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Abstract

With the objective of developing computational methods for stability analysis of switched systems, we consider the problem of finding the minimal lower bounds on average dwell-time that guarantee global asymptotic stability of the origin. Analytical results in the literature quantifying such lower bounds assume existence of multiple Lyapunov functions that satisfy some inequalities. For our purposes, we formulate an optimization problem that searches for the optimal value of the parameters in those inequalities and includes the computation of the associated Lyapunov functions. In its generality, the problem is nonconvex and difficult to solve numerically, so we fix some parameters which results in a linear program (LP). For linear vector fields described by Hurwitz matrices, we prove that such programs are feasible and the resulting solution provides a lower bound on the average dwell-time for exponential stability. Through some experiments, we compare our results with the bounds obtained from other methods in the literature and we report some improvements in the results obtained using our method.

Keywords: Switched systems, continuous piecewise-affine Lyapunov functions, average dwell-time, linear programs

1. Introduction

Switched systems comprise a family of dynamical subsystems orchestrated by a switching signal that activates one of these subsystems at a given time. This abstract framework has been useful in modeling a class of hybrid systems with

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continuous and discrete dynamics. Another common source of switched systems is uncertainty quantification in continuous-time systems and the associated differential inclusions. Stability analysis of switched systems, therefore, has gathered a lot of attention in the literature. The references [16, 25] provide a comprehensive overview of the different approaches on this topic.

When analyzing stability under arbitrary switching, existence of a common Lyapunov function is a necessary and sufficient condition for the asymptotic stability of an equilibrium of the switched system [6]. Thus, over the years, a lot of attention in the literature has been given to computing a common Lyapunov function for the switched system under different hypotheses. For some results in this direction, the reader may refer to [1] for discrete-time systems, and [22] for continuous-time systems. Particularly relevant to this paper is the technique based on the construction of continuous and piecewise affine (CPA) Lyapunov functions, which is reviewed in [11]. The papers [3, 13] present the adaptation of computing CPA Lyapunov functions in case of arbitrarily switching systems. However, such methods have not yet been used in the context of constrained, or dwell-time based, switched systems.

For certain applications, existence of common Lyapunov function is a stringent requirement, and may not hold for the given system data. For that reason, when the individual subsystems are asymptotically stable and one can not compute a common Lyapunov function, it is natural to ask how we can guarantee stability for a certain class of switching signals (which is smaller than the set of switching signals with arbitrary switching). The works [21] and [14] studied the stability of switched systems by putting a bound on how fast the switches can occur. Depending on the system data, lower bounds were derived on the (average) dwell-time which ensures global asymptotic stability if the length of interval between two consecutive switches (on average) is greater than the derived lower bound. A tutorial like exposition of these concepts also appears in [16, Chapter 3]. Several works have followed up to extend this idea in several directions. Some generalizations have been addressed in the recent papers [18, 23] with nonlinearities in the system data.

Computational methods with multiple Lyapunov functions for getting best possible lower bounds on the dwell-time have not received much attention in the literature. The references [4, 7, 17, 20] provide some algorithms for calculating lower bounds on the dwell-time in the linear case. Among these, the papers [17, 20] build on dwell-time bounds obtained from multiple Lyapunov functions, which is also the case for this article. The authors of [20] developed optimization-based methods for the automatic verification of dwell-time properties. On the

other hand, [17] proposes some relaxations in the form of sequential convex programs to compute lower bounds on the average dwell-time. With similar motivation, this article studies computational methods for computing best possible lower bounds on the (average) dwell-time using linear programming (LP) methods. In fact, our approach uses techniques based on the construction of CPA Lyapunov functions, under the constraints that are normally imposed for dwell-time based stability conditions. For a given family of dynamical subsystems with asymptotically stable origin, the question of interest is to find the *smallest* lower bound on the dwell-time, which ensures asymptotic stability of the switched system under the so-called compatibility constraints. Such questions can be formulated as an optimization problem and in its full generality, it is a nonconvex problem, even when dealing with linear subsystems and quadratic Lyapunov functions for individual subsystems.

