# The Hankel-type $L_q/L_p$ Induced Norms of Positive Systems Across Switching

**Y. Ebihara** [*] **D. Peaucelle** [**] **D. Arzelier** [**]

[*] *Department of Electrical Engineering, Graduate School of Engineering, Kyoto University, Kyotodaigaku-Katsura, Nishikyo-ku, Kyoto 615-8510, Japan (e-mail: ebihara.yoshio.5r@kyoto-u.ac.jp).*
[**] *LAAS-CNRS, Univ. Toulouse, CNRS, Toulouse, France (e-mail: peaucelle,arzelier@laas.fr).*

**Abstract:** The Hankel-type $L_q/L_p$ induced norms across a single switching over two linear time-invariant (LTI) positive systems are discussed. The norms are defined as the induced norms from vector-valued $L_p$-past inputs to vector valued $L_q$-future outputs across a switching at the time instant zero. The Hankel-type $L_2/L_2$ induced norm across a single switching for general LTI systems is studied in details to evaluate the performance deterioration caused by switching. Thanks to the strong positivity property, we successfully characterize the Hankel-type $L_q/L_p$ induced norms for the positive system switching even for $p, q$ being 1, 2, $\infty$. In particular, we will show that some of them are given in the form of linear program (LP) and semidefinite program (SDP). The SDP-based characterizations are useful for the analysis of the Hankel-type $L_q/L_p$ induced norms where the systems of interest are affected by parametric uncertainties.

*Keywords:* switched positive system, Hankel-type $L_q/L_p$ induced norm, linear programming, semidefinite programming.

## 1. INTRODUCTION

As the theory of linear time-invariant (LTI) positive systems becomes mature, there has been a growing interest in the analysis and synthesis of switched positive systems. An LTI system is said to be (internally) positive if its state and output are nonnegative for any nonnegative initial state and any nonnegative input [Farina and Rinaldi [2000], Kaczorek [2001]]. The studies on switched positive systems originate from the stability and the stabilizability analysis under arbitrary switching and switching with certain dwell time over finitely many positive systems. To this date, fruitful results have been obtained for instance by Gurvits et al. [2007], Mason and Shorten [2007], Blanchini et al. [2012], Fornasini and Valcher [2012]. These results are beautifully summarized by Blanchini et al. [2015] with a plenty of stimulating practical examples. In addition, those relatively new results on the $L_1/L_1$ and $L_\infty/L_\infty$ induced norms for LTI positive systems by Rantzer [2015], Briat [2013], Shen and Lam [2014], Ebihara et al. [2017] are successfully extended to switched positive systems again by Blanchini et al. [2015]. In the switched case, however, there are inevitable difficulties in computing these induced norms exactly since they are characterized by infinite dimensional linear differential inequalities.

Even though the input-to-output properties of dynamical systems are usually investigated by *induced norms* and this direction has been naturally pursued in the studies of switched positive systems as briefly summarized above, there is another promising attempt to evaluate the performance deterioration caused by switching quantitatively. Namely, Asai [2002, 2005] considered the case where a general (not necessarily positive) LTI system switches to another LTI system at the time instant zero, and introduced the *Hankel-type $L_2/L_2$ induced norm* as the induced norm from vector-valued $L_2$-past inputs to vector valued $L_2$-future outputs. Here, the past input is injected to the system before switching, driving the initial state of the system after switching to some nonzero values along with the state transition at the time instant zero, and the future output corresponds to the initial response of the system after switching. As intuitively deduced from the standard Hankel norm results summarized in Green and Limebeer [1995], this norm is readily characterized by using the controllability gramian of the system before switching and the observability gramian of the system after switching. We could say that this norm is tailored to purely evaluate the performance deterioration caused by switching.

The objective of this paper is to derive the explicit characterizations of the Hankel-type $L_q/L_p$ induced norms of positive systems across a single switching. These norms are defined by exactly the same manner as Asai [2002, 2005], even though we evaluate the past inputs with $L_p$ norm and the future outputs with $L_q$ norm where $p, q$ being $1, 2$ or $\infty$. This study is obviously related to the analysis of the $L_q/L_p$ Hankel norms [1] in the standard non-

[1] Precisely speaking, in Wilson [1989] and Lu and Balas [1998], the authors discussed the $L_{q,s}/L_{p,r}$ Hankel and induced norms with the proper definition of the function space $L_{p,r}$. The $L_q/L_p$ Hankel (induced) norm in the current paper corresponds to the special case of the $L_{q,q}/L_{p,p}$ Hankel (induced) norm studied in Wilson [1989] and Lu and Balas [1998].

switching setting dealt with by Wilson [1989] and Lu and Balas [1998]. In particular, it is worth mentioning that Lu and Balas [1998] provided closed-form formulas of the $L_q/L_p$ Hankel norms for the case where $p$, $q$ are 1, 2, or $\infty$. However, there are still unavoidable difficulties in computing the $L_q/L_p$ Hankel norms of general LTI systems if we follow Lu and Balas [1998]. The difficulties stem from the facts that (a) for the computation of the $L_1/L_1$ and $L_\infty/L_\infty$ Hankel norms, we unavoidably need to compute the absolute integral of impulse responses; (b) for the computation of the "norm-induced initial conditions" that are necessary in dealing with the $L_1/L_2$, $L_1/L_\infty$, and $L_2/L_\infty$ Hankel norms, we need to deal with implicit functions. It has been shown recently by Ebihara et al. [2019] that we can circumvent these difficulties when dealing with *externally* positive systems. It is definitely obvious that we can carry out the absolute integral of impulse responses without any ado in the case of externally positive systems. Beyond that, the main contribution of Ebihara et al. [2019] lies in that (a) they showed that the difficulties that stem from the "norm-induced initial conditions" can also be circumvented by the positivity property, (b) for the $L_1/L_p$ and $L_q/L_\infty$ Hankel norms with $p, q$ being $1, 2, \infty$, they provided explicit characterizations in the form of linear program (LP) and semidefinite program (SDP).

The results in this paper can be regarded as generalization of those in Ebihara et al. [2019]. However, such generalization cannot be performed directly since the time-varying nature caused by switching makes the analysis more involved. It is also true that we have to confine ourselves to the case where the systems before and after switching are both *internally* positive to handle their state transition across switching. Nevertheless, we eventually clarify that we can explicitly characterize the Hankel-type $L_q/L_p$ induced norms even for $p, q$ being 1, 2, $\infty$ partially again thanks to the strong positivity property. In particular, we will show that some of them are given in the form of linear program (LP) and semidefinite program (SDP), the latter of which enables us to analyze the Hankel-type $L_q/L_p$ induced norms in the case where the systems of interest are affected by parametric uncertainties.

We use the following notation. The set of $n \times m$ real matrices is denoted by $\mathbb{R}^{n\times m}$, and the set of $n \times m$ entrywise nonnegative (strictly positive) matrices is denoted by $\mathbb{R}_+^{n\times m}$ ($\mathbb{R}_{++}^{n\times m}$). For a matrix $A$, we also write $A \geq 0$ ($A > 0$) to denote that $A$ is entrywise nonnegative (strictly positive). We denote by $\mathbf{1}^n \in \mathbb{R}^n$ the all-ones vector. The set of $n\times n$ Hurwitz matrices is denoted by $\mathbb{H}^n$, and the set of $n\times n$ Metzler matrices (real square matrices whose off-diagonal entries are nonnegative) is denoted by $\mathbb{M}^n$. The set of $n\times n$ real symmetric matrices is denoted by $\mathbb{S}^n$. For a matrix $A \in \mathbb{S}^n$, we write $A \succ 0$ ($A \prec 0$) to denote that $A$ is positive (negative) definite. For a matrix $A \in \mathbb{S}^n$, we also denote by $\lambda_{\max}(A)$ and $d_{\max}(A)$ the maximum eigenvalue and the maximum diagonal entry of $A$, respectively. Finally, for $A \in \mathbb{R}^{n\times n}$, we define $\mathrm{He}\{A\} = A + A^T$.

