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Linear Programming based Lower Bounds on Average Dwell-Time via Multiple Lyapunov Functions

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Abstract

With the objective of developing computational methods for stability analysis of switched systems, we consider the problem of finding the minimal lower bounds on average dwell-time that guarantee global asymptotic stability of the origin. Analytical results in the literature quantifying such lower bounds assume existence of multiple Lyapunov functions that satisfy some inequalities. For our purposes, we formulate an optimization problem that searches for the optimal value of the parameters in those inequalities and includes the computation of the associated Lyapunov functions. In its generality, the problem is nonconvex and difficult to solve numerically, so we fix some parameters which results in a linear program (LP). For linear vector fields described by Hurwitz matrices, we prove that such programs are feasible and the resulting solution provides a lower bound on the average dwell-time for exponential stability. Through some experiments, we compare our results with the bounds obtained from other methods in the literature and we report some improvements in the results obtained using our method.

Keywords: Switched systems, continuous piecewise-affine Lyapunov functions, average dwell-time, linear programs

1. Introduction

Switched systems comprise a family of dynamical subsystems orchestrated by a switching signal that activates one of these subsystems at a given time. This abstract framework has been useful in modeling a class of hybrid systems with

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5 continuous and discrete dynamics. Another common source of switched systems
6 is uncertainty quantification in continuous-time systems and the associated differ-
7 ential inclusions. Stability analysis of switched systems, therefore, has gathered a
8 lot of attention in the literature. The references [16, 25] provide a comprehensive
9 overview of the different approaches on this topic.

10 When analyzing stability under arbitrary switching, existence of a common
11 Lyapunov function is a necessary and sufficient condition for the asymptotic sta-
12 bility of an equilibrium of the switched system [6]. Thus, over the years, a lot
13 of attention in the literature has been given to computing a common Lyapunov
14 function for the switched system under different hypotheses. For some results in
15 this direction, the reader may refer to [1] for discrete-time systems, and [22] for
16 continuous-time systems. Particularly relevant to this paper is the technique based
17 on the construction of continuous and piecewise affine (CPA) Lyapunov functions,
18 which is reviewed in [11]. The papers [3, 13] present the adaptation of computing
19 CPA Lyapunov functions in case of arbitrarily switching systems. However, such
20 methods have not yet been used in the context of constrained, or dwell-time based,
21 switched systems.

22 For certain applications, existence of common Lyapunov function is a strin-
23 gent requirement, and may not hold for the given system data. For that reason,
24 when the individual subsystems are asymptotically stable and one can not com-
25 pute a common Lyapunov function, it is natural to ask how we can guarantee
26 stability for a certain class of switching signals (which is smaller than the set of
27 switching signals with arbitrary switching). The works [21] and [14] studied the
28 stability of switched systems by putting a bound on how fast the switches can
29 occur. Depending on the system data, lower bounds were derived on the (aver-
30 age) dwell-time which ensures global asymptotic stability if the length of interval
31 between two consecutive switches (on average) is greater than the derived lower
32 bound. A tutorial like exposition of these concepts also appears in [16, Chap-
33 ter 3]. Several works have followed up to extend this idea in several directions.
34 Some generalizations have been addressed in the recent papers [18, 23] with non-
35 linearities in the system data.

36 Computational methods with multiple Lyapunov functions for getting best
37 possible lower bounds on the dwell-time have not received much attention in the
38 literature. The references [4, 7, 17, 20] provide some algorithms for calculat-
39 ing lower bounds on the dwell-time in the linear case. Among these, the papers
40 [17, 20] build on dwell-time bounds obtained from multiple Lyapunov functions,
41 which is also the case for this article. The authors of [20] developed optimization-
42 based methods for the automatic verification of dwell-time properties. On the

43 other hand, [17] proposes some relaxations in the form of sequential convex pro-
 44 grams to compute lower bounds on the average dwell-time. With similar motiva-
 45 tion, this article studies computational methods for computing best possible lower
 46 bounds on the (average) dwell-time using linear programming (LP) methods. In
 47 fact, our approach uses techniques based on the construction of CPA Lyapunov
 48 functions, under the constraints that are normally imposed for dwell-time based
 49 stability conditions. For a given family of dynamical subsystems with asymptoti-
 50 cally stable origin, the question of interest is to find the *smallest* lower bound on
 51 the dwell-time, which ensures asymptotic stability of the switched system under
 52 the so-called compatibility constraints. Such questions can be formulated as an
 53 optimization problem and in its full generality, it is a nonconvex problem, even
 54 when dealing with linear subsystems and quadratic Lyapunov functions for indi-
 55 vidual subsystems.

56 In this paper, we provide a new technique for solving the optimization prob-
 57 lem that corresponds to the computation of a minimum average dwell-time that
 58 ensures stability. The intermediate step in getting this bound is to first compute
 59 the Lyapunov functions for individual subsystems satisfying certain inequalities.
 60 In our work, we search for these Lyapunov functions from the family of continu-
 61 ous piecewise affine functions, in contrast to the quadratic ones. This is done by
 62 discretizing the state space into simplices and solving for the values of the Lya-
 63 punov functions at the vertices of the simplices, using some inequality constraints.
 64 The resulting optimization problem actually turns out to be a linear program. The
 65 solution to this linear program provides us with a Lyapunov function for each
 66 subsystem and also a dwell-time bound.

67 The remainder of the paper is organized as follows: we recall some basic re-
 68 sults on (average) dwell-time stability in Section 2 and describe the problem being
 69 studied in this paper. Section 3 provides the LP formulation of the proposed prob-
 70 lem along with some results about the feasibility of these programs. We provide
 71 some simulations and comparisons with other methods in Section 4, followed by
 72 concluding remarks in Section 5.