In this paper, we provide a new technique for solving the optimization problem that corresponds to the computation of a minimum average dwell-time that ensures stability. The intermediate step in getting this bound is to first compute the Lyapunov functions for individual subsystems satisfying certain inequalities. In our work, we search for these Lyapunov functions from the family of continuous piecewise affine functions, in contrast to the quadratic ones. This is done by discretizing the state space into simplices and solving for the values of the Lyapunov functions at the vertices of the simplices, using some inequality constraints. The resulting optimization problem actually turns out to be a linear program. The solution to this linear program provides us with a Lyapunov function for each subsystem and also a dwell-time bound.

The remainder of the paper is organized as follows: we recall some basic results on (average) dwell-time stability in Section 2 and describe the problem being studied in this paper. Section 3 provides the LP formulation of the proposed problem along with some results about the feasibility of these programs. We provide some simulations and comparisons with other methods in Section 4, followed by concluding remarks in Section 5.

2. Problem Setup

We consider time-dependent switched dynamical systems described as

$$\dot{\mathbf{x}} = \mathbf{f}_\sigma(\mathbf{x}) \tag{1}$$

where, for some given index set $\mathcal{P} \subset \mathbb{N}$, the function $\sigma : [0, \infty) \to \mathcal{P}$ is piecewise constant and right-continuous, called the *switching signal*. The discontinuities of

σ, called the *switching times*, are assumed to be locally finite. The vector fields $\mathbf{f}_i : \mathbb{R}^n \to \mathbb{R}^n$, for each $i \in \mathcal{P}$, are assumed to be locally Lipschitz and with $\mathbf{f}_i(\mathbf{0}) = \mathbf{0}$. We say that a switching signal σ has an average dwell-time $\tau_a > 0$, if there exists $N_0 > 0$, such that

$$N_\sigma(t, s) \leq N_0 + \frac{t - s}{\tau_a},$$

where $N_\sigma(t, s)$ denotes the number of switches over the interval (s, t). The set of all switching signals with average dwell-time τ_a is denoted by Σ_{τ_a}.

For the stability of the origin for such systems, let us recall the following result, which follows from [14], [16, Chapter 3], and [18, Theorem 1]:

Theorem 1. *Suppose that there exist $\mathcal{C}^1$ Lyapunov functions $V_p : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $i \in \mathcal{P}$, satisfying the following:*

(L1) *There exist $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ such that*

$$\underline{\alpha}(\|\mathbf{x}\|) \leq V_i(\mathbf{x}) \leq \overline{\alpha}(\|\mathbf{x}\|), \quad \forall \mathbf{x} \in \mathbb{R}^n, i \in \mathcal{P}. \tag{2}$$

(L2) *There exists a Lipschitz function $\rho \in \mathcal{K}$, such that, for every $i \in \mathcal{P}$,*

$$\nabla V_i(\mathbf{x}) \cdot \mathbf{f}_i(\mathbf{x}) \leq -\rho(V_i(\mathbf{x})) \quad \forall \mathbf{x} \in \mathbb{R}^n, \tag{3}$$

(L3) *There exists $\chi \in \mathcal{K}_\infty$ such that, for every $i, j \in \mathcal{P}$,*

$$V_j(\mathbf{x}) \leq \chi(V_i(\mathbf{x})) \quad \forall \mathbf{x} \in \mathbb{R}^n, i \neq j. \tag{4}$$

Then the origin is globally asymptotically stable for the switched system (1)*, uniformly over the set Σ_{τ_a}, for τ_a satisfying*

$$\tau_a > \sup_{s>0} \int_s^{\chi(s)} \frac{1}{\rho(r)} dr. \tag{5}$$

The lower bound on the average dwell-time given in (5), with nonlinear functions, has also appeared in the context of impulsive systems in [24], [5]. In what follows, we will restrict ourselves to linear subsystems where the functions α and ρ can be taken as linear. Writing (5) for such cases allows us to better understand the degrees of freedom at our disposal for minimizing the lower bounds on τ_a.

2.1. Corollaries and special cases

Let us present two corollaries to this result depending on the class of functions chosen for V_i and the vector fields $\mathbf{f}_i$, $i \in \mathcal{P}$.

