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# Converging approximations of attractors via almost Lyapunov functions and semidefinite programming

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Corbinian Schlosser<sup>1</sup>

## Abstract

In this paper we combine the approaches from [21] and [11] for approximating global attractors. In [21] the global attractors is arbitrarily well approximated by sets that are not necessarily positively invariant. On the contrary, the method from [11] provides supersets of the global attractor which are positively invariant but not necessarily converging. In this paper we marry both approaches by combining their techniques and get converging outer approximations of the global attractor consisting of positively invariant sets. Because both the methods from [21] and [11] are based on convex optimization via sum-of-squares techniques the same is true for our proposed method. The method is easy to use and numerical examples illustrate the procedure.

**Keywords:** Global attractor, Lyapunov function, outer approximations, dynamical systems, infinite-dimensional linear programming, sum-of-squares, semidefinite programming.

## 1 Introduction

Global attractors and Lyapunov functions are intimately connected (see for example [3]). Hence it is only natural that Lyapunov functions play a central role in control and asymptotic analysis of dynamical systems. Computing Lyapunov functions is therefore an intriguing but complicated task and often restricted to the case of finding a Lyapunov function for an a-priori given set. That finds excessive application in verification of asymptotic stability of an equilibrium point  $x^*$  which lays at the base of many control problems. There are several techniques available to search for Lyapunov functions for a given set  $A$  (such as  $A = \{x^*\}$ ), see [7]. Among these are sum-of-squares (SOS) techniques that recently gained more and more popularity, beginning with the work of Parillo [17], and have further developed since

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[2]. SOS techniques typically result in feasibility problems for linear matrix inequalities for which fast numerical methods exist.

In the case of exponentially asymptotically stable fixed points searching for polynomial Lyapunov functions is sufficient [2]. But already dropping the assumption of exponentially fast decay causes that global asymptotic stability of a single equilibrium point can not any longer be verified by a polynomial Lyapunov function, i.e. there exists a dynamical system with globally asymptotically stable equilibrium point for which no polynomial Lyapunov function exists [1]. That shows that the general use of SOS polynomial Lyapunov functions is limited and extensions of the SOS methods are needed in order to treat more general systems. This gets even more drastic when we search for the attractor of the system, that is we do not a-priori know for which set  $A$  we want to check if it is an attractor.

Except for Lyapunov based techniques, several methods for computing/approximating the global attractor have been proposed. Some approximate the attractor by following trajectories for long but finite time  $T \in [0, \infty)$  or using set oriented methods as in [5]. Another approach motivated by a relaxation of the notion of Lyapunov functions was given in [11]. There the authors showed that the weaker concept of Lyapunov function they use is still sufficient to envelope the attractor by positively invariant sets while at the same time SOS polynomials can be used. In order to make their method computationally tractable – via a SOS hierarchy of finite dimensional (convex) semidefinite programs (SDPs) – the authors relaxed their problem at the cost of guaranteed convergence. Related to this approach is [21] where the global attractor is characterized by an infinite dimensional linear programming problem (LP), also solved via a hierarchy of finite dimensional SDPs. In [21] the authors showed convergence but the obtained sets lack the good property of being positively invariant. In this paper we merge both methods into one, maintaining both their advantages (using positively invariant sets and guaranteed convergence). We use that, for both, the basis is built by an infinite dimensional optimization problem. This similarity in nature can be well combined, results in an infinite dimensional LP and can be solved by sequence of finite dimensional SPDs.

## 2 Notations

We denote by  $\mathbb{N}$  the natural numbers. The non-negative real numbers are denoted by  $\mathbb{R}_+$ . We denote the euclidean inner product of two vectors  $a, b \in \mathbb{R}^n$  by  $a \cdot b$ . The function  $\text{dist}(\cdot, K)$  denotes the Euclidean distance function to a set  $K \subset \mathbb{R}^n$  and  $\text{dist}(K_1, K_2)$  denotes the Hausdorff distance of two subsets of  $\mathbb{R}^n$ . The space of polynomials (in  $n$  variables) is denoted by  $\mathbb{R}[x_1, \dots, x_n]$  or shorter  $\mathbb{R}[x]$ . The degree of  $p \in \mathbb{R}[x]$  is denoted by  $\text{deg}(p)$  and the set polynomials of degree at most  $k \in \mathbb{N}$  by  $\mathbb{R}[x]_d$ . The space of continuous functions on  $X$  is denoted by  $\mathcal{C}(X)$  and the space of continuously differentiable functions on  $\mathbb{R}^n$  by  $\mathcal{C}^1(\mathbb{R}^n)$  and we denote the gradient of  $g \in \mathcal{C}^1(\mathbb{R}^n)$  by  $\nabla g$ . The Lebesgue measure is denoted by  $\lambda$ .

### 3 Setting and preliminary definitions

We consider ordinary differential equations

$$\dot{x} = f(x), \quad x(0) = x_0 \in \mathbb{R}^n. \quad (1)$$

for a locally Lipschitz continuous vector field  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ . Further we restrict to a compact constraint set  $X \subset \mathbb{R}^n$ . By  $\varphi_t(x_0)$  we denote the solution of (1) at time  $t$ .

Next we define what it means for a set to be positively invariant for (1).

**Definition 1** *A set  $S$  is called positively invariant for (1) if for all  $x \in S$  and  $t \in \mathbb{R}_+$  the solution  $\varphi_t(x)$  exists and is located in  $S$ .*

The main object in this paper are attractors. There are several different notions of attractors, we use the concept of global attractors from [19].