73 2. Problem Setup

74 We consider time-dependent switched dynamical systems described as

$$\dot{\mathbf{x}} = \mathbf{f}_\sigma(\mathbf{x}) \tag{1}$$

75 where, for some given index set $\mathcal{P} \subset \mathbb{N}$, the function $\sigma : [0, \infty) \rightarrow \mathcal{P}$ is piecewise
 76 constant and right-continuous, called the *switching signal*. The discontinuities of

77 σ , called the *switching times*, are assumed to be locally finite. The vector fields
 78 $\mathbf{f}_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$, for each $i \in \mathcal{P}$, are assumed to be locally Lipschitz and with
 79 $\mathbf{f}_i(\mathbf{0}) = \mathbf{0}$. We say that a switching signal σ has an average dwell-time $\tau_a > 0$, if
 80 there exists $N_0 > 0$, such that

$$N_\sigma(t, s) \leq N_0 + \frac{t - s}{\tau_a},$$

81 where $N_\sigma(t, s)$ denotes the number of switches over the interval (s, t) . The set of
 82 all switching signals with average dwell-time τ_a is denoted by Σ_{τ_a} .

83 For the stability of the origin for such systems, let us recall the following
 84 result, which follows from [14], [16, Chapter 3], and [18, Theorem 1]:

85 **Theorem 1.** *Suppose that there exist C^1 Lyapunov functions $V_p : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$,*
 86 *$i \in \mathcal{P}$, satisfying the following:*

87 **(L1)** *There exist $\underline{\alpha}, \bar{\alpha} \in \mathcal{K}_\infty$ such that*

$$\underline{\alpha}(\|\mathbf{x}\|) \leq V_i(\mathbf{x}) \leq \bar{\alpha}(\|\mathbf{x}\|), \quad \forall \mathbf{x} \in \mathbb{R}^n, i \in \mathcal{P}. \quad (2)$$

88 **(L2)** *There exists a Lipschitz function $\rho \in \mathcal{K}$, such that, for every $i \in \mathcal{P}$,*

$$\nabla V_i(\mathbf{x}) \cdot \mathbf{f}_i(\mathbf{x}) \leq -\rho(V_i(\mathbf{x})) \quad \forall \mathbf{x} \in \mathbb{R}^n, \quad (3)$$

89 **(L3)** *There exists $\chi \in \mathcal{K}_\infty$ such that, for every $i, j \in \mathcal{P}$,*

$$V_j(\mathbf{x}) \leq \chi(V_i(\mathbf{x})) \quad \forall \mathbf{x} \in \mathbb{R}^n, i \neq j. \quad (4)$$

90 *Then the origin is globally asymptotically stable for the switched system (1), uni-*
 91 *formly over the set Σ_{τ_a} , for τ_a satisfying*

$$\tau_a > \sup_{s>0} \int_s^{\chi(s)} \frac{1}{\rho(r)} dr. \quad (5)$$

92 The lower bound on the average dwell-time given in (5), with nonlinear func-
 93 tions, has also appeared in the context of impulsive systems in [24], [5]. In what
 94 follows, we will restrict ourselves to linear subsystems where the functions α and
 95 ρ can be taken as linear. Writing (5) for such cases allows us to better understand
 96 the degrees of freedom at our disposal for minimizing the lower bounds on τ_a .

97 *2.1. Corollaries and special cases*

98 Let us present two corollaries to this result depending on the class of functions
99 chosen for V_i and the vector fields $\mathbf{f}_i, i \in \mathcal{P}$.

Corollary 1. *Assume that for each $i \in \mathcal{P}$, there exist symmetric, positive definite matrices $P_i \succ 0$, such that*

$$A_i^\top P_i + P_i A_i + \alpha P_i \preceq 0, \quad \text{for every } i \in \mathcal{P}, \quad (6a)$$

$$P_j \preceq \mu P_i, \quad \text{for every } j \neq i \quad (6b)$$

100 *for some $\alpha > 0$ and $\mu \geq 1$. Then the switched system (1) with $\mathbf{f}_i(\mathbf{x}) = A_i \mathbf{x}, i \in \mathcal{P}$,*
101 *has a globally exponentially stable equilibrium at the origin, uniformly over the*
102 *set Σ_{τ_a} , where τ_a satisfies*

$$\tau_a > \frac{\ln(\mu)}{\alpha}. \quad (7)$$

103 Thus, for linear dynamics and quadratic Lyapunov functions, we can get a
104 lower bound on the average dwell-time by solving matrix inequalities (6). If we
105 take, $\alpha > 0, \mu \geq 1$, and $P_i \succ 0$ as the unknowns in (6), then these inequalities are
106 not linear with respect to the unknowns, and it is difficult to compute a solution.
107 The article [17] addresses the problem of minimizing $\frac{\ln(\mu)}{\alpha}$ subject to inequalities
108 (6) by proposing convex relaxations.