## 2. DEFINITION OF THE HANKEL-TYPE $L_q/L_p$ INDUCED NORMS

Suppose two stable LTI systems $\Sigma_\mathrm{p}$ and $\Sigma_\mathrm{f}$ are given, which are the models of the system before and after switching at the time $t = 0$, respectively (see Fig. 1). We assume that the state space realizations of $\Sigma_\mathrm{p}$ and $\Sigma_\mathrm{f}$ are given respectively by

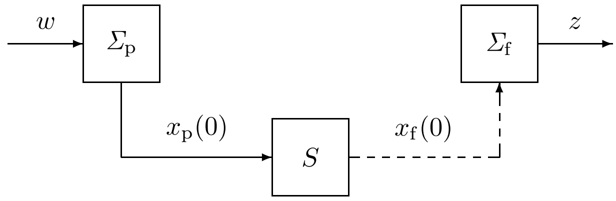


Fig. 1. Switching from $\Sigma_\mathrm{p}$ to $\Sigma_\mathrm{f}$ along with state transition.

$$\Sigma_\mathrm{p}: \ \dot{x}_\mathrm{p}(t) = A_\mathrm{p}x_\mathrm{p}(t) + B_\mathrm{p}w(t) \quad (t \leq 0) \tag{1}$$

and

$$\Sigma_\mathrm{f}: \begin{cases} \dot{x}_\mathrm{f}(t) = A_\mathrm{f}x_\mathrm{f}(t), \\ z(t) = C_\mathrm{f}x_\mathrm{f}(t) \end{cases} \quad (t \geq 0). \tag{2}$$

Here $A_\mathrm{p} \in \mathbb{H}^{n_\mathrm{p}}$, $B_\mathrm{p} \in \mathbb{R}^{n_\mathrm{p}\times n_w}$, $A_\mathrm{f} \in \mathbb{H}^{n_\mathrm{f}}$ and $C_\mathrm{f} \in \mathbb{R}^{n_z\times n_\mathrm{f}}$. We consider the case where the system $\Sigma_\mathrm{p}$ switches to the system $\Sigma_\mathrm{f}$ at $t = 0$ along with the state transition described by

$$x_\mathrm{f}(0) = Sx_\mathrm{p}(0). \tag{3}$$

Here, $S \in \mathbb{R}^{n_\mathrm{f}\times n_\mathrm{p}}$ is a given matrix.

For the input signal $w$ and the output signal $z$, we define

$$\|w\|_{1-} := \int_{-\infty}^{0} |w(t)|_1 dt, \quad \|z\|_{1+} := \int_{0}^{\infty} |z(t)|_1 dt,$$
$$\|w\|_{2-} := \sqrt{\int_{-\infty}^{0} |w(t)|_2^2 dt}, \ \|z\|_{2+} := \sqrt{\int_{0}^{\infty} |z(t)|_2^2 dt},$$
$$\|w\|_{\infty-} := \operatorname*{ess\,sup}_{-\infty<t\leq 0} |w(t)|_\infty, \ \|z\|_{\infty+} := \operatorname*{ess\,sup}_{0\leq t<\infty} |z(t)|_\infty$$

where for $v \in \mathbb{R}^{n_v}$ we define

$$|v|_1 := \sum_{j=1}^{n_v} |v_j|, \quad |v|_2 := \sqrt{\sum_{j=1}^{n_v} v_j^2}, \quad |v|_\infty := \max_{1\leq j\leq n_v} |v_j|.$$

For $p, q = 1, 2, \infty$ we also define

$$\begin{aligned} L_{p-} &:= \{w : \ \|w\|_{p-} < \infty\}, \\ L_{p-}^+ &:= \{w : \ w \in L_{p-}, \ w(t) \geq 0 \ (\forall t \leq 0)\}, \\ L_{q+} &:= \{z : \ \|z\|_{q+} < \infty\}, \\ L_{q+}^+ &:= \{z : \ z \in L_{q+}, \ z(t) \geq 0 \ (\forall t \geq 0)\}. \end{aligned}$$

Then, the Hankel-type $L_q/L_p$ induced norm across switching from $\Sigma_\mathrm{p}$ to $\Sigma_\mathrm{f}$ with the state transition matrix $S \in \mathbb{R}^{n_\mathrm{f}\times n_\mathrm{p}}$ is defined by

$$\gamma_{q/p} := \sup_{w\in L_{p-}, \ \|w\|_{p-}=1} \|z\|_{q+} \ \text{ s.t. } (1), (2), (3). \tag{4}$$

Note that $x_\mathrm{p}(-\infty) = 0$ is tacitly assumed. In the following, we partition $B_\mathrm{p} \in \mathbb{R}^{n_\mathrm{p}\times n_w}$ and $C_\mathrm{f} \in \mathbb{R}^{n_z\times n_\mathrm{f}}$ as follows:

$$B_\mathrm{p} = [\, B_{\mathrm{p},1} \ \cdots \ B_{\mathrm{p},n_w} \,] \ (B_{\mathrm{p},j} \in \mathbb{R}^{n_\mathrm{p}\times 1}, \ j = 1, \ldots, n_w),$$
$$C_\mathrm{f}^T = [\, C_{\mathrm{f},1}^T \ \cdots \ C_{\mathrm{f},n_z}^T \,] \ (C_{\mathrm{f},i} \in \mathbb{R}^{1\times n_\mathrm{f}}, \ i = 1, \ldots, n_z).$$

## 3. $Lq/Lp$ HANKEL NORMS OF POSITIVE SYSTEMS

In this section, the definition of positive systems and the condition for LTI systems to be positive are reviewed, followed by the recent results on the $L_q/L_p$ *Hankel norms* of (non-switched) positive systems by Ebihara et al. [2019].

### 3.1 Positive Systems

Let us consider the stable LTI system $G$ described by

$$G: \left\{ \begin{array}{l} \dot{x}(t) = Ax(t) + Bw(t), \\ z(t) = Cx(t) \end{array} \right. \tag{5}$$

where $A \in \mathbb{H}^n$, $B \in \mathbb{R}^{n\times n_w}$, $C \in \mathbb{R}^{n_z \times n}$. The impulse response $g$ of the system $G$ is given by

$$g(t) = \left\{ \begin{array}{cl} 0 & (t<0), \\ C\exp(At)B & (t \geq 0). \end{array} \right. \tag{6}$$

The definition of the positivity of $G$ and its characterization are given as follows.

*Definition 1.* (Farina and Rinaldi [2000]). The LTI system $G$ given by (5) is called *internally* positive if its state $x(t)$ and output $z(t)$ are nonnegative for $t \geq 0$ for any nonnegative input $w(t)$ for $t \geq 0$ and nonnegative initial state $x(0)$.

*Proposition 2.* (Farina and Rinaldi [2000]). The system $G$ given by (5) is internally positive if and only if

$$A \in \mathbb{M}^n,\ B \in \mathbb{R}_+^{n\times n_w},\ C \in \mathbb{R}_+^{n_z\times n}. \tag{7}$$

*Definition 3.* (Farina and Rinaldi [2000]). The LTI system $G$ given by (5) is called *externally* positive if its output $z(t)$ is nonnegative for $t \geq 0$ for any nonnegative input $w(t)$ for $t \geq 0$ and the zero initial state $x(0) = 0$.

*Proposition 4.* (Farina and Rinaldi [2000]). The system $G$ given by (5) is externally positive if and only if its impulse response given by (6) is nonnegative, i.e., $g(t) \geq 0$ $(\forall t \geq 0)$.

In the following, we simply use the shortcut "positive" system to denote an "internally positive" system.

