it is enough to prove that the integral $\int_U\tilde{g}(\mathbf{z})^{-n/d}\,\mathrm{d}\mathbf{z}$ over some neighborhood of $\mathbf{y}\in\mathbb{R}^{n-1}$ converges. Take $U=B(\mathbf{y},\delta)\subset\mathbb{R}^{n-1}$ to be the open ball of radius $\delta>0$ centered at $\mathbf{y}$. Since the real zero $\mathbf{y}$ of the non-negative polynomial $\tilde{g}$ is also its singular point, that is, $\frac{\partial\tilde{g}}{\partial y_i}(\mathbf{y})=0$, $i=1,\dots,n-1$, the Taylor expansion of $\tilde{g}$ at the point of the ball $\mathbf{y}+t\mathbf{v}$, $\mathbf{v}\in\mathbb{S}^{n-2}$, $t\in[0,\delta)$, reads

$$\tilde{g}(\mathbf{y}+t\mathbf{v})\;=\;t^2\sum_{i,j=1}^{n-1}\frac{\partial^2\tilde{g}}{\partial y_i\partial y_j}(\mathbf{y})v_iv_j+o(t^3)\;\geq\;t^2(\varepsilon+o(t)),$$

where we invoke (5.4). For a sufficiently small $\delta>0$ there exists $\varepsilon'>0$ so that $\varepsilon+o(t)>\varepsilon'$ for any $\mathbf{y}+t\mathbf{v}\in B(\mathbf{y},\delta)$. This leads to an estimate

$$\begin{aligned}\int_{B(\mathbf{y},\delta)}\frac{1}{\tilde{g}(\mathbf{z})^{n/d}}\,\mathrm{d}\mathbf{z}\;&=\;\int_{\mathbb{S}^{n-2}}\int_0^\delta\frac{t^{n-2}}{\tilde{g}(\mathbf{y}+t\mathbf{v})^{n/d}}\,\mathrm{d}t\,\mathrm{d}\mathbb{S}^{n-2}(\mathbf{v})\\&\leq\;\frac{\mathrm{vol}\,(\mathbb{S}^{n-2})}{\varepsilon'^{\,n/d}}\int_0^\delta t^{-2n/d+n-2}\,\mathrm{d}t,\end{aligned}$$

where we performed a spherical change of variables. Finally, the last integral is finite, since $-2n/d+n-2 > -n/2+n-2 > -1$ whenever $d \geq 4$ and $n \geq 3$. $\square$

5.1. **On the denseness of generic forms in the boundary of $\mathcal{C}_{d,n}$.** We conclude this section with a proposition, that justifies the term "generic". Note that our proof of this result exploits facts from classical algebraic geometry.

Proposition 5.4. *Non-generic non-negative forms in $\partial\mathcal{C}_{d,n}$ form a semialgebraic subset of $\mathcal{H}_{d,n}$ of codimension at least 2. Moreover, this set is nowhere dense in $\partial\mathcal{C}_{d,n} \subset \mathcal{H}_{d,n}$, endowed with the topology induced from $(\mathcal{H}_{d,n}, \langle\cdot,\cdot\rangle)$.*

Proof. Throughout this proof, $\mathcal{H}^{\mathbb{C}}_{d,n} = \mathcal{H}_{d,n} \otimes_{\mathbb{R}} \mathbb{C}$ denotes the space of complex n-variate forms of degree d. Those forms that are singular at some point of *the complex projective* $(n-1)$*-space* $\mathbb{C}\mathrm{P}^{n-1}$ form an irreducible algebraic hypersurface

$$D^{\mathbb{C}}_{d,n} \;=\; \left\{ g \in \mathcal{H}^{\mathbb{C}}_{d,n} \,:\, \frac{\partial g}{\partial x_1}(\mathbf{x}) = \cdots = \frac{\partial g}{\partial x_n}(\mathbf{x}) = 0 \text{ for some } \mathbf{x} \in \mathbb{C}\mathrm{P}^{n-1} \right\},$$

that can be retrieved as the Zariski closure of the boundary $\partial\mathcal{C}_{d,n} \subset \mathcal{H}_{d,n} \subset \mathcal{H}^{\mathbb{C}}_{d,n}$ of the cone of non-negative forms, see [17, Thm. 4.1]. Let us consider the set