109 For the algorithms proposed in this paper, we first need a corollary to Theo-
110 rem 1 with continuous Lyapunov functions, and norm-like bounds on the growth
111 and Dini-derivative of such functions. For a continuous function $V : \mathbb{R}^n \rightarrow \mathbb{R}$, we
112 define the Dini-derivative along the solutions of the system $\dot{\mathbf{x}} = \mathbf{f}_i(\mathbf{x})$ as

$$D^+V(\mathbf{x}, \mathbf{f}_i(\mathbf{x})) := \limsup_{h \rightarrow 0^+} \frac{V(\phi_i(h, \mathbf{x})) - V(\mathbf{x})}{h}$$

113 where $t \mapsto \phi_i(t, \mathbf{x})$ is an absolutely continuous function that satisfies $\phi_i(0, \mathbf{x}) =$
114 \mathbf{x} and $\dot{\phi}_i(t, \mathbf{x}) = \mathbf{f}_i(\phi_i(t, \mathbf{x}))$ almost surely for $t \geq 0$. We use this notion to state
115 the following corollary to Theorem 1, obtained by taking $\underline{\alpha}, \bar{\alpha}, \rho$ to be homoge-
116 nous functions, and χ being linear.

Corollary 2. *Assume that for each $i \in \mathcal{P}$, there exist continuous functions $V_i : \mathbb{R}^n \rightarrow \mathbb{R}$, such that*

$$\underline{\alpha} \|\mathbf{x}\|^d \leq V_i(\mathbf{x}) \leq \bar{\alpha} \|\mathbf{x}\|^d, \quad \text{for every } i \in \mathcal{P}, \quad (8a)$$

$$D^+V_i(\mathbf{x}, \mathbf{f}_i(\mathbf{x})) \leq -\alpha \|\mathbf{x}\|^d, \quad \text{for every } i \in \mathcal{P}, \quad (8b)$$

$$V_i(\mathbf{x}) \leq \mu V_j(\mathbf{x}), \quad \text{for every } j \neq i. \quad (8c)$$

117 for some $d > 0$, $\bar{a} \geq \underline{a} > 0$, $\alpha > 0$ and $\mu \geq 1$; Then the switched system
 118 $\dot{\mathbf{x}} = \mathbf{f}_\sigma(\mathbf{x})$ has a globally exponentially stable equilibrium at the origin, uniformly
 119 over the set Σ_{τ_a} , with τ_a satisfying,

$$\tau_a > \frac{\bar{a} \ln(\mu)}{\alpha}. \quad (9)$$

120 In contrast to Theorem 1, the proof of Corollary 2 using Dini derivative requires
 121 some care but essentially follows similar concepts. In this paper, we will build on
 122 the statement of Corollary 2 and, in particular, address the following problem:

123 *Problem statement.* With $d = 1$ and for fixed values of $\underline{a} > 0$, $\bar{a} > 0$, and
 124 $\mu \geq 1$, find piecewise linear functions V_i , for each $i \in \mathcal{P}$, that satisfy (8) while
 125 maximizing α .

126 The reason for fixing the constants $\underline{a} > 0$, $\bar{a} > 0$, and $\mu \geq 1$ is that the
 127 foregoing problem then transforms into a linear program. We will provide the
 128 formulation of this linear program and discuss its feasibility in the next section.
 129 For the sake of clarity in this conference paper, we present our ideas for the linear
 130 vector fields but similar concepts can be extended to nonlinear systems.

131 2.2. Quadratic functions and matrix inequalities

132 Before discussing the LP problem and CPA Lyapunov functions, let us first
 133 look at the inequalities (8) for the case $d = 2$ more carefully. In this case, we let
 134 $V_i(\mathbf{x}) = \mathbf{x}^\top P_i \mathbf{x}$, with symmetric and positive definite $P_i \in \mathbb{R}^{n \times n}$. In particular,
 135 (8) takes the following form, where I is the identity matrix:

$$\begin{cases} \underline{a}I \preceq P_i \preceq \bar{a}I, & i \in \mathcal{P}, \\ A_i^\top P_i + P_i A_i \preceq -\alpha I, & i \in \mathcal{P}, \\ P_i \preceq \mu P_j, & i, j \in \mathcal{P}. \end{cases} \quad (10)$$

In (10), if we fix $\underline{a}, \bar{a} > 0$ and $\mu \geq 1$, then the inequalities result in LMIs with unknowns P_i , $i \in \mathcal{P}$, which can be solved to maximize α . Practically one can select \underline{a} small, e.g. $\underline{a} = 10^{-5}$ as we do in our examples, and then a large enough $\bar{a} > 0$ will ensure that $\tau_a = \bar{a} \ln(\mu)/\alpha$ can be made minimal for the given $\mu \geq 1$ by maximizing α . Indeed, assume the conditions (10) are fulfilled for some positive constants $\underline{a} = \underline{a}^*$, $\bar{a} = \bar{a}^*$, $\alpha = \alpha^*$ and $V_i(\mathbf{x}) = \mathbf{x}^\top P_i^* \mathbf{x}$, $P_i^* \in \mathbb{R}^{n \times n}$ symmetric, $i \in \mathcal{P}$. Then we have the lower bound $\tau_a^* = \bar{a}^* \ln(\mu)/\alpha^*$ on the average dwell-time. Now fix new constants $\underline{a}, \bar{a} > 0$ such that $\bar{a}/\underline{a} \geq \bar{a}^*/\underline{a}^*$ and set $\alpha =$

$(\bar{a}/\underline{a}^*)\alpha^*$ and $P_i = (\bar{a}/\underline{a}^*)P_i^*$. Then, for $V_i(\mathbf{x}) := \mathbf{x}^\top P_i \mathbf{x} = (\bar{a}/\underline{a}^*)\mathbf{x}^\top P_i^* \mathbf{x}$, we have $V_i(\mathbf{x}) \leq \mu V_j(\mathbf{x})$, and

$$\underline{a}\|\mathbf{x}\|^2 \leq \underline{a}^* \frac{\bar{a}}{\underline{a}^*} \|\mathbf{x}\|^2 \leq V_i(\mathbf{x}) = (\bar{a}/\underline{a}^*)\mathbf{x}^\top P_i^* \mathbf{x} \leq \bar{a}\|\mathbf{x}\|^2$$

for $i, j \in \mathcal{P}$. That is, the constraints (10) are fulfilled with these values of $\underline{a}, \bar{a}, \alpha, P_i$ and further, for the lower bound on the average dwell-time we have

$$\tau_a = \frac{\bar{a} \ln(\mu)}{\alpha} = \frac{\bar{a} \ln(\mu)}{\frac{\bar{a}\alpha^*}{\underline{a}^*}} = \frac{\bar{a}^* \ln(\mu)}{\alpha^*} = \tau_a^*.$$