### 3.2 $Lq/Lp$ Hankel Norms of Positive Systems

In the case where

$$\begin{array}{l} A_{\rm p} = A_{\rm f} = A \in \mathbb{R}^{n\times n},\ B_{\rm p} = B \in \mathbb{R}^{n\times n_w}, \\ C_{\rm f} = C \in \mathbb{R}^{n_z\times n},\ S = I_n, \end{array} \tag{8}$$

we can see that the Hankel-type $L_q/L_p$ induced norm $\gamma_{q/p}$ defined by (4) reduces to the standard $L_q/L_p$ Hankel norm of the system $G$ which is denoted by $\|G\|_{q/p}$. The $L_q/L_p$ Hankel norms of general (i.e., nonpositive) LTI systems are studied by Wilson [1989] and Lu and Balas [1998], and recently those results are refined for positive systems by Ebihara et al. [2019]. We summarize the results for positive systems in the next proposition, where $X \in \mathbb{S}^n$ and $P \in \mathbb{S}^n$ stand for the controllability and the observability gramians of the system $G$ given by (5), respectively. These are the unique solutions of the Lyapunov equations

$$AX + XA^T + BB^T = 0, \quad PA + A^TP + C^TC = 0. \tag{9}$$

*Proposition 5.* Let us consider the stable and positive LTI system $G$ given by (5) and (7). Then, we have

$$\|G\|_{1/1} = |-\mathbf{1}_{n_z}^T CA^{-1}B|_\infty. \tag{10}$$

$$\|G\|_{2/1} = \sqrt{d_{\max}(B^TPB)}. \tag{11}$$

$$\|G\|_{\infty/1} = \max_{t\geq 0}\max_{i,j} |g_{i,j}(t)|. \tag{12}$$

$$\|G\|_{1/2} = \sqrt{\mathbf{1}_{n_z}^T CA^{-1}XA^{-T}C^T\mathbf{1}_{n_z}} \tag{13}$$

$$\|G\|_{2/2} = \sqrt{\lambda_{\max}(XP)}. \tag{14}$$

$$\|G\|_{\infty/2} = \sqrt{d_{\max}(CXC^T)}. \tag{15}$$

$$\|G\|_{1/\infty} = \mathbf{1}_{n_z}^T CA^{-2}B\mathbf{1}_{n_w}. \tag{16}$$

$$\|G\|_{2/\infty} = \sqrt{\mathbf{1}_{n_w}^T B^TA^{-T}PA^{-1}B\mathbf{1}_{n_w}}. \tag{17}$$

$$\|G\|_{\infty/\infty} = |-CA^{-1}B\mathbf{1}_{n_w}|_\infty. \tag{18}$$

□

The well known characterizations (12) and (14) as well as (11) and (15) shown in Wilson [1989] are valid even for general LTI systems. The characterizations (10) and (18) are shown by Ebihara et al. [2019] on the basis of fundamental results shown in Desoer and Vidyasagar [1975]. These are valid even for *externally* positive systems. The rest characterizations, (13), (16) and (17), are shown by Ebihara et al. [2019] where the treatment of implicit functions needed for the computation of "norm-induced initial conditions" in Lu and Balas [1998] for general system case is successfully circumvented by strong positivity property. Again, (13), (16) and (17) are valid even for *externally* positive systems. Finally, from Lu and Balas [1998], we see that the $L_q/L_p$ Hankel norm is identical to the corresponding $L_q/L_p$ *induced* norm for the cases where $(q,p) \in \{(1,1),(2,1),(\infty,1),(\infty,2),(\infty,\infty)\}$.

## 4. THE HANKEL-TYPE $Lq/Lp$ INDUCED NORMS

### 4.1 Preliminary Results

In considering the Hankel-type $L_q/L_p$ induced norm $\gamma_{q/p}$ for the positive system switching, the underlying assumptions will be used:

(i) Both systems $\Sigma_{\rm p}$ and $\Sigma_{\rm f}$ are stable and positive, i.e.,

$$\begin{array}{l} A_{\rm p} \in \mathbb{H}^{n_{\rm p}} \cap \mathbb{M}^{n_{\rm p}},\ B_{\rm p} \in \mathbb{R}_+^{n_{\rm p}\times n_w}, \\ A_{\rm f} \in \mathbb{H}^{n_{\rm f}} \cap \mathbb{M}^{n_{\rm f}},\ C_{\rm f} \in \mathbb{R}_+^{n_z\times n_{\rm f}}. \end{array} \tag{19}$$

(ii) The matrix $S$ in (3) is nonnegative, i.e.,

$$S \in \mathbb{R}_+^{n_{\rm f}\times n_{\rm p}}. \tag{20}$$

The assumption (20) implies that the positivity of the "future" state $x_{\rm f}$ is inherited from the one of the "past" state $x_{\rm p}$. The next lemma plays a key role in analyzing the Hankel-type $L_q/L_p$ induced norm $\gamma_{q/p}$ for the positive system switching.

*Lemma 6.* For the positive system switching from $\Sigma_{\rm p}$ to $\Sigma_{\rm f}$ described by (1), (2), (3), (19), and (20), suppose an input $w \in L_{p-}$ yields an output $z \in L_{q+}$ where $p, q$ being $1, 2, \infty$. Define the input associated to $w \in L_{p-}$ by $\widehat{w} \in L_{p-}^+$ such that

$$\widehat{w}_j(t) := |w_j(t)|\ (t \leq 0,\ j = 1,\ldots,n_w).$$

Then, the output $\widehat{z} \in L_{q+}$ corresponding to the input $\widehat{w} \in L_{p-}^+$ satisfies

$$\widehat{z}_i(t) \geq |z_i(t)|\ (\forall t \geq 0,\ i = 1,\ldots,n_z).$$

□

*Proof of Lemma 6:* We denote by $\Sigma$ the linear operator from $w \in L_{p-}$ to $z \in L_{q+}$. Namely,

$$\Sigma:\ L_{p-} \ni w \mapsto z \in L_{q+},$$

$$\begin{array}{rl} (\Sigma w)(t) & = C_{\rm f}\exp(A_{\rm f}t)S\displaystyle\int_{-\infty}^0 \exp(A_{\rm p}(-\tau))B_{\rm p}w(\tau)d\tau \\ & = z(t)\ (t \geq 0). \end{array} \tag{21}$$

For the input $w \in L_{p-}$, let us define $w_+ \in L_{p-}^+$ and $w_- \in L_{p-}^+$ as

$$w_{+,j}(t)=\begin{cases} w_j(t) & (w_j(t)\ge 0)\\ 0 & (w_j(t)<0)\end{cases},\ w_{-,j}(t)=\begin{cases} 0 & (w_j(t)\ge 0)\\ -w_j(t) & (w_j(t)<0)\end{cases}.$$

Then we have $w = w_+ - w_-$ and $\widehat{w} = w_+ + w_-$. From the linearity of $\Sigma$ and the triangular inequality, we have

$$\begin{aligned}|z_i(t)| &= |(\Sigma w)(t)_i|\\ &= |(\Sigma(w_+ - w_-))(t)_i|\\ &= |(\Sigma w_+)(t)_i - (\Sigma w_-)(t)_i|\\ &\le |(\Sigma w_+)(t)_i| + |(\Sigma w_-)(t)_i| = (\Sigma w_+)(t)_i + (\Sigma w_-)(t)_i\\ &= (\Sigma(w_+ + w_-))(t)_i\\ &= (\Sigma\widehat{w})(t)_i\\ &= \widehat{z}_i(t).\end{aligned}$$

Here in the fourth equality we used the fact that $(\Sigma w_\pm)(t) \ge 0$ $(\forall t \ge 0)$ holds for $w_\pm \in L_{p-}^+$, which is obvious from (19), (20), and (21). This completes the proof. ■

The next result follows from Lemma 6.

*Lemma 7.* For the positive system switching from $\Sigma_{\rm p}$ to $\Sigma_{\rm f}$ described by (1), (2), (3), (19), and (20), suppose there exists an input $w \in L_{p-}$ with $\|w\|_{p-} = 1$ such that the corresponding output $z \in L_{q+}$ satisfies $\|z\|_{q+} = \gamma$ for a given $\gamma > 0$. Then, there exists an input $\widehat{w} \in L_{p-}^+$ with $\|\widehat{w}\|_{p-} = 1$ such that the corresponding output $\widehat{z} \in L_{q+}^+$ satisfies $\|\widehat{z}\|_{q+} \ge \gamma$. □

We also note that the next fundamental result holds.

*Lemma 8.* For the positive system switching from $\Sigma_{\rm p}$ to $\Sigma_{\rm f}$ described by (1), (2), (3), (19), and (20), suppose inputs $w_1, w_2 \in L_{p-}^+$ yield outputs $z_1, z_2 \in L_{q+}^+$, respectively. Then, if $w_1(t) \ge w_2(t)$ $(\forall t \le 0)$, we have $\|z_1\|_{q+} \ge \|z_2\|_{q+}$. □

Lemma 8 follows from the linearity and positivity of the operator (21). The results in Lemmas 6-8 play important roles in characterizing the Hankel-type $L_q/L_p$ induced norms across a single switching from the positive system $\Sigma_{\rm p}$ to the positive system $\Sigma_{\rm f}$ in the next subsections. In the following, we denote by $X_{\rm p} \in \mathbb{S}^{n_{\rm p}}$ and $P_{\rm f} \in \mathbb{S}^{n_{\rm f}}$ the controllability gramian of $\Sigma_{\rm p}$ and the observability gramian of $\Sigma_{\rm f}$, respectively. These are the unique solutions of the Lyapunov equations

$$A_{\rm p}X_{\rm p} + X_{\rm p}A_{\rm p}^T + B_{\rm p}B_{\rm p}^T = 0,\ P_{\rm f}A_{\rm f} + A_{\rm f}^TP_{\rm f} + C_{\rm f}^TC_{\rm f} = 0.$$

### *4.2 The Hankel-type induced norms $\gamma_{q/p}$ with $p = \infty$*

In the case where we consider the Hankel-type induced norm $\gamma_{q/p}$ with $p = \infty$, we can readily see from Lemmas 6-8 that the next strong result holds.

