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Viability and Exponentially Stable Trajectories for Differential Inclusions in Wasserstein Spaces

Benoît Bonnet-Weill* and Hélène Frankowska†

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Abstract

In this article, we prove a general viability theorem for continuity inclusions in Wasserstein spaces, and provide an application thereof to the existence of exponentially stable trajectories obtained via the second method of Lyapunov.

1 Introduction

During the past decade, the study of continuity equations in the space of measures has gained a tremendous amount of steam. Originally motivated by applications to crowd motion [13, 15, 22], opinion propagation [1, 23] and game theory [11, 19, 20], the investigation of the mathematical properties of multi-agent systems – mostly studied via optimal transport techniques in the mean-field setting – has become a broad field of research at the intersection between pure, applied and computational mathematics. In this context, the literature devoted to the analysis of control problems formulated in the so-called *Wasserstein spaces* has been steadily growing for several years (see e.g. [5, 6, 7, 8, 9, 10, 12, 14] and the references therein).

The aim of this paper is to present a novel viability result for set-valued dynamics in Wasserstein spaces, following the terminology introduced in our previous work [6]. The concept of viability, which goes back to the eighties for differential inclusions, has tremendous applications in control theory, e.g. to ensure the existence of solutions to state-constrained control systems [4, Chapter 10], to derive optimality conditions in the form of Hamilton-Jacobi-Bellman equations [17] or to characterise the existence of Lyapunov stable trajectories [16]. Motivated by these aspects, we derive a general viability result for set-valued dynamics in Wasserstein spaces in Theorem 4.2. The latter relies strongly on Theorem 3.1, which is a technical prerequisite ensuring the existence of solutions with prescribed initial velocities to continuity inclusions. Finally, we apply these results in Theorem 5.2, where we illustrate how they can be used to obtain exponentially stable trajectories in terms of a given Lyapunov function.

The organisation of the paper is the following. After recollecting preliminary notions of optimal transport and functional analysis in Section 2, we prove the existence of solutions to continuity inclusions with prescribed initial velocities in Section 3, and use the corresponding result to prove the viability theorem in Section 4. Then, in Section 5, we discuss the existence of exponentially stable trajectories via the second method of Lyapunov.

2 Preliminaries

In this section, we recall preliminary notions of measure theory, optimal transport and set-valued analysis, for which we refer to [2, 3] and [4] respectively.

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2.1 Optimal transport and calculus in Wasserstein spaces

In the sequel, $\mathscr{P}(\Omega)$ will denote the space of Borel probability measures over a Borel measurable set $\Omega \subset \mathbb{R}^d$, endowed with the standard *narrow topology* (see [3, Chapter 5]). Let $\mathscr{P}_2(\mathbb{R}^d)$ be the subset of probability measures whose 2-momentum $\mathcal{M}_2^2(\mu) := \int_{\mathbb{R}^d} |x|^2 \mathrm{d}\mu(x)$ is finite, and $\mathscr{P}_c(\mathbb{R}^d)$ be that of measures with compact support. Given $\mu \in \mathscr{P}(\mathbb{R}^d)$ and $p \in [1, +\infty)$, we denote by $(L^p(\mathbb{R}^d, \mathbb{R}^d; \mu), \|\cdot\|_{L^p(\mu)})$ the Banach space of maps from $\mathbb{R}^d$ into itself that are p-summable with respect to μ, and by $L^\infty(\mathbb{R}^d, \mathbb{R}^d; \mu)$ that of μ-essentially bounded maps. We will also denote by $\mathscr{L}^1$ the standard Lebesgue measure over $\mathbb{R}$, and use the shorter notation $(L^p(I, \mathbb{R}_+), \|\cdot\|_p)$ with $p \in [1, +\infty]$ for the Lebesgue spaces of maps going from an interval $I \subset \mathbb{R}$ into $\mathbb{R}_+$.

Given a Borel map $f : \mathbb{R}^d \to \mathbb{R}^d$, we define the *pushforward* $f_\sharp \mu \in \mathscr{P}(\mathbb{R}^d)$ of μ through f as the unique measure satisfying $f_\sharp \mu(B) := \mu(f^{-1}(B))$ for every Borel set $B \subset \mathbb{R}^d$. Using this notation, we can define the set of *transport plans* between two elements $\mu, \nu \in \mathscr{P}(\mathbb{R}^d)$ as

$$\Gamma(\mu, \nu) := \Big\{ \gamma \in \mathscr{P}(\mathbb{R}^{2d}) \text{ s.t. } \pi^1_\sharp \gamma = \mu \text{ and } \pi^2_\sharp \gamma = \nu \Big\},$$

where $\pi^1, \pi^2 : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ are the projections onto the first and second factors respectively. Leveraging this notion, we can in turn recall the notion of *Wasserstein distance* between measures.

Definition 2.1 (Wasserstein distance)**.** *The quantity*

$$W_2(\mu, \nu) := \min_{\gamma \in \Gamma(\mu,\nu)} \left(\int_{\mathbb{R}^{2d}} |x - y|^2 \mathrm{d}\gamma(x, y) \right)^{1/2},$$

defines a distance between any two measures $\mu, \nu \in \mathscr{P}_2(\mathbb{R}^d)$, and we denote by $\Gamma_o(\mu, \nu)$ the (nonempty) set of transport plans at which the minimum is attained.

Following [3, 21], the complete separable metric space $(\mathscr{P}_2(\mathbb{R}^d), W_2)$ – usually called *Wasserstein space* – can be formally endowed with the structure of a differentiable manifold. In the sequel, we will also consider the (non-complete) metric space $(\mathscr{P}_c(\mathbb{R}^d), W_2)$ of compactly supported measures equipped with the W_2-metric. We end this first preliminary section by stating a simplified version of a pivotal result of Wasserstein calculus, allowing to describe the superdifferential of the squared Wasserstein distance (see e.g. [3, Theorem 10.2.2]).