136 In other words, for a fixed $\mu \geq 1$, if there is a solution to (10) for some choice of
 137 $\underline{a}^*, \bar{a}^*, \alpha^*$ which yields the bound τ_a^* for the average dwell-time, then by choosing
 138 \bar{a}/\underline{a} large enough, one can always find another solution to (10) which gives at least
 139 as good a bound on average dwell-time as τ_a^* . Thus, given $\mu \geq 1$, maximizing
 140 $\alpha > 0$ under the constraints (10) for a fixed $\underline{a}, \bar{a} > 0$ delivers as good lower bounds
 141 on the average dwell-time τ_a as minimizing $\tau_a = \bar{a} \ln(\mu)/\alpha$, where both \bar{a} and α
 142 are variables, given that \bar{a}/\underline{a} is large enough.

143 Note that in the setting of the LMI problem (10) we are searching for quadratic
 144 Lyapunov functions for the individual subsystems $\dot{\mathbf{x}} = A_i \mathbf{x}$, $i \in \mathcal{P}$, which can be
 145 conservative. In the next section we consider a similar approach for modeling the
 146 conditions (8) using piecewise linear Lyapunov functions and an LP formulation
 147 to compute them. Due to the foregoing observation, when solving (8) using an LP
 148 formulation, we will fix \underline{a} and \bar{a} with \bar{a}/\underline{a} large enough and maximize α .

149 **3. Continuous Piecewise Affine Lyapunov Functions and Linear Program-** 150 **ming Formulation**

151 Our LP approach to compute piecewise linear Lyapunov functions fulfilling
 152 the conditions (8) is based on the so-called CPA method to compute Lyapunov
 153 functions, see e.g. [19, 3, 10, 13]. Its description is somewhat more involved
 154 than the LMI approach, because it is based on partitioning a neighborhood of the
 155 origin into simplices and the underlying idea behind constructing this collection
 156 of simplices, called *triangulation*, is described in the next subsection.

157 *3.1. The Triangulation \mathcal{T}_K^F*

Roughly speaking, a triangulation is the subdivision of a subset of \mathbb{R}^n into simplices. A suitable concrete triangulation for our aim of parameterizing Lyapunov

functions for the individual subsystems is the triangular-fan of the triangulation in [8], where its efficient implementation is also discussed. In its definition, we use the functions $\mathbf{R}^{\mathcal{J}} : \mathbb{R}^n \rightarrow \mathbb{R}^n$, defined for every $\mathcal{J} \subset \{1, 2, \dots, n\}$ by

$$\mathbf{R}^{\mathcal{J}}(\mathbf{x}) := \sum_{i=1}^n (-1)^{\mathbb{1}_{\mathcal{J}}(i)} x_i \mathbf{e}_i,$$

158 where \mathbf{e}_i is the standard i th unit vector in \mathbb{R}^n , and $\mathbb{1}_{\mathcal{J}}(i) = 1$ if $i \in \mathcal{J}$ and 0
 159 otherwise. Thus, $\mathbf{R}^{\mathcal{J}}(\mathbf{x})$ is the vector \mathbf{x} , except for a minus has been put in front
 160 of the coordinate x_i whenever $i \in \mathcal{J}$.

161 We first define the triangulation \mathcal{T}^{std} and use it to construct the intermediate
 162 triangulation \mathcal{T}_K , which in turn is used to define our desired triangulation $\mathcal{T}_K^{\mathbf{F}}$.

The standard triangulation \mathcal{T}^{std} consists of the simplices

$$\mathfrak{S}_{\mathbf{z}\mathcal{J}\rho} := \text{co} \left\{ \mathbf{x}_0^{\mathbf{z}\mathcal{J}\rho}, \mathbf{x}_1^{\mathbf{z}\mathcal{J}\rho}, \dots, \mathbf{x}_n^{\mathbf{z}\mathcal{J}\rho} \right\},$$

163 where co denotes the convex hull, and

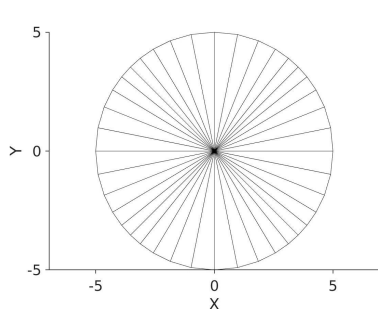
$$\mathbf{x}_j^{\mathbf{z}\mathcal{J}\rho} := \mathbf{R}^{\mathcal{J}} \left(\mathbf{z} + \sum_{i=1}^j \mathbf{e}_{\rho(i)} \right), \quad (11)$$

164 for all $\mathbf{z} \in \mathbb{N}_0^n = \{0, 1, \dots\}^n$, all $\mathcal{J} \subset \{1, 2, \dots, n\}$, all $\rho \in S_n$, and $j =$
 165 $0, 1, \dots, n$. Here, S_n denotes the set of all permutations of $\{1, 2, \dots, n\}$.

