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ON A PARTICULAR EXTENSION OF THE CHRISTOFFEL FUNCTION

JBL

1. INTRODUCTION

A certain variant of the Christoffel function (see below) allows to relate to the underlying density without scaling factor. This variant considers balls $\mathbf{B}(\boldsymbol{\xi}, \delta)$ around each point $\boldsymbol{\xi} \in \text{supp}(\mu)$, as also do people in topological data analysis (see the interesting paper [1]).

A preliminary result. Let $\Omega \subset \mathbb{R}^d$ be compact with nonempty interior and such that $\Omega = \overline{\text{int}(\Omega)}$.

Lemma 1.1. *Let $\phi, \psi \in \mathcal{M}(\Omega)_+$ be two finite Borel measures on Ω with $\mathbf{M}_n(\phi), \mathbf{M}_n(\psi) > 0$ for all $n \in \mathbb{N}$. Let $\lambda > 0$ be such that $\mathbf{M}_n(\psi) \preceq \lambda \mathbf{M}_n(\phi)$ for all $n \in \mathbb{N}$. Then $\psi \leq \lambda \phi$ and there exists $0 \leq h \in L^\infty(\Omega, \phi)$ such that*

$$(1.1) \quad d\psi = h(\mathbf{x}) d\phi \quad \text{and} \quad \|h\|_\infty \leq \lambda.$$

Moreover,

$$(1.2) \quad \|h\|_\infty = \inf_{\lambda} \{ \lambda : \mathbf{M}_n(\psi) \preceq \lambda \mathbf{M}_n(\phi), \quad \forall n \in \mathbb{N} \}.$$

2. MAIN RESULT

Let μ be a finite Borel probability measure on Ω with density $f \in L^\infty(\Omega)$, i.e., $d\mu = f(\mathbf{x}) 1_\Omega(\mathbf{x}) d\mathbf{x}$ and $\int_\Omega f(\mathbf{x}) d\mathbf{x} = 1$.

Next, given $\boldsymbol{\xi} \in \mathbb{R}^d$ let $\mathbf{B}(\boldsymbol{\xi}; \delta) := \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x} - \boldsymbol{\xi}\| \leq \delta\}$, and let $d\phi_\xi = \frac{1}{\text{vol}(\mathbf{B}(\boldsymbol{\xi}, \delta))} 1_{\mathbf{B}(\boldsymbol{\xi}, \delta)} d\mathbf{x}$. Given $h \in L^\infty(\Omega)$ let $\|h\|_\infty^{\boldsymbol{\xi}, \delta}$ be the infinity-norm on $\Omega \cap \mathbf{B}(\boldsymbol{\xi}, \delta)$.

Consider the extended Christoffel function:

$$(2.1) \quad \Lambda_n^\mu(\phi_\xi) := \min_{p \in \mathbb{R}^{[\mathbf{x}]_n}} \left\{ \int_\Omega p^2 d\mu : \int_{\mathbf{B}(\boldsymbol{\xi}, \delta)} p^2 d\phi_\xi = 1 \right\},$$

and observe that if $\delta = 0$ then $\Lambda_n^\mu(\phi_\xi) = \Lambda_n^\mu(\delta_\xi)$ coincides with the standard Christoffel function $\boldsymbol{\xi} \mapsto \Lambda_n^\mu(\boldsymbol{\xi})$.

Lemma 2.1. *Let $\Lambda_n^\mu(\phi_\xi)$ be as in (2.1) and $\boldsymbol{\xi} \in \text{int}(\Omega)$. Then:*

$$(2.2) \quad \Lambda_n^\mu(\phi_\xi) = \lambda_{\min}(\mathbf{M}_n(\mu), \mathbf{M}_n(\phi_\xi)), \quad \forall n \in \mathbb{N}.$$

Equivalently:

$$(2.3) \quad \begin{aligned} \Lambda_n^\mu(\phi_\xi)^{-1} &= \lambda_{\max}(\mathbf{M}_n(\phi_\xi), \mathbf{M}_n(\mu)) \\ &= \inf_{\lambda} \{ \lambda : \mathbf{M}_n(\phi_\xi) \preceq \lambda \mathbf{M}_n(\mu) \} \leq \|f\|_{\infty}^{\xi, \delta}, \end{aligned}$$

and moreover,

$$(2.4) \quad \lim_{n \rightarrow \infty} \Lambda_n^\mu(\phi_\xi)^{-1} = \|f\|_{\infty}^{\xi, \delta}.$$

Proof. Observe that (2.1) reads

$$(2.5) \quad \Lambda_n^\mu(\phi_{\xi, \delta}) = \min_{\mathbf{p} \in \mathbb{R}^{s(n)}} \{ \langle \mathbf{M}_n(\mu), \mathbf{p}\mathbf{p}^T \rangle : \langle \mathbf{M}_n(\phi_\xi), \mathbf{p}\mathbf{p}^T \rangle = 1 \}.$$

By Barvinok \square the semidefinite relaxation

$$\theta := \min_{\mathbf{X} \succeq 0} \{ \langle \mathbf{M}_n(\mu), \mathbf{X} \rangle : \langle \mathbf{M}_n(\phi_\xi), \mathbf{X} \rangle = 1 \},$$

of (2.5) has a rank-one optimal solution $\mathbf{X}^* = \mathbf{q}\mathbf{q}^T \succeq 0$ for some $\mathbf{q} \in \mathbb{R}^{s(n)}$, and therefore $\mathbf{x} \mapsto q(\mathbf{x}) := \mathbf{q}^T \mathbf{v}_n(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]_n$ is an optimal solution of (2.1), which also yields $\theta = \Lambda_n^\mu(\phi_\xi)$. As $0 \prec \mathbf{X} := \mathbf{I}/\text{trace}(\mathbf{M}_n(\phi_\xi))$ is a feasible solution, Slater's condition holds and therefore by standard conic duality:

$$\Lambda_n^\mu(\phi_\xi) = \sup_{\lambda} \{ \lambda : \lambda \mathbf{M}_n(\phi_\xi) \preceq \mathbf{M}_n(\mu) \} = \lambda_{\min}(\mathbf{M}_n(\mu), \mathbf{M}_n(\phi_\xi)),$$

which yields (2.2) and also the two equalities in (2.3). To get the inequality just use Lemma 1.1.

Next by (2.3), the sequence $(\Lambda_n^\mu(\phi_\xi)^{-1})_{n \in \mathbb{N}}$ is monotone increasing and is bounded above. Therefore $\lim_{n \rightarrow \infty} \Lambda_n^\mu(\phi_\xi)^{-1} = \rho^*$ for some $\rho^* \leq \|f\|_{\infty}^{\xi, \delta}$. But then by construction, $\mathbf{M}_n(\phi_\xi) \preceq \rho^* \mathbf{M}_n(\mu)$ for all $n \in \mathbb{N}$, which implies $\rho^* \geq \|f\|_{\infty}^{\xi, \delta}$, and which in turn yields the desired result. \square

As an immediate consequence, we obtain the following result.

Lemma 2.2. *For every $n \in \mathbb{N}$ define:*

$$(2.6) \quad \Lambda_n^\mu(\phi_\xi) := \min_{p \in \mathbb{R}[\mathbf{x}]_n} \left\{ \int_{\Omega} p^2 d\mu : \int_{\mathbf{B}(\xi, 1/n)} p^2 d\phi_\xi = 1 \right\}.$$

If f is piecewise continuous and $\xi \in \text{int}(\Omega)$ is not a point of discontinuity then:

$$(2.7) \quad \liminf_{n \rightarrow \infty} \Lambda_n^\mu(\phi_\xi) \geq \frac{1}{f(\xi)}.$$

Proof. If $\xi \in \text{int}(\Omega)$ then $\mathbf{B}(\xi, 1/n) \subset \text{int}(\Omega)$ provided that n is sufficiently large, say $n \geq n_0$. So let $\Lambda_n^\mu(\xi)$ be as in (2.6) with $n \geq n_0$. If f is not a point of discontinuity then $\|f\|_{\infty}^{\xi, 1/n} \downarrow f(\xi)$ as n increases. Moreover by (2.3):

$$\limsup_{n \rightarrow \infty} \Lambda_n^\mu(\phi_\xi)^{-1} \leq \limsup_{n \rightarrow \infty} \|f\|_{\infty}^{\xi, 1/n} = f(\xi),$$

which yields the desired result (2.7). \square

In topological data analysis, they analyze clouds \mathcal{N} of finitely many points by approximating the cloud with $\bigcup_{\xi \in \mathcal{N}} \mathbf{B}(\xi, \varepsilon)$.

Notice that from KKT-optimality conditions, an optimal solution \mathbf{p} of (2.1) satisfies: $\mathbf{M}_n(\mu) \mathbf{p} = \Lambda_n^\mu(\phi_\xi) \mathbf{M}_n(\phi_\xi) \mathbf{p}$ and so

$$\mathbf{p} = \Lambda_n^\mu(\phi_\xi) \mathbf{M}_n(\mu)^{-1} \int_{\Omega} \mathbf{v}_n(\mathbf{y}) \underbrace{\mathbf{v}_n(\mathbf{y})^T \mathbf{p}}_{=p(\mathbf{y})} d\phi_\xi(\mathbf{y}).$$

and using $\mathbf{v}_n(\mathbf{x})^T \mathbf{M}_n(\mu)^{-1} \mathbf{v}_n(\mathbf{y}) = \sum_{\alpha} P_\alpha(\mathbf{x}) P_\alpha(\mathbf{y})$, yields:

$$\begin{aligned} \mathbf{x} \mapsto p(\mathbf{x}) = \mathbf{v}_n(\mathbf{x})^T \mathbf{p} &= \Lambda_n^\mu(\phi_\xi) \sum_{\alpha \in \mathbb{N}_n^d} P_\alpha(\mathbf{x}) \int_{\mathbf{B}(\xi, \delta)} P_\alpha(\mathbf{y}) p(\mathbf{y}) d\phi_\xi(\mathbf{y}). \\ &= \Lambda_n^\mu(\phi_\xi) \int_{\mathbf{B}(\xi, \delta)} \sum_{\alpha \in \mathbb{N}_n^d} P_\alpha(\mathbf{x}) P_\alpha(\mathbf{y}) p(\mathbf{y}) d\phi_\xi(\mathbf{y}) \\ &= \Lambda_n^\mu(\phi_\xi) \int_{\mathbf{B}(\xi, \delta)} K(\mathbf{x}, \mathbf{y}) p(\mathbf{y}) d\phi_\xi(\mathbf{y}) \end{aligned}$$

So the optimal solution $p \in \mathbb{R}[\mathbf{x}]_n$ is reproduced by the kernel $K(\mathbf{x}, \mathbf{y})$ on $\mathbf{B}(\xi, \delta)$ (whereas K is a RKHS on $L^2(\mu)$ (and not on $L^2(\phi_\xi)$). Moreover:

$$\Lambda_n^\mu(\phi_\xi)^{-1} = \sum_{\alpha \in \mathbb{N}_n^d} \left(\int_{\mathbf{B}(\xi, \delta)} P_\alpha(\mathbf{y}) p(\mathbf{y}) d\phi_\xi(\mathbf{y}) \right)^2,$$

which coincides with $\Lambda_n^\mu(\xi)^{-1} = K(\xi, \xi)$ when $\phi_\xi = \delta_\xi$ (as then $p(\xi) = 1$).

REFERENCES

- [1] F. CHAZAL, D. COHEN-STEINER, Q. MÉRIGOT Geometric Inference for Probability Measures, *FoCM* **11** (2011), pp. 733–751.