Proposition 2.2. *For every $\mu, \nu \in \mathscr{P}_2(\mathbb{R}^d)$ and each $\gamma \in \Gamma_o(\mu, \nu)$, it holds that*

$$\begin{aligned} &\tfrac{1}{2} W_2^2((\mathrm{Id} + h\xi)_\sharp \mu, \nu) - \tfrac{1}{2} W_2^2(\mu, \nu) \\ &\qquad \leq h \int_{\mathbb{R}^{2d}} \langle \xi(x), x - y \rangle \mathrm{d}\gamma(x, y) + h^2 \, \|\xi\|^2_{L^2(\mu)} \end{aligned}$$

for every $\xi \in L^2(\mathbb{R}^d, \mathbb{R}^d; \mu)$ and each $h > 0$.

2.2 Elements of set-valued analysis

In what follows, given a complete separable metric space $(\mathscr{S}, d_{\mathscr{S}}(\cdot, \cdot))$ and a Fréchet space $(E, d_E(\cdot, \cdot))$ (see e.g. [18]), we shall write $\mathcal{F} : \mathscr{S} \rightrightarrows E$ to denote *set-valued maps* from $\mathscr{S}$ into E. We will also denote by $C^0(\mathbb{R}^d, \mathscr{S})$ and $\mathrm{AC}([0, T], \mathscr{S})$ the spaces of continuous and absolutely continuous maps from $\mathbb{R}^d$ and $[0, T]$ into $\mathscr{S}$ respectively, and write $\mathrm{Lip}(\phi\,; K)$ for the Lipschitz constant of a map $\phi : \mathbb{R}^d \to \mathscr{S}$ over some set $K \subset \mathbb{R}^d$.

In the coming definitions, we recall the notions of *measurability* and *lower-semicontinuity* for multifunctions. Therein and in general, $\mathbb{B}_{\mathscr{S}}(s, r)$ will stand for the ball of radius $r > 0$ centered at $s \in \mathscr{S}$.

Definition 2.3 (Measurability)**.** *A set-valued mapping $\mathcal{F} : [0, T] \rightrightarrows E$ is $\mathscr{L}^1$-measurable provided that*

$$\mathcal{F}^{-1}(\mathcal{O}) := \Big\{ s \in \mathscr{S} \ \text{ s.t. } \mathcal{F}(s) \cap \mathcal{O} \neq \emptyset \Big\}$$

is $\mathscr{L}^1$-measurable for each open set $\mathcal{O} \subset E$. We then say that an $\mathscr{L}^1$-measurable function $t \in [0, T] \mapsto f(t) \in \mathcal{F}(t)$ is a measurable selection.

Definition 2.4 (Lower-semicontinuity)**.** *A set-valued mapping $\mathcal{F} : \mathscr{S} \rightrightarrows E$ is lower-semicontinuous at $s \in \mathscr{S}$ if for each open set $\mathcal{U} \subset E$ such that $\mathcal{F}(s) \cap \mathcal{U} \neq \emptyset$, there exists $\delta > 0$ such that $\mathcal{F}(s') \cap \mathcal{U} \neq \emptyset$ for all $s' \in \mathbb{B}_{\mathscr{S}}(s, \delta)$.*

In what follows, we will denote by $\overline{\mathrm{co}}\, B$ the *closed convex hull* of a set $B \subset E$, defined by

$$\overline{\mathrm{co}}\, B := \overline{\bigcup_{N \geq 1} \left\{ \sum_{i=1}^{N} \alpha_i b_i \ \text{ s.t. } \alpha_i \geq 0,\ b_i \in B,\ \sum_{i=1}^{N} \alpha_i = 1 \right\}}^{E}.$$

We will also use the standard notations $\mathrm{int}(\mathcal{Q})$ and $\partial \mathcal{Q} := \mathcal{Q} \setminus \mathrm{int}(\mathcal{Q})$ to refer to the interior and topological boundary of a closed set $\mathcal{Q} \subset \mathscr{S}$ respectively, as well as

$$\mathrm{dist}_{\mathscr{S}}(\mathcal{Q}\,; \mathcal{R}) := \inf \left\{ d_{\mathscr{S}}(s, s') \ \text{ s.t. } s \in \mathcal{Q} \text{ and } s' \in \mathcal{R} \right\}$$

for the distance between two closed subsets of $\mathscr{S}$. We finally recall that $(C^0(K, \mathbb{R}^d), \|\cdot\|_{C^0(K,\mathbb{R}^d)})$ is a separable Banach space whenever $K \subset \mathbb{R}^d$ is compact.

2.3 Continuity equations and inclusions in $(\mathscr{P}_c(\mathbb{R}^d), W_2)$

In the sequel, we will consider measure dynamics described by *continuity equations* of the form

$$\partial_t \mu(t) + \mathrm{div}_x(v(t)\mu(t)) = 0, \tag{2.1}$$

whose solutions are understood in the sense of distributions, namely

$$\int_0^T \int_{\mathbb{R}^d} \Big(\partial_t \phi(t,x) + \langle \nabla_x \phi(t,x), v(t,x) \rangle \Big) \mathrm{d}\mu(t)(x) \mathrm{d}t = 0,$$

for each $\phi \in C_c^\infty((0,T) \times \mathbb{R}^d)$. Here and in what follows, we shall assume that the *velocity-fields* $v : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ satisfy the following assumptions.

Hypotheses (CE).

(i) The application $t \in [0,T] \mapsto v(t,x) \in \mathbb{R}^d$ is $\mathscr{L}^1$-measurable for all $x \in \mathbb{R}^d$, and there exists a map $m(\cdot) \in L^1([0,T], \mathbb{R}_+)$ such that

$$|v(t,x)| \leq m(t)(1 + |x|),$$

for $\mathscr{L}^1$-almost every $t \in [0,T]$ and all $x \in \mathbb{R}^d$.

(ii) For each compact set $K \subset \mathbb{R}^d$, there exists a map $l_K(\cdot) \in L^1([0,T], \mathbb{R}_+)$ such that

$$\mathrm{Lip}(v(t)\,; K) \leq l_K(t),$$

for $\mathscr{L}^1$-almost every $t \in [0,T]$.

