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Analytical Bounds for an Interval Kalman Filter

Quoc-Hung Lu¹, Soheib Fergani¹, Carine Jauberthie¹

Abstract—This paper is concerned with analytical developments of results firstly introduced by the authors in [1]. These developments are devoted to the optimization of upper bounds of the interval covariance matrices appearing in the Interval Kalman Filter [2]. The proposed study is mainly highlighted through two aspects. Firstly, the optimization is further performed by considering a class of upper bounds and minimizing the traces of these bounds in two stages (in terms of a gain matrix and then with respect to a scalar parameter). Secondly, the paper provides conditions under which the optimal trace value is controlled and hence the proposed Algorithm in [1], namely Optimal Upper Bound Interval Kalman Filter (OUBIKF), is ensured to perform with stability (i.e. without width explosion of the resulting interval estimators). Also under these conditions, the OUBIKF Algorithm, having a similar structure of the Standard Kalman Filter (SKF), is ensured to get a smaller trace upper bound of the covariance matrices in the correction step than the one in the prediction step. Numerical simulations based on an automotive model is performed to illustrate the developed results.

I. INTRODUCTION

With the growth of the industrial automatization and the fast development of intelligent system applications, the necessity of efficient control strategies have risen to higher levels. Nevertheless, the main problems to the synthesis of such solutions have been the cost and the feasibility. Indeed, all efficient control approaches are based on reliable information either from high precision sensors (expensive and not always easy to embed) or high fidelity information reconstruction (estimators, observers). For this sake, in both industry and academia, Kalman Filter introduced in [3] has always been interested by its elegant form and result characteristics (optimal estimator, on-line implementation,...). This is a kind of stochastic approach for estimation and referred as Standard Kalman Filter (SKF). Since then, many extensions of the SKF have been presented to improve its applicability and performance when dealing additionally with bounded uncertainties, of which the two major derivations are robust and interval Kalman filtering.

The robust Kalman filtering, [4]–[7], provides essentially point estimators (of the real states) attempting to limit the disturbance effects to the filter performance. For instance, in [6] and [7], finite-horizon robust Kalman filters for discrete time-varying uncertain systems with additive uncertain covariance white noises are studied without and with missing measurements respectively. Both papers concern an minimization of the trace of a chosen upper bound of all admissible error estimation covariances with respect to (w.r.t.) some design scalar parameters selected (or tuned) adequately, says a point-wise optimization approach.

The interval Kalman filtering provides essentially intervals containing all admissible estimators (of the real states) consistent with considered uncertainties and usually being

used as interval estimators for bounds of the real states. It may have a relation with the robust approach when using an element (usually the center) of the yielded interval as a robust estimator in some sense to be precised, however this is not the initial objective of the interval approach. The Interval Kalman Filter (IKF) was first introduced in [2] with an optimal solution and a suboptimal scheme for the purpose of real-time implementation. Then, authors have tried to further investigate this interesting research by its simplicity (although with conservatism) in computation thanks to interval computations (Section II-A) and the similar structure of the SKF with two steps (prediction and correction) in which the later would improve the estimator obtained from the former via the stake of a gain matrix [1], [8], [9].

[8] and [9] study enhancing methods for IKF and [1] proposes an optimal solution for the conservatism problem due to the choice of the IKF bounds. In [8], the proposed method consists in adding some positivity constraints together with the SIVIA algorithm to obtain the interval matrix $[K_k]$ containing all potential optimal gains and hence yielding guaranteed estimation results (without missing some admissible estimations as in the suboptimal case proposed by [2]). In [9], the interval $[K_k]$ of [8] is replaced by a point matrix K_k minimizing the trace of an upper bound of the estimation error covariances, thanks to which the computation time is reduced and the resulted estimators are less conservative. In [1], an optimal upper bound of all symmetric positive semidefinite matrices belonging to a given interval is provided under the form $\alpha^* I$ with $\alpha^* \in \mathbb{R}_+$, thanks to which upper bound expressions are simplified and suitable for advanced optimizations and the computation time is also reduced. Then considering a large class of upper bounds characterized by two real parameters and including the one used in [9], [1] also proposes a point-wise optimization for each choice of these scalar parameters.

The present work is a development of [1]. The first motivation drives our researches is to find an uniformly optimized solution of the error estimation covariance upper bounds in terms of their characterized scalar parameters. Furthermore, in the interval approach, a major issue is the conservatism of the resulted estimators due to the one of interval computations accumulated in algorithm iterations. In the worst case, the width of the resulted estimators may explode with a very high value. No study in the above papers addresses the conditions under which the provided algorithms can be controlled to perform with stability, i.e. without explosion in width of the resulted estimators. This is another motivation for our work.

The paper is organized as follows: Section II starts with a brief preliminary part, then introduces the OUBIKF with two stages optimization and guaranteed conditions. In Section III, simulations using the proposed estimation filter applied on an automotive model are provided to emphasize its efficiency. Finally, Section IV is the paper conclusions.

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II. OPTIMALLY BOUNDED INTERVAL KALMAN FILTER

A. Preliminaries

Denote $S(n) = \{M \in \mathbb{R}^{n \times n} : M = M^T\}$ and $S_+(n) = \{M \in S(n) : M \succeq 0\}$, where $M \succeq 0$ stands for a real positive semi-definite matrix M . For an $n \times n$ matrix $M = (m_{ij})$, the notations $\sigma_i(M)$, $\lambda_i(M)$ ($i = 1, \dots, n$) indicate resp. the singular values and eigenvalues of M among which $\sigma_{\max}(M)$ and $\lambda_{\max}(M)$ are the corresponding maximum values. $\text{tr}(M)$, $\|M\|$, $\|M\|_*$, $\|M\|_F$ are denoted resp. for the trace, operator norm, nuclear norm and Frobenius norm of M .

Let M, N be two real squared matrices of the same size. Define an order between M and N denoted by $N \preceq M$ if and only if $M - N \succeq 0$. M is called an upper bound of N , and N a lower bound of M . This partial order is extended to the notion of bounds for a non empty set Ω of real squared matrices: L is an upper (lower) bound of Ω , denoted by $\Omega \preceq L$ ($L \preceq \Omega$) if $M \preceq L$ ($L \preceq M$), $\forall M \in \Omega$. If P and Q are two upper (lower) bounds of Ω , then P is better than Q if the norm of P is smaller (greater) than or equal to the norm of Q .