Corollary 1. *Assume that for each $i \in \mathcal{P}$, there exist symmetric, positive definite matrices $P_i \succ 0$, such that*

$$A_i^\top P_i + P_i A_i + \alpha P_i \preceq 0, \quad \textit{for every } i \in \mathcal{P}, \tag{6a}$$

$$P_j \preceq \mu P_i, \quad \textit{for every } j \neq i \tag{6b}$$

for some $\alpha > 0$ and $\mu \geq 1$. Then the switched system (1) *with $\mathbf{f}_i(\mathbf{x}) = A_i\mathbf{x}$, $i \in \mathcal{P}$, has a globally exponentially stable equilibrium at the origin, uniformly over the set Σ_{τ_a}, where τ_a satisfies*

$$\tau_a > \frac{\ln(\mu)}{\alpha}. \tag{7}$$

Thus, for linear dynamics and quadratic Lyapunov functions, we can get a lower bound on the average dwell-time by solving matrix inequalities (6). If we take, $\alpha > 0$, $\mu \geq 1$, and $P_i \succ 0$ as the unknowns in (6), then these inequalities are not linear with respect to the unknowns, and it is difficult to compute a solution. The article [17] addresses the problem of minimizing $\frac{\ln(\mu)}{\alpha}$ subject to inequalities (6) by proposing convex relaxations.

For the algorithms proposed in this paper, we first need a corollary to Theorem 1 with continuous Lyapunov functions, and norm-like bounds on the growth and Dini-derivative of such functions. For a continuous function $V : \mathbb{R}^n \to \mathbb{R}$, we define the Dini-derivative along the solutions of the system $\dot{\mathbf{x}} = \mathbf{f}_i(\mathbf{x})$ as

$$D^+V(\mathbf{x}, \mathbf{f}_i(\mathbf{x})) := \limsup_{h \to 0+} \frac{V(\boldsymbol{\phi}_i(h, \mathbf{x})) - V(\mathbf{x})}{h}$$

where $t \mapsto \boldsymbol{\phi}_i(t, \mathbf{x})$ is an absolutely continuous function that satisfies $\boldsymbol{\phi}_i(0, \mathbf{x}) = \mathbf{x}$ and $\dot{\boldsymbol{\phi}}_i(t, \mathbf{x}) = \mathbf{f}_i(\boldsymbol{\phi}_i(t, \mathbf{x}))$ almost surely for $t \geq 0$. We use this notion to state the following corollary to Theorem 1, obtained by taking $\underline{\alpha}$, $\overline{\alpha}$, ρ to be homogenous functions, and χ being linear.

Corollary 2. *Assume that for each $i \in \mathcal{P}$, there exist continuous functions $V_i : \mathbb{R}^n \to \mathbb{R}$, such that*

$$\underline{a}\|\mathbf{x}\|^d \leq V_i(\mathbf{x}) \leq \overline{a}\|\mathbf{x}\|^d, \quad \textit{for every } i \in \mathcal{P}, \tag{8a}$$

$$D^+V_i(\mathbf{x}, \mathbf{f}_i(\mathbf{x})) \leq -\alpha\|\mathbf{x}\|^d, \quad \textit{for every } i \in \mathcal{P}, \tag{8b}$$

$$V_i(\mathbf{x}) \leq \mu V_j(\mathbf{x}), \quad \textit{for every } j \neq i. \tag{8c}$$

for some $d > 0$, $\overline{a} \geq \underline{a} > 0$, $\alpha > 0$ and $\mu \geq 1$; Then the switched system $\dot{\mathbf{x}} = \mathbf{f}_\sigma(\mathbf{x})$ has a globally exponentially stable equilibrium at the origin, uniformly over the set Σ_{τ_a}, with τ_a satisfying,

$$\tau_a > \frac{\overline{a}\ln(\mu)}{\alpha}. \tag{9}$$

In contrast to Theorem 1, the proof of Corollary 2 using Dini derivative requires some care but essentially follows similar concepts. In this paper, we will build on the statement of Corollary 2 and, in particular, address the following problem:

Problem statement. With $d = 1$ and for fixed values of $\underline{a} > 0$, $\overline{a} > 0$, and $\mu \geq 1$, find piecewise linear functions V_i, for each $i \in \mathcal{P}$, that satisfy (8) while maximizing α.

The reason for fixing the constants $\underline{a} > 0$, $\overline{a} > 0$, and $\mu \geq 1$ is that the foregoing problem then transforms into a linear program. We will provide the formulation of this linear program and discuss its feasibility in the next section. For the sake of clarity in this conference paper, we present our ideas for the linear vector fields but similar concepts can be extended to nonlinear systems.