Under these Cauchy-Lipschitz assumptions, we have the following well-posedness result for (2.1) (see e.g. [6]).

Theorem 2.5 (Well-posedness of (2.1))**.** *Let $r > 0$ and suppose that $v : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies hypotheses* **(CE)**. *Then for each $(\tau, \mu_\tau) \in [0,T] \times \mathscr{P}(B(0,r))$, there exists a unique solution $\mu(\cdot) \in \mathrm{AC}([\tau,T], \mathscr{P}_c(\mathbb{R}^d))$ to*

$$\begin{cases} \partial_t \mu(t) + \mathrm{div}_x(v(t)\mu(t)) = 0, \\ \mu(\tau) = \mu_\tau. \end{cases} \tag{2.2}$$

Moreover, the curve $\mu(\cdot)$ satisfies

$$\mathrm{supp}(\mu(t)) \subset B(0, R_r), \quad W_2(\mu(t), \mu(s)) \leq c_r \int_s^t m(\zeta) \mathrm{d}\zeta, \tag{2.3}$$

for all $\tau \leq s \leq t \leq T$, where $R_r, c_r > 0$ depend only on the magnitudes of $r, \|m(\cdot)\|_1$. Furthermore, the latter can be represented explicitly as

$$\mu(t) = \Phi^v_{(\tau,t)}(\cdot)_\sharp \mu(\tau), \tag{2.4}$$

for every $t \in [\tau, T]$, where $(\Phi_{(\tau,t)}(\cdot))_{t\in[0,T]}$ are the flows of diffeomorphisms defined as the unique solution of

$$\begin{cases} \partial_t \Phi^v_{(\tau,t)}(x) = v(t, \Phi^v_{(\tau,t)}(x)), \\ \Phi^v_{(\tau,\tau)}(x) = x, \end{cases}$$

for all $x \in \mathbb{R}^d$.

In our previous work [6], we proposed a set-valued generalisation of Cauchy problems of the form (2.1), for which the right-hand sides are *set-valued maps*

$$V : [0,T] \times \mathscr{P}_2(\mathbb{R}^d) \rightrightarrows C^0(\mathbb{R}^d, \mathbb{R}^d),$$

whose images typically lie within subsets of locally Lipschitz and sublinear vector fields.

Definition 2.6 (Continuity inclusions)**.** *A curve of measures $\mu(\cdot) \in \mathrm{AC}([\tau, T], \mathscr{P}_c(\mathbb{R}^d))$ is a solution of*

$$\begin{cases} \partial_t \mu(t) \in -\mathrm{div}_x\Big(V(t, \mu(t))\mu(t)\Big), \\ \mu(\tau) = \mu_\tau, \end{cases} \tag{2.5}$$

if there exists a map $t \in [0,T] \mapsto v(t) \in V(t, \mu(t))$ such that $\mu(\cdot)$ solves (2.2), with $t \in [0,T] \mapsto v(t)_{|K} \in C^0(K, \mathbb{R}^d)$ being $\mathscr{L}^1$-measurable for each compact set $K \subset \mathbb{R}^d$.

Throughout this article, we will impose the following set of assumptions on the set-valued map $V(\cdot,\cdot)$, and will sometimes use the notation

$$V(t,\mu)_{|K} := \Big\{ v_{|K} \in C^0(K, \mathbb{R}^d) \text{ s.t. } v \in V(t,\mu) \Big\}$$

where $K \subset \mathbb{R}^d$ is a compact set.

Hypotheses (CI).

- (i) *The set-valued map $t \in [0,T] \rightrightarrows V(t,\mu)_{|K} \subset C^0(K, \mathbb{R}^d)$ is lower-semicontinuous with closed non-empty images for all $\mu \in \mathscr{P}(K)$ whenever $K \subset \mathbb{R}^d$ is a compact set.*
- (ii) *There exists a map $m(\cdot) \in L^1([0,T], \mathbb{R}_+)$ such that for all $\mu \in \mathscr{P}_2(\mathbb{R}^d)$ and each $v \in V(t,\mu)$, it holds*
$$|v(x)| \leq m(t)\Big(1 + |x| + \mathcal{M}_2(\mu)\Big),$$
for $\mathscr{L}^1$-almost every $t \in [0,T]$ and all $x \in \mathbb{R}^d$.
- (iii) *There exists $l(\cdot) \in L^1([0,T], \mathbb{R}_+)$ such that for $\mathscr{L}^1$-almost every $t \in [0,T]$, all $\mu \in \mathscr{P}_2(\mathbb{R}^d)$ and each $v \in V(t,\mu)$, it holds*
$$\mathrm{Lip}(v\,; \mathbb{R}^d) \leq l(t).$$
- (iv) *There exists $L(\cdot) \in L^1([0,T], \mathbb{R}_+)$ such that for $\mathscr{L}^1$-almost every $t \in [0,T]$, all $\mu, \nu \in \mathscr{P}_2(\mathbb{R}^d)$ and each $v \in V(t,\mu)$, there exists $w \in V(t,\nu)$ such that*
$$\sup_{x\in\mathbb{R}^d} |v(x) - w(x)| \leq L(t) W_2(\mu,\nu).$$

We would like to stress that hypothesis **(CI)**-(i) is not sharp compared to its natural measurability counterparts in **(CE)**-(i) or [6, p.608], but greatly simplifies the proofs of Section 3. Similarly, one could opt for localised versions of **(CI)**-(iii) and (iv), at the price of extra technicalities. For the sake of simplicity and readability, we defer the investigation of viability properties in such a general context to a subsequent article.

In the following theorem, we recall a condensed version of the Filippov estimates derived in [6, Theorem 4].