A real interval, denoted by $[x]$, is a closed connected subset of \mathbb{R} . A real interval matrix $[X]$ of dimension $p \times q$ is a matrix with real interval components $[x_{ij}]$, $i \in \{1, \dots, p\}$, $j \in \{1, \dots, q\}$. Write $X \in [X]$ to indicate a *point matrix* $X = (x_{ij})$ belonging element-wise to $[X]$. Define

$$\begin{aligned} \sup([X]) &\triangleq (\sup([x_{ij}])) \text{ and } \inf([X]) \triangleq (\inf([x_{ij}])) \\ \text{as element-wise operators and} \\ \text{mid}([X]) &\triangleq (\sup([X]) + \inf([X]))/2 = (\text{mid}([x_{ij}])), \\ \text{rad}([X]) &\triangleq (\sup([X]) - \inf([X]))/2 = (\text{rad}([x_{ij}])), \\ \text{width}([X]) &\triangleq \sup([X]) - \inf([X]) = (\text{width}([x_{ij}])), \end{aligned}$$

where the last three matrices are called resp. the *midpoint matrix*, the *radius matrix* and the *width matrix* of $[X]$. Denote also $\overline{X} = \sup([X])$, $\underline{X} = \inf([X])$, $[X] = [\underline{X}, \overline{X}]$, where \overline{X} and \underline{X} will be called resp. the *largest* and *smallest matrix* of $[X]$ to distinguish with the notion of upper/lower bound matrices defined above. Let $[X]$ be an $n \times n$ real interval matrix, denote: $S([X]) \triangleq \{X \in [X] : X = X^T\}$ $S_+([X]) \triangleq \{X \in S([X]) : X \succeq 0\}$.

The basic interval operators $\diamond \in \{+, -, \times, \div\}$ defined in [10] can be used to compute directly all operations $[u] \diamond [v]$ and $\alpha \diamond [u]$, for real intervals $[u]$, $[v]$ and $\alpha \in \mathbb{R}$, without any further approximation algorithm. Then, interval matrix computations are defined similarly to matrix computations using the basic operators and more general operators are constructed by means of inclusion function [10]. In practice, the package Intlab developed for Matlab is used for these computations.

Again, we recall that the conservatism of interval computations is a major issue of all interval filtering and their objective is to find (interval) bounds for real states rather than their point estimators, so it is worthy to define

Definition 1. An interval filter is called C-stable if the widths of interval estimators for all time instant k are upper bounded by a common constant C.

B. First stages optimization of the Filter

Consider the following linear discrete time dynamical system

$$\begin{cases} x_k &= A_k x_{k-1} + B_k u_k + w_k, \\ y_k &= C_k x_k + D_k u_k + v_k, \end{cases} \quad k \in \mathbb{N}^*, \quad (1)$$

in which $x_k \in \mathbb{R}^{n_x}$ and $y_k \in \mathbb{R}^{n_y}$ represent state variables and measurements respectively, $u_k \in \mathbb{R}^{n_u}$ inputs, $w_k \in \mathbb{R}^{n_x}$ and $v_k \in \mathbb{R}^{n_y}$ stochastic noises with covariance Q_k and R_k respectively.

For any $K_k \in \mathbb{R}^{n_x \times n_y}$, $k \geq 1$, define:

$$\varphi_k(K_k) \triangleq (I - K_k C_k) P_{k|k-1} (I - K_k C_k)^T + K_k R_k K_k^T, \quad (2)$$

then $\varphi_k(K_k) = P_{k|k}$, $\varphi_k(\mathbf{0}) = P_{k|k-1}$, where $\mathbf{0}$ is the zero matrix whose dimension is appropriate to the context and will be clarified if necessary under the notation like $\mathbf{0}_{m \times n}$. $P_{k|k-1}$ and $P_{k|k}$ are respectively *prediction* and *estimation error covariance* in the SKF:

$$P_{k|k-i} \triangleq \mathbb{E}[(x_k - \hat{x}_{k|k-i})(x_k - \hat{x}_{k|k-i})^T], \quad i \in \{0, 1\},$$

where $\hat{x}_{k|k-1}$ and $\hat{x}_{k|k}$ are respectively *a priori* and *a posteriori estimates* of the real states x_k .

Let $S_k = C_k P_{k|k-1} C_k^T + R_k$ and assume that S_k is nonsingular¹. Let $K_k^* = P_{k|k-1} C_k^T S_k^{-1}$, then

$$\varphi_k(K_k^*) = (I - K_k^* C_k) \varphi_k(\mathbf{0}) = (I - K_k^* C_k) P_{k|k-1}.$$

Theorem 1. Consider system (1) with SKF assumptions. Let $k \geq 1$. Then:

$$0 \preceq \varphi_k(K_k^*) \preceq \varphi_k(K_k), \quad \forall K_k \in \mathbb{R}^{n_x \times n_y}, \quad (3)$$

$$K_k^* = \text{argmin}_{K_k} \text{tr}\{\varphi_k(K_k)\} = \text{argmin}_{K_k} \text{tr}\{P_{k|k}\}. \quad (4)$$

Proof. Since any covariance matrix is positive semidefinite, then $\varphi_k(K_k) \succeq 0$, $\forall K_k$, and hence $\varphi_k(K_k^*) \succeq 0$. By assumptions, $S_k \in S_+(n_y)$ and is nonsingular, then

$$0 \preceq (K_k - P_{k|k-1} C_k^T S_k^{-1}) S_k (K_k - P_{k|k-1} C_k^T S_k^{-1})^T$$

where the right hand side (RHS) of the above expression equals $\varphi_k(K_k) - \varphi_k(K_k^*)$. Thus, (3) is implied. Then, (4) is concluded using Proposition 2 of [1]. \square

Assumptions A1: Matrices A_k, B_k, C_k, D_k are unknown and belonging to given interval matrices $[A], [B], [C], [D]$ respectively. w_k, v_k are centered Gaussian vectors with covariance matrices Q_k and R_k belonging respectively to given interval matrices $[Q]$ and $[R]$. The initial state x_0 is also Gaussian with mean μ_0 and covariance matrix P_0 . In addition, $x_0, \{w_1, \dots, w_k\}$ and $\{v_1, \dots, v_k\}$ are mutually independent.

Aim: The aim of the Filter is to get estimate intervals $[\hat{x}_{k|k}]$ which contain all admissible estimates $\hat{x}_{k|k}$ of states x_k induced by mixed uncertainties. Then related confidence intervals are yielded according to the 3- σ rule since x_k 's follow the multivariate normal distribution.