2.2. Quadratic functions and matrix inequalities

Before discussing the LP problem and CPA Lyapunov functions, let us first look at the inequalities (8) for the case $d = 2$ more carefully. In this case, we let $V_i(\mathbf{x}) = \mathbf{x}^\top P_i \mathbf{x}$, with symmetric and positive definite $P_i \in \mathbb{R}^{n\times n}$. In particular, (8) takes the following form, where I is the identity matrix:

$$\begin{cases} \underline{a} I \preceq P_i \preceq \overline{a} I, & i \in \mathcal{P}, \\ A_i^\top P_i + P_i A_i \preceq -\alpha I, & i \in \mathcal{P}, \\ P_i \preceq \mu P_j, & i, j \in \mathcal{P}. \end{cases} \tag{10}$$

In (10), if we fix $\underline{a}, \overline{a} > 0$ and $\mu \geq 1$, then the inequalities result in LMIs with unknowns P_i, $i \in \mathcal{P}$, which can be solved to maximize α. Practically one can select $\underline{a}$ small, e.g. $\underline{a} = 10^{-5}$ as we do in our examples, and then a large enough $\overline{a} > 0$ will ensure that $\tau_a = \overline{a}\ln(\mu)/\alpha$ can be made minimal for the given $\mu \geq 1$ by maximizing α. Indeed, assume the conditions (10) are fulfilled for some positive constants $\underline{a} = \underline{a}^*, \overline{a} = \overline{a}^*, \alpha = \alpha^*$ and $V_i(\mathbf{x}) = \mathbf{x}^\top P_i^* \mathbf{x}$, $P_i^* \in \mathbb{R}^{n\times n}$ symmetric, $i \in \mathcal{P}$. Then we have the lower bound $\tau_a^* = \overline{a}^* \ln(\mu)/\alpha^*$ on the average dwell-time. Now fix new constants $\underline{a}, \overline{a} > 0$ such that $\overline{a}/\underline{a} \geq \overline{a}^*/\underline{a}^*$ and set $\alpha =$

$(\overline{a}/\overline{a}^*)\alpha^*$ and $P_i = (\overline{a}/\overline{a}^*)P_i^*$. Then, for $V_i(\mathbf{x}) := \mathbf{x}^\top P_i \mathbf{x} = (\overline{a}/\overline{a}^*)\mathbf{x}^\top P_i^* \mathbf{x}$, we have $V_i(\mathbf{x}) \leq \mu V_j(\mathbf{x})$, and

$$\underline{a}\|\mathbf{x}\|^2 \leq \underline{a}^* \frac{\overline{a}}{\overline{a}^*}\|\mathbf{x}\|^2 \leq V_i(\mathbf{x}) = (\overline{a}/\overline{a}^*)\mathbf{x}^\top P_i^* \mathbf{x} \leq \overline{a}\|\mathbf{x}\|^2$$

for $i, j \in \mathcal{P}$. That is, the constraints (10) are fulfilled with these values of $\underline{a}, \overline{a}, \alpha, P_i$ and further, for the lower bound on the average dwell-time we have

$$\tau_a = \frac{\overline{a}\ln(\mu)}{\alpha} = \frac{\overline{a}\ln(\mu)}{\frac{\overline{a}\alpha^*}{\overline{a}^*}} = \frac{\overline{a}^*\ln(\mu)}{\alpha^*} = \tau_a^*.$$

In other words, for a fixed $\mu \geq 1$, if there is a solution to (10) for some choice of $\underline{a}^*$, $\overline{a}^*$, α^* which yields the bound τ_a^* for the average dwell-time, then by choosing $\overline{a}/\underline{a}$ large enough, one can always find another solution to (10) which gives at least as good a bound on average dwell-time as τ^*. Thus, given $\mu \geq 1$, maximizing $\alpha > 0$ under the constraints (10) for a fixed $\underline{a}, \overline{a} > 0$ delivers as good lower bounds on the average dwell-time τ_a as minimizing $\tau_a = \overline{a}\ln(\mu)/\alpha$, where both $\overline{a}$ and α are variables, given that $\overline{a}/\underline{a}$ is large enough.

Note that in the setting of the LMI problem (10) we are searching for quadratic Lyapunov functions for the individual subsystems $\dot{\mathbf{x}} = A_i\mathbf{x}$, $i \in \mathcal{P}$, which can be conservative. In the next section we consider a similar approach for modeling the conditions (8) using piecewise linear Lyapunov functions and an LP formulation to compute them. Due to the foregoing observation, when solving (8) using an LP formulation, we will fix $\underline{a}$ and $\overline{a}$ with $\underline{a}/\overline{a}$ large enough and maximize α.