Theorem 2.7 (Filippov estimates)**.** *Let $w : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ be a velocity field satisfying hypotheses* **(CE)**, *$\nu^0 \in \mathscr{P}_c(\mathbb{R}^d)$ and $\nu(\cdot)$ be the unique solution of*

$$\partial_t \nu(t) + \mathrm{div}_x(w(t)\nu(t)) = 0, \qquad \nu(0) = \nu^0.$$

Moreover, let $r > 0$ be such that $\mathrm{supp}(\nu(t)) \subset B(0,r)$ for all $t \in [0,T]$, and consider the Lebesgue integrable map

$$\eta_\nu : t \in [0,T] \mapsto \mathrm{dist}_{C^0(B(0,r),\mathbb{R}^d)}\Big(w(t)\,; V(t,\nu(t))_{|B(0,r)}\Big).$$

Then for every $(\tau,\mu_\tau) \in [0,T] \times \mathscr{P}(B(0,r))$, there exists a solution $\mu(\cdot) \in \mathrm{AC}([\tau,T],\mathscr{P}_c(\mathbb{R}^d))$ to (2.5) such that $\mathrm{supp}(\mu(t)) \subset B(0,R_r)$ and

$$W_2(\mu(t),\nu(t)) \le C_r\Big(W_2(\mu_\tau,\nu(\tau)) + \int_\tau^t \eta_\nu(s)\mathrm{d}s\Big)$$

for all times $t \in [\tau,T]$. Therein, $R_r > 0$ only depends on the magnitudes of $r, \|m(\cdot)\|_1$, while $C_r > 0$ only depends on those of $r, \|m(\cdot)\|_1, \|l(\cdot)\|_1$ and $\|L(\cdot)\|_1$.

In what follows, we will often work with the *reachable* and *solution* sets of the Cauchy problem (2.5).

Definition 2.8 (Reachable and solution sets)**.** *Given $\mu^0 \in \mathscr{P}_c(\mathbb{R}^d)$, we define the solution set of (2.5) as*

$$\mathcal{S}_{[0,T]}(\mu^0) := \Big\{\mu(\cdot) \in \mathrm{AC}([0,T],\mathscr{P}_c(\mathbb{R}^d)) \text{ solution of (2.5)} \\ \text{with } (\tau,\mu_\tau) = (0,\mu^0)\Big\}$$

and similarly, we denote the underlying reachable set at time $t \in [0,T]$ by

$$\mathcal{R}_t(\mu^0) := \Big\{\mu(t) \text{ such that } \mu(\cdot) \in \mathcal{S}_{[0,T]}(\mu^0)\Big\}. \tag{2.6}$$

Theorem 2.9 (Properties of $\mathcal{S}_{[0,T]}(\mu^0)$ and $\mathcal{R}_t(\mu^0)$)**.** *Let $V : [0,T] \times \mathscr{P}_2(\mathbb{R}^d) \rightrightarrows C^0(\mathbb{R}^d,\mathbb{R}^d)$ be a set-valued map with convex images satisfying hypotheses* **(CI)**.

Then for each $r > 0$ and any $\mu^0 \in \mathscr{P}(B(0,r))$, the sets $\mathcal{S}_{[0,T]}(\mu^0) \subset C^0([0,T],\mathscr{P}_2(B(0,R_r)))$ and $\mathcal{R}_t(\mu^0) \subset \mathscr{P}_2(B(0,R_r))$ are compact, with $R_r > 0$ being as in Theorem 2.7. Moreover, the reachable sets satisfy the semigroup property

$$\mathcal{R}_t(\mu^0) = \mathcal{R}_{t-\tau}(\mathcal{R}_\tau(\mu^0)), \tag{2.7}$$

for all times $0 \le \tau \le t \le T$.

Proof. See the arguments in [6, Theorem 6]. □

3 Existence of measure curves with prescribed initial velocities

In this section, we establish the existence of solutions to (2.5) with prescribed initial velocities. This fairly non-trivial result will be instrumental in the proof of the viability theorem of Section 4.

Theorem 3.1 (Curves with given initial velocities)**.** *Let $V : [0,T] \times \mathscr{P}_2(\mathbb{R}^d) \rightrightarrows C^0(\mathbb{R}^d,\mathbb{R}^d)$ be a set-value map satisfying hypotheses* **(CI)**.

Then, there exists a set $\mathscr{T} \subset (0,T)$ of full $\mathscr{L}^1$-measure such that for each $r > 0$, any $(\tau,\mu_\tau) \in \mathscr{T} \times \mathscr{P}(B(0,r))$ and $v_\tau \in V(\tau,\mu_\tau)$, there exists a solution $\mu(\cdot) \in \mathrm{AC}([\tau,T],\mathscr{P}_c(\mathbb{R}^d))$ of (2.5) such that

$$W_2(\mu(\tau+h),(\mathrm{Id}+hv_\tau)_\sharp\mu_\tau) = o_\tau(h), \tag{3.1}$$

for all $h > 0$ sufficiently small.

Proof. First and foremost, let it be noted that by applying Theorem 2.7 with a constant curve of measures $\nu(\cdot) \equiv \nu \in \mathscr{P}(B(0,r))$, there exists $R_r > 0$ such that

$$\mathrm{supp}(\mu(t)) \subset B(0, R_r),$$

for any solution of (2.5) starting from $\mu_\tau \in \mathscr{P}(B(0,r))$ at time $\tau \in [0,T]$. We also define $\mathscr{T} \subset (0,T)$ of full $\mathscr{L}^1$-measure as the intersection of the sets of Lebesgue points (see e.g. [2, Corollary 2.23]) of $m(\cdot), l(\cdot)$ and $L(\cdot)$ at which hypotheses **(CI)**-(ii), (iii) and (iv) hold.