*Lemma 9.* For the positive system switching from $\Sigma_{\rm p}$ to $\Sigma_{\rm f}$ described by (1), (2), (3), (19), and (20), the Hankel-type induced norms $\gamma_{q/\infty}$ with $q$ being $1, 2, \infty$ are attained by the input $w^\star \in L_{\infty-}^+$ given by $w^\star(t) = \mathbf{1}_{n_w}$ $(\forall t \le 0)$. This input leads to the initial condition before switching

$$x_{\rm p}(0) = -A_{\rm p}^{-1}B_{\rm p}\mathbf{1}_{n_w} \in \mathbb{R}_+^{n_{\rm p}} \tag{22}$$

and the initial condition after switching

$$x_{\rm f}(0) = -SA_{\rm p}^{-1}B_{\rm p}\mathbf{1}_{n_w} \in \mathbb{R}_+^{n_{\rm f}}. \tag{23}$$

□

From this lemma we can obtain the next theorem. The proof of this theorem is given in the appendix section.

*Theorem 10.* For the positive system switching from $\Sigma_{\rm p}$ to $\Sigma_{\rm f}$ described by (1), (2), (3), (19), and (20), we have

$$\gamma_{1/\infty} = \mathbf{1}_{n_z}^T C_{\rm f}A_{\rm f}^{-1}SA_{\rm p}^{-1}B_{\rm p}\mathbf{1}_{n_w}, \tag{24}$$

$$\gamma_{2/\infty} = \sqrt{\mathbf{1}_{n_w}^T B_{\rm p}^T A_{\rm p}^{-T}S^TP_{\rm f}SA_{\rm p}^{-1}B_{\rm p}\mathbf{1}_{n_w}}, \tag{25}$$

$$\gamma_{\infty/\infty} = \max_{t_{\rm f}\ge 0} |-C_{\rm f}\exp(A_{\rm f}t_{\rm f})SA_{\rm p}^{-1}B_{\rm p}\mathbf{1}_{n_w}|_\infty. \tag{26}$$

Moreover, the following conditions (a-i)-(a-v) are equivalent for a given $\gamma > 0$, and similarly for the conditions (b-i) and (b-ii).

(a-i) $\gamma_{1/\infty} < \gamma$.

(a-ii) There exists $F \in \mathbb{R}^{(n_{\rm p}+n_{\rm f})\times(n_{\rm p}+n_{\rm f}+1)}$ such that

$$\begin{bmatrix} -2\gamma & 0 & \mathbf{1}_{n_w}^T B_{\rm p}^T\\ 0 & 0_{n_{\rm f}} & 0\\ B_{\rm p}\mathbf{1}_{n_w} & 0 & 0_{n_{\rm p}}\end{bmatrix} + {\rm He}\left\{\begin{bmatrix} \mathbf{1}_{n_z}^T C_{\rm f} & 0\\ A_{\rm f} & S\\ 0 & A_{\rm p}\end{bmatrix} F\right\} \prec 0. \tag{27}$$

(a-iii) There exists $H \in \mathbb{R}^{(n_{\rm p}+n_{\rm f})\times(n_{\rm p}+n_{\rm f}+1)}$ such that

$$\begin{bmatrix} -2\gamma & 0 & \mathbf{1}_{n_z}^T C_{\rm f}\\ 0 & 0_{n_{\rm p}} & 0\\ C_{\rm f}^T\mathbf{1}_{n_z} & 0 & 0_{n_{\rm f}}\end{bmatrix} + {\rm He}\left\{\begin{bmatrix} \mathbf{1}_{n_w}^T B_{\rm p}^T & 0\\ A_{\rm p}^T & S^T\\ 0 & A_{\rm f}^T\end{bmatrix} H\right\} \prec 0. \tag{28}$$

(a-iv) There exist $f_{\rm p} \in \mathbb{R}_{++}^{n_{\rm p}}$, $f_0 \in \mathbb{R}_{++}^{n_{\rm f}}$, and $f_{\rm f} \in \mathbb{R}_{++}^{n_{\rm f}}$ such that

$$\begin{aligned} &A_{\rm p}f_{\rm p} + B_{\rm p}\mathbf{1}_{n_w} < 0, \quad Sf_{\rm p} < f_0,\\ &A_{\rm f}f_{\rm f} + f_0 < 0, \quad \mathbf{1}_{n_z}^T C_{\rm f}f_{\rm f} < \gamma.\end{aligned} \tag{29}$$

(a-v) There exist $h_{\rm f} \in \mathbb{R}_{++}^{n_{\rm f}}$, $h_0 \in \mathbb{R}_{++}^{n_{\rm p}}$, and $h_{\rm p} \in \mathbb{R}_{++}^{n_{\rm p}}$ such that

$$\begin{aligned} &h_{\rm f}^TA_{\rm f} + \mathbf{1}_{n_z}^T C_{\rm f} < 0, \quad h_{\rm f}^TS < h_0^T,\\ &h_{\rm p}^TA_{\rm p} + h_0^T < 0, \quad h_{\rm p}^TB_{\rm p}\mathbf{1}_{n_w} < \gamma.\end{aligned} \tag{30}$$

(b-i) $\gamma_{2/\infty} < \gamma$.

(b-ii) There exist $Q_{\rm f} \in \mathbb{S}_{++}^{n_{\rm f}}$, $F_1 \in \mathbb{R}^{(n_{\rm p}+n_{\rm f})\times(n_{\rm p}+n_{\rm f}+1)}$ and $F_2 \in \mathbb{R}^{n_{\rm f}\times 2n_{\rm f}}$ such that

$$\begin{aligned} &\begin{bmatrix} -\gamma^2 & 0 & \\ 0 & 0_{n_{\rm p}} & 0\\ 0 & 0 & Q_{\rm f}\end{bmatrix} + {\rm He}\left\{\begin{bmatrix} \mathbf{1}_{n_w}^T B_{\rm p}^T & 0\\ A_{\rm p}^T & S^T\\ 0 & I_{n_{\rm f}}\end{bmatrix} F_1\right\} \prec 0,\\ &\begin{bmatrix} C_{\rm f}^TC_{\rm f} & Q_{\rm f}\\ Q_{\rm f} & 0\end{bmatrix} + {\rm He}\left\{\begin{bmatrix} A_{\rm f}^T\\ -I_{n_{\rm f}}\end{bmatrix}\right\} F_2 \prec 0.\end{aligned} \tag{31}$$

□

Important remarks on Theorem 10 are as follows.

*Remark 11.* (i) It is clear that (24) and (25) reduce to (16) and (17), respectively, in the case of (8). On the other hand, (26) looks much more complicated than (18), and we see that (18) can be obtained by assuming (8) and the maximum in (26) is attained at $t_{\rm f} = 0$. In the time-invariant case (8), it is allowed to consider the "shift" of input signal $w$ due to the time-invariant nature of the system and this intuitively explains the reason why the maximum is attained at $t_{\rm f} = 0$. More rigorously, in the case where (8) holds in (26), we see that

$$\frac{d}{dt_{\rm f}}(-C\exp(At_{\rm f})A^{-1}B\mathbf{1}_{n_w}) = -C\exp(At_{\rm f})B\mathbf{1}_{n_w} \le 0$$

and hence the maximum is actually attained at $t_{\rm f} = 0$. However, in the switching case, the intrinsic time-varying nature of the system does not allow us to conclude in such a way and we have to take the maximum over $t_{\rm f} \geq 0$ as in (26).

(ii) The LMI-based characterizations (27), (28), and (31) are useful in analyzing the Hankel-type induced norm $\gamma_{q/p}$ where the positive systems $\Sigma_{\rm p}$ and $\Sigma_{\rm f}$ as well as the matrix $S$ are affected by parametric uncertainties. See Section 5 for concrete examples.