3. Continuous Piecewise Affine Lyapunov Functions and Linear Programming Formulation

Our LP approach to compute piecewise linear Lyapunov functions fulfilling the conditions (8) is based on the so-called CPA method to compute Lyapunov functions, see e.g. [19, 3, 10, 13]. Its description is somewhat more involved than the LMI approach, because it is based on partitioning a neighborhood of the origin into simplices and the underlying idea behind constructing this collection of simplices, called *triangulation*, is described in the next subsection.

3.1. The Triangulation $\mathcal{T}_K^{\mathbf{F}}$

Roughly speaking, a triangulation is the subdivision of a subset of $\mathbb{R}^n$ into simplices. A suitable concrete triangulation for our aim of parameterizing Lyapunov

functions for the individual subsystems is the triangular-fan of the triangulation in [8], where its efficient implementation is also discussed. In its definition, we use the functions $\mathbf{R}^{\mathcal{J}} : \mathbb{R}^n \to \mathbb{R}^n$, defined for every $\mathcal{J} \subset \{1, 2, \ldots, n\}$ by

$$\mathbf{R}^{\mathcal{J}}(\mathbf{x}) := \sum_{i=1}^{n} (-1)^{\mathbb{1}_{\mathcal{J}}(i)} x_i \mathbf{e}_i,$$

where $\mathbf{e}_i$ is the standard ith unit vector in $\mathbb{R}^n$, and $\mathbb{1}_{\mathcal{J}}(i) = 1$ if $i \in \mathcal{J}$ and 0 otherwise. Thus, $\mathbf{R}^{\mathcal{J}}(\mathbf{x})$ is the vector $\mathbf{x}$, except for a minus has been put in front of the coordinate x_i whenever $i \in \mathcal{J}$.

We first define the triangulation $\mathcal{T}^{\text{std}}$ and use it to construct the intermediate triangulation $\mathcal{T}_K$, which in turn is used to define our desired triangulation $\mathcal{T}_K^{\mathbf{F}}$.

The standard triangulation $\mathcal{T}^{\text{std}}$ consists of the simplices

$$\mathfrak{S}_{\mathbf{z}\mathcal{J}\rho} := \text{co}\left\{\mathbf{x}_0^{\mathbf{z}\mathcal{J}\rho}, \mathbf{x}_1^{\mathbf{z}\mathcal{J}\rho}, \ldots, \mathbf{x}_n^{\mathbf{z}\mathcal{J}\rho}\right\},$$

where co denotes the convex hull, and

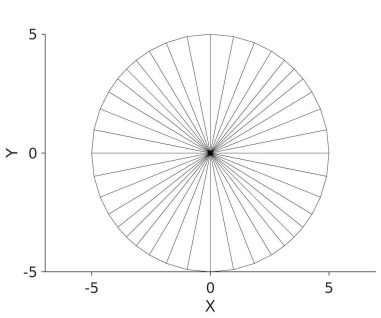
$$\mathbf{x}_j^{\mathbf{z}\mathcal{J}\rho} := \mathbf{R}^{\mathcal{J}}\left(\mathbf{z} + \sum_{i=1}^{j} \mathbf{e}_{\rho(i)}\right), \tag{11}$$

for all $\mathbf{z} \in \mathbb{N}_0^n = \{0, 1, \ldots\}^n$, all $\mathcal{J} \subset \{1, 2, \ldots, n\}$, all $\rho \in S_n$, and $j = 0, 1, \ldots, n$. Here, S_n denotes the set of all permutations of $\{1, 2, \ldots, n\}$.