Fix $\tau \in \mathscr{T}$ and observe that, as a consequence of hypotheses **(CI)**, every $v_\tau \in V(\tau, \mu_\tau)$ satisfies hypotheses **(CE)**. Whence, there exists a unique solution $\nu(\cdot) \in \mathrm{AC}([\tau,T], \mathscr{P}_c(\mathbb{R}^d))$ to the Cauchy problem

$$\begin{cases} \partial_t \nu(t) + \mathrm{div}_x(v_\tau \nu(t)) = 0, \\ \nu(\tau) = \mu_\tau, \end{cases}$$

which can be represented as

$$\nu(t) = \Phi^{v_\tau}_{(\tau,t)}(\cdot)_\sharp \mu_\tau.$$

Under hypotheses **(CE)**, it follows from standard linearisation techniques for characteristic flows (see e.g. [8, Appendix A]) that

$$\Phi^{v_\tau}_{(\tau,\tau+h)}(x) = x + h v_\tau(x) + o_{\tau,x}(h), \tag{3.2}$$

where $\int_0^T \sup_{x \in B(0,R)} |o_{\tau,x}(h)| \mathrm{d}\tau = o_R(h)$ for each $R > 0$. Furthermore, upon noticing that

$$\Big(\mathrm{Id} + h v_\tau, \Phi^{v_\tau}_{(\tau,\tau+h)}\Big)_\sharp \mu_\tau \in \Gamma\Big((\mathrm{Id} + h v_\tau)_\sharp \mu_\tau, \nu(\tau+h)\Big)$$

we can easily get from (3.2) the following distance estimate

$$\begin{aligned} W_2(\nu(\tau+h), &(\mathrm{Id} + h v_\tau)_\sharp \mu_\tau) \\ &\leq \|\Phi^{v_\tau}_{(\tau,\tau+h)} - \mathrm{Id} - h v_\tau\|_{L^2(\mu_\tau)} = o_\tau(h). \end{aligned} \tag{3.3}$$

Setting $K := B(0,R_r)$, observe that under hypothesis **(CI)**-(i), the set-valued map $t \in [0,T] \rightrightarrows V(t,\mu_\tau)_{|K}$ is lower-semicontinuous. Thus, for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$v_\tau \in V(t,\mu_\tau)_{|K} + \varepsilon \mathbb{B}_{C^0(K,\mathbb{R}^d)},$$

for all times $t \in [\tau, \tau+\delta]$. In turn, by using hypothesis **(CI)**-(iv) in the previous identity, it further holds that

$$v_\tau \in V(t,\nu(t))_{|K} + \Big(\varepsilon + L(t) W_2(\mu_\tau, \nu(t))\Big) \mathbb{B}_{C^0(K,\mathbb{R}^d)}. \tag{3.4}$$

Noticing that by construction, the curve $\nu(\cdot)$ satisfies

$$W_2(\mu_\tau, \nu(t)) \leq c_r m(\tau)(t-\tau)$$

for all times $t \in [\tau, T]$, it then follows from (3.4) that

$$\begin{aligned} \int_\tau^{\tau+h} \mathrm{dist}_{C^0(K,\mathbb{R}^d)}\Big(v_\tau\,; V(t,\nu(t))_{|K}\Big) \mathrm{d}t \\ \leq \varepsilon h + c_r m(\tau)\, h \int_\tau^{\tau+h} L(t) \mathrm{d}t \end{aligned} \tag{3.5}$$

for each $h > 0$ sufficiently small. By applying Theorem 2.7 in conjunction with (3.5) while recalling that $\tau \in \mathscr{T}$ is a Lebesgue point of $L(\cdot)$, we obtain the existence of a solution $\mu(\cdot) \in \mathrm{AC}([\tau,T], \mathscr{P}_c(\mathbb{R}^d))$ of (2.5) such that

$$W_2(\mu(\tau+h), \nu(\tau+h)) \leq C_r \varepsilon h + o_\tau(h), \tag{3.6}$$

for every $\varepsilon > 0$, whenever $h > 0$ is sufficiently small. Whence, by merging the estimates of (3.3) and (3.6), we can finally conclude that the curve $\mu(\cdot) \in \mathrm{AC}([\tau,T], \mathscr{P}_c(\mathbb{R}^d))$ satisfies (3.1). ∎

4 Viability for continuity inclusions

In this section, we prove a general viability result for (2.5) involving the *contingent cone* to the set of constraints.

Definition 4.1 (Contingent cones)**.** *The contingent cone to a set $\mathcal{Q} \subset \mathscr{P}_2(\mathbb{R}^d)$ at some $\mu \in \mathcal{Q}$ is defined by*

$$T_{\mathcal{Q}}(\mu) := \Big\{ \xi \in L^2(\mathbb{R}^d, \mathbb{R}^d; \mu) \text{ s.t. there exists } h_i \to 0^+ \\ \textit{for which } \mathrm{dist}_{\mathscr{P}_2}\big((\mathrm{Id} + h_i\xi)_\sharp \mu \,;\, \mathcal{Q}\big) = o(h_i) \Big\}.$$

In what follows, we shall say that a subset $\mathcal{Q} \subset \mathscr{P}_2(\mathbb{R}^d)$ is *proper* if $\mathcal{Q} \cap \mathbb{B}_{\mathscr{P}_2}(\mu, r)$ is compact for every $\mu \in \mathcal{Q}$ and each $r > 0$.

Theorem 4.2 (Viability for proper constraints)**.** *Let $V : [0,T] \times \mathscr{P}_2(\mathbb{R}^d) \rightrightarrows C^0(\mathbb{R}^d, \mathbb{R}^d)$ be a set-valued map with convex images satisfying hypotheses* **(CI)***, and $\mathcal{Q} \subset \mathscr{P}_2(\mathbb{R}^d)$ be a proper set such that*

$$V(t, \nu) \cap \overline{\mathrm{co}}\, T_{\mathcal{Q}}(\nu) \neq \emptyset$$

for $\mathscr{L}^1$-almost every $t \in [0,T]$ and each $\nu \in \mathcal{Q}$. Then for each $\mu^0 \in \mathcal{Q} \cap \mathscr{P}_c(\mathbb{R}^d)$, there exists a curve $\mu(\cdot) \in \mathcal{S}_{[0,T]}(\mu^0)$ such that $\mu(t) \in \mathcal{Q}$ for all times $t \in [0,T]$.