**Definition 2** *The set*

$$M_+ = \{x_0 \in X : \varphi_t(x_0) \in X \text{ for all } t \in \mathbb{R}_+\} \quad (2)$$

*is called the maximum positively invariant (MPI) set. A compact set  $\mathcal{A} \subset X$  is called a global attractor if it is minimal uniformly attracting, i.e., it is the smallest compact set  $\mathcal{A}$  such that*

$$\lim_{t \rightarrow \infty} \text{dist}(\varphi_t(M_+), \mathcal{A}) = 0.$$

Since  $X$ , and hence also  $M_+$  is compact, the global attractor exists and is unique [19].

It remains to define (smooth) Lyapunov functions.

**Definition 3** *Let  $M_+$  be the maximum positively invariant set. A non-negative function  $V \in C^1(\mathbb{R}^n)$  is called a smooth Lyapunov function for the dynamical system induced by  $f$  if for all  $x \in M_+$  we have*

$$\nabla V(x) \cdot f(x) \leq -V(x). \quad (3)$$

*We say  $V$  is a Lyapunov function for a set  $A \subset M_+$  if  $V$  is a Lyapunov function and  $A := V^{-1}(\{0\}) \cap M_+$ .*

Lyapunov functions and global attractors are intimately related, one reason is the following theorem.

**Theorem 1** ([3] **Theorem 2.7.1**, [23]) *A compact set  $A \subset M_+$  is asymptotically stable for (1) with constraint set  $X$  if and only if there exists a smooth Lyapunov function  $V$  for  $A$ . On the contrary, if  $V$  is a (smooth) Lyapunov function then  $V^{-1}(\{0\})$  is asymptotically stable and contains the global attractor.*

## 4 Two infinite dimensional optimization problems for the global attractor

We begin with presenting the infinite dimensional optimization problems for global attractors from [21] and [11].

The first one is the LP presented in [21] which reads

$$\begin{aligned}
 p_1^* = \inf & \int_X w(x) dx \\
 \text{s.t.} & (w, v_1, v_2) \in \mathcal{C}(\mathbb{R}^n) \times \mathcal{C}^1(\mathbb{R}^n) \times \mathcal{C}^1(\mathbb{R}^n) \\
 & -v^1 - v^2 + w \geq 1 && \text{on } X \\
 & w \geq 0 && \text{on } X \\
 & \beta v^1 - \nabla v^1 \cdot f \geq 0 && \text{on } X \\
 & \beta v^2 + \nabla v^2 \cdot f \geq 0 && \text{on } X
 \end{aligned} \tag{4}$$

where  $\beta > 0$  is a parameter. In [21] the authors showed  $\lambda(\mathcal{A}) = p_1^*$  where  $\lambda$  denotes the Lebesgue measure. Each feasible triple  $(w, v_1, v_2)$  induces a set  $A := w^{-1}([1, \infty))$  that satisfies

$$\mathcal{A} \subset A \text{ and } \lambda(A \setminus \mathcal{A}) \leq \int_X w d\lambda - p_1^*, \tag{5}$$

and hence, as  $(w, v_1, v_2)$  gets optimal,  $A$  converges to  $\mathcal{A}$  with respect to the Lebesgue measure [21].

The optimization problem from [11] reads

$$\begin{aligned}
 p_2^* = \inf & \lambda(J^{-1}([0, 1])) \\
 \text{s.t.} & J \in \mathcal{C}^1(\mathbb{R}^n) \\
 & J(x) \geq 0 \text{ for all } x \in X \\
 & \nabla J \cdot f \leq -(J - 1) \text{ on } X \\
 & \emptyset \neq J^{-1}([0, 1]) \subset \overset{\circ}{X}
 \end{aligned} \tag{6}$$

The reason for replacing the Lyapunov constraint  $\nabla J \cdot f \leq -J$  by  $\nabla J \cdot f \leq -(J - 1)$  is that there always exist polynomials satisfying the second constraint but, as mentioned before, this might not be the case for the first. In [11] the authors showed that  $p_2^* = \lambda(\mathcal{A})$  and that the set  $J^{-1}([0, 1]) \subset X$  is positively invariant, contains the global attractor and converges towards the global attractor with respect to the Lebesgue measure when  $J$  gets optimal.

Compared to the method from [21] this has the advantage that the sets obtained as in [11] are always positively invariant. On the other hand (6) is not linear and the evaluation of the cost is difficult, while evaluating the cost in (4) is easy. A heuristic for the cost term in (6) was proposed in order to simplify the cost term but this came at the cost of exactness and convergence could not be guaranteed anymore.

We will formulate an optimization problem that combines the advantages of both methods without sacrificing convergence. Afterwards we solve the optimization problem as in [11] and [21] using sum-of-squares methods.

## 5 A combined approach

Here we will merge the idea of the relaxed Lyapunov equation

$$\nabla J \cdot f \leq -(J - 1) \quad (7)$$

and the linear structure of (4) with an easy evaluation of the cost term.

We first note the following result, which is only a reformulation of [11, Proposition 5], about a perturbed Lyapunov condition.

**Lemma 1** *Let  $\varepsilon > 0$  and  $0 \leq J \in \mathcal{C}^1(\mathbb{R}^n)$  satisfy*

$$\nabla J \cdot f \leq -(J - \varepsilon) = -J + \varepsilon \text{ on } M_+ \quad (8)$$

*then the set  $S := J^{-1}([0, \varepsilon])$  is positively invariant and contains the global attractor (for  $X$ ). Further, if  $S \cap X \subset \bar{X}$  then  $S \cap X \subset M_+$ .*

**Proof:** This follows from [11, Proposition 5] by considering the function  $\varepsilon^{-1}V$  which satisfies (7).  $\square$

We refer to almost Lyapunov functions for functions  $0 \leq J$  that satisfy (8). Because of the close relation between Lyapunov functions and functions solving (8) we start with an infinite dimensional LP motivated by (4) and (6) but involving only Lyapunov functions before turning to the perturbed version (note that the following LP considers  $M_+$  and not  $X$  yet)

$$\begin{aligned} p_3^* = \inf & \int_X w(x) dx \\ \text{s.t.} & (w, V) \in \mathcal{C}(M_+) \times \mathcal{C}^1(\mathbb{R}^n) \\ & w + V \geq 1 && \text{on } M_+ \\ & w \geq 0 && \text{on } M_+ \\ & V \geq 0 && \text{on } M_+ \\ & \nabla V \cdot f \leq -V && \text{on } M_+ \end{aligned} \quad (9)$$

This is also an infinite dimensional LP and clearly, any feasible  $V$  is a Lyapunov function and hence by Theorem 1 we have  $V = 0$  on the global attractor  $\mathcal{A}$ . In particular from  $w + V \geq 1$  on  $M_+$  it follows  $w \geq 1$  on  $\mathcal{A}$  and by non-negativity of  $w$ ,

$$\int_{M_+} w(x) dx \geq \lambda(\mathcal{A}).$$

That means  $p_3^* \geq \lambda(\mathcal{A})$ . On the other hand let  $V$  be a Lyapunov function for  $\mathcal{A}$ , i.e.  $V^{-1}(\{0\}) = \mathcal{A}$ , satisfying  $\nabla V \cdot f \leq -V$ , according to Theorem 1. For  $k \in \mathbb{N}$  define the function  $w_k := \max\{0, 1 - k \cdot V\} \in \mathcal{C}(M_+)$ . The pair  $(w_k, k \cdot V)$  is feasible for all  $k \in \mathbb{N}$  and for all  $x \in M_+$

$$w_k(x) \rightarrow \begin{cases} 1, & x \in \mathcal{A} \\ 0, & \text{else.} \end{cases}$$

By the Beppo-Levi monotone convergence theorem it follows  $\int_{M_+} w_k(x) dx \rightarrow \lambda(\mathcal{A})$ , hence  $p_3^* \leq \lambda(\mathcal{A})$ . We have shown

**Proposition 1** *We have  $p_3^* = \lambda(\mathcal{A})$ .*

In the next step we want to replace the last constraint in the LP (9) by (8) because this will allow us to search for feasible  $J$  that are polynomial. Since (8) is a relaxation of the Lyapunov condition (3) we penalize a function for not being a Lyapunov function. The corresponding LP reads

$$\begin{aligned}
p_4^* &= \inf_X \int w(x) dx + \varepsilon \lambda(X) \\
\text{s.t.} \quad & (w, J, \varepsilon) \in \mathcal{C}(\mathbb{R}^n) \times \mathcal{C}^1(\mathbb{R}^n) \times [0, \infty) \\
& w + J \geq 1 && \text{on } M_+ \\
& w \geq 0 && \text{on } M_+ \\
& J \geq 0 && \text{on } M_+ \\
& \nabla J \cdot f \leq -J + \varepsilon && \text{on } M_+
\end{aligned} \tag{10}$$

Since any feasible pair  $(w, V)$  for the LP (9) induces a feasible triple  $(w, V, 0)$  for (10) with the same cost we get  $p_4^* \leq p_3^*$ . On the other hand for any feasible  $(w, J, \varepsilon)$  we have by Lemma 1 that  $V \leq \varepsilon$  on  $\mathcal{A}$  and hence by the first constraint in (10) we have  $w \geq 1 - \varepsilon$  on  $\mathcal{A}$  and  $w \geq 0$  by the second constraint, so that we get

$$\int_X w(x) dx + \varepsilon \lambda(X) \geq (1 - \varepsilon) \lambda(\mathcal{A}) + \varepsilon \lambda(X) \geq (1 - \varepsilon) \lambda(\mathcal{A}) + \varepsilon \lambda(\mathcal{A}) = \lambda(\mathcal{A}).$$

This proves

**Proposition 2** *We have  $p_4^* = \lambda(\mathcal{A})$ .*

The last obstacle that we have to overcome is that we do not want to include  $M_+$  explicitly because the set  $M_+$  is typically not known a-priori. Therefore we make use of what already entered the LP (4) (note that in the LP (4) only  $X$  but not  $M_+$  appears) where the MPI set enters implicitly through the constraint  $\beta v - \nabla v \cdot f \geq 0$ , see [13]. That leads to the final LP with discounting factor  $\beta > 0$

$$\begin{aligned}
p_5^* &= \inf_X \int w(x) dx + \varepsilon \lambda(X) \\
\text{s.t.} \quad & (w, J, \varepsilon, v) \in \mathcal{C}(\mathbb{R}^n) \times \mathcal{C}^1(\mathbb{R}^n) \times [0, \infty) \times \mathcal{C}^1(\mathbb{R}^n) \\
& w + J - v \geq 1 && \text{on } X \\
& w \geq 0 && \text{on } X \\
& J \geq 0 && \text{on } X \\
& \nabla J \cdot f + J + v \leq \varepsilon && \text{on } X \\
& \beta v - \nabla v \cdot f \geq 0 && \text{on } X
\end{aligned} \tag{11}$$

We will show that also for this LP we have that the optimal value is given by the volume of the global attractor  $\lambda(\mathcal{A})$ .

**Theorem 2** *Let  $X$  be compact and  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a locally Lipschitz continuous vector field. Let  $\mathcal{A}$  be the global attractor for the dynamical system induced by  $f$  with constraint set  $X$ . Then we have for any  $\beta > 0$  in (11)*

$$p_5^* = \lambda(\mathcal{A}).$$

Further for any feasible  $(w, J, \varepsilon, v)$  we have  $J^{-1}([0, \varepsilon])$  is positively invariant and

$$\mathcal{A} \subset K := J^{-1}([0, \varepsilon]) \cap v^{-1}([0, \infty)) \cap X \quad (12)$$

with

$$\lambda(K \setminus \mathcal{A}) \leq \int_X w(x) dx + \varepsilon \lambda(X) - p_5^* \quad (13)$$

which converges to zero as  $(w, J, \varepsilon, v)$  gets optimal for (11).

**Proof:** The essential observation is that any feasible  $(w, J, \varepsilon, v)$  satisfies  $v \geq 0$  on  $M_+$ . This is implied by the last constraint in (11) [13, Lemma 3]. It follows that  $\nabla J \cdot f \leq -J + \varepsilon$  on  $M_+$  as well as  $w + J \geq 1$  on  $M_+$ , i.e. that  $(w, J, \varepsilon)$  is feasible for the LP (10),  $p_5^* \geq p_4^* = \lambda(\mathcal{A})$  and that the set  $J^{-1}([0, \varepsilon])$  is positively invariant and contains  $\mathcal{A}$  by Lemma 1. To show the remaining inequality,  $\lambda(\mathcal{A}) \leq p_5^*$ , we use a construction from [21] to find a function  $v \in \mathcal{C}^1(\mathbb{R}^n)$  with

$$\beta v - \nabla v \cdot f = 0, \quad v = 0 \text{ on } M_+ \text{ and } v < 0 \text{ on } X \setminus M_+. \quad (14)$$

In [21] it was shown that such a  $v$  exists for  $\beta$  larger than  $\text{Lip}(f, X)$  the Lipschitz constant of  $f$  on  $X$ . If  $\beta$  is smaller than  $\text{Lip}(f, X)$  then there exists still a function  $v$  having properties (arbitrarily) close to (14) (in the sense that  $\beta v - \nabla v \cdot f \geq 0$ ,  $v \leq r$  on  $M_+$  and  $v < 0$  on a relative open set  $U \subset X \setminus M_+$  where  $r > 0$  is arbitrary and  $U$  can be chosen arbitrarily close to  $X \setminus M_+$ , see [21]). The following arguments then remains the same, just gets a bit more technical – therefore we assume to have a function  $v$  satisfying (14). We show that for any point  $(w, J, \varepsilon)$  feasible for (10) and  $\delta > 0$  we can find  $k \in \mathbb{N}$  such that  $(w + \delta, J, \varepsilon + \delta, kv)$  is feasible for (11). The corresponding cost for  $(w + \delta, J, \varepsilon + \delta, \tilde{v})$  differs from the cost of  $(w, J, \varepsilon)$  for the LP (10) only by  $2\delta\lambda(X)$ . Then  $\lambda(\mathcal{A}) = p_4^* \leq p_5^*$  follows once we have found a  $k \in \mathbb{N}$  such  $(w + \delta, J, \varepsilon + \delta, kv)$  is feasible for (10). The non-negativity of  $w$  and  $J$  is not (negatively) affected by  $\delta$  we focus on the first and fourth constraint in the LP (11). For any  $k \in \mathbb{N}$  and all  $x \in X$  where

$$w(x) + J(x) + \delta \geq 1 \text{ and } \nabla J(x) \cdot f(x) + J(x) \leq \varepsilon + \delta \quad (15)$$

we have  $w(x) + \delta + J(x) - kv \geq 1$  and  $\nabla J(x) \cdot f(x) + J(x) + kv \leq \varepsilon + \delta$  as well, by non-positivity of  $kv$ . Therefore let us focus on the points where (15) is not satisfied, i.e. on the set

$$U = \{x \in X : w + J < 1 - \delta \text{ or } \nabla J \cdot f + J > \varepsilon + \delta\}. \quad (16)$$

By feasibility of  $(w, J, \varepsilon)$  for (10) we have  $U \cap M_+ = \emptyset$  and from  $\delta > 0$  and continuity of  $w$  and  $J, \nabla J$  and  $f$  we even have  $\overline{U} \cap M_+ = \emptyset$ . Hence there exists  $\rho > 0$  such that  $-v > \rho$  on  $U$ . Let now  $k \in \mathbb{N}$  with

$$k \geq \rho^{-1} \max_{x \in U} \{1 - \delta - w(x) - J(x), \nabla J(x) \cdot f(x) + J(x) - \varepsilon\}.$$

For this choice of  $k$  we get for all  $x \in U$

$$w(x) + \delta + J(x) - kv(x) \geq w(x) + \delta + J(x) - k\rho \geq 1$$



and

$$\nabla J(x) \cdot f + J(x) + kv(x) \leq \nabla J(x) \cdot f + J(x) - k\rho \leq \varepsilon.$$

Hence  $(w + \delta, J, \varepsilon + \delta, \tilde{v})$  is feasible and because  $\delta > 0$  was arbitrary we conclude  $p_5^* \leq p_4^* = \lambda(\mathcal{A})$ . It remains to show (13). The first constraint in the LP (11) gives for  $K$  from (12)

$$w \geq 1 - J + v \geq 1 - \varepsilon \text{ on } K \quad (17)$$

which contains  $\mathcal{A}$  due to  $\mathcal{A} \subset M_+ \subset v^{-1}([0, \infty))$  as mentioned at the beginning of the proof. Non-negativity of  $w$  now implies

$$\int_X w(x) dx + \varepsilon\lambda(X) \geq \int_K 1 - \varepsilon dx + \varepsilon\lambda(X) = (1 - \varepsilon)\lambda(K) + \varepsilon\lambda(X) \geq \lambda(K).$$

Subtracting  $p_5^* = \lambda(\mathcal{A})$  on both sides finishes the proof.  $\square$

From Lemma 1 we derive the following corollary.

**Corollary 1** *In Theorem 2, if  $J^{-1}([0, \varepsilon]) \subset \overset{\circ}{X}$  then the sets  $X$  and  $v^{-1}([0, \infty))$  can be omitted in (12).*

**Proof:** By Lemma 1 we have  $J^{-1}([0, \varepsilon]) \cap X \subset M_+$  and the crucial estimate (17) used to show (13) holds even on  $J^{-1}([0, \varepsilon])$ .  $\square$

Note that by [11] there exist minimizing sequences for (11) with almost Lyapunov functions  $J$  such that  $J^{-1}([0, \varepsilon]) \subset \overset{\circ}{X}$  whenever  $\mathcal{A} \subset \overset{\circ}{X}$ .

**Remark 1** *The dual problem of the LP (11) acts on the space of Borel measures on  $X$ . We did not include the dual problem here for two reasons: First, it demands further technical notation. Second, the dual problem gives less insight into the global attractor – this is pointed out in [21] – as for instance When the global attractor has vanishing Lebesgue volume then one optimal solution to the dual problem is trivial, i.e. all decision variables are zero.*

**Remark 2** *Discrete time systems can be handled in the same way – only the continuous time objects have to be replaced by their discrete counterparts. We refer to [21] for details about a similar treatment for discrete time systems.*

*A discrete time system is of the form  $x^+ = f(x)$ , and the notions of (maximum) positive invariant sets and global attractors adapt ([19], [21]). The LP corresponding to the LP (11) has the following form for discounting factor  $\alpha$  and decay rate  $\gamma$  with  $\alpha, \gamma \in (0, 1)$*

$$\begin{aligned} p_{discrete}^* &= \inf_X \int w(x) dx + \varepsilon\lambda(X) \\ \text{s.t.} & \quad (w, J, \varepsilon, v) \in \mathcal{C}(\mathbb{R}^n) \times \mathcal{C}(\mathbb{R}^n) \times [0, \infty) \times \mathcal{C}(\mathbb{R}^n) \\ & \quad w + J - v \geq 1 && \text{on } X \\ & \quad w \geq 0 && \text{on } X \\ & \quad J \geq 0 && \text{on } X \\ & \quad \gamma J \circ f - J + v \leq \varepsilon && \text{on } X \\ & \quad v - \alpha v \circ f \geq 0 && \text{on } X \end{aligned} \quad (18)$$

*Similar to Theorem 2 we obtain that  $p_{discrete}^* = \lambda(\mathcal{A})$  for discrete time systems.*

## 6 Solving the linear programs

Here we assume algebraic structure of the problem in order to get access to the polynomial optimization framework that has also been used in [11] and [21] to solve the corresponding optimization problems by a hierarchy of finite dimensional semidefinite programs. The idea is to replace the decision variables  $w, J, v$  by polynomials (this will be justified by the Stone-Weierstraß theorem) and the non-negativity will be algebraically certified by a SOS condition. This is a standard procedure and we refer to [15] and [13] for details.

**Assumption 1:** The vector field  $f$  is polynomial and  $X$  is a compact basic semi-algebraic set, that is, there exist polynomials  $g_1, \dots, g_j \in \mathbb{R}[x_1, \dots, x_n]$  such that  $X = \{x \in \mathbb{R}^n : g_i(x) \geq 0 \text{ for } i = 1, \dots, j\}$ . Further we assume that one of the  $p_i$  is given by  $g_i(x) = R_X^2 - \|x\|_2^2$  for some large enough  $R_X \in \mathbb{R}$ .

For each fixed degree  $k$  of the appearing polynomials we get the following SDP

$$\begin{aligned}
 d_k &:= \inf \mathbf{w}'\mathbf{l} + \varepsilon \mathbf{l}_1 \\
 \text{s.t.} \quad & w, J, v \in \mathbb{R}[x]_k, \varepsilon \in \mathbb{R}, p_0, q_0, r_0, s_0 \in \mathbb{R}[x]_{k/2} \\
 & p_i, q_i, r_i, s_i \in \mathbb{R}[x]_{(k-\deg(g_i))/2} \text{ for } i = 1, \dots, j \\
 & \varepsilon \geq 0 \\
 & w + J - v - 1 = p_0^2 + \sum_{i=1}^j p_i^2 g_i \\
 & w(x) = q_0^2 + \sum_{i=1}^j q_i^2 g_i \\
 & \nabla J \cdot f + J + v - \varepsilon = t_0^2 + \sum_{i=1}^j t_i^2 g_i \\
 & \beta v - \nabla v \cdot f = s_0^2 + \sum_{i=1}^j s_i^2 g_i
 \end{aligned} \tag{19}$$

where  $\mathbf{w}'$  is the vector of coefficients of the polynomial  $w$  and  $\mathbf{l}$  is the vector of the moments of the Lebesgue measure over  $X$  (i.e.,  $\mathbf{l}_\alpha = \int_X x^\alpha dx$ ,  $\alpha \in \mathbb{N}^n$ ,  $\sum_i \alpha_i \leq k$ ), both indexed in the same basis of  $\mathbb{R}[x]_k$  such that  $l_1$  corresponds to  $\alpha = 0$ , i.e.  $l_1 = \lambda(X)$ , which gives  $\mathbf{w}'\mathbf{l} = \int_X w(x) dx$ .

The following theorem shows that the properties of the infinite dimensional LP are inherited to the finite dimensional SPDs and that their optimal values converge to  $\lambda(\mathcal{A})$  as well.

**Theorem 3** *Under Assumption 1 we have  $d_k \nearrow \lambda(\mathcal{A})$  as  $k \rightarrow \infty$ , for any feasible  $(w_k, J_k, \varepsilon_k, v_k)$  (and corresponding  $p_i, q_i, r_i, s_i$ ) we have  $J_k^{-1}([0, \varepsilon_k])$  is positively invariant and*

$$J_k^{-1}([0, \varepsilon_k]) \cap v_k^{-1}([0, \infty)) \cap X \supset \mathcal{A} \tag{20}$$

*with convergence with respect to the Lebesgue measure as  $k \rightarrow \infty$  when  $(w_k, J_k, \varepsilon_k, v_k)$  get optimal for (19).*

**Proof:** Since the SPDs (19) are tightenings of the LP (11) we only need to show convergence. The proof is very similar to the proof of [21, Theorem 5 and 6]. The important difference

to the LP (9), where the semidefinite relaxation is doomed to fail when there exists no polynomial Lyapunov function, is that the additional perturbation parameter  $\varepsilon$  guarantees that we can always find (almost) optimal polynomials  $J$  satisfying the constraint  $\nabla J \cdot f + J \leq \varepsilon$  by increasing  $\varepsilon$  slightly if necessary. To see this let  $(w, J, \varepsilon, v)$  be feasible for the LP (11). Then for any  $\delta > 0$  the quadrupel  $(w + \delta, J + \delta, \varepsilon + 3\delta, v + \delta)$  is strictly feasible, that is

$$\begin{aligned} (w + \delta) + (J + \delta) - (v + \delta) &\geq 1 + \delta, \quad w + \delta \geq \delta, \quad J + \delta > \delta, \\ \nabla(J + \delta) \cdot f + (J + \delta) + v + \delta &\leq \varepsilon + 2\delta < \varepsilon + 3\delta, \\ \beta(v + \delta) - \nabla(v + \delta) \cdot f &= \beta v - \nabla v \cdot f + \beta\delta \geq \beta\delta 0. \end{aligned} \tag{21}$$

and the cost only changes by  $4\delta\lambda(X)$ . By the Stone-Weierstraß theorem we can find polynomials  $(p_w, p_J, p_v)$  close enough to  $(w + \delta/2, J + \delta/2, v + \delta/2)$  in the topology of  $\mathcal{C}^1$  such that  $(p_w, p_J, \varepsilon + 3\delta, p_v)$  is still strictly feasible for the LP (11) (due to (21)). An SOS representation of  $(p_w, p_J, p_v)$  follows then from Putinar's positivstellensatz [18] which shows that  $(p_w, p_J, \varepsilon + 3\delta, p_v)$  is feasible for the SPD (19) for large enough  $k$  with the same cost of  $(w, J, \varepsilon, v)$  plus the additional term  $4\delta\lambda(X)$ . Since  $\delta > 0$  was arbitrary it follows that  $d_k \rightarrow p_5^* = \lambda(\mathcal{A})$ .  $\square$

An convergence estimate as in Theorem 2 and similarly an extension as in Corollary 1 can be obtained by the same arguments.

**Corollary 2** *In Theorem 3, for  $k \in \mathbb{N}$  let  $(w_k, J_k, \varepsilon_k, v_k)$  be optimal for (19) then we have*

$$\lambda((J_k^{-1}([0, \varepsilon_k]) \cap v_k^{-1}([0, \infty))) \cap X \setminus A) \leq d_k - \lambda(\mathcal{A}) \rightarrow 0 \text{ as } k \rightarrow \infty.$$

*Further if  $J_k^{-1}([0, \varepsilon_k]) \subset \overset{\circ}{X}$  then also  $\lambda(J_k^{-1}([0, \varepsilon_k]) \setminus \mathcal{A}) \leq d_k \setminus \lambda(\mathcal{A})$ .*

**Proof:** This holds by the same arguments as in Theorem 2 and Corollary 1 because the SPD (19) is a tightening of the LP (11).  $\square$

The cost of having guaranteed bounds and finding the global optima using SOS methods comes at the price of expensive scaling, that is, high degree  $d$  or dimension  $n$  makes the SDP intractable for current solvers/memory/computation power. Therefore, further structure should be exploited, such as sparsity [20], [22], [24] or symmetry [8].

## 7 Numerical examples

We illustrate our approach by three numerical examples that have been used in [21] and [11], namely by the following globally asymptotically stable system with attractor  $\mathcal{A} = \{(0, 0)\}$ , which does not allow for a polynomial Lyapunov function[1]

$$\begin{aligned} \dot{x} &= -2y(-x^4 + 2x^2y^2 + y^4) - 2x(x^2 + y^2)(x^4 + 2x^2y^2 - y^2) \\ \dot{y} &= 2x(x^4 + 2x^2y^2 - y^4) - 2y(x^2 + y^2)(-x^4 + 2x^2y^2 + y^4), \end{aligned} \tag{22}$$

the Van-der-Pol oscillator

$$\dot{x} = 2y, \quad \dot{y} = -0.8x - 10(x^2 - 0.21)y \tag{23}$$

and the Hénon map as an example of discrete time systems, given by

$$x_{m+1} = \frac{2}{3}(1 + y_m) - 2.1x_m^2, \quad y_{m+1} = 0.45x_m. \quad (24)$$

Van–der–Pol oscillator we observe as in [11] and [21] that the proposed method works very well and comparable with the method in [11], see Figure 1. We notice that the approximation from [21] seems to perform slightly better than our approach in terms of volume discrepancy with the real attractor.

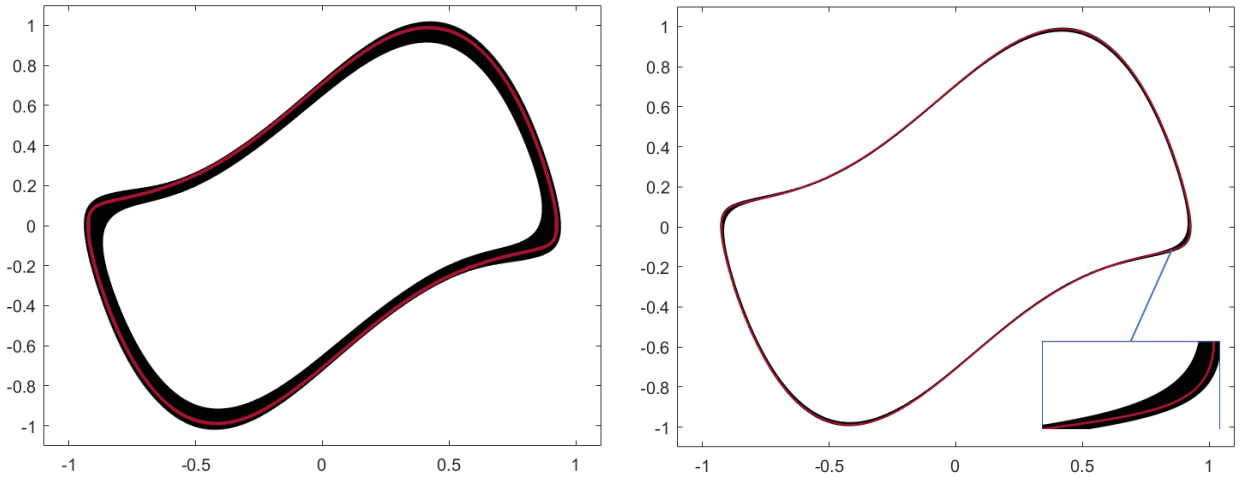


Figure 1: Outer approximations (black) of the attractor (red) for the Van–der–Pol oscillator for  $X = \{x : 0.4 \leq \|x\|_2 \leq 2\}$ . Left: approximation, degree 12 polynomials and  $\beta = 0.2$ . Right: approximation for polynomials up to degree 16 and  $\beta = 0.2$ .

For the Hénon map the situation is similar (Figure 2) compared to [21] – it takes higher degree polynomials to capture the complex topology of the Hénon attractor via (18) compared to [21].

In the case of the system (22) we notice some numerical instabilities in particular with the decision variable  $\varepsilon$  in (19), that penalizes the deviation from a Lyapunov function. We observed that the solutions obtained using Yalmip [16] and Mosek showed some robustness issues. The optimal value  $\varepsilon^*$  obtained by the solver was too small in some cases, while the function  $J$  still showed the correct behaviour. The set on the right in Figure (3) gives an outer approximation consisting of a positively invariant set and is consistent with the set obtained in [11]. The picture on the left illustrates the numerical problem concerning the decision variable  $\varepsilon$  in the SDP (19) – namely, that the optimal  $\varepsilon^*$  found by the solver was too small (order  $10^{-7}$ ) and created the holes that can be observed in the picture on the left in Figure (3). A correction, by increasing  $\varepsilon^*$  by 8 times leads to the picture on the right in Figure (3), which shows that the function  $J$  found by the solver provides the correct geometry.

The appearance of numerical issues at the example of (22) can be explained by reminding that the polynomial  $J$  tries to mimic a Lyapunov functions for the global attractor. Since there exists no polynomial Lyapunov function for (22), naturally, the problem gets numerically difficult.

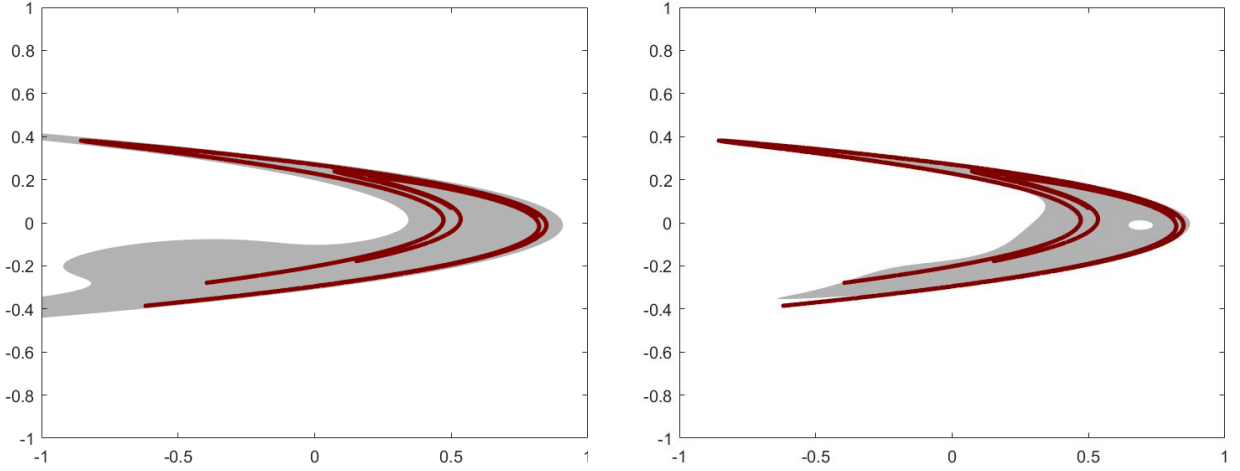


Figure 2: Outer approximations (black) of the attractor (red) for the Hénon map for  $X = [0, 1]^2$ . Left: approximation, degree 6 polynomials and  $\alpha = 0.002$ ,  $\gamma = 0.05$ . Right: approximation for polynomials up to degree 14 and with the same  $\alpha, \gamma$ .

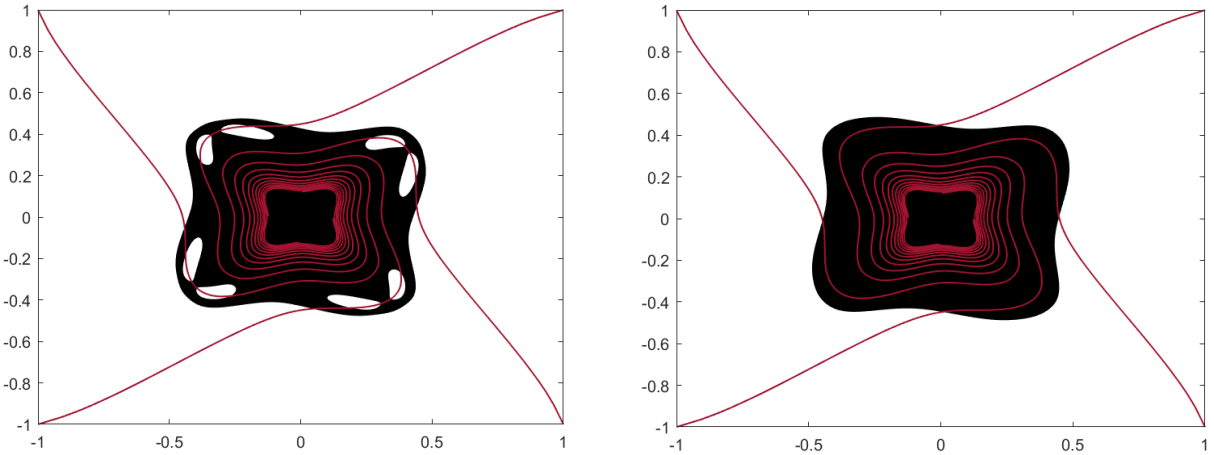


Figure 3: Outer approximations (black) of the attractor  $\mathcal{A} = \{(0, 0)\}$  and trajectories starting from  $(1, 1), (1, -1), (-1, 1), (-1, -1)$  (red) for (22) for  $X = [-1, 1]^2$ . Left: approximation by degree 16 polynomials and  $\beta = 0.2$ , the obtained  $\varepsilon^*$  in (19) is too small and causes incorrect behaviour of the set  $J^{-1}([0, \varepsilon^*])$ , see white “holes”. Right: Outer approximation for polynomials up to degree 16 and  $\beta = 0.2$  obtained by  $J^{-1}([0, 8 \cdot \varepsilon^*])$ .

**Remark 3** *The discounting parameter  $\beta > 0$  can be tuned and several solutions corresponding to different values of  $\beta$  can be intersected to improve the quality of the approximation [21].*

## 8 Conclusion

We presented a linear programming approach to outer approximations of global attractors via positively invariant sets. This builds on the recent works in [11] and [21]. We combine both methods by keeping their fundamental advantages. That is: We use the approximation via positively invariant sets from [11] by using their method of perturbed Lyapunov equation and we overcome their difficulty in evaluating the cost function by maintaining the linear structure of the optimization problem from [21].

This leads to an infinite dimensional linear programming problem characterizing (up to Lebesgue measure discrepancy zero) of global attractors via certain preimages. Applying sum-of-squares techniques as in the [11] and [21] allows us to formulate a converging hierarchy of semidefinite programs. This gives rise to convergent outer approximations of the global attractor by positively invariant semialgebraic sets.

We illustrated the approach by numerical examples, including one of a vector field which does not allow for a polynomial Lyapunov function. For this case we observed numerical issues which can be explained by the antagonizing structure of non-existence of polynomial Lyapunov functions but the search for a polynomial “almost” Lyapunov function.

Therefore, with regard to applications we think that our approach should be understood as a practical extension of [11] and as a qualitative extension of the previous work in [21], where it seems that the global attractor is approximated with less Lebesgue measure discrepancy but not necessarily by positively invariant sets are produced.

Appart from approximating the global attractor this work can be extended to bounding extreme events on attractors based on the work [6].

Future work could be devoted to use a similar technique to merge the works [10] and [9] in a similar way. Another possible direction could be towards data based algorithms for attractors as in [12] or towards attractors of partial differential equations – where Lyapunov functions also provide a powerful tool – via for instance [14], [4].

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