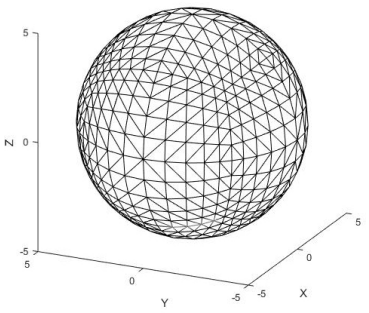
Now fix a $K \in \mathbb{N}_+ = \{1, 2, \ldots\}$ and define the hypercube $\mathcal{H}_K := [-K, K]^n$. Consider the simplices $\mathfrak{S}_{\mathbf{z}\mathcal{J}\rho} \subset \mathcal{H}_K$ in $\mathcal{T}^{\text{std}}$, that intersect the boundary of $\mathcal{H}_K$. We are only interested in those intersections that are $(n-1)$-simplices, i.e. we take every simplex with vertices $\mathbf{x}_j := \mathbf{R}^{\mathcal{J}}\left(\mathbf{z} + \sum_{i=1}^{j} \mathbf{e}_{\rho(i)}\right)$, $j \in \{0, 1, \ldots, n\}$, where exactly one vertex $\mathbf{x}_{j^*}$ satisfies $\|\mathbf{x}_{j^*}\|_\infty < K$ and the other n of the $n+1$ vertices satisfy $\|\mathbf{x}_j\|_\infty = K$, i.e. for $j \in \{0, 1, \ldots, n\} \setminus \{j^*\}$. Then we replace the vertex $\mathbf{x}_{j^*}$ by $\mathbf{0}$; it is not difficult to see that j^* is necessarily equal to 0. The collection of such vertices triangulates $\mathcal{H}_K$ and this new triangulation of $\mathcal{H}_K$ is our desired triangulation $\mathcal{T}_K$.

It has been shown [2] that it is often advantageous in the CPA method to map the vertices of the triangulation by the mapping $\mathbf{F}\colon \mathbb{R}^n \to \mathbb{R}^n$, $\mathbf{F}(\mathbf{0}) = \mathbf{0}$ and

$$\mathbf{F}(\mathbf{x}) := \frac{\|\mathbf{x}\|}{\|\mathbf{x}\|_\infty}\mathbf{x}, \quad \text{if } \mathbf{x} \neq \mathbf{0}. \tag{12}$$



(a) The triangulation $\mathcal{T}_5^{\mathbf{F}}$ in two dimensions



(b) The triangulation $\mathcal{T}_5^{\mathbf{F}}$ in three dimension; note that the origin is a vertex of all the tetrahedra in $\mathcal{T}_5^{\mathbf{F}}$.

Figure 1: Our proposed triangulation in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Note that $\mathbf{F}$ maps the hypercubes $\{\mathbf{x} \in \mathbb{R}^n \colon \|\mathbf{x}\|_\infty = r\}$ to the spheres $\{\mathbf{x} \in \mathbb{R}^n \colon \|\mathbf{x}\| = r\}$.

Finally, we define the triangulation $\mathcal{T}_K^{\mathbf{F}}$ that will be used in the LP problem to parameterize CPA Lyapunov functions. Let $\mathcal{T}_K^{\mathbf{F}}$ be the triangulation consisting of the simplices

$$\mathfrak{S}_\nu := \operatorname{co}\{\mathbf{0}, \mathbf{F}(\mathbf{x}_1^\nu), \mathbf{F}(\mathbf{x}_2^\nu), \ldots, \mathbf{F}(\mathbf{x}_n^\nu)\},$$

where

$$\operatorname{co}\left\{\mathbf{0}, \mathbf{x}_1^{\mathbf{z}^{\mathcal{J}\rho}}, \mathbf{x}_2^{\mathbf{z}^{\mathcal{J}\rho}}, \ldots, \mathbf{x}_n^{\mathbf{z}^{\mathcal{J}\rho}}\right\} \in \mathcal{T}_K.$$

The subset of $\mathbb{R}^n$ subdivided into simplices by the triangulation $\mathcal{T}_K^{\mathbf{F}}$ is denoted by

$$\mathcal{D}_{\mathcal{T}_K^{\mathbf{F}}} := \bigcup_{\mathfrak{S}_\nu \in \mathcal{T}_K^{\mathbf{F}}} \mathfrak{S}_\nu.$$

Figure 1 depicts two exemplary triangulations of the type $\mathcal{T}_K^{\mathbf{F}}$ for two and three dimension with $K = 5$. The implementation of the triangulation is discussed in [12, 9].

3.2. LP Problem

We are now ready to state our LP problem to parameterize piecewise linear Lyapunov functions for the switched system fulfilling the conditions in (8). For formulating this LP, and showing that its feasibility provides us the lower bound on average dwell-time, we focus our attention on the switched linear systems:

$$\dot{\mathbf{x}} = A_\sigma \mathbf{x} \tag{13}$$

with $\sigma : [0, \infty) \to \mathcal{P}$ being the switching signal, and $A_i \in \mathbb{R}^{n \times n}$, for each $i \in \mathcal{P}$.