*4.3 The Hankel-type induced norms $\gamma_{q/p}$ with $q=1$*

When considering the Hankel-type induced norm $\gamma_{q/p}$ for positive system switching, we can confine ourselves to nonnegative input signals from Lemma 7. This leads to $x_{\rm p}(0) \in \mathbb{R}_+^{n_{\rm p}}$, $x_{\rm f}(0) = Sx_{\rm p}(0) \in \mathbb{R}_+^{n_{\rm f}}$, and hence $z(t) = C_{\rm f}\exp(A_{\rm f}t)Sx_{\rm p}(0) \in \mathbb{R}_+^{n_z}$ ($\forall t \geq 0$) holds. It follows that $\|z\|_{1+} = -\mathbf{1}_{n_z}^T C_{\rm f} A_{\rm f}^{-1} S x_{\rm p}(0)$. Namely, we can characterize $\gamma_{1/p}$ as follows:

$$\gamma_{1/p} = \sup_{w \in L_{p-}^+,\ \|w\|_{p-}=1} -\mathbf{1}_{n_z}^T C_{\rm f} A_{\rm f}^{-1} S \int_{-\infty}^{0} \exp(-A_{\rm p}\tau) B_{\rm p} w(\tau) d\tau.$$

From this expression, we can see that $\gamma_{1/p}$ is identical to the $L_\infty/L_p$ Hankel norm $\|\widehat{G}\|_{\infty/p}$ of the single-output, stable and positive LTI system $\widehat{G}$ given by

$$\widehat{G}(s) := \left[\begin{array}{c|c} A_{\rm p} & B_{\rm p} \\ \hline -\mathbf{1}_{n_z}^T C_{\rm f} A_{\rm f}^{-1} S & 0 \end{array}\right].$$

From this key observation and Proposition 5, we can obtain the next theorem.

*Theorem 12.* For the positive system switching from $\Sigma_{\rm p}$ to $\Sigma_{\rm f}$ described by (1), (2), (3), (19), and (20), we have

$$\gamma_{1/1} = \max_{t_{\rm p} \geq 0} |-\mathbf{1}_{n_z}^T C_{\rm f} A_{\rm f}^{-1} S \exp(A_{\rm p} t_{\rm p}) B_{\rm p}|_\infty, \quad (32)$$

$$\gamma_{1/2} = \sqrt{\mathbf{1}_{n_z}^T C_{\rm f} A_{\rm f}^{-1} S X_{\rm p} S^T A_{\rm f}^{-T} C_{\rm f}^T \mathbf{1}_{n_z}}. \quad (33)$$

Moreover, the following conditions (c-i) and (c-ii) are equivalent for a given $\gamma > 0$.

(c-i) $\gamma_{1/2} < \gamma$.

(c-ii) There exist $Y_{\rm p} \in \mathbb{S}_{++}^{n_{\rm p}}$, $F_1 \in \mathbb{R}^{(n_{\rm f}+n_{\rm p})\times(n_{\rm f}+n_{\rm p}+1)}$ and $F_2 \in \mathbb{R}^{n_{\rm p}\times 2n_{\rm p}}$ such that

$$\begin{bmatrix} -\gamma^2 & 0 & 0 \\ 0 & 0_{n_{\rm p}} & 0 \\ 0 & 0 & Y_{\rm p} \end{bmatrix} + {\rm He}\left\{ \begin{bmatrix} \mathbf{1}_{n_z}^T C_{\rm f} & 0 \\ A_{\rm f} & S \\ 0 & I_{n_{\rm p}} \end{bmatrix} F_1 \right\} \prec 0, \quad (34)$$

$$\begin{bmatrix} B_{\rm p}B_{\rm p}^T & Y_{\rm p} \\ Y_{\rm p} & 0 \end{bmatrix} + {\rm He}\left\{ \begin{bmatrix} A_{\rm p} \\ -I_{n_{\rm p}} \end{bmatrix} \right\} F_2 \prec 0.$$

□

*Proof of Theorem 12:* It is clear from $\gamma_{1/p} = \|\widehat{G}\|_{\infty/p}$ and (12), (15) that (32) and (33) hold. The equivalence of (c-i) and (c-ii) follows by a similar argument used in the proof of the equivalence of (b-i) and (b-ii) in Theorem 10. ■

It should be noted that the expression of $\gamma_{1/\infty}$ given by (24) can also be obtained from the fact that $\gamma_{1/\infty} = \|\widehat{G}\|_{\infty/\infty}$ and (18). Important remarks on Theorem 12 are as follows.

*Remark 13.* (i) We can see that (33) reduces to (13) in the case of (8). On the other hand, we see that (32) can be reduced to (10) by assuming (8) and the maximum in (32) is attained at $t_{\rm p}=0$. In fact, in the case where (8) holds in (32), we see that

$$\frac{d}{dt_{\rm p}}(-\mathbf{1}_{n_z}^T C A^{-1} \exp(At_{\rm p})B) = -\mathbf{1}_{n_z}^T C \exp(At_{\rm p}) B \leq 0$$

and hence the maximum is actually attained at $t_{\rm p}=0$.

(ii) The worst case input $w^\star \in L_{2-}$ that attains (33) can be given explicitly by

$$w^\star(t) = \frac{B_{\rm p}^T \exp(-A_{\rm p}^T t) S^T A_{\rm f}^{-T} C_{\rm f}^T \mathbf{1}_{n_z}}{\sqrt{\mathbf{1}_{n_z}^T C_{\rm f} A_{\rm f}^{-1} S X_{\rm p} S^T A_{\rm f}^{-T} C_{\rm f}^T \mathbf{1}_{n_z}}} \ (t \leq 0).$$

On the other hand, regarding (32), let us define $t_{\rm p}^\star$ and $j^\star$ such that $\gamma_{1/1} = -\mathbf{1}_{n_z}^T C_{\rm f} A_{\rm f}^{-1} S \exp(A_{\rm p} t_{\rm p}^\star) B_{{\rm p},j^\star}$. We further define $w_\alpha(t) \in L_{1-}^+$ ($\alpha = 1,2,\ldots$) by $w_\alpha(t) = v_\alpha(t) e_{j^\star}$ where $e_j \in \mathbb{R}^{n_w}$ is the $j$-th vector in the standard basis in $\mathbb{R}^{n_w}$ and $\{v_\alpha\}$ is a sequence of functions in $L_{1-}^+$ with unit norm $\|v_\alpha\|_{1-} = 1$, which tends to the delta function $\delta(t+t_{\rm p}^\star)$ as $\alpha \to \infty$. Then, we see that the output $z_\alpha \in L_{1+}$ corresponding to the input $w_\alpha \in L_{1-}^+$ satisfies

$$\|z_\alpha\|_{1+} = \|\Sigma(w_\alpha)\|_{1+} \to \gamma_{1/1}\ (\alpha \to \infty).$$

(iii) Obviously, the duality holds between $\gamma_{2/\infty}$ given by (25) and $\gamma_{1/2}$ given by (33). Namely, we see that the Hankel-type $L_2/L_\infty$ induced norm on the positive system switching from $\Sigma_{\rm p}$ to $\Sigma_{\rm f}$ via $S \in \mathbb{R}_+^{n_{\rm f}\times n_{\rm p}}$ is equivalent to the Hankel-type $L_1/L_2$ induced norm on the positive system switching from $\widetilde{\Sigma}_{\rm f}$ to $\widetilde{\Sigma}_{\rm p}$ via $S^T \in \mathbb{R}_+^{n_{\rm p}\times n_{\rm f}}$ where

$$\widetilde{\Sigma}_{\rm f}:\ \dot{\xi}_{\rm p}(t) = A_{\rm f}^T \xi_{\rm p}(t) + C_{\rm f}^T \widetilde{w}(t) \quad (t \leq 0) \quad (35)$$

and

$$\widetilde{\Sigma}_{\rm p}:\ \left\{ \begin{array}{rcl} \dot{\xi}_{\rm f}(t) &=& A_{\rm p}^T \xi_{\rm f}(t), \\ \widetilde{z}(t) &=& B_{\rm p}^T \xi_{\rm f}(t) \end{array} \right. \quad (t \geq 0). \quad (36)$$

*4.4 The Hankel-type induced norms $\gamma_{\infty/1}$, $\gamma_{\infty/2}$, $\gamma_{2/1}$, $\gamma_{2/2}$*

In this section, explicit characterizations of $\gamma_{\infty/1}$, $\gamma_{\infty/2}$, $\gamma_{2/1}$, and $\gamma_{2/2}$ are given, where the result for $\gamma_{2/2}$ has already been shown in Asai [2005]. The results in this subsection can be derived without relying on the positivity and hence they are valid even for general (i.e., nonpositive) switching cases.

