A modified Christoffel function and its asymptotic properties

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ON A PARTICULAR EXTENSION OF THE CHRISTOFFEL FUNCTION

JBL

1. Introduction

A certain variant of the Christoffel function (see below) allows to relate to the underlying density without scaling factor. This variant considers balls $\mathbf{B}(\boldsymbol{\xi},\delta)$ around each point $\boldsymbol{\xi}\in\mathrm{supp}(\mu)$, as also do people in topological data analysis (see the interesting paper [1]).

A preliminary result. Let $\boldsymbol{\Omega}\subset\mathbb{R}^d$ be compact with nonempty interior and such that $\boldsymbol{\Omega}=\overline{\mathrm{int}(\boldsymbol{\Omega})}$.

Lemma 1.1. *Let $\phi,\psi\in\mathscr{M}(\boldsymbol{\Omega})_+$ be two finite Borel measures on $\boldsymbol{\Omega}$ with $\mathbf{M}_n(\phi),\mathbf{M}_n(\psi)\succ 0$ for all $n\in\mathbb{N}$. Let $\lambda>0$ be such that $\mathbf{M}_n(\psi)\preceq\lambda\,\mathbf{M}_n(\phi)$ for all $n\in\mathbb{N}$. Then $\psi\leq\lambda\,\phi$ and there exists $0\leq h\in L^\infty(\boldsymbol{\Omega},\phi)$ such that*

$$d\psi\,=\,h(\mathbf{x})\,d\phi\quad\textit{and}\quad\|h\|_\infty\,\leq\,\lambda. \tag{1.1}$$

Moreover,

$$\|h\|_\infty\,=\,\inf_{\lambda}\{\,\lambda:\,\mathbf{M}_n(\psi)\,\preceq\,\lambda\,\mathbf{M}_n(\phi),\quad\forall n\in\mathbb{N}\,\}. \tag{1.2}$$

2. Main result

Let μ be a finite Borel probability measure on $\boldsymbol{\Omega}$ with density $f\in L^\infty(\boldsymbol{\Omega})$, i.e., $d\mu=f(\mathbf{x})\,1_{\boldsymbol{\Omega}}(\mathbf{x})d\mathbf{x}$ and $\int_{\boldsymbol{\Omega}}f(\mathbf{x})\,d\mathbf{x}=1$.

Next, given $\boldsymbol{\xi}\in\mathbb{R}^d$ let $\mathbf{B}(\boldsymbol{\xi};\delta):=\{\mathbf{x}\in\mathbb{R}^d:\|\mathbf{x}-\boldsymbol{\xi}\|\leq\delta\}$, and let $d\phi_{\boldsymbol{\xi}}=\frac{1}{\mathrm{vol}(\mathbf{B}(\boldsymbol{\xi},\delta))}1_{\mathbf{B}(\boldsymbol{\xi},\delta)}d\mathbf{x}$. Given $h\in L^\infty(\boldsymbol{\Omega})$ let $\|h\|_\infty^{\boldsymbol{\xi};\delta}$ be the infinity-norm on $\boldsymbol{\Omega}\cap\mathbf{B}(\boldsymbol{\xi},\delta)$.

Consider the extended Christoffel function:

$$\Lambda^\mu_n(\phi_{\boldsymbol{\xi}})\,:=\,\min_{p\in\mathbb{R}[\mathbf{x}]_n}\{\int_{\boldsymbol{\Omega}}p^2d\mu\,:\,\int_{\mathbf{B}(\boldsymbol{\xi},\delta)}p^2\,d\phi_{\boldsymbol{\xi}}=1\,\}, \tag{2.1}$$

and observe that if $\delta=0$ then $\Lambda^\mu_n(\phi_{\boldsymbol{\xi}})=\Lambda^\mu_n(\delta_{\boldsymbol{\xi}})$ coincides with the standard Christoffel function $\boldsymbol{\xi}\mapsto\Lambda^\mu_n(\boldsymbol{\xi})$.

Lemma 2.1. *Let $\Lambda^\mu_n(\phi_{\boldsymbol{\xi}})$ be as in* (2.1) *and $\boldsymbol{\xi}\in\mathrm{int}(\boldsymbol{\Omega})$. Then:*

$$\Lambda^\mu_n(\phi_{\boldsymbol{\xi}})\,=\,\lambda_{\min}(\mathbf{M}_n(\mu),\mathbf{M}_n(\phi_{\boldsymbol{\xi}})),\quad\forall n\in\mathbb{N}. \tag{2.2}$$

Equivalently:

$$\begin{aligned}\Lambda_n^\mu(\phi_{\boldsymbol{\xi}})^{-1} &= \lambda_{\max}(\mathbf{M}_n(\phi_{\boldsymbol{\xi}}),\mathbf{M}_n(\mu))\\ &= \inf_\lambda\{\,\lambda:\ \mathbf{M}_n(\phi_{\boldsymbol{\xi}})\,\preceq\,\lambda\,\mathbf{M}_n(\mu)\,\}\ \leq\ \|f\|_\infty^{\boldsymbol{\xi},\delta},\end{aligned}\tag{2.3}$$

and moreover,

$$\lim_{n\to\infty}\Lambda_n^\mu(\phi_{\boldsymbol{\xi}})^{-1}\,=\,\|f\|_\infty^{\boldsymbol{\xi},\delta}.\tag{2.4}$$

Proof. Observe that (2.1) reads

$$\Lambda_n^\mu(\phi_{\boldsymbol{\xi},\delta})\,=\,\min_{\mathbf{p}\in\mathbb{R}^{s(n)}}\{\langle\mathbf{M}_n(\mu),\mathbf{p}\mathbf{p}^T\rangle:\ \langle\mathbf{M}_n(\phi_{\boldsymbol{\xi}}),\mathbf{p}\mathbf{p}^T\rangle\,=\,1\,\}.\tag{2.5}$$

By Barvinok [] the semidefinite relaxation

$$\theta\,:=\,\min_{\mathbf{X}\succeq 0}\{\,\langle\mathbf{M}_n(\mu),\mathbf{X}\rangle:\ \langle\mathbf{M}_n(\phi_{\boldsymbol{\xi}}),\mathbf{X}\rangle\,=\,1\,\},$$

of (2.5) has a rank-one optimal solution $\mathbf{X}^*=\mathbf{q}\mathbf{q}^T\succeq 0$ for some $\mathbf{q}\in\mathbb{R}^{s(n)}$, and therefore $\mathbf{x}\mapsto q(\mathbf{x}):=\mathbf{q}^T\mathbf{v}_n(\mathbf{x})\in\mathbb{R}[\mathbf{x}]_n$ is an optimal solution of (2.1), which also yields $\theta=\Lambda_n^\mu(\phi_{\boldsymbol{\xi}})$. As $0\prec\mathbf{X}:=\mathbf{I}/\mathrm{trace}(\mathbf{M}_n(\phi_{\boldsymbol{\xi}}))$ is a feasible solution, Slater's condition holds and therefore by standard conic duality:

$$\Lambda_n^\mu(\phi_{\boldsymbol{\xi}})\,=\,\sup_\lambda\{\,\lambda:\ \lambda\,\mathbf{M}_n(\phi_{\boldsymbol{\xi}})\,\preceq\,\mathbf{M}_n(\mu)\,\}\,=\,\lambda_{\min}(\mathbf{M}_n(\mu),\mathbf{M}_n(\phi_{\boldsymbol{\xi}})),$$

which yields (2.2) and also the two equalities in (2.3). To get the inequality just use Lemma 1.1.

Next by (2.3), the sequence $(\Lambda_n^\mu(\phi_{\boldsymbol{\xi}})^{-1})_{n\in\mathbb{N}}$ is monotone increasing and is bounded above. Therefore $\lim_{n\to\infty}\Lambda_n^\mu(\phi_{\boldsymbol{\xi}})^{-1}=\rho^*$ for some $\rho^*\leq\|f\|_\infty^{\boldsymbol{\xi},\delta}$. But then by construction, $\mathbf{M}_n(\phi_{\boldsymbol{\xi}})\preceq\rho^*\,\mathbf{M}_n(\mu)$ for all $n\in\mathbb{N}$, which implies $\rho^*\geq\|f\|_\infty^{\boldsymbol{\xi},\delta}$, and which in turn yields the desired result. □

As an immediate consequence, we obtain the following result.

Lemma 2.2. *For every $n\in\mathbb{N}$ define:*

$$\Lambda_n^\mu(\phi_{\boldsymbol{\xi}})\,:=\,\min_{p\in\mathbb{R}[\mathbf{x}]_n}\{\int_{\boldsymbol{\Omega}}p^2d\mu\ :\ \int_{\mathbf{B}(\boldsymbol{\xi},1/n)}p^2\,d\phi_{\boldsymbol{\xi}}=1\,\}.\tag{2.6}$$

If f is piecewise continuous and $\boldsymbol{\xi}\in\mathrm{int}(\boldsymbol{\Omega})$ is not a point of discontinuity then:

$$\liminf_{n\to\infty}\Lambda_n^\mu(\phi_{\boldsymbol{\xi}})\,\geq\,\frac{1}{f(\boldsymbol{\xi})}.\tag{2.7}$$

Proof. If $\boldsymbol{\xi}\in\mathrm{int}(\boldsymbol{\Omega})$ then $\mathbf{B}(\boldsymbol{\xi},1/n)\subset\mathrm{int}(\boldsymbol{\Omega})$ provided that n is sufficiently large, say $n\geq\ n_0$. So let $\Lambda_n^\mu(\boldsymbol{\xi})$ be as in (2.6) with $n\geq n_0$. If f is not a point of discontinuity then $\|f\|_\infty^{\boldsymbol{\xi},1/n}\downarrow f(\boldsymbol{\xi})$ as n increases. Moreover by (2.3):

$$\limsup_{n\to\infty}\Lambda_n^\mu(\phi_{\boldsymbol{\xi}})^{-1}\,\leq\,\limsup_{n\to\infty}\|f\|_\infty^{\boldsymbol{\xi},1/n}\,=\,f(\boldsymbol{\xi}),$$

which yields the desired result (2.7). □

In topological data analysis, they analyze clouds $\mathcal{N}$ of finitely many points by approximating the cloud with $\bigcup_{\boldsymbol{\xi}\in\mathcal{N}} \mathbf{B}(\boldsymbol{\xi},\varepsilon)$.

Notice that from KKT-optimality conditions, an optimal solution $\mathbf{p}$ of (2.1) satisfies: $\mathbf{M}_n(\mu)\,\mathbf{p} = \Lambda^\mu_n(\phi_{\boldsymbol{\xi}})\,\mathbf{M}_n(\phi_{\boldsymbol{\xi}})\,\mathbf{p}$ and so

$$\mathbf{p} \,=\, \Lambda^\mu_n(\phi_{\boldsymbol{\xi}})\,\mathbf{M}_n(\mu)^{-1}\int_{\mathbf{\Omega}} \mathbf{v}_n(\mathbf{y})\,\underbrace{\mathbf{v}_n(\mathbf{y})^T\mathbf{p}}_{=p(\mathbf{y})}\,d\phi_{\boldsymbol{\xi}}(\mathbf{y}).$$

and using $\mathbf{v}_n(\mathbf{x})^T\mathbf{M}_n(\mu)^{-1}\mathbf{v}_n(\mathbf{y}) = \sum_\alpha P_\alpha(\mathbf{x})P_\alpha(\mathbf{y})$, yields:

$$\begin{aligned}
\mathbf{x}\mapsto p(\mathbf{x}) \,=\, \mathbf{v}_n(\mathbf{x})^T\mathbf{p} \;&=\; \Lambda^\mu_n(\phi_{\boldsymbol{\xi}})\sum_{\alpha\in\mathbb{N}^d_n} P_\alpha(\mathbf{x})\int_{\mathbf{B}(\boldsymbol{\xi},\delta)} P_\alpha(\mathbf{y})\,p(\mathbf{y})\,d\phi_{\boldsymbol{\xi}}(\mathbf{y}).\\
&=\; \Lambda^\mu_n(\phi_{\boldsymbol{\xi}})\int_{\mathbf{B}(\xi,\delta)}\sum_{\alpha\in\mathbb{N}^d_n} P_\alpha(\mathbf{x})\,P_\alpha(\mathbf{y})\,p(\mathbf{y})\,d\phi_{\boldsymbol{\xi}}(\mathbf{y})\\
&=\; \Lambda^\mu_n(\phi_{\boldsymbol{\xi}})\int_{\mathbf{B}(\xi,\delta)} K(\mathbf{x},\mathbf{y})\,p(\mathbf{y})\,d\phi_{\boldsymbol{\xi}}(\mathbf{y})
\end{aligned}$$

So the optimal solution $p\in\mathbb{R}[\mathbf{x}]_n$ is reproduced by the kernel $K(\mathbf{x},\mathbf{y})$ on $\mathbf{B}(\boldsymbol{\xi},\delta)$ (whereas K is a RKHS on $L^2(\mu)$ (and not on $L^2(\phi_{\boldsymbol{\xi}})$). Moreover:

$$\Lambda^\mu_n(\phi_{\boldsymbol{\xi}})^{-1} \,=\, \sum_{\alpha\in\mathbb{N}^d_n}\left(\int_{\mathbf{B}(\boldsymbol{\xi},\delta)} P_\alpha(\mathbf{y})\,p(\mathbf{y})\,d\phi_{\boldsymbol{\xi}}(\mathbf{y})\right)^2,$$

which coincides with $\Lambda^\mu_n(\boldsymbol{\xi})^{-1} = K(\boldsymbol{\xi},\boldsymbol{\xi})$ when $\phi_{\boldsymbol{\xi}} = \delta_{\boldsymbol{\xi}}$ (as then $p(\boldsymbol{\xi}) = 1$).

References

[1] F. Chazal, D. Cohen-Steiner, Q. Mérigot Geometric Inference for Probability Measures, *FoCM* **11** (2011), pp. 733–751.