$$X^{\mathbb{C}}_{d,n} \;=\; \left\{ g \in \mathcal{H}^{\mathbb{C}}_{d,n} \,:\, \begin{array}{l} \dfrac{\partial g}{\partial x_1}(\mathbf{x}) = \cdots = \dfrac{\partial g}{\partial x_n}(\mathbf{x}) = 0 \text{ and} \\ \operatorname{rank}\left(\operatorname{Hess}_{\mathbf{x}}(g)\right) \leq n-2 \text{ for some } \mathbf{x} \in \mathbb{C}\mathrm{P}^{n-1} \end{array} \right\} \tag{5.5}$$

of forms with a singular point $\mathbf{x} \in \mathbb{C}\mathrm{P}^{n-1}$, at which the Hessian matrix $\operatorname{Hess}_{\mathbf{x}}(g) = \left(\frac{\partial^2 g}{\partial x_i \partial x_j}(\mathbf{x})\right)$ has corank at least 2. Note that $X^{\mathbb{C}}_{d,n}$ is obtained by projecting to the first coordinate the algebraic variety of pairs $(g,\mathbf{x}) \in \mathcal{H}^{\mathbb{C}}_{d,n} \times \mathbb{C}\mathrm{P}^{n-1}$ satisfying conditions in (5.5). Thus, the set $X^{\mathbb{C}}_{d,n} \subset \mathcal{H}^{\mathbb{C}}_{d,n}$ is algebraic by [23, Thm. 1.11].

Let $g(\mathbf{x}) = x_n^{d-2}\sum_{i=1}^{n-1} x_i^2 + \frac{2}{d}\sum_{i=1}^{n-1} x_i^d$ be a degree d form that we considered in Example 2. Apart from the unique real zero $(0:\cdots:0:1)$, the form g (in general) has other singular points in $\mathbb{C}\mathrm{P}^{n-1}$. All these points are described by the equations

$$x_1(x_1^{d-2}+x_n^{d-2}) \;=\; 0,\; \ldots,\; x_{n-1}(x_{n-1}^{d-2}+x_n^{d-2}) \;=\; 0, \quad \sum_{i=1}^{n-1} x_i^2 \;=\; 0. \tag{5.6}$$

One verifies directly that the Hessian matrix $\operatorname{Hess}_{\mathbf{x}} g$ at any solution $\mathbf{x} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ of (5.6) has corank one. This implies that $g \in D^{\mathbb{C}}_{d,n} \setminus X^{\mathbb{C}}_{d,n}$. Since the variety $X^{\mathbb{C}}_{d,n}$ is properly (as just shown) contained in the irreducible hypersurface $D^{\mathbb{C}}_{d,n} \subset \mathcal{H}^{\mathbb{C}}_{d,n}$, its codimension in $\mathcal{H}^{\mathbb{C}}_{d,n}$ must be at least 2. In particular, the real part $X_{d,n} := X^{\mathbb{C}}_{d,n} \cap \mathcal{H}_{d,n}$ is of (real) codimension at least 2 in $\mathcal{H}_{d,n}$. By definition, a non-negative form $g \in \partial\mathcal{C}_{d,n}$ is non-generic if and only if it is in $X_{d,n}$. Thus, the semialgebraic subset $X_{d,n} \cap \partial\mathcal{C}_{d,n} \subset \mathcal{H}_{d,n}$ of non-generic forms has codimension at least 2.

To prove the second claim, let $U \subset \mathcal{H}_{d,n}$ be any open subset with $U \cap \partial\mathcal{C}_{d,n} \neq \varnothing$. By [24, Rem. 2.4] and the proof of [24, Lemma 2.5], the open subset $U \cap \partial\mathcal{C}_{d,n}$ of the boundary of $\mathcal{C}_{d,n}$ has codimension 1. Since the set $U \cap (X_{d,n} \cap \partial\mathcal{C}_{d,n})$ is of codimension at least 2, it is not dense in $U \cap \partial\mathcal{C}_{d,n}$ and hence $X_{d,n} \cap \partial\mathcal{C}_{d,n}$ is nowhere dense in $\partial\mathcal{C}_{d,n}$. $\square$

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