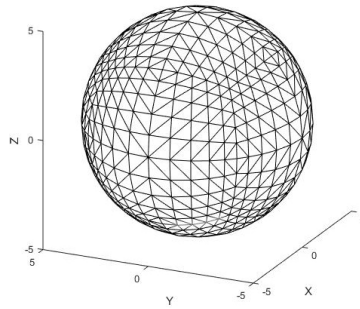
166 Now fix a $K \in \mathbb{N}_+ = \{1, 2, \dots\}$ and define the hypercube $\mathcal{H}_K := [-K, K]^n$.
 167 Consider the simplices $\mathfrak{S}_{\mathbf{z}\mathcal{J}\rho} \subset \mathcal{H}_K$ in \mathcal{T}^{std} , that intersect the boundary of \mathcal{H}_K .
 168 We are only interested in those intersections that are $(n-1)$ -simplices, i.e. we
 169 take every simplex with vertices $\mathbf{x}_j := \mathbf{R}^{\mathcal{J}} \left(\mathbf{z} + \sum_{i=1}^j \mathbf{e}_{\rho(i)} \right)$, $j \in \{0, 1, \dots, n\}$,
 170 where exactly one vertex \mathbf{x}_{j^*} satisfies $\|\mathbf{x}_{j^*}\|_{\infty} < K$ and the other n of the $n+1$
 171 vertices satisfy $\|\mathbf{x}_j\|_{\infty} = K$, i.e. for $j \in \{0, 1, \dots, n\} \setminus \{j^*\}$. Then we replace
 172 the vertex \mathbf{x}_{j^*} by $\mathbf{0}$; it is not difficult to see that j^* is necessarily equal to 0. The
 173 collection of such vertices triangulates \mathcal{H}_K and this new triangulation of \mathcal{H}_K is
 174 our desired triangulation \mathcal{T}_K .

175 It has been shown [2] that it is often advantageous in the CPA method to map
 176 the vertices of the triangulation by the mapping $\mathbf{F} : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $\mathbf{F}(\mathbf{0}) = \mathbf{0}$ and

$$\mathbf{F}(\mathbf{x}) := \frac{\|\mathbf{x}\|}{\|\mathbf{x}\|_{\infty}} \mathbf{x}, \quad \text{if } \mathbf{x} \neq \mathbf{0}. \quad (12)$$



(a) The triangulation $\mathcal{T}_5^{\mathbf{F}}$ in two dimensions



(b) The triangulation $\mathcal{T}_5^{\mathbf{F}}$ in three dimension; note that the origin is a vertex of all the tetrahedra in $\mathcal{T}_5^{\mathbf{F}}$.

Figure 1: Our proposed triangulation in \mathbb{R}^2 and \mathbb{R}^3 .

177 Note that \mathbf{F} maps the hypercubes $\{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_\infty = r\}$ to the spheres $\{\mathbf{x} \in$
 178 $\mathbb{R}^n : \|\mathbf{x}\| = r\}$.

Finally, we define the triangulation $\mathcal{T}_K^{\mathbf{F}}$ that will be used in the LP problem to parameterize CPA Lyapunov functions. Let $\mathcal{T}_K^{\mathbf{F}}$ be the triangulation consisting of the simplices

$$\mathfrak{S}_\nu := \text{co}\{\mathbf{0}, \mathbf{F}(\mathbf{x}_1^\nu), \mathbf{F}(\mathbf{x}_2^\nu), \dots, \mathbf{F}(\mathbf{x}_n^\nu)\},$$

where

$$\text{co}\{\mathbf{0}, \mathbf{x}_1^{\mathbf{z}\mathcal{J}\rho}, \mathbf{x}_2^{\mathbf{z}\mathcal{J}\rho}, \dots, \mathbf{x}_n^{\mathbf{z}\mathcal{J}\rho}\} \in \mathcal{T}_K.$$

The subset of \mathbb{R}^n subdivided into simplices by the triangulation $\mathcal{T}_K^{\mathbf{F}}$ is denoted by

$$\mathcal{D}_{\mathcal{T}_K^{\mathbf{F}}} := \bigcup_{\mathfrak{S}_\nu \in \mathcal{T}_K^{\mathbf{F}}} \mathfrak{S}_\nu.$$

179 Figure 1 depicts two exemplary triangulations of the type $\mathcal{T}_K^{\mathbf{F}}$ for two and three
 180 dimension with $K = 5$. The implementation of the triangulation is discussed in
 181 [12, 9].

182 3.2. LP Problem

183 We are now ready to state our LP problem to parameterize piecewise linear
 184 Lyapunov functions for the switched system fulfilling the conditions in (8). For
 185 formulating this LP, and showing that its feasibility provides us the lower bound
 186 on average dwell-time, we focus our attention on the switched linear systems:

$$\dot{\mathbf{x}} = A_\sigma \mathbf{x} \tag{13}$$

187 with $\sigma : [0, \infty) \rightarrow \mathcal{P}$ being the switching signal, and $A_i \in \mathbb{R}^{n \times n}$, for each $i \in \mathcal{P}$.

188 We use three constants $\underline{a}, \bar{a} > 0$ and $\mu \geq 1$ in the LP problem. We want the
 189 ratio \bar{a}/\underline{a} to be large, as discussed in the last section, and then we want to try out
 190 different $\mu \geq 1$ to obtain as good a lower bound on the average dwell-time as
 191 possible.

192 The variables of the LP problem are $\alpha \in \mathbb{R}$ and $V_{\mathbf{x},i} \in \mathbb{R}$ for every vertex \mathbf{x} of
 193 a simplex in $\mathcal{T}_K^{\mathbb{F}}$ and every $i \in \mathcal{P}$.