Proof. The proof of this result relies on an estimate "à la Grönwall" on the distance between $\mathcal{R}_t(\mu^0)$ defined as in (2.6) and $\mathcal{Q}$. Consider $\mu^0 \in \mathcal{Q} \cap \mathscr{P}_c(\mathbb{R}^d)$ and fix $r > 0$ such that $\mu^0 \in \mathscr{P}(B(0,r))$. Let then $R_r \geq r > 0$ be as in Theorem 2.7 and choose $R \geq R_r > 0$ in such a way that

$$\begin{aligned} &\mathrm{dist}_{\mathscr{P}_2}\Big(\mathcal{R}_t(\mu^0) \,;\, \partial\mathbb{B}_{\mathscr{P}_2}(\delta_0, R)\Big) \\ &\qquad \geq \mathrm{dist}_{\mathscr{P}_2}(\mathcal{R}_t(\mu^0) \,;\, \mathcal{Q}) + 1. \end{aligned} \tag{4.1}$$

for all times $t \in [0,T]$. We then define the restricted constraints set $\mathcal{Q}_R := \mathcal{Q} \cap \mathbb{B}_{\mathscr{P}_2}(\delta_0, R)$, which is compact since $\mathcal{Q}$ is proper, along with the distance function

$$g : t \in [0,T] \mapsto \mathrm{dist}_{\mathscr{P}_2}\Big(\mathcal{R}_t(\mu^0) \,;\, \mathcal{Q}_R\Big). \tag{4.2}$$

It can be checked easily that $g(\cdot) \in \mathrm{AC}([0,T], \mathbb{R}_+)$, and we denote by $\mathscr{D}_g \subset (0,T)$ the set of full $\mathscr{L}^1$-measure where it is differentiable.

Step 1 – Distance estimate Noticing at first that $g(0) = 0$ by construction, we claim that $g(\cdot) \equiv 0$ on $[0,T]$. Indeed otherwise, by the continuity of $g(\cdot)$, there exists some $t \in [0,T]$ and $\delta > 0$ such that $g(t) = 0$ and while $g(\tau) > 0$ for each $\tau \in (t, t+\delta)$. Let $\tau \in (t, t+\delta) \cap \mathscr{T} \cap \mathscr{D}_g$ with $\mathscr{T} \subset (0,T)$ being defined as in Theorem 3.1, and observe that since $\mathcal{R}_\tau(\mu^0)$ and $\mathcal{Q}_R$ are both compact – the former by Theorem 2.9 and the latter by construction –, it then holds that

$$g(\tau) = W_2(\mu_\tau, \nu_\tau),$$

for some $\mu_\tau \in \mathcal{R}_\tau(\mu^0)$ and $\nu_\tau \in \mathcal{Q}_R$. Moreover by (4.1), one necessarily has that

$$\nu_\tau \in \mathcal{Q} \cap \mathrm{int}(\mathbb{B}_{\mathscr{P}_2}(\delta_0, R)).$$

Thus $T_{\mathcal{Q}_R}(\nu_\tau) = T_{\mathcal{Q}}(\nu_\tau)$ and for each $\xi_\tau \in T_{\mathcal{Q}}(\nu_\tau)$, there exists a sequence $h_i \to 0^+$ such that

$$\begin{aligned} W_2(\mu_\tau, (\mathrm{Id} + h_i\xi_\tau)_\sharp \nu_\tau) &\geq \mathrm{dist}_{\mathscr{P}_2}(\mu_\tau \,;\, \mathcal{Q}_R) + o_\tau(h_i) \\ &= W_2(\mu_\tau, \nu_\tau) + o_\tau(h_i). \end{aligned} \tag{4.3}$$

Hence, by fixing an arbitrary $\gamma_\tau \in \Gamma_o(\mu_\tau, \nu_\tau)$ and applying Proposition 2.2 with $(\pi^2, \pi^1)_\sharp \gamma_\tau \in \Gamma_o(\nu_\tau, \mu_\tau)$, it holds

$$\begin{aligned} o_\tau(1) &\leq \tfrac{1}{2h_i} W_2^2(\mu_\tau, (\mathrm{Id} + h_i \xi_\tau)_\sharp \nu_\tau) - \tfrac{1}{2h_i} W_2^2(\mu_\tau, \nu_\tau) \\ &\leq \int_{\mathbb{R}^{2d}} \langle \xi_\tau(y), y - x \rangle \mathrm{d}\gamma_\tau(x, y) + h_i \, \|\xi_\tau\|_{L^2(\nu_\tau)}, \end{aligned}$$

and we subsequently obtain upon letting $i \to +\infty$ that

$$\int_{\mathbb{R}^{2d}} \langle \xi_\tau(y), x - y \rangle \mathrm{d}\gamma_\tau(x, y) \leq 0, \tag{4.4}$$

for each $\xi_\tau \in T_{\mathcal{Q}}(\nu_\tau)$ and all $\gamma_\tau \in \Gamma_o(\mu_\tau, \nu_\tau)$.

On another note, by Theorem 3.1, there exists for every $v_\tau \in V(\tau, \mu_\tau)$ a solution $\mu(\cdot)$ of (2.5) such that

$$W_2(\mu(\tau + h), (\mathrm{Id} + h v_\tau)_\sharp \mu_\tau) = o_\tau(h)$$

for any small $h > 0$. Whence, we can estimate the forward difference quotient of $\frac{1}{2} g^2(\cdot)$ at $\tau \in \mathscr{T} \cap \mathscr{D}_g$ as

$$\begin{aligned} \tfrac{1}{2h}(g^2(\tau + h) - g^2(\tau)) \leq \tfrac{1}{2h} W_2^2((\mathrm{Id} + h v_\tau)_\sharp \mu_\tau, \nu_\tau) \\ - \tfrac{1}{2h} W_2^2(\mu_\tau, \nu_\tau) + o_\tau(1), \end{aligned} \tag{4.5}$$

where we used the fact that $\nu_\tau \in \mathcal{Q}_R$. Besides, by Proposition 2.2, it holds for each $\gamma_\tau \in \Gamma_o(\mu_\tau, \nu_\tau)$ that

$$\begin{aligned} &\tfrac{1}{2h} W_2^2((\mathrm{Id} + h v_\tau)_\sharp \mu_\tau, \nu_\tau) - \tfrac{1}{2h} W_2^2(\mu_\tau, \nu_\tau) \\ &\qquad \leq \int_{\mathbb{R}^{2d}} \langle v_\tau(x), x - y \rangle \mathrm{d}\gamma_\tau(x, y) + h \, \|v_\tau\|_{L^2(\mu_\tau)} \, . \end{aligned} \tag{4.6}$$