*Characterization of $\gamma_{\infty/1}$ and $\gamma_{\infty/2}$* For the characterization of $\gamma_{\infty/p}$ ($p=1,2$), for each $t_{\rm f} \geq 0$, let us define

$$\nu_{\infty/p}(t_{\rm f}) := \sup_{w \in L_{p-},\ \|w\|_{p-}=1} \|z(t_{\rm f})\|_\infty \ \text{ s.t. } (1), (2), (3). \quad (37)$$

Then, we have

$$\gamma_{\infty/p} = \max_{t_{\rm f} \geq 0} \nu_{\infty/p}(t_{\rm f}). \quad (38)$$

On the other hand, in view of the fact that $z(t)$ ($t \geq 0$) can be written explicitly as (21), let us define for each $t_{\rm f} \geq 0$ the LTI positive system $\widehat{F}_{t_{\rm f}}$ by

$$\widehat{F}_{t_{\rm f}}(s) := \left[\begin{array}{c|c} A_{\rm p} & B_{\rm p} \\ \hline C_{\rm f}\exp(A_{\rm f}t_{\rm f})S & 0 \end{array}\right] \ (t_{\rm f} \geq 0). \quad (39)$$

Then, it may be deduced from (12), (15), (21), (37) and (39) that

$$\begin{aligned} \nu_{\infty/1}(t_{\rm f}) &= \|\widehat{F}_{t_{\rm f}}\|_{\infty/1} \\ &= \max_{t_{\rm p}\geq 0} \max_{i,j} |C_{{\rm f},i} \exp(A_{\rm f}t) S \exp(A_{\rm p}t_{\rm p}) B_{{\rm p},j}|, \end{aligned} \quad (40)$$

$$\nu_{\infty/2}(t_{\rm f}) = \|\widehat{F}_{t_{\rm f}}\|_{\infty/2} = \sqrt{d_{\max}(C_{\rm f}\exp(A_{\rm f}t_{\rm f})SX_{\rm p}S^T\exp(A_{\rm f}^Tt_{\rm f})C_{\rm f}^T)}. \quad (41)$$

It follows from (38), (40) and (41) that the next results hold.

*Theorem 14.* For the system switching from $\Sigma_{\rm p}$ to $\Sigma_{\rm f}$ described by (1), (2), and (3), we have

$$\gamma_{\infty/1} = \max_{t_{\rm f}\geq 0}\max_{t_{\rm p}\geq 0}\max_{i,j}|C_{{\rm f},i}\exp(A_{\rm f}t)S\exp(A_{\rm p}t_{\rm p})B_{{\rm p},j}|, \quad (42)$$

$$\gamma_{\infty/2} = \max_{t_{\rm f}\geq 0}\sqrt{d_{\max}(C_{\rm f}\exp(A_{\rm f}t_{\rm f})SX_{\rm p}S^T\exp(A_{\rm f}^Tt_{\rm f})C_{\rm f}^T)}. \quad (43)$$

□

*Remark 15.* (i) We note that the expression of $\gamma_{\infty/\infty}$ given by (26) can also be obtained from the fact that

$$\gamma_{\infty/\infty} = \max_{t_{\rm f}\geq 0}\ \nu_{\infty/\infty}(t_{\rm f}) = \max_{t_{\rm f}\geq 0}\ \|\widehat{F}_{t_{\rm f}}\|_{\infty/\infty}$$

and (18).

(ii) We can see that (42) reduces to (12) in the case of (8). On the other hand, we see that (43) can be reduced to (15) by assuming (8) and the maximum in (43) is attained at $t_{\rm f} = 0$. In fact, in the case where (8) holds in (43), we see that

$$\begin{aligned}&\frac{d}{dt_{\rm f}}C\exp(At_{\rm f})X\exp(A^Tt_{\rm f})C^T\\ &= C\exp(At_{\rm f})(AX+XA^T)\exp(A^Tt_{\rm f})C^T \leq 0\end{aligned} \quad (44)$$

and hence the maximum is actually attained at $t_{\rm f} = 0$.

(iii) We can obtain similar results to (ii) of Remark 13 regarding the construction of the worst case input $w^\star \in L_{p-}^+$ for $\gamma_{\infty/p}$ $(p = 1, 2)$.

*Characterization of $\gamma_{2/1}$* We next consider the characterization of $\gamma_{2/1}$. To this end, we recall the next lemma from Chellaboina et al. [1999]. In the following, we define

$$\|z\|_{(\infty,2)+} := \operatorname*{ess\,sup}_{0\leq t<\infty}|z(t)|_2,$$
$$L_{(\infty,2)+} := \left\{z:\ \|z\|_{(\infty,2)+} < \infty\right\}.$$

*Lemma 16.* (Chellaboina et al. [1999]). Let us consider the stable LTI system $G$ given by (5) with $x(0) = 0$ and define its *induced norm* $\|G\|_{(\infty,2)/1}^{\rm ind}$ from $w \in L_{1+}$ to $z \in L_{(\infty,2)+}$ by

$$\|G\|_{(\infty,2)/1}^{\rm ind} := \sup_{w\in L_{1+},\ \|w\|_{1+}=1}\|z\|_{(\infty,2)+}.$$

Then, we have

$$\|G\|_{(\infty,2)/1}^{\rm ind} = \max_{t\geq 0}\sqrt{d_{\max}(B^T\exp(A^Tt)C^TC\exp(At)B)}.$$

We now go back to the analysis of the Hankel-type induced norm $\gamma_{2/1}$. If we define $\widehat{C}_{\rm f} := P_{\rm f}^{1/2}S \in \mathbb{R}^{n_{\rm f}\times n_{\rm p}}$, we can see from (21) that

$$\begin{aligned}\gamma_{2/1} &= \sup_{w\in L_{1-},\|w\|_{1-}=1}\left|\widehat{C}_{\rm f}\int_{-\infty}^{0}\exp(A_{\rm p}(-\tau))B_{\rm p}w(\tau)d\tau\right|_2\\ &= \sup_{w\in L_{1+},\|w\|_{1+}=1}\left|\widehat{C}_{\rm f}\int_{0}^{\infty}\exp(A_{\rm p}(\tau))B_{\rm p}w(\tau)d\tau\right|_2\\ &= \|\widehat{H}\|_{(\infty,2)/1}^{\rm ind}\end{aligned}$$

where

$$\widehat{H}(s) := \left[\begin{array}{c|c}A_{\rm p} & B_{\rm p}\\ \hline \widehat{C}_{\rm f} & 0\end{array}\right].$$

From this key observation and Lemma 16, the next theorem follows.

*Theorem 17.* For the system switching from $\Sigma_{\rm p}$ to $\Sigma_{\rm f}$ described by (1), (2), and (3), we have

$$\gamma_{2/1} = \max_{t_{\rm p}\geq 0}\sqrt{d_{\max}(B_{\rm p}^T\exp(A_{\rm p}^Tt_{\rm p})S^TP_{\rm f}S\exp(A_{\rm p}t_{\rm p})B_{\rm p})}. \quad (45)$$

□

*Remark 18.* (i) We can see that (45) reduces to (11) by assuming (8) and the maximum in (45) is attained at $t_{\rm p} = 0$. We can verify this similarly to (44). It is also true that we can obtain similar results to (ii) of Remark 13 on the construction of the worst case input $w^\star \in L_{1-}^+$ for $\gamma_{2/1}$.

(ii) Obviously, the duality holds between $\gamma_{\infty/2}$ given by (43) and $\gamma_{2/1}$ given by (45).

*Characterization of $\gamma_{2/2}$* We finally note from Asai [2005] that $\gamma_{2/2}$ is given by

$$\gamma_{2/2} = \sqrt{\lambda_{\max}(S^TP_{\rm f}SX_{\rm p})}. \quad (46)$$

As pointed by Ebihara and Hagiwara [2005], this result directly follows if we note that the adjoint operator of $\Sigma$ given by (21) for $(q,p) = (2,2)$ can be represented by

$$\Sigma^*:\ L_{2+}\ni\zeta\mapsto\xi\in L_{2-},$$
$$(\Sigma^*\zeta)(t) = B_{\rm p}^T\exp(-A_{\rm p}^Tt)S^T\int_0^\infty\exp(A_{\rm f}^T\tau)C_{\rm f}^T\zeta(\tau)d\tau = \xi(t)\ (t\leq 0).$$

The worst case input $w^\star \in L_{2-}$ that attains (46) can be given explicitly by

$$w^\star(t) = B_{\rm p}^T\exp(-A_{\rm p}^Tt)v^\star \quad (47)$$

where $v^\star \in \mathbb{R}^{n_{\rm p}}$ is the eigenvector corresponding to the eigenvalue $\lambda_{\max}(S^TP_{\rm f}SX_{\rm p})$ with $v^{\star T}X_{\rm p}v^\star = 1$.

### 4.5 Usefulness of the LP-based Characterizations

In the case where we have tunable parameters in $\Sigma_{\rm p}$ and $\Sigma_{\rm f}$ in a specific form, the LP-based characterizations (29) and (30) enable us to synthesize the Hankel-type $L_1/L_\infty$ induced-norm-optimal-parameters by solving geometric programming problems (GPs). We have a definite prospect that this GP-based-synthesis method can be applied to the optimal parameter tuning problem of the Foschini-Miljanic (FM) algorithm [Foschini and Miljanic [1993]] for power control in wireless network communication. Recently, the FM algorithm for power control in wireless network communication received renewed interests from the viewpoint of positive system theory, see, for instance Zappavigna et al. [2012], Colombino and Smith [2016], Ogura et al. [2019]. The wireless network communication of interest consists of multiple receiver-transmitter pairs, and the quality of the signal transmission is measured by the Signal-to-Noise-and-Interference-Ratio (SNIR) of each receiver. The FM algorithm controls the powers of transmitters so that the SNIR of each receiver converges to its reference value. The reference values can be regarded as tuning parameters in FM algorithm, and we have in mind to “increase” them as much as possible, under upper power limit in each transmitter even in switching (changing) network topology. To that end, the standard steady state synthesis along the line of the GP-based treatment by Boyd et al. [2007] is deficient in general, since in transient the power levels may go beyond the upper limits due to the bumpy responses across switching. Our prospect is to suppress such bumpy responses directly by using the Hankel-type $L_1/L_\infty$ induced norm characterization in the form of GP, thereby obtaining optimal reference values that satisfy the power constraint. This topic is currently under investigation.

## 5. NUMERICAL EXAMPLES

### 5.1 Problem Setting

Let us consider the case where the systems $\Sigma_{\mathrm{p}}$, $\Sigma_{\mathrm{f}}$, and the matrix $S$ in (1), (2), and (3), respectively, are affected by polytopic-type uncertainty of the form

$$\begin{bmatrix} C_{\mathrm{f}} & 0 & 0 \\ A_{\mathrm{f}} & S & 0 \\ 0 & A_{\mathrm{p}} & B_{\mathrm{p}} \end{bmatrix} \in \left\{ \sum_{l=1}^{N} \alpha_l \begin{bmatrix} C_{\mathrm{f}}^{[l]} & 0 & 0 \\ A_{\mathrm{f}}^{[l]} & S^{[l]} & 0 \\ 0 & A_{\mathrm{p}}^{[l]} & B_{\mathrm{p}}^{[l]} \end{bmatrix} : \ \alpha \in \boldsymbol{\alpha}_{\mathcal{P}} \right\},$$

$$\boldsymbol{\alpha}_{\mathcal{P}} = \left\{ \alpha \in \mathbb{R}_+^N : \ \sum_{l=1}^{N} \alpha_l = 1 \right\}.$$

Here, we assume that the given matrices $A_{\mathrm{p}}^{[l]}$, $B_{\mathrm{p}}^{[l]}$, $A_{\mathrm{f}}^{[l]}$, $C_{\mathrm{f}}^{[l]}$, and $S^{[l]}$ $(l = 1, \ldots, N)$ that define the vertices of the polytope satisfy $A_{\mathrm{p}}^{[l]} \in \mathbb{M}^{n_{\mathrm{p}}}$, $B_{\mathrm{p}}^{[l]} \in \mathbb{R}_+^{n_{\mathrm{p}} \times n_w}$, $A_{\mathrm{f}}^{[l]} \in \mathbb{M}^{n_{\mathrm{f}}}$, $C_{\mathrm{f}}^{[l]} \in \mathbb{R}_+^{n_z \times n_{\mathrm{f}}}$, and $S^{[l]} \in \mathbb{R}_+^{n_{\mathrm{f}} \times n_{\mathrm{p}}}$. In the following, we denote by $\Sigma_{\mathrm{p},\alpha}$, $\Sigma_{\mathrm{f},\alpha}$, and $S_\alpha$ the positive systems and the nonnegative matrix corresponding to the parameter $\alpha \in \boldsymbol{\alpha}_{\mathcal{P}}$. We assume that both $\Sigma_{\mathrm{p},\alpha}$ and $\Sigma_{\mathrm{f},\alpha}$ are stable for any $\alpha \in \boldsymbol{\alpha}_{\mathcal{P}}$. Under these assumptions, we denote by $\gamma_{q/p}(\alpha)$ the Hankel-type $L_q/L_p$ induced norm on the positive system switching from $\Sigma_{\mathrm{p},\alpha}$ to $\Sigma_{\mathrm{f},\alpha}$ via $S_\alpha$.

The problem we consider in this section is to compute the worst case Hankel-type $L_q/L_p$ induced norm $\gamma_{q/p}^\star$ defined by $\gamma_{q/p}^\star := \max_{\alpha \in \boldsymbol{\alpha}_{\mathcal{P}}} \gamma_{q/p}(\alpha)$. Even though exact and efficient computation of $\gamma_{q/p}^\star$ is hard, we can compute its upper bound efficiently by using the SDP characterizations provided in the preceding section. For instance, we consider the analysis of $\gamma_{1/\infty}^\star$ in the next subsection.

### 5.2 Computation Results for $\gamma_{1/\infty}^\star$

From (27) and (28), it may be seen that we can obtain upper bounds of $\gamma_{1/\infty}^\star$ by solving the following SDPs.

$$\overline{\gamma}_{1/\infty}^{\star \mathrm{p}} := \inf_{\gamma, F} \quad \gamma \quad \text{subject to}$$

$$\begin{bmatrix} -2\gamma & 0 & \mathbf{1}_{n_w}^T {B_{\mathrm{p}}^{[l]}}^T \\ 0 & 0 & 0 \\ B_{\mathrm{p}}^{[l]} \mathbf{1}_{n_w} & 0 & 0 \end{bmatrix} + \mathrm{He} \left\{ \begin{bmatrix} \mathbf{1}_{n_z}^T C_{\mathrm{f}}^{[l]} & 0 \\ A_{\mathrm{f}}^{[l]} & S^{[l]} \\ 0 & A_{\mathrm{p}}^{[l]} \end{bmatrix} F \right\} \prec 0 \tag{48}$$
$(l = 1, \ldots, N)$.

$$\overline{\gamma}_{1/\infty}^{\star \mathrm{d}} := \inf_{\gamma, H} \quad \gamma \quad \text{subject to}$$

$$\begin{bmatrix} -2\gamma & 0 & \mathbf{1}_{n_z}^T C_{\mathrm{f}}^{[l]} \\ 0 & 0 & 0 \\ {C_{\mathrm{f}}^{[l]}}^T \mathbf{1}_{n_z} & 0 & 0 \end{bmatrix} + \mathrm{He} \left\{ \begin{bmatrix} \mathbf{1}_{n_w}^T {B_{\mathrm{p}}^{[l]}}^T & 0 \\ {A_{\mathrm{p}}^{[l]}}^T & S^{[l]T} \\ 0 & {A_{\mathrm{f}}^{[l]}}^T \end{bmatrix} H \right\} \prec 0 \tag{49}$$
$(l = 1, \ldots, N)$.

As a concrete example, let us consider the case where $N = 2$ and

$$\begin{bmatrix} C_{\mathrm{f}}^{[1]} & 0 & 0 \\ A_{\mathrm{f}}^{[1]} & S^{[1]} & 0 \\ 0 & A_{\mathrm{p}}^{[1]} & B_{\mathrm{p}}^{[1]} \end{bmatrix} = \left[\begin{array}{rr:rr:r} 0.98 & 0.29 & 0 & 0 & 0 \\ \hdashline -1.27 & 0.30 & 0.80 & 0.60 & 0 \\ 0.56 & -0.36 & 0.90 & 0.88 & 0 \\ \hdashline 0 & 0 & -0.91 & 0.79 & 0.67 \\ 0 & 0 & 0.36 & -0.60 & 0.13 \end{array}\right],$$

$$\begin{bmatrix} C_{\mathrm{f}}^{[2]} & 0 & 0 \\ A_{\mathrm{f}}^{[2]} & S^{[2]} & 0 \\ 0 & A_{\mathrm{p}}^{[2]} & B_{\mathrm{p}}^{[2]} \end{bmatrix} = \left[\begin{array}{rr:rr:r} 0.83 & 0.85 & 0 & 0 & 0 \\ \hdashline -0.81 & 0.37 & 0.37 & 0.87 & 0 \\ 0.52 & -0.52 & 0.59 & 0.93 & 0 \\ \hdashline 0 & 0 & -0.68 & 0.73 & 0.03 \\ 0 & 0 & 0.55 & -1.04 & 0.45 \end{array}\right].$$

From (24), we find on the two vertices that $\gamma_{1/\infty}(e_1) = 10.9644$ and $\gamma_{1/\infty}(e_2) = 12.4567$. We then next solve the SDPs (48) and (49) to evaluate the worst case Hankel-type $L_1/L_\infty$ induced norm $\gamma_{1/\infty}^\star$. It turns out that $\overline{\gamma}_{1/\infty}^{\star \mathrm{p}} = \overline{\gamma}_{1/\infty}^{\star \mathrm{d}} = 12.7535$. On the other hand, by a brute force gridding search, we confirm that $\gamma_{1/\infty}^\star \approx 12.7535$ and this is attained by $\alpha = [\ 0.2799 \ 0.7201\ ]^T$. Namely, in this particular example, the result obtained from the SDPs (48) and (49) is numerically verified to be exact. We can also confirm the exactness of the result obtained by the SDPs (48) and (49) by duality-based arguments, see Lemma 3.5 of Ebihara et al. [2015] for details.

## 6. CONCLUSION

In this paper, we analyzed the Hankel-type $L_q/L_p$ induced norms across a single switching over two LTI positive systems. We derived explicit representations of the Hankel-type $L_q/L_p$ induced norms for $p, q$ being $1, 2, \infty$, where those new results for $(q, p) = \{(\infty, 1), (\infty, 2), (2, 1)\}$ are valid even for general (nonpositive) switching cases. In particular, for $(q, p) = \{(1, \infty), (2, \infty), (1, 2)\}$, we provided LP- and SDP-based characterizations. By numerical examples, we illustrated the usefulness of the SDP-based characterizations for the analysis of the Hankel-type $L_q/L_p$ induced norms where the systems of interest are affected by parametric uncertainties. Future topics include the application of the LP-based results for the Hankel-type $L_1/L_\infty$ induced norm to the optimal parameter tuning problem of the Foschini-Miljanic algorithm for power control in wireless network communication.

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## Appendix A. PROOF OF THEOREM 10

In the following, for a matrix $E \in \mathbb{R}^{n\times m}$ with $n \geq m$ and $\mathrm{rank}(E) = r$, we denote by $E^\perp$ a full-rank matrix such that $E^\perp \in \mathbb{R}^{(n-r)\times n}$ and $E^\perp E = 0$.

*Proof of Theorem 10:* Once the initial condition is given by (23), we can derive (24)-(26) by standard procedure. Therefore we prove the equivalence of (a-i)-(a-v) and (b-i) and (b-ii) in the following.

(a-i)⇔(a-ii) In view of the fact that

$$\begin{bmatrix} \mathbf{1}_{n_z}^T C_\mathrm{f} & 0 \\ A_\mathrm{f} & S \\ 0 & A_\mathrm{p} \end{bmatrix}^\perp = \begin{bmatrix} 1 & -\mathbf{1}_{n_z}^T C_\mathrm{f} A_\mathrm{f}^{-1} & \mathbf{1}_{n_z}^T C_\mathrm{f} A_\mathrm{f}^{-1} S A_\mathrm{p}^{-1} \end{bmatrix},$$

the equivalence (a-i)⇔(a-ii) follows from (24) and $\mathcal{S}$-variable LMI results shown in Ebihara et al. [2015]. The proof for (a-i)⇔(a-iii) follows similarly.

(a-i)⇔(a-iv) We first prove (a-i)⇒(a-iv). To this end suppose (a-i) holds, i.e., $\mathbf{1}_{n_z}^T C_\mathrm{f} A_\mathrm{f}^{-1} S A_\mathrm{p}^{-1} B_\mathrm{p} \mathbf{1}_{n_w} < \gamma$. Then, there exist sufficiently small $\varepsilon_i > 0$ $(i = 1, 2, 3)$ and $\hat{f}_\mathrm{p} \in \mathbb{R}_{++}^{n_\mathrm{p}}$, $\hat{f}_0 \in \mathbb{R}_{++}^{n_\mathrm{f}}$, and $\hat{f}_\mathrm{f} \in \mathbb{R}_{++}^{n_\mathrm{f}}$ such that

$$\begin{aligned} &A_\mathrm{p}\hat{f}_\mathrm{p} = -B_\mathrm{p}\mathbf{1}_{n_w} - \varepsilon_1 \mathbf{1}_{n_\mathrm{p}},\ \hat{f}_0 = S\hat{f}_\mathrm{p} + \varepsilon_2 \mathbf{1}_{n_\mathrm{f}}, \\ &A_\mathrm{f}\hat{f}_\mathrm{f} = -\hat{f}_0 - \varepsilon_3 \mathbf{1}_{n_\mathrm{f}}\ \mathbf{1}_{n_z}^T C_\mathrm{f} \hat{f}_\mathrm{f} < \gamma. \end{aligned}$$

Here we used the fact that $A_\mathrm{p}^{-1} \leq 0$ and $A_\mathrm{f}^{-1} \leq 0$. It is clear that $f_\mathrm{p} = \hat{f}_\mathrm{p}$, $f_0 = \hat{f}_0$, and $f_\mathrm{f} = \hat{f}_\mathrm{f}$ satisfy (29) and hence (a-iv) hold. Then, we prove that (a-iv)⇒(a-i). To this end, we note from (29) that

$$\mathbf{1}_{n_z}^T C_\mathrm{f} f_\mathrm{f} < \gamma,\ f_\mathrm{f} > -A_\mathrm{f}^{-1} f_0,\ S f_\mathrm{p} < f_0,\ f_\mathrm{p} > -A_\mathrm{p}^{-1} B_\mathrm{p} \mathbf{1}_{n_w}.$$

It follows that the proof can be completed by

$$\begin{aligned} \gamma &> \mathbf{1}_{n_z}^T C_\mathrm{f} f_\mathrm{f} \geq -\mathbf{1}_{n_z}^T C_\mathrm{f} A_\mathrm{f}^{-1} f_0 \geq -\mathbf{1}_{n_z}^T C_\mathrm{f} A_\mathrm{f}^{-1} S f_\mathrm{p} \\ &\geq \mathbf{1}_{n_z}^T C_\mathrm{f} A_\mathrm{f}^{-1} S A_\mathrm{p}^{-1} B_\mathrm{p} \mathbf{1}_{n_w}. \end{aligned}$$

The proof for (a-i)⇔(a-v) follows similarly.

(b-i)⇔(b-ii) In view of the fact that

$$\begin{bmatrix} \mathbf{1}_{n_w}^T B_\mathrm{p}^T & 0 \\ A_\mathrm{p}^T & S^T \\ 0 & I_{n_\mathrm{f}} \end{bmatrix}^\perp = \begin{bmatrix} 1 & -\mathbf{1}_{n_w}^T B_\mathrm{p}^T A_\mathrm{p}^{-T} & \mathbf{1}_{n_w}^T B_\mathrm{p}^T A_\mathrm{p}^{-T} S^T \end{bmatrix},$$

$$\begin{bmatrix} A_\mathrm{f}^T \\ -I_{n_\mathrm{f}} \end{bmatrix}^\perp = \begin{bmatrix} I_{n_\mathrm{f}} & A_\mathrm{f}^T \end{bmatrix},$$

the equivalence (b-i)⇔(b-ii) follows from (25) and Ebihara et al. [2015]. ■