194 The objective of the LP problem is to maximize α .

195 The constraints of the LP problem are:

196 **(C1)** The first set of constraints is that, for every $i \in \mathcal{P}$, we set $V_{\mathbf{0},i} = 0$, and for
 197 every vertex \mathbf{x} of a simplex in $\mathcal{T}_K^{\mathbb{F}}$ and for every $i \in \mathcal{P}$:

$$\underline{a}\|\mathbf{x}\| \leq V_{\mathbf{x},i} \leq \bar{a}\|\mathbf{x}\| \quad (14)$$

198 **(C2)** The second set of constraints is more involved. For every simplex $\mathfrak{S}_\nu :=$
 199 $\text{co}\{\mathbf{0}, \mathbf{x}_1^\nu, \mathbf{x}_2^\nu, \dots, \mathbf{x}_n^\nu\} \in \mathcal{T}_K^{\mathbb{F}}$, we define the matrix $X_\nu = (\mathbf{x}_1^\nu \ \mathbf{x}_2^\nu \ \cdots \ \mathbf{x}_n^\nu)$,
 200 i.e. \mathbf{x}_k^ν is the k th column of X_ν . Further, we define for every $i \in \mathcal{P}$, the
 201 vector of variables $\mathbf{v}_{\nu,i} = (V_{\mathbf{x}_1^\nu,i} \ V_{\mathbf{x}_2^\nu,i} \ \cdots \ V_{\mathbf{x}_n^\nu,i})^\top$.

202 The constraints are: for every simplex $\mathfrak{S}_\nu \in \mathcal{T}_K^{\mathbb{F}}$, for all $j = 1, \dots, n$ and
 203 all $i \in \mathcal{P}$:

$$\mathbf{v}_{\nu,i}^\top X_\nu^{-1} A_i \mathbf{x}_j^\nu \leq -\alpha \|\mathbf{x}_j^\nu\|. \quad (15)$$

204 Note that these constraints are automatically fulfilled for $j = 0$, i.e. $\mathbf{x}_j^\nu = \mathbf{0}$.

205 **(C3)** The third set of constraints is: for every vertex \mathbf{x} of a simplex in $\mathcal{T}_K^{\mathbb{F}}$ and for
 206 every $i, j \in \mathcal{P}$,

$$V_{\mathbf{x},j} \leq \mu V_{\mathbf{x},i}. \quad (16)$$

207 3.3. Solution to LP delivers lower bounds on dwell-time

208 In the previous subsection, we formulated an LP which basically specified
 209 the constraints in (8) at the vertices of the simplices contained in the triangula-
 210 tion. Here we prove that the feasibility of such a program provides us with piece-
 211 wise linear Lyapunov functions for the individual subsystems over the entire state
 212 space that additionally fulfill (8c), thereby providing a lower bound on the average
 213 dwell-time.

214 Toward this end, assume that the LP problem in Section 3.2 has a solution
 215 with $\alpha > 0$. We then define the piecewise linear function $V_i : \mathcal{D}_{\mathcal{T}_K^{\mathbb{F}}} \rightarrow \mathbb{R}$, for every
 216 $i \in \mathcal{P}$, in the following way:

- 217 • For every $\mathbf{x} \in \mathcal{D}_{\mathcal{T}_K^{\mathbf{F}}}$ there exists a simplex $\mathfrak{S}_\nu = \text{co}\{\mathbf{0}, \mathbf{x}_1^\nu, \mathbf{x}_2^\nu, \dots, \mathbf{x}_n^\nu\} \in \mathcal{T}_K^{\mathbf{F}}$
 218 such that $\mathbf{x} \in \mathfrak{S}_\nu$ and there exist a unique $\lambda \in [0, 1]^n$, $\sum_{j=1}^n \lambda_j \leq 1$, such
 219 that $\mathbf{x} = \sum_{j=1}^n \lambda_j \mathbf{x}_j^\nu$. We define

$$V_i(\mathbf{x}) = \sum_{j=1}^n \lambda_j V_{\mathbf{x}_j^\nu, i}.$$

It is not difficult to see that the functions V_i , $i \in \mathcal{P}$, are continuous functions that are linear on each simplex $\mathfrak{S}_\nu \in \mathcal{T}_K^{\mathbf{F}}$, in particular each V_i has the constant gradient $\nabla V_{\nu, i} := \mathbf{v}_{\nu, i}^\top X_\nu^{-1}$ (row vector) on the interior of \mathfrak{S}_ν , see e.g. [10, Rem. 9]. Hence, for any $\mathbf{x} \in \mathfrak{S}_\nu \in \mathcal{T}_K^{\mathbf{F}}$, $\mathbf{x} = \sum_{j=1}^n \lambda_j \mathbf{x}_j^\nu$, we have for any $i \in \mathcal{P}$ by **(C1)** and **(C2)** that

$$\begin{aligned} \nabla V_{\nu, i} \cdot A_i \mathbf{x} &= \mathbf{v}_{\nu, i}^\top X_\nu^{-1} A_i \sum_{j=1}^n \lambda_j \mathbf{x}_j^\nu = \sum_{j=1}^n \lambda_j \mathbf{v}_{\nu, i}^\top X_\nu^{-1} A_i \mathbf{x}_j^\nu \\ &\leq -\alpha \sum_{j=1}^n \lambda_j \|\mathbf{x}_j^\nu\| \leq -\frac{\alpha}{a} \sum_{j=1}^n \lambda_j V_i(\mathbf{x}_j^\nu) \\ &= -\frac{\alpha}{a} V_i\left(\sum_{j=1}^n \lambda_j \mathbf{x}_j^\nu\right) = -\frac{\alpha}{a} V_i(\mathbf{x}). \end{aligned} \quad (17)$$