We use three constants $\underline{a}, \overline{a} > 0$ and $\mu \geq 1$ in the LP problem. We want the ratio $\overline{a}/\underline{a}$ to be large, as discussed in the last section, and then we want to try out different $\mu \geq 1$ to obtain as good a lower bound on the average dwell-time as possible.

The variables of the LP problem are $\alpha \in \mathbb{R}$ and $V_{\mathbf{x},i} \in \mathbb{R}$ for every vertex $\mathbf{x}$ of a simplex in $\mathcal{T}_K^{\mathbf{F}}$ and every $i \in \mathcal{P}$.

The objective of the LP problem is to maximize α.

The constraints of the LP problem are:

(C1) The first set of constraints is that, for every $i \in \mathcal{P}$, we set $V_{\mathbf{0},i} = 0$, and for every vertex $\mathbf{x}$ of a simplex in $\mathcal{T}_K^{\mathbf{F}}$ and for every $i \in \mathcal{P}$:

$$\underline{a}\|\mathbf{x}\| \leq V_{\mathbf{x},i} \leq \overline{a}\|\mathbf{x}\| \tag{14}$$

(C2) The second set of constraints is more involved. For every simplex $\mathfrak{S}_\nu := \mathrm{co}\{\mathbf{0}, \mathbf{x}_1^\nu, \mathbf{x}_2^\nu \ldots, \mathbf{x}_n^\nu\} \in \mathcal{T}_K^{\mathbf{F}}$, we define the matrix $X_\nu = (\mathbf{x}_1^\nu\ \mathbf{x}_2^\nu\ \cdots \mathbf{x}_n^\nu)$, i.e. $\mathbf{x}_k^\nu$ is the kth column of X_ν. Further, we define for every $i \in \mathcal{P}$, the vector of variables $\mathbf{v}_{\nu,i} = \left(V_{\mathbf{x}_1^\nu,i}\ V_{\mathbf{x}_2^\nu,i}\ \cdots\ V_{\mathbf{x}_n^\nu,i}\right)^\top$.

The constraints are: for every simplex $\mathfrak{S}_\nu \in \mathcal{T}_K^{\mathbf{F}}$, for all $j = 1, \ldots, n$ and all $i \in \mathcal{P}$:

$$\mathbf{v}_{\nu,i}^\top X_\nu^{-1} A_i \mathbf{x}_j^\nu \leq -\alpha\|\mathbf{x}_j^\nu\|. \tag{15}$$

Note that these constraints are automatically fulfilled for $j = 0$, i.e. $\mathbf{x}_j^\nu = \mathbf{0}$.

(C3) The third set of constraints is: for every vertex $\mathbf{x}$ of a simplex in $\mathcal{T}_K^{\mathbf{F}}$ and for every $i, j \in \mathcal{P}$,

$$V_{\mathbf{x},j} \leq \mu V_{\mathbf{x},i}. \tag{16}$$

3.3. Solution to LP delivers lower bounds on dwell-time

In the previous subsection, we formulated an LP which basically specified the constraints in (8) at the vertices of the simplices contained in the triangulation. Here we prove that the feasibility of such a program provides us with piecewise linear Lyapunov functions for the individual subsystems over the entire state space that additionally fulfill (8c), thereby providing a lower bound on the average dwell-time.

Toward this end, assume that the LP problem in Section 3.2 has a solution with $\alpha > 0$. We then define the piecewise linear function $V_i \colon \mathcal{D}_{\mathcal{T}_K^{\mathbf{F}}} \to \mathbb{R}$, for every $i \in \mathcal{P}$, in the following way:

- For every $\mathbf{x} \in \mathcal{D}_{\mathcal{T}_K^{\mathbf{F}}}$ there exists a simplex $\mathfrak{S}_\nu = \mathrm{co}\{\mathbf{0}, \mathbf{x}_1^\nu, \mathbf{x}_2^\nu \ldots, \mathbf{x}_n^\nu\} \in \mathcal{T}_K^{\mathbf{F}}$ such that $\mathbf{x} \in \mathfrak{S}_\nu$ and there exist a unique $\lambda \in [0,1]^n$, $\sum_{j=1}^n \lambda_j \leq 1$, such that $\mathbf{x} = \sum_{j=1}^n \lambda_j \mathbf{x}_j^\nu$. We define

$$V_i(\mathbf{x}) = \sum_{j=1}^n \lambda_j V_{\mathbf{x}_j^\nu, i}.$$

It is not difficult to see that the functions V_i, $i \in \mathcal{P}$, are continuous functions that are linear on each simplex $\mathfrak{S}_\nu \in \mathcal{T}_K^{\mathbf{F}}$, in particular each V_i has the constant gradient $\nabla V_{\nu,i} := \mathbf{v}_{\nu,i}^\top X_\nu^{-1}$ (row vector) on the interior of $\mathfrak{S}_\nu$, see e.g. [10, Rem. 9]. Hence, for any $\mathbf{x} \in \mathfrak{S}_\nu \in \mathcal{T}_K^{\mathbf{F}}$, $\mathbf{x} = \sum_{j=1}^n \lambda_j \mathbf{x}_j^\nu$, we have for any $i \in \mathcal{P}$ by **(C1)** and **(C2)** that

$$\begin{aligned}
\nabla V_{\nu,i} \bullet A_i \mathbf{x} &= \mathbf{v}_{\nu,i}^\top X_\nu^{-1} A_i \sum_{j=1}^n \lambda_j \mathbf{x}_j^\nu = \sum_{j=1}^n \lambda_j \mathbf{v}_\nu^\top X_\nu^{-1} A_i \mathbf{x}_j^\nu \\
&\leq -\alpha \sum_{j=1}^n \lambda_j \|\mathbf{x}_j^\nu\| \leq -\frac{\alpha}{\overline{a}} \sum_{j=1}^n \lambda_j V_i(\mathbf{x}_j^\nu) \\
&= -\frac{\alpha}{\overline{a}} V_i(\sum_{j=1}^n \lambda_j \mathbf{x}_j^\nu) = -\frac{\alpha}{\overline{a}} V(\mathbf{x}). \qquad (17)
\end{aligned}$$

Now, for any $\mathbf{x}$ in the interior of $\mathcal{D}_{\mathcal{T}_K^{\mathbf{F}}}$, we have that, for any $i \in \mathcal{P}$, there exists a simplex $\mathfrak{S}_\nu \in \mathcal{T}_K^{\mathbf{F}}$ and an $h > 0$, such that

$$\mathbf{x} + [0, h] A_i \mathbf{x} \subset \mathfrak{S}_\nu,$$

where ν can depend on both $\mathbf{x}$ and i. Because V_i is linear on $\mathfrak{S}_\nu$ we have

$$\limsup_{h \to 0+} \frac{V_i(\mathbf{x} + h A_i \mathbf{x}) - V_i(\mathbf{x})}{h} = \nabla V_{\nu,i} \bullet A_i \mathbf{x} \leq -\frac{\alpha}{\overline{a}} V_i(\mathbf{x})$$

and since this holds true for all $i \in \mathcal{P}$, we have $D^+ V_i(\mathbf{x}, A_i \mathbf{x}) \leq -\frac{\alpha}{\overline{a}} V_i(\mathbf{x})$. Since

$$V_i(\mathbf{x}) = \sum_{j=1}^n \lambda_j V_{\mathbf{x}_j^\nu, i} \geq \underline{a} \sum_{j=1}^n \lambda_j \|\mathbf{x}_j^\nu\| \geq \underline{a} \|\mathbf{x}\|$$

by the constraints **(C1)**, and for all $i, k \in \mathcal{P}$,

$$V_i(\mathbf{x}) = \sum_{j=0}^n \lambda_j V_{\mathbf{x}_j^\nu, i} \leq \sum_{j=0}^n \lambda_j \mu V_{\mathbf{x}_j^\nu, k} = \mu V_k(\mathbf{x}),$$

it is clear that the V_i fulfill the constraints (8) in the interior of $\mathcal{D}_{\mathcal{T}_K^{\mathbf{F}}}$. Just define

$$\overline{a}' := \max_{\|\mathbf{x}\|=1} \max_{i=1,2,\ldots,N} V_i(\mathbf{x})$$

and we have

$$\underline{a}\|\mathbf{x}\| \leq V_i(\mