Validated Numerics:Algorithms and Practical Applications in Aerospace

Mioara Joldes

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Abstract

My lecture will survey some classical and recent validated computing algorithms based on the theory of set-valued analysis, in suitable functional spaces, as well as by combining symbolic and numerical computations. These techniques are illustrated with some applications which appear in practical space mission analysis and design. This is only a short summary of the talk.

1 Introduction to rigorous computing

The field of *rigorous computing* [13], sometimes called *validated computing* as well, uses numerical computations, yet is able to provide rigorous mathematical statements about the obtained results, such as sure and reasonably tight error bounds. Examples where the accuracy of numeric results has to be guaranteed range from computer-assisted proofs in dynamical systems to more practical engineering problems like the control of critical systems (spacecraft engines, surgical robots and so on).

Traditional validated computing methods are based on arbitrary precision libraries and interval arithmetic computations [7], a simple set arithmetic, which always returns an interval guaranteed to contain the correct result. To further bridge the gap between *experimental mathematics* and *computer proofs*, the symbolic-numeric aspect is considered: modern computer algebra algorithms (rooted in commutative and differential algebra) are employed via approximation theory (in suitable functional spaces) to obtain efficient approximations and analytic error bounds [2].

In this context, we provide an introduction to the theory of set-valued analysis and to some related symbolic-numeric algorithms. Firstly, we focus on effectively computing and manipulating various kinds of ball arithmetic (e.g., made by a polynomial approximation together with a validated error bound). We consider Taylor series expansions as well as series expansions based on orthogonal polynomials and associated approximation algorithms (Taylor and Chebyshev models) [1, 3]. Broadly speaking, the idea of working with polynomial approximations instead of functions is analogous to using floating-point arithmetic instead of real numbers [4, 12]: various *generalized Fourier series*, including Chebyshev series, play the role of floating-point numbers. However, one comprises a function space counterpart of interval arithmetic by providing rigorous

*LAAS-CNRS, Univ de Toulouse, CNRS, Toulouse, France (joldes@laas.fr)

truncation error bounds. The main appeal of this approach is the ability to solve functional equations rigorously using enclosure methods [6, 8].

Secondly, we exploit approximation algorithms mainly related to D-finite functions i.e., solutions of linear differential equations with polynomial coefficients. This property allows for developing a uniform theoretic and algorithmic treatment of these functions, an idea that has led to many symbolic computation applications in recent years [9].

Finally, we present the basic principles of a posteriori Newton-like validation methods, which allow for proving particular existence statements and explicit a posteriori bounds, for actual solutions to certain differential equations, using a combination of analytical arguments and numerical computations [2].

2 An example: the computation of orbital collision probability

An illustration of these techniques is related to the efficient finite precision evaluation of certain numerical functions. For instance, in [10], we deal with the computation of the orbital collision probability between two space objects involved in a so-called short-term encounter. In brief, a dramatically increasing number of space debris, located mainly in Low Earth Orbits, constitute a serious hazard for operational satellites. It is thus critical to provide adequate mitigation and avoidance strategies, when a conjunction includes at least one active satellite. Usually, the relative debris–satellite positions and velocities are only approximately known, hence the risk of an on-orbit collision is modeled as a collision probability. Under some additional hypotheses, the probability of collision is reduced to the integral of a 2D Gaussian probability density function over a disk. If it is evaluated to be sufficiently high, a collision avoidance maneuver is decided, but each such maneuver reduces the remaining satellite fuel and thus its active in-orbit life. Yet, a wrong computation which underestimates the risk, could result in the satellite loss. This implies that both efficient and accurate algorithms are needed for this evaluation.

The computational method presented in [10] is based on a power series expression for the integral, derived by use of Laplace transform and D-finite functions properties. Furthermore, a finite precision evaluation method with reduced cancellation [5] is employed. Analytic bounds on the truncation error are also derived and are used to obtain a very accurate algorithm.

First implemented and tested on a very large number of practical cases from a database by the French Space Agency (CNES), an implementation of this algorithm has been embedded and tested on board on the ESOC OPS-SAT 3-Units CubeSat satellite via the CNES ASTERIA software [11].

3 Conclusion

The previous example shows that computer algebra can provide the theoretical and algorithmic tools needed to design highly efficient symbolic-numeric algorithms and implementations, with practical reach for critical systems, where the accuracy of numeric results has to be guaranteed, like those arising in attitude and orbit control systems of spacecraft.

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