Now, for any \mathbf{x} in the interior of $\mathcal{D}_{\mathcal{T}_K^{\mathbf{F}}}$, we have that, for any $i \in \mathcal{P}$, there exists a simplex $\mathfrak{S}_\nu \in \mathcal{T}_K^{\mathbf{F}}$ and an $h > 0$, such that

$$\mathbf{x} + [0, h] A_i \mathbf{x} \subset \mathfrak{S}_\nu,$$

where ν can depend on both \mathbf{x} and i . Because V_i is linear on \mathfrak{S}_ν we have

$$\limsup_{h \rightarrow 0^+} \frac{V_i(\mathbf{x} + h A_i \mathbf{x}) - V_i(\mathbf{x})}{h} = \nabla V_{\nu, i} \cdot A_i \mathbf{x} \leq -\frac{\alpha}{a} V_i(\mathbf{x})$$

and since this holds true for all $i \in \mathcal{P}$, we have $D^+ V_i(\mathbf{x}, A_i \mathbf{x}) \leq -\frac{\alpha}{a} V_i(\mathbf{x})$. Since

$$V_i(\mathbf{x}) = \sum_{j=1}^n \lambda_j V_{\mathbf{x}_j^\nu, i} \geq \underline{a} \sum_{j=1}^n \lambda_j \|\mathbf{x}_j^\nu\| \geq \underline{a} \|\mathbf{x}\|$$

by the constraints **(C1)**, and for all $i, k \in \mathcal{P}$,

$$V_i(\mathbf{x}) = \sum_{j=0}^n \lambda_j V_{\mathbf{x}_j^\nu, i} \leq \sum_{j=0}^n \lambda_j \mu V_{\mathbf{x}_j^\nu, k} = \mu V_k(\mathbf{x}),$$

it is clear that the V_i fulfill the constraints (8) in the interior of $\mathcal{D}_{\mathcal{T}_K^F}$. Just define

$$\bar{a}' := \max_{\|\mathbf{x}\|=1} \max_{i=1,2,\dots,N} V_i(\mathbf{x})$$

and we have

$$\begin{aligned} \underline{a}\|\mathbf{x}\| &\leq V_i(\mathbf{x}) \leq \bar{a}'\|\mathbf{x}\|, \\ D^+V_i(\mathbf{x}, A_i\mathbf{x}) &\leq -\frac{\alpha}{\bar{a}}V_i(\mathbf{x}) \end{aligned}$$

220 for all \mathbf{x} in the interior of $\mathcal{D}_{\mathcal{T}_K^F}$. Note that we proved (17) directly from the con-
 221 straints and did not go through constraints (8) with $\bar{a} = \bar{a}'$, which would lead to a
 222 worse estimate on τ_a .

223 By extending V_i to \mathbb{R}^n in the obvious way, i.e. for every $\mathbf{x} \in \mathbb{R}^n$ there exists
 224 a \mathfrak{S}_ν and unique numbers $\lambda_j \geq 0$ such that $\mathbf{x} = \sum_{j=1}^n \lambda_j \mathbf{x}_j^\nu$ (a cone defined by
 225 the vertices of \mathfrak{S}_ν) and we set $V_i(\mathbf{x}) = \sum_{j=1}^n \lambda_j V_{\mathbf{x}_j^\nu, i}$, we see that the V_i fulfill the
 226 constraints (8) on \mathbb{R}^n , for each $i \in \mathcal{P}$.

227 4. Simulations

228 We will now test our LP algorithm for several examples from the literature.
 229 The class of systems for these simulations is (13). The set \mathcal{P} and the matrices
 230 $\{A_i\}_{i \in \mathcal{P}}$ will be specified differently for the examples considered here. In the
 231 examples, we always fix $\underline{a} = 10^{-5}$ and $\bar{a} = 10$. We used YALMIP/sdpt3 and
 232 Gurobi to solve the LMI and LP problems, respectively.

233 4.1. Example 1: Dwell-time stable but not under arbitrary switching

Consider the switched system (13) with

$$A_1 = \begin{pmatrix} -0.1 & -1 \\ 2 & -0.1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -0.1 & -2 \\ 1 & -0.1 \end{pmatrix}$$

234 This example is taken from [16, p. 26]. It is stable for a certain minimum value
 235 for the average dwell-time, but it is not stable under arbitrary switching. Solving
 236 (10) using the LMI approach, the best τ_a obtained is 5.1929 with $\mu = 2$. Using the
 237 LP approach, the best τ_a obtained is 4.5283 with $\mu = 1.4$ and using $K = 500$ in
 238 the triangulation. Using triangulations with fewer triangles delivers a higher lower
 239 bound τ_a for the average dwell time; $K = 50$ gives $\tau_a = 5.16493$ with $\mu = 1.45$,
 240 $K = 100$ gives $\tau_a = 4.79315$ with $\mu = 1.4$, and $K = 200$ gives $\tau_a = 4.62407$
 241 with $\mu = 1.4$. In all cases they are better than the bounds from the LMI approach.

242

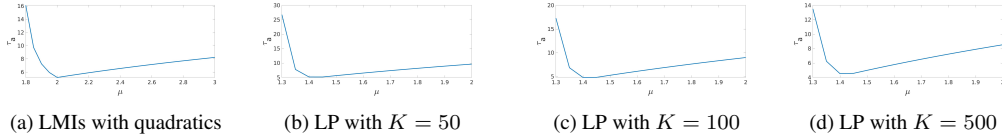


Figure 2: Plots for μ versus τ_a in Example 1.