Thus, upon merging (4.5) and (4.6) while letting $h \to 0^+$, we further obtain

$$g(\tau)\dot{g}(\tau) \leq \int_{\mathbb{R}^{2d}} \langle v_\tau(x), x - y \rangle \mathrm{d}\gamma_\tau(x, y), \tag{4.7}$$

for any $\tau \in (t, t + \delta) \cap \mathscr{T} \cap \mathscr{D}_g$ and each $\gamma_\tau \in \Gamma_o(\mu_\tau, \nu_\tau)$. Then, it follows by inserting crossed terms in (4.7) that

$$\begin{aligned} g(\tau)\dot{g}(\tau) &\leq \int_{\mathbb{R}^{2d}} \langle v_\tau(x) - v_\tau(y), x - y \rangle \mathrm{d}\gamma_\tau(x, y) \\ &\quad + \int_{\mathbb{R}^{2d}} \langle v_\tau(y) - \xi_\tau(y), x - y \rangle \mathrm{d}\gamma_\tau(x, y) \\ &\quad + \int_{\mathbb{R}^{2d}} \langle \xi_\tau(y), x - y \rangle \mathrm{d}\gamma_\tau(x, y) \\ &\leq l(\tau) g^2(\tau) + \int_{\mathbb{R}^{2d}} \langle v_\tau(y) - \xi_\tau(y), x - y \rangle \mathrm{d}\gamma_\tau(x, y), \end{aligned} \tag{4.8}$$

where we used (4.2), (4.4) and hypothesis **(CI)**-(iii). Recall now that by hypothesis **(CI)**-(iv) along with the definition of $\mathscr{T}$, there exists for every $w_\tau \in V(\tau, \nu_\tau)$ some other element $v_\tau \in V(\tau, \mu_\tau)$ for which

$$\sup_{x \in \mathbb{R}^d} |v_\tau(x) - w_\tau(x)| \leq L(\tau) W_2(\mu_\tau, \nu_\tau).$$

Observing that (4.8) holds for every $v_\tau \in V(\tau, \mu_\tau)$, this further implies that

$$\begin{aligned} g(\tau)\dot{g}(\tau) \leq (l(\tau) + L(\tau)) g^2(\tau) \\ + \int_{\mathbb{R}^{2d}} \langle w_\tau(y) - \xi_\tau(y), x - y \rangle \mathrm{d}\gamma_\tau(x, y), \end{aligned} \tag{4.9}$$

for all $\xi_\tau \in T_{\mathcal{Q}}(\nu_\tau)$ and each $w_\tau \in V(\tau, \nu_\tau)$. Noticing in turn that the right-hand side of the previous identity is both linear and continuous with respect to $\xi_\tau \in L^2(\mathbb{R}^d, \mathbb{R}^d; \nu_\tau)$, one can deduce that (4.9) in fact holds for each $\xi_\tau \in \overline{\mathrm{co}}\, T_{\mathcal{Q}}(\nu_\tau)$. Choosing in particular

$$\xi_\tau = w_\tau \in V(\tau, \nu_\tau) \cap \overline{\mathrm{co}}\, T_{\mathcal{Q}}(\nu_\tau) \neq \emptyset,$$

we finally recover the differential inequality

$$\dot{g}(\tau) \leq (l(\tau) + L(\tau))g(\tau)$$

that holds for all $\tau \in (t, t+\delta) \cap \mathscr{T} \cap \mathscr{D}_g$. Since $g(t) = 0$, it follows from Grönwall's lemma that $g(\cdot) \equiv 0$ on $(t, t+\delta)$, which contradicts our working assumption.

Step 2 – Existence of a viable curve As a consequence of Step 1, it holds that

$$\mathcal{R}_t(\mu^0) \cap \mathcal{Q} \neq \emptyset$$

for all times $t \in [0, T]$. For each $n \geq 1$, consider the dyadic subdivision $[0, T] = \cup_{k=0}^{2^n-1}[t_k, t_{k+1}]$ of the interval $[0, T]$, where $t_k := kT/2^n$. By using the semigroup property (2.7) of the reachable sets, one can prove by a simple induction argument that there exists a solution $\mu_n(\cdot) \in \mathcal{S}_{[0,T]}(\mu^0)$ such that

$$\mu_n(t_k) \in \mathcal{Q}, \tag{4.10}$$

for all $k \in \{0, \ldots, 2^n\}$ and each $n \geq 1$.

By repeating this process for arbitrary integers $n \geq 1$, we can find a sequence of trajectories $(\mu_n(\cdot)) \subset \mathcal{S}_{[0,T]}(\mu^0)$ for which (4.10) holds. By the compactness result of Theorem 2.9, there exists a curve $\mu(\cdot) \in \mathcal{S}_{[0,T]}(\mu^0)$ such that

$$\sup_{t\in[0,T]} W_2(\mu_n(t), \mu(t)) \underset{n\to+\infty}{\longrightarrow} 0,$$

along an adequate subsequence. By construction, the limit curve $\mu(\cdot) \in \mathrm{AC}([0, T], \mathscr{P}_c(\mathbb{R}^d))$ is such that

$$\mu(t_k) \in \mathcal{Q}$$

for every $k \in \{0, \ldots, 2^n\}$ and each $n \geq 1$, which yields the thesis by a classical density argument. □

5 Application to the existence of exponentially stable trajectories

In this last section, we provide an application of Theorem 4.2 to obtain the existence of trajectories for which a Lyapunov functional $\mathcal{W} : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}_+ \cup \{+\infty\}$ with domain $\mathrm{dom}(\mathcal{W}) \subset \mathscr{P}_2(\mathbb{R}^d)$ decays exponentially. This result will involve a suitable class of *directional lower-derivatives*, defined by

$$\mathrm{D}_\uparrow \mathcal{W}(\mu)(\xi) := \liminf_{\substack{h\to 0^+,\ \mu_h\in\mathrm{dom}(\mathcal{W}) \\ W_2(\mu_h,(\mathrm{Id}+h\xi)_\sharp\mu)=o(h)}} \frac{\mathcal{W}(\mu_h) - \mathcal{W}(\mu)}{h}$$

for any μ with $\mathcal{W}(\mu) < \infty$ and each $\xi \in L^2(\mathbb{R}^d, \mathbb{R}^d; \mu)$.

Definition 5.1 (Strict Lyapunov functions)**.** *A map $\mathcal{W} : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}_+ \cup \{+\infty\}$ is a strict Lyapunov function for $V : \mathbb{R}_+ \times \mathscr{P}_2(\mathbb{R}^d) \rightrightarrows C^0(\mathbb{R}^d, \mathbb{R}^d)$ if the following holds.*

(i) $\mathcal{W}(\cdot)$ has compact sublevels in $\mathscr{P}_2(\mathbb{R}^d)$.

(ii) For $\mathscr{L}^1$-almost every $t \geq 0$ and all $\mu \in \mathrm{dom}(\mathcal{W})$, there exists a $v \in V(t, \mu)$ for which

$$\mathrm{D}_\uparrow \mathcal{W}(\mu)(v) \leq -\rho\mathcal{W}(\mu),$$

where $\rho > 0$ is a fixed constant.

Theorem 5.2 (Exponentially stable trajectories)**.** *Let $V : \mathbb{R}_+ \times \mathscr{P}_2(\mathbb{R}^d) \rightrightarrows C^0(\mathbb{R}^d, \mathbb{R}^d)$ be a set-valued map with convex images satisfying hypotheses* **(CI)** *wherein $[0, T]$ is replaced by $[0, +\infty)$, and $\mathcal{W} : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}_+ \cup \{+\infty\}$ be a strict Lyapunov function for $V(\cdot, \cdot)$. Then for each $\mu^0 \in \mathscr{P}_c(\mathbb{R}^d)$, there exists a curve $\mu(\cdot) \in \mathcal{S}_{[0,+\infty)}(\mu^0)$ such that*

$$\mathcal{W}(\mu(t)) \leq \mathcal{W}(\mu^0)e^{-\rho t},$$

for all times $t \geq 0$.

Proof. We consider the extended dynamical system

$$\begin{cases} \partial_t \boldsymbol{\mu}(t) \in -\mathrm{div}_{(x,y)}\Big(\mathscr{V}(t,\boldsymbol{\mu}(t))\boldsymbol{\mu}(t)\Big), \\ \boldsymbol{\mu}(0) = \mu^0 \times \delta_{\mathcal{W}(\mu^0)}, \end{cases} \tag{5.1}$$

whose right-hand side is defined by

$$\mathscr{V}(t,\boldsymbol{\mu}) := \Big\{\big(v, -\rho \textstyle\int_{\mathbb{R}^{d+1}} y \,\mathrm{d}\boldsymbol{\mu}(x,y)\big) \text{ s.t. } v \in V(t, \boldsymbol{\pi}^d_\sharp \boldsymbol{\mu})\Big\},$$

with $\boldsymbol{\pi}^d : (x,y) \in \mathbb{R}^d \times \mathbb{R} \mapsto x \in \mathbb{R}^d$. Fixing an arbitrary $T > 0$, it can be checked that this velocity field satisfies hypotheses **(CI)**. We consider the constraints defined by

$$\mathcal{Q} := \Big\{\boldsymbol{\mu} \in \mathscr{P}_2(\mathbb{R}^{d+1}) \text{ s.t. } \boldsymbol{\mu} = \mu \times \delta_y \text{ and } y \geq \mathcal{W}(\mu)\Big\},$$

which is proper in $\mathscr{P}_2(\mathbb{R}^{d+1})$ under our assumptions. By adapting existing results of non-smooth analysis following e.g. [5, Proposition 2.17], it can be verified that

$$\mathrm{D}_\uparrow \mathcal{W}(\mu)(v) \leq -\rho \mathcal{W}(\mu) \iff (v, -\rho y) \in T_{\mathcal{Q}}(\mu \times \delta_y),$$

for all $\mu \in \mathscr{P}_2(\mathbb{R}^d)$, any $v \in V(t,\mu)$ and all $y \geq \mathcal{W}(\mu)$, which equivalently means that

$$\mathscr{V}(t,\boldsymbol{\mu}) \cap T_{\mathcal{Q}}(\boldsymbol{\mu}) \neq \emptyset$$

for $\mathscr{L}^1$-almost every $t \in [0,T]$ and each $\boldsymbol{\mu} \in \mathcal{Q}$. Thus by Theorem 4.2, there exists a solution $\boldsymbol{\mu}(\cdot)$ of (5.1) such that $\boldsymbol{\mu}(t) \in \mathcal{Q}$ for all times $t \in [0,T]$. Moreover by (2.4) of Theorem 2.5, the curve $\boldsymbol{\mu}(\cdot)$ admits the decomposition

$$\boldsymbol{\mu}(t) = \mu(t) \times \delta_{y(t)}$$

where $\mu(\cdot) \in \mathcal{S}_{[0,T]}(\mu^0)$ satisfies

$$\mathcal{W}(\mu(t)) \leq y(t) = e^{-\rho t}\mathcal{W}(\mu^0),$$

for all times $t \in [0,T]$.

We can then extend $\mu(\cdot)$ to $\mathbb{R}_+$ by repeating this process on intervals of the form $[nT,(n+1)T]$ while replacing $\mathcal{W}(\mu^0)$ by $\mathcal{W}(\mu(nT))$ in (5.1) for $n \geq 1$, yielding

$$\mathcal{W}(\mu(t)) \leq \mathcal{W}(\mu^0)e^{-\rho t}$$

for all times $t \geq 0$, and thus concluding our proof. □

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