Principle: OUBIKF follows the same structure of the SKF. In the prediction step, thanks to interval computations, the *a priori* estimate $[\hat{x}_{k|k-1}] = [A][\hat{x}_{k-1|k-1}] + [B]u_k$ is provided. It contains all admissible estimates $\hat{x}_{k|k-1} = A_k \hat{x}_{k-1|k-1} + B_k u_k$ for all values of $A_k \in [A]$, $B_k \in [B]$ and $\hat{x}_{k-1|k-1} \in [\hat{x}_{k-1|k-1}]$. In the correction step, an interval estimator $[\hat{x}_{k|k}] = [\hat{x}_{k|k-1}] + K_k(y_k - [\hat{y}_k])$ is provided, in which $[\hat{y}_k] = [C][\hat{x}_{k|k-1}] + [D]u_k$ and the gain K_k is a point matrix chosen in an optimal way. The gain K_k is chosen by a two stages optimization considering the class of

¹The nonsingularity of S_k can be assured if R_k is assumed to be positive definite or more strictly measurement noises are assumed to be vectors of independent random components. In practice, the pseudo-inverse S_k^+ is used instead with notice that $S_k^+ = S_k^{-1}$ when the later exists.

upper bounds $\Gamma \triangleq \{\bar{\varphi}_k(K_k, \beta) : S_+([P_{k|k}]) \preceq \bar{\varphi}_k(K_k, \beta)\}$, where $[P_{k|k}]$ is the interval matrix containing all admissible estimation error covariances $P_{k|k}$ and the form of $\bar{\varphi}_k(K_k, \beta)$ will be clarified in Theorem 2c). Γ also includes the upper bound of $S_+([P_{k|k}])$ used in [9]. Each upper bound in Γ is seen as a function of two arguments: gain matrix K_k and real parameter $\beta > 0$. The optimization is performed in two stages, first in terms of K_k and then with respect to β in order to get the optimal bound $\bar{\varphi}_k^*$ of $S_+([P_{k|k}])$ among others in Γ . Then, the Filter is developed and applied with the guaranteed conditions under which the model should be designed to obtain the Filter stability (Definition 1) thanks to the fact that the $\text{tr}\{\bar{\varphi}_k^*\}$ is non-asymptotically and asymptotically bounded.

Theorem 2. Consider system (1) with assumptions A1. Let $M = \text{mid}([C])$, R_{ij} be a matrix whose elements are zeros except its ij -th element equals to that of $\text{rad}([C])$, n_0 the number of non null radius of $\text{rad}([C])$ and $\Sigma = \sum_{i,j} R_{ij} R_{ij}^T = \text{diag}\{\text{rad}([C])\text{rad}([C])^T\}$. The following statements hold:

a) $\forall k \geq 1, \forall A_k \in [A], \forall C_k \in [C], \forall Q_k \in [Q], \forall R_k \in [R], \forall \hat{x}_{k|k} \in [\hat{x}_{k|k}], \forall \beta > 0$ and $\forall K_k \in \mathbb{R}^{n_x \times n_y}$:

$$P_{k|k} \preceq (1 + \beta^{-1}n_0) (I - K_k M) P_{k|k-1} (I - K_k M)^T + K_k \left[(\beta + n_0) \sum_{i=1}^{n_y} \sum_{j=1}^{n_x} R_{ij} P_{k|k-1} R_{ij}^T + R_k \right] K_k^T, \quad (5)$$

b) If $P_{k|k-1} \in [P_{k|k-1}]$, $S_+([P_{k|k-1}]) \preceq \alpha_k I$ and $S_+([R]) \preceq \gamma I$, then $\forall k \geq 1, \forall A_k \in [A], \forall C_k \in [C], \forall Q_k \in [Q], \forall R_k \in [R], \forall \hat{x}_{k|k} \in [\hat{x}_{k|k}], \forall \beta > 0$ and $\forall K_k \in \mathbb{R}^{n_x \times n_y}$:

$$P_{k|k} \preceq \alpha_k (1 + \beta^{-1}n_0) (I - K_k M) (I - K_k M)^T + K_k [\alpha_k (\beta + n_0) \Sigma + \gamma I] K_k^T. \quad (6)$$

c) Let $\bar{\varphi}_k(K_k, \beta)$ be RHS of (6) and $\bar{K}_{k,\beta}^* = M^T S_{k,\beta}^{-1}$ with $S_{k,\beta} = MM^T + \beta \Sigma + \gamma / [\alpha_k (1 + n_0/\beta)] I_{n_y}$. Then:

$$\bar{\varphi}_k(\bar{K}_{k,\beta}^*, \beta) = \alpha_k (1 + n_0 \beta^{-1}) (I_{n_x} - \bar{K}_{k,\beta}^* M), \quad (7)$$

$$0 \preceq \bar{\varphi}_k(\bar{K}_{k,\beta}^*, \beta) \preceq \bar{\varphi}_k(K_k, \beta), \forall K_k \in \mathbb{R}^{n_x \times n_y}, \forall \beta > 0. \quad (8)$$

$$\bar{K}_{k,\beta}^* = \text{argmin}_{K_k} \text{tr}\{\bar{\varphi}_k(K_k, \beta)\}. \quad (9)$$

Proof. The RHS of (5) is upper bounded by the RHS of (6) implied by using the property: $A \preceq B \Rightarrow XAX^T \preceq XBX^T$ (with appropriate dimension). Thus, the statement b) is proved. The proof of statement c) can be derived in a similar way as the one of Theorem 1. Let's now prove the statement a).

Let $C_k \in [C]$. Applying the decomposition $C_k = M + \Delta_k$ where $\Delta_k = \sum_{i=1}^{n_y} \sum_{j=1}^{n_x} \alpha_{ij}(k) R_{ij}$ for appropriate $\alpha_{ij}(k) \in [-1, 1]$ to (2), one gets

$$P_{k|k} = \Lambda_1 + \Lambda_2 + \Lambda_3 + K_k R_k K_k^T,$$

where $\Lambda_1 = (I - K_k M) P_{k|k-1} (I - K_k M)^T$, $\Lambda_2 = (K_k \Delta_k) P_{k|k-1} (K_k \Delta_k)^T$ and $\Lambda_3 = Z + Z^T$ with $Z = (K_k M - I) P_{k|k-1} (K_k \Delta_k)^T$.

Writing $P_{k|k-1} = P_{k|k-1}^{1/2} \left(P_{k|k-1}^{1/2} \right)^T$ and using (14),

$$\Lambda_3 \preceq \sum_{i,j} T_{ij} \left\{ \beta_{ij}^{-1} \Lambda_1 + \beta_{ij} K_k R_{ij} P_{k|k-1} (K_k R_{ij})^T \right\},$$

for any $\beta_{ij} > 0$, where $T_{ij} = 1$ when $\text{rad}([C])_{ij} > 0$ and null otherwise. Then, applying (15), $\forall \sigma_{i,j,u,v} = \sigma_{u,v,i,j}^{-1} > 0$:

$$\Lambda_2 \preceq K_k \left[\sum_{i,j} T_{ij} \left(\sum_{u,v} T_{uv} \sigma_{i,j,u,v} \right) R_{ij} P_{k|k-1} R_{ij}^T \right] K_k^T.$$

Choose $\beta_{ij} = \beta$ and $\sigma_{i,j,u,v} = 1$ for all i, j, u, v and get

$$P_{k|k} \preceq \left(1 + \sum_{i,j} \beta^{-1} T_{ij} \right) \Lambda_1 + K_k R_k K_k^T + K_k \left[\sum_{i,j} T_{ij} \left(\beta + \sum_{u,v} T_{uv} \right) R_{ij} P_{k|k-1} R_{ij}^T \right] K_k^T.$$

Then, (5) holds noting that $\sum_{i=1}^{n_y} \sum_{j=1}^{n_x} T_{ij} = n_0$. \square

Remark 1. In [9], the choice $\beta_{ij} = \sigma_{i,j,u,v} = 1$ is used. In [1], beside choosing $\beta_{ij} = \beta > 0$, the choice $\sigma_{i,j,u,v} = \sigma > 0, \forall i, j, u, v$, is used regardless the condition $\sigma_{i,j,u,v} = \sigma_{u,v,i,j}^{-1}$. In the present work, a circumspect study of the minimization of $\text{tr}\{\bar{\varphi}_k(\bar{K}_{k,\beta}^*, \beta)\}$ is provided, noting that $1 = \inf \sup \{\sigma_{i,j,u,v} > 0 : \sigma_{i,j,u,v} = \sigma_{u,v,i,j}^{-1}\}$ is used.

C. Second stage of optimization and Guaranteed conditions

In this stage, instead of finding directly the optimal upper bound $\bar{\varphi}_k^* = \inf_{\beta > 0} \bar{\varphi}_k(\bar{K}_{k,\beta}^*, \beta)$, the function

$$\phi_k(\beta) \triangleq \text{tr}\{\bar{\varphi}_k(\bar{K}_{k,\beta}^*, \beta)\}, \quad \beta > 0,$$

is considered to find its minimum $\Phi_k^* \triangleq \inf_{\beta > 0} \phi_k(\beta)$. The following notations are used:

- o All notations defined in Theorems 1 and 2:
 $\varphi_k, K_k^*, S_k, M, R_{ij}, \Sigma, n_0, \alpha_k, \gamma, \bar{\varphi}_k, \bar{K}_{k,\beta}^*, S_{k,\beta},$
- o $r = \text{rank}(M)$, $\{\lambda_i\}_{i=1,\dots,r}$ are non null eigenvalues of MM^T , $d_{\min} = \min\{\Sigma_{ii} \neq 0, i = 1, \dots, n_y\}$, $d_{\max} = \max\{\Sigma_{ii} \neq 0, i = 1, \dots, n_y\}$.

Lemma 1. Let $\alpha > 0, c > 0, \beta > 0$, $a(\beta) = \alpha(1 + n_0/\beta)$, $\xi(\beta) = a(\beta) \left[n_x - \text{tr}\{M^T (MM^T + \beta c I_{n_y})^{-1} M\} \right]$. Then

$$\begin{aligned} \lim_{\beta \rightarrow \infty} \xi(\beta) &= \alpha n_x = \alpha \text{tr}\{I_{n_x}\}, \\ \lim_{\beta \rightarrow 0} \xi(\beta) &= \alpha c n_0 \text{tr}\{(MM^T)^+\}, \quad \text{if } n_x = r \\ &= \infty, \quad \text{if } n_x > r. \end{aligned}$$

Furthermore:

a) If $n_x = r$ and $\lambda_i \geq n_0 c, \forall i = 1, \dots, r$, then $\xi(\beta)$ is non-decreasing and

$$\inf_{\beta > 0} \xi(\beta) = \lim_{\beta \rightarrow 0} \xi(\beta), \quad \sup_{\beta > 0} \xi(\beta) = \lim_{\beta \rightarrow \infty} \xi(\beta).$$

b) If $n_x > r$ and $\lambda_i \leq n_0 c, \forall i = 1, \dots, r$, then $\xi(\beta)$ is non-increasing and

$$\sup_{\beta > 0} \xi(\beta) = \lim_{\beta \rightarrow 0} \xi(\beta), \quad \inf_{\beta > 0} \xi(\beta) = \lim_{\beta \rightarrow \infty} \xi(\beta).$$

Proposition 1. Let $k \geq 1, \epsilon > 0, c_{1,k} = d_{\min} + \frac{\gamma}{\alpha_k(\epsilon + n_0)}$ and $c_{2,k} = d_{\max} + \frac{\gamma}{\alpha_k n_0}$. Let $h(\beta) = \alpha_k (1 + n_0/\beta) [n_x - r]$, $g(\beta) = \alpha_k (1 + n_0/\beta) n_x$ and for $i \in \{1, 2\}$, $\xi_{i,k}(\beta) = \alpha_k (1 + n_0/\beta) \left[n_x - \text{tr}\{M^T (MM^T + \beta c_{i,k} I_{n_y})^{-1} M\} \right]$. Then for all $0 < \beta \leq \epsilon$:

$$0 \leq h(\beta) \leq \xi_{1,k}(\beta) \leq \phi_k(\beta) \leq \xi_{2,k}(\beta) \leq g(\beta), \quad (10)$$

Furthermore:

a) If $n_x = r$ and $\lambda_i \geq n_0 d_{\max} + \frac{\gamma}{\alpha_k}, \forall i = 1, \dots, r$, then

$$0 < \underline{c}_k \cdot \text{tr}\{(MM^T)^+\} \leq \Phi_k^* \leq \lim_{\beta \rightarrow 0} \phi_k(\beta) \leq \bar{c}_k \cdot \text{tr}\{(MM^T)^+\} < \alpha_k n_x,$$

where $\underline{c}_k = \alpha_k n_0 d_{\min} + \gamma$ and $\bar{c}_k = \alpha_k n_0 d_{\max} + \gamma$.

b) If $n_x > r$ and $\lambda_i \leq n_0 d_{\min} + \frac{\gamma}{\alpha_k}, \forall i = 1, \dots, r$, then

$$\begin{aligned} \infty &= \lim_{\beta \rightarrow 0} \phi_k(\beta) \geq \xi_{2,k}(\beta) \geq \phi_k(\beta) \geq \\ &\geq \xi_{1,k}(\beta) \geq \lim_{\beta \rightarrow \infty} \phi_k(\beta) = \alpha_k n_x = \Phi_k^*. \end{aligned}$$

Remark 2. Figure 1 illustrates Proposition 1 in which (10) is highlighted. Lemma 1 is technically needed for Proposition 1,

while the last one provides the bounds of $\phi_k(\beta)$ together with its infimum value Φ_k^* in two accessible cases.

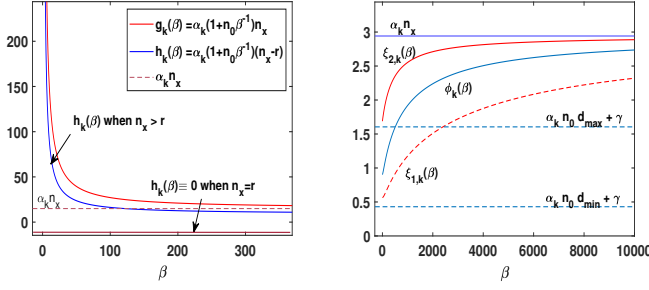


Figure 1: *Left:* $h_k(\beta)$ and $g_k(\beta)$ at a fixed time $k \geq 1$. *Right:* An example of $\phi_k(\beta)$ for the case $n_x = r$ and $\lambda_i \geq n_0 d_{\max} + \gamma/\alpha_k, \forall i = 1, \dots, r$ at a fixed time $k \geq 1$.

Guaranteed conditions: It is difficult to get the exact infimum of $\phi_k(\beta)$, noted Φ_k^* , but its bounds and limits can be determined conditioning:

$$\mathbf{C1}: \begin{cases} n_x = r \\ \lambda_i \geq n_0 d_{\max} + \gamma/\alpha_k, \quad \forall i = 1, \dots, r. \end{cases}$$

Using Proposition 1 under conditions C1, the following expression can be obtained:

$$0 \leq \left| \lim_{\beta \rightarrow 0} \phi_k(\beta) - \Phi_k^* \right| \leq \alpha_k n_0 (d_{\max} - d_{\min}) \text{tr}\{(MM^T)^+\},$$

in which d_{\max}, d_{\min} are controllable. So, the minimization of $\phi_k(\beta)$ (exactly or approximately) consists in how to design the system under consideration to reach conditions C1 and control d_{\max}, d_{\min} in an appropriate way.

Design of conditions C1:

Condition 1: $n_x = r$. Since $M \in \mathbb{R}^{n_y \times n_x}$, $r \leq \min\{n_x, n_y\}$, to get $r = n_x$, it requires:

- $n_y \geq n_x$,
- M has n_x linearly independent columns.

Note that the number of output measurements n_y is not necessary the number of physical sensors n_s . In view of the system design, it is not prohibited (and it is possible) to design more measurement outputs than the number of states, although the number of sensors can be not sufficient. The missing measurements due to the lack of sensor can be (not always) estimated by several ways, e.g. by an observer considered as a virtual sensor or by known constraint conditions on the bounds of these measurements. This situation also leads to the rationality of using the interval matrix $[C]$ to model the bounded measurement uncertainty.

The second requirement can be regularized numerically in particular for the interval context. A regularization $[C] \leftarrow [C] + [\underline{\epsilon}, \bar{\epsilon}]$ with appropriate small (point matrices) $\underline{\epsilon}, \bar{\epsilon}$ can be used, if it is necessary, so that the (new) matrix $[C]$ still verified all constraint conditions of the system and $M = \text{mid}([C])$ has n_x linearly independent columns.

Condition 2: $\lambda_i \geq n_0 d_{\max} + \gamma/\alpha_k, \quad \forall i = 1, \dots, r$.

This condition is equivalent to $\lambda_{\min} \geq n_0 d_{\max} + \gamma/\alpha_k$, where $\lambda_{\min} = \min_{i=1, \dots, r} \{\lambda_i\}$. This condition is achievable thanks to the Lemma 2 below.

Lemma 2. Let $\delta = \max_{i,j} \{\text{rad}([c_{ij}])\}$ with $[C] = ([c_{ij}])$. If for some $s \in (0, 1)$, the following two expressions hold

- $0 \leq \delta \leq \sqrt{s \frac{\lambda_{\min}}{n_0 n_x}}$,
- $\alpha_k \geq \max \left\{ \frac{\gamma}{(1-s)\lambda_{\min}}, \sup_{P \in S_+([P_{k|k-1}])} \{\lambda_{\max}(P)\} \right\}$

then the condition $\lambda_{\min} \geq n_0 d_{\max} + \gamma/\alpha_k$ is verified.

Proof. By assumptions, $d_{\max} \leq (s \lambda_{\min}/n_0)$ and $1/\alpha_k \leq (1-s)\lambda_{\min}/\gamma$, implying that $n_0 d_{\max} + \frac{\gamma}{\alpha_k} \leq \lambda_{\min}$. \square

The following lemma is used for Theorem 3 computations. Only its third statement needs the first condition of C1.

Lemma 3. The following statements hold:

- $\bar{K}^* \triangleq M^+ = M^T(MM^T)^+ = (M^T M)^+ M^T \in \mathbb{R}^{n_x \times n_y}$.
- $\lim_{\beta \rightarrow 0} \bar{K}_{k,\beta}^* = \bar{K}^*$ and $\lim_{\beta \rightarrow 0} \varphi_k(\bar{K}_{k,\beta}^*) = \varphi_k(\bar{K}^*)$.
- If $\text{rank}(M) = n_x$ then $I_{n_x} - \bar{K}^* M = \mathbf{0}_{n_x \times n_x}$.

Theorem 3. Assume that $\text{rank}(M) = n_x$ and assumptions of Lemma 2 are verified. Then

$$\begin{aligned} \varphi_k(\bar{K}^*) &\leq \lim_{\beta \rightarrow 0} \bar{\varphi}_k(\bar{K}_{k,\beta}^*, \beta) \leq \lim_{\beta \rightarrow 0} \bar{\varphi}_k(\bar{K}^*, \beta) \leq \\ &\leq (\alpha_k n_0 \delta^2 n_x + \gamma)(M^T M)^+ \leq \alpha_k I_{n_x} \end{aligned} \quad (11)$$

where $P_{k|k}^{\bar{K}^*} \triangleq \varphi_k(\bar{K}^*)$ is the error covariance matrix associated with the use of $K_k = \bar{K}^*$.

Assume further that $S_+([Q]) \leq \lambda_1 I_{n_x}$, $S_+([A][A]^T) \leq \lambda_2 I_{n_x}$, $S_+([P_0|0]) \leq \bar{\alpha}_0 I_{n_x}$ and $\frac{\lambda_2 n_0 n_x \delta^2}{\lambda_{\min}} \leq L < 1$. Let

$$\Psi_L = \frac{\lambda_1 n_0 n_x \delta^2 + \gamma}{\lambda_{\min}(1-L)} \quad \text{and} \quad \Psi_L(k) = \Psi_L + (\bar{\alpha}_0 - \Psi_L)L^k,$$

then

$$P_{k|k}^{\bar{K}^*} \leq \Psi_L(k) I_{n_x}, \quad \text{MSE}^{\bar{K}^*} = \text{tr}\{P_{k|k}^{\bar{K}^*}\} \leq \Psi_L(k) n_x, \quad (12)$$

$$P_{k+1|k}^{\bar{K}^*} \leq (\lambda_2 \Psi_L(k) + \lambda_1) I_{n_x}. \quad (13)$$

Remark 3. Since $\Psi_L(k) = \Psi_L + (\bar{\alpha}_0 - \Psi_L)L^k$, then $\Psi_L(k) \downarrow \Psi_L$ if $\bar{\alpha}_0 \geq \Psi_L$ and $\Psi_L(k) \uparrow \Psi_L$ if $\bar{\alpha}_0 < \Psi_L$. In the later case, $P_{k|k}^{\bar{K}^*} \leq \Psi_L I_{n_x}, \forall k \geq 1$. Furthermore, Ψ_L can be precomputed and controlled before the algorithm starts. For instance, it can be controlled the choice of L, s, δ so that $\Psi_L \leq \Psi$ with a given constant $\Psi > 0$. Concretely, the constraint $\frac{L}{\lambda_2} \geq s \geq \frac{n_0 n_x \delta^2}{\lambda_{\min}}$ can be reduce to $s = L/\lambda_2 = n_0 n_x \delta^2/\lambda_{\min}$ which implies $\Psi_L = \frac{\lambda_1 L}{\lambda_2(1-L)} + \frac{\gamma}{\lambda_{\min}(1-L)}$. Let $\Psi_L = \Psi > \frac{\gamma}{\lambda_{\min}}$ and get

$$L = \frac{\Psi - \gamma/\lambda_{\min}}{\Psi + \lambda_1/\lambda_2}, \quad s = L/\lambda_2, \quad \delta^2 = s \lambda_{\min}/(n_0 n_x).$$

Proof. From Theorems 1-2, one gets:

$$\begin{aligned} 0 \leq \varphi_k(K_k^*) &\leq \lim_{\beta \rightarrow 0} \varphi_k(\bar{K}_{k,\beta}^*) = \varphi_k(\bar{K}^*) \leq \\ &\leq \lim_{\beta \rightarrow 0} \bar{\varphi}_k(\bar{K}_{k,\beta}^*, \beta) \leq \lim_{\beta \rightarrow 0} \bar{\varphi}_k(\bar{K}^*, \beta), \end{aligned}$$

By assumptions of the theorem, the conditions C1 holds, then

$$\begin{aligned} \lim_{\beta \rightarrow 0} \bar{\varphi}_k(\bar{K}^*, \beta) &= \bar{K}^* [\alpha_k n_0 \Sigma + \gamma I_{n_y}] \bar{K}^{*T} \quad (\text{Lemma 3c}) \\ &\leq (\alpha_k n_0 \delta^2 n_x + \gamma)(M^T M)^+, \end{aligned}$$

in which $\Sigma \leq \delta^2 n_x I_{n_y}$ and $\bar{K}^* \bar{K}^{*T} = (M^T M)^+$.

Since MM^T and $M^T M$ have the common non null eigenvalues then they have the same λ_{\min} (positive). So we get $\bar{K}^* \bar{K}^{*T} = (M^T M)^+ \leq \frac{1}{\lambda_{\min}} I_{n_x}$ and hence

$$\lim_{\beta \rightarrow 0} \bar{\varphi}_k(\bar{K}^*, \beta) \leq (\alpha_k n_0 \delta^2 n_x + \gamma) \frac{1}{\lambda_{\min}} I_{n_x} \leq \alpha_k I_{n_x},$$

where the last inequality holds thanks to $\lambda_{\min} \geq n_0 \delta^2 n_x + \frac{\gamma}{\alpha_k}$. By recursion, we get

$$P_{k|k}^{\bar{K}^*} = (\bar{A}_{k,1}^{\otimes}) P_{0|0} (\bar{A}_{k,1}^{\otimes})^T + \sum_{i=1}^k (\bar{A}_{k,i+1}^{\otimes} \bar{C}_i) Q_i (\bar{A}_{k,i+1}^{\otimes} \bar{C}_i)^T + \sum_{i=1}^k (\bar{A}_{k,i+1}^{\otimes} \bar{K}^*) R_i (\bar{A}_{k,i+1}^{\otimes} \bar{K}^*)^T,$$

where $\bar{A}_{k,s}^{\otimes} = \bar{A}_k \bar{A}_{k-1} \dots \bar{A}_{s+1} \bar{A}_s$ if $s \leq k$ and $\bar{A}_{k,s}^{\otimes} = I$ if $s > k$, $\bar{C}_k = I - \bar{K}^* C_k$, $\bar{A}_k = \bar{C}_k A_k$.

For any $p \geq 1$, $C_p \in [C]$ is decomposed as $C_p = M + \Delta_p$, $\Delta_p = \sum_{i,j} \alpha_{ij}(p) R_{ij}$, $\alpha_{ij}(p) \in [-1, 1]$ and hence, using Lemma 3 and (15), we get

$$\begin{aligned} (I - \bar{K}^* C_p) (I - \bar{K}^* C_p)^T &= \bar{K}^* \Delta_p \Delta_p^T \bar{K}^{*T} \leq \\ &\leq \bar{K}^* (n_0 \Sigma) \bar{K}^{*T} \leq n_0 \delta^2 n_x \bar{K}^* \bar{K}^{*T} \leq n_0 \delta^2 n_x \frac{1}{\lambda_{\min}} I_{n_x}, \end{aligned}$$

implying that $\bar{A}_p \bar{A}_p^T \leq \lambda_2 n_0 \delta^2 n_x \frac{1}{\lambda_{\min}} I_{n_x}$.

Substituting these results into $P_{k|k}^{\bar{K}^*}$'s expression above, then

$$P_{k|k}^{\bar{K}^*} \leq \bar{\alpha}_0 \underline{L}^k I_{n_x} + \frac{\lambda_1 n_0 n_x \delta^2 + \gamma}{\lambda_{\min}} \sum_{i=1}^k \underline{L}^{k-i} I_{n_x}$$

and the conclusion holds noting that $\underline{L} = \frac{\lambda_2 n_0 n_x \delta^2}{\lambda_{\min}} \leq L < 1$ and $\sum_{i=0}^{k-1} L^i = \frac{1-L^k}{1-L}$. In addition, $\lim_{k \rightarrow \infty} \Psi_L(k) = \Psi_L$. \square

Algorithm 1 OUBIKF ALGORITHM

Initialization: $[\hat{x}_{0|0}]$, $\mathcal{P}_{0|0}$, $[A]$, $[B]$, $[C]$, $[D]$, $[Q]$, $[R]$, s , λ_{\min} , u_k , y_k , $k = 1, 2, 3, \dots, N$

Find n_0 and find γ such that $S_+([R]) \leq \gamma I$

$\bar{K}^* = \text{mid}([C])^+$; $\text{radC} = \text{rad}([C])$; $\Sigma = \text{diag}\{\text{radC} * \text{radC}^T\}$.

for $k = 1, 2, 3, \dots, N$ **do**

Prediction step:

$$[\hat{x}_{k|k-1}] = [A][\hat{x}_{k-1|k-1}] + [B]u_k$$

$$[P_{k|k-1}] = [A]P_{k-1|k-1}[A]^T + [Q]$$

Find α_k such that $S_+([P_{k|k-1}]) \leq \alpha_k I$

$$\alpha_k = \max\{\gamma / [(1-s)\lambda_{\min}], \alpha_k\}$$

Correction step:

$$[\hat{x}_{k|k}] = \bar{K}^* ([-1, 1] * \text{radC}) [\hat{x}_{k|k-1}] + \bar{K}^* (y_k - [D]u_k)$$

$$P_{k|k} = \bar{K}^* [\alpha_k n_0 \Sigma + \gamma I_{n_y}] \bar{K}^{*T}$$

end for

(*) γ, α_k are chosen using Proposition 6 of [1]; s, λ_{\min} must satisfy **C1**; n_0 the number of non zero radius of $[C]$.

Remark 4. Confidence intervals are defined by

$\text{CI}_k^i = \left[\inf([\hat{x}_{k|k}^i]) - h \sqrt{\mathcal{P}_{k|k}^{ii}}, \sup([\hat{x}_{k|k}^i]) + h \sqrt{\mathcal{P}_{k|k}^{ii}} \right]$, for $i = 1, \dots, n_x$ and $h = 1, 2, 3$, which contain the states x_k with probabilities at least 68%, 95%, 99.7% according to h .

III. SIMULATION

In this section, the OUBIKF Algorithm is applied to a model taken from automotive domain presented in [11]. This model is a nonlinear continuous-time model which has been discretized/linearized and thus given under the form (1). A discretization is applied with a sampling time $T = 0.05s$ to get matrices A_d, B_d, C_d, D_d (non interval and independent of time instant k) according to equations in (1). Then, interval matrices $[A]$, $[B]$, $[D]$ are generated as follow: for $F \in \{A_d, B_d, D_d\}$, let $F = \text{mid}([F])$ and choose the radii $\text{rad}([F])$ at random in $[0, 0.5]$. $[Q]$ and $[R]$ are generated in the same way, their diagonal elements being intervals of positive real numbers.

Choose $M = \text{mid}([C]) = C_d$. With this choice, $\text{rank}(M) = n_x$, so the first part of conditions **C1** is satisfied. The second part of conditions **C1** is reached using Remark 3 to compute L, s, δ with the choices $\Psi = 10\gamma/\lambda_{\min}$ and $n_0 = n_x n_y$. Then $[C]$ is generated in the same way of $[A]$ so that $\max_{i,j} \{\text{rad}([c_{ij}])\} \leq \delta$.

Ψ_L	λ_{\min}	λ_1	λ_2	γ	s	L	δ
3.84	3.49	0.91	2.30	1.34	0.24	0.82	0.23

Table I: Parameter computation results.

Inputs u_k are simulated according to a dynamic change for $N = 864$ iterations. The initial state is chosen at $x_0 = (0, 0)^T$. At each step k , chose $A_k, B_k, C_k, D_k, Q_k, R_k$ in corresponding interval matrices and so that Q_k and R_k are positive semi-definite. Then w_k, v_k are simulated and $\{x_k, y_k\}_{k \in 1:N}$ are computed according to system (1). The Algorithm is initialized at $[\hat{x}_0] = ([-0.5, 0.5], [-0.5, 0.5])^T$ and $\mathcal{P}_{0|0} = \max\{\text{diag}(\text{sup}([Q]))\} I$.

The 95% confidence intervals CI_k contain all real states x_k as shown in Figure 2. The computation time using the OUBIKF with the new setting of the present work is improved against the OUBIKF with the setting proposed in [1] (Table II), while the last one has been shown by simulation to be more efficient in computation time against its precursor [9].

OUBIKF	With new settings	With settings of [1]
Computation time (s)	2.33	3.02

Table II: Computation time of OUBIKF with two settings.

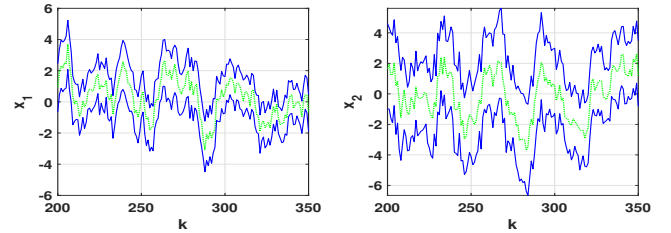


Figure 2: Estimation results. For $i = 1, 2$, the center green line: real states x_k^i , the blue lines: 95% confidence intervals CI_k^i .

IV. CONCLUSION

The OUBIKF Algorithm proposed in [1] is enhanced theoretically and practically by the two stages optimization and the guaranteed conditions **C1**.

Under these conditions, the optimal upper bound is reached approximately by the chosen $\mathcal{P}_{k|k}$ whose trace is highly closed to the optimal value Φ_k^* . Also, the considered Algorithm is ensured to perform with stability in the sense that the trace of $\mathcal{P}_{k|k}$ is non-asymptotically and asymptotically bounded and can be controlled, implying that there is no width explosion of the yielded estimators. In addition, the trace of $\mathcal{P}_{k|k}$ is smaller than the one of the upper bound ($\alpha_k I_{n_x}$) of $S_+([P_{k|k-1}])$ in the prediction step.

Thanks to deep analysis in limit results, expressions of the correction step are simplified and many factors of them can be computed off-line. It reduces the algorithm computation time in comparison Algorithm 1 with others used in [2], [8], [9],

[1] depending on the complexity of the gain expressions and of the corresponding method of finding the gain.

V. APPENDIX

Proposition 2. a) (Proposition 1 in [9]) Let M, N be two real matrices of the same dimension then

$$MN^T + NM^T \preceq t^{-1}MM^T + tNN^T, \forall t > 0. \quad (14)$$

b) If $\{M_u\}_{u=1:n}$ is a sequence of real matrices, then

$$\sum_{u=1}^n M_u \sum_{u=1}^n M_u^T \preceq \sum_{u=1}^n \left(1 + \sum_{v=1, v \neq u}^n \sigma_{u,v} \right) M_u M_u^T \quad (15)$$

provided that $\sigma_{u,v} = \sigma_{v,u}^{-1} > 0, \forall u, v \in \{1:n\}, v \neq u$.

Proof. $\sum_{u=1}^n M_u \sum_{u=1}^n M_u^T = \sum_{u=1}^n M_u M_u^T + \sum_{u \neq v} M_u M_v^T$.

Applying (14), one gets

$$\begin{aligned} \sum_{u \neq v} M_u M_v^T &= \sum_{u < v} (M_u M_v^T + M_v M_u^T), \\ &\preceq \sum_{u < v} (\sigma_{u,v} M_u M_u^T + \sigma_{v,u} M_v M_v^T), \forall \sigma_{u,v} = \sigma_{v,u}^{-1} > 0. \end{aligned}$$

Then by rewriting the last RHS of the above inequality in the form $\sum_{v=1, v \neq u}^n \sigma_{u,v} M_u M_u^T$, (15) is concluded. \square

Proof of Lemma 1. Since $MM^T \in S_+(n_y)$, it can be decomposed as $MM^T = Q\Lambda Q^T$, $QQ^T = I$ and $\Lambda = \text{diag}\{\lambda_i(MM^T), i = 1, \dots, n_y\}$ with r positive eigenvalues λ_i . It implies $(MM^T + sI)^{-1} = Q(\Lambda + sI)^{-1}Q^T, \forall s \in \mathbb{R}$. Thus:

$$\text{tr}\{M^T (MM^T + \beta c I_{n_y})^{-1} M\} = \sum_{i=1}^r \frac{\lambda_i}{\lambda_i + \beta c}$$

$$\text{and } \xi(\beta) = a(\beta) \left[n_x - \sum_{i=1}^r \frac{\lambda_i}{\lambda_i + \beta c} \right] = a(\beta) \sum_{i=1}^r \left[\frac{n_x}{r} - \frac{\lambda_i}{\lambda_i + \beta c} \right].$$

Since $0 < r = \text{rank}(M) \leq \min\{n_x, n_y\}$ then $\frac{n_x}{r} = 1 + \delta$ for some $\delta \geq 0$, and the lemma is concluded using

$$\begin{aligned} \xi(\beta) &= (1 + n_0/\beta)\alpha\delta r + \alpha c \sum_{i=1}^r \frac{\beta + n_0}{c\beta + \lambda_i}, \\ \frac{d\xi}{d\beta}(\beta) &= \frac{-\alpha\delta r}{\beta^2} + \alpha c \sum_{i=1}^r \frac{\lambda_i - n_0 c}{(c\beta + \lambda_i)^2}. \quad \square \end{aligned}$$

Proof of Proposition 1. Using following facts

- $0 \leq MM^T + \beta c_{1,k} I_{n_y} \preceq MM^T + \beta \Sigma + \frac{\gamma\beta}{\alpha_k(\beta+n_0)} I_{n_y} \preceq MM^T + \beta c_{2,k} I_{n_y}$,
- $A, B \in S_+(n)$ and $0 \leq A \preceq B$ imply that
 - + $0 \leq B^+ \preceq A^+$ (note that $X^+ \equiv X^{-1}$ if X^{-1} exists),
 - + $0 \leq M^T A M \preceq M^T B M, \forall M \in \mathbb{R}^{n \times p}$,
 - + $0 \leq P + A \preceq P + B, \forall P \in \mathbb{R}^{n \times n}$,
 - + $0 \leq sA \preceq sB, \forall s > 0$ and $tB \preceq tA \preceq 0, \forall t < 0$,

and get for all $0 < \beta \leq \epsilon$ (note that $\xi_{1,k}(\cdot)$ depends on ϵ):

$$0 \leq \xi_{1,k}(\beta) \leq \phi_k(\beta) \leq \xi_{2,k}(\beta) \leq \alpha_k(1 + n_0/\beta)n_x.$$

It remains to prove $0 \leq h_k(\beta) \leq \xi_{1,k}(\beta)$ for (10) to be true. It is obvious that $h_k(\beta) \geq 0, \forall \beta > 0$ since $r = \text{rank}(M) \leq \min\{n_x, n_y\}$. $h_k(\beta) \leq \xi_{1,k}(\beta)$ follows from the fact that

$$r \geq \sum_{i=1}^r \frac{\lambda_i}{\lambda_i + \beta c_1} = \text{tr}\{M^T (M^T + \beta c_1 I_{n_y})^{-1} M\}, \forall \beta > 0.$$

By (10), we get for all $\epsilon > 0$:

$$0 \leq \alpha_k(n_x - r) \leq \inf_{\beta > 0} \xi_{1,k}(\beta) \leq$$

$$\leq \Phi_k^* \leq \inf_{\beta > 0} \xi_{2,k}(\beta) \leq \alpha_k n_x,$$

• $0 \leq \lim_{\beta \rightarrow 0} \xi_{1,k}(\beta) \leq \lim_{\beta \rightarrow 0} \phi_k(\beta) \leq \lim_{\beta \rightarrow 0} \xi_{2,k}(\beta)$, and, beside, $\Phi_k^* \leq \lim_{\beta \rightarrow 0} \phi_k(\beta)$. Then using Lemma 1, the two statements a) and b) of the proposition are concluded. \square

Proof of Lemma 3. a) The first expression is verified thanks to Proposition 3.2 of [12].

b) Since $A^+ = A^{-1}$ when the later exists and applying the Tikhonov's regularization from Theorem 4.3 of [12], we get

$$\begin{aligned} \lim_{\beta \rightarrow 0} \bar{K}_{k,\beta}^* &= \lim_{\beta \rightarrow 0} M^T \left(MM^T + \beta \Sigma + \frac{\gamma\beta}{\alpha_k(\beta+n_0)} I \right)^+ = \\ &= \lim_{\beta \rightarrow 0} \lim_{\eta \rightarrow 0} M^T \left(MM^T + \beta \Sigma + \frac{\gamma\beta}{\alpha_k(\beta+n_0)} I \right)^T \times \\ &\quad \times \left[\left(MM^T + \beta \Sigma + \frac{\gamma\beta}{\alpha_k(\beta+n_0)} I \right)^2 + \eta I \right]^{-1} = \\ &= \lim_{\eta \rightarrow 0} M^T (MM^T) \left[(MM^T)^2 + \eta I \right]^{-1} = \\ &= M^T (MM^T)^+ = \bar{K}^*. \end{aligned}$$

$$\begin{aligned} \lim_{\beta \rightarrow 0} \varphi_k(\bar{K}_{k,\beta}^*) &= \varphi_k(\bar{K}^*) \text{ thanks to } \lim_{\beta \rightarrow 0} \bar{K}_{k,\beta}^* = \bar{K}^* \text{ and} \\ \varphi_k(\bar{K}_{k,\beta}^*) - \varphi_k(\bar{K}^*) &= (\bar{K}_{k,\beta}^* - \bar{K}^*) (S_k \bar{K}^{*T} - C_k P_{k|k-1}) \\ &\quad + (\bar{K}_{k,\beta}^* S_k - P_{k|k-1} C_k^T) (\bar{K}_{k,\beta}^* - \bar{K}^*)^T. \end{aligned}$$

c) By definition of Moore-Penrose pseudoinverse, we get $MM^+M = M$ and hence $M(I_{n_x} - M^+M) = \mathbf{0}_{n_y \times n_x}$.

Let $X = I_{n_x} - M^+M$. By assumption $\text{rank}(M) = n_x$, the null space of M is $\{\mathbf{0}_{n_x \times 1}\}$. It follows that all columns of X equal $\mathbf{0}_{n_x \times 1}$ and hence $X = I_{n_x} - \bar{K}^* M = \mathbf{0}_{n_x \times n_x}$. \square

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