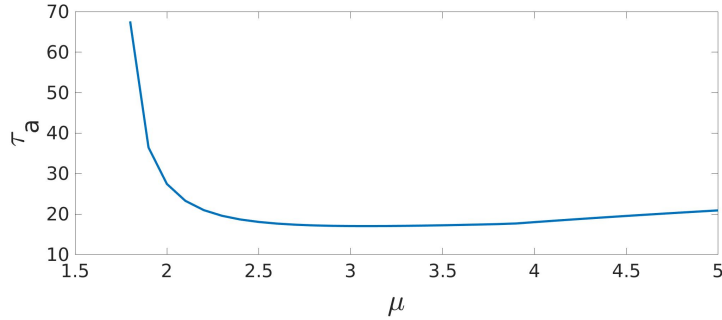


Figure 3: LMI based plot for μ vs τ_a in Example 2.

243 *4.2. Example 2: Stable under arbitrary switching but no common quadratic Lyapunov function*
 244

Take $\mathcal{P} = \{1, 2\}$, with the matrices

$$A_1 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -1 & -10 \\ 0.1 & -1 \end{pmatrix}$$

245 This example is taken from [6]. It is stable under arbitrary switching but the
 246 matrices A_1 and A_2 do not share a common quadratic Lyapunov function. This
 247 example helps us see the limitation of using LMIs because searching for quadratic
 248 certificates in this case is not the best choice.

249 Solving (10) using LMIs, the minimum value for τ_a is $\tau_a = 17.0394$ with
 250 $\mu = 3.1$. Whereas, with our LP approach, $K = 20$ gives a solution with $\mu =$
 251 1 . Hence, the origin is stable under arbitrary switching ($\tau_a = 0$), and we get
 252 a common piecewise linear Lyapunov function although no quadratic Lyapunov
 253 function exists.

254 **4.3. Example 3: Exponentially stable system under arbitrary switching with 5**
 255 **modes**

We consider the switched system (13) with $\mathcal{P} = \{1, 2, 3, 4, 5\}$, where

$$\begin{aligned} A_1 &= \begin{pmatrix} -5 & 1 & 2 \\ 0 & -5 & 1 \\ 0 & 1 & -2 \end{pmatrix}, & A_2 &= \begin{pmatrix} -1 & 3 & 1 \\ 0 & -2 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \\ A_3 &= \begin{pmatrix} 0 & 0 & 3 \\ -2 & -1 & -3 \\ -1 & 0 & -2 \end{pmatrix}, & A_4 &= \begin{pmatrix} -4 & 0 & -3 \\ 2 & -2 & 4 \\ 1 & 0 & -1 \end{pmatrix}, \\ A_5 &= \begin{pmatrix} -1 & 0 & 0 \\ -1 & -1 & -1 \\ -3 & 0 & -4 \end{pmatrix} \end{aligned}$$

256 This example was also considered in [17] with a graph that determines the switch-
 257 ing sequence, and in this particular, we have the star topology. With $K = 6$, we
 258 get a solution with $\mu = 1$, i.e. the origin is exponentially stable under arbitrary
 259 switching (which is then arbitrary without the graph too).

260 With the LMI approach in (10), the minimum value for the average dwell-time
 261 is $\tau_a = 4.6870$ with $\mu = 2.7$.

262

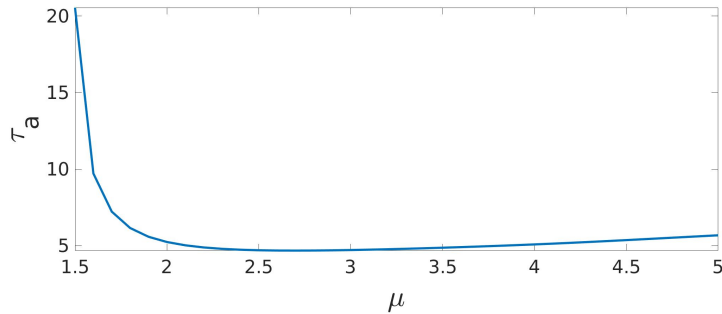


Figure 4: LMI based plot for μ vs τ_a in Example 3.

263 **5. Conclusions**

264 We considered a linear programming (LP) based computational algorithm for
 265 computing lower bounds on average dwell-times that ensure asymptotic stabil-
 266 ity of switched systems. The algorithm is essentially based on gridding the state
 267 space into simplices and computing values for the corresponding Lyapunov func-
 268 tions at the vertices of these simplices. By choosing appropriate values of the
 269 parameters in the inequalities defining the linear program, the solution provides

270 us lower bounds on the average dwell-time necessary to assure stability. From the
271 simulations, we see in several case studies, that LP based bounds are better than
272 the ones based on linear matrix inequalities (LMIs). This is not really surprising
273 since LMIs restrict the Lyapunov functions to be quadratic, whereas the proposed
274 LPs can potentially approximate a broader class of Lyapunov function templates.
275 Computing dwell-time via inequalities in (8) introduces some conservatism be-
276 cause we optimize over a single parameter α while keeping μ fixed. The same
277 conservatism is observed in going from (6) to (10). As a topic of ongoing investi-
278 gation, we are working out algorithm to optimize α and μ simultaneously directly
279 using an LP version of (6). Other than understanding the complexity of the pro-
280 posed algorithm, we also aim to study the extensions of the proposed algorithm
281 in different directions, which includes the study of generalized lower bounds on
282 average dwell-time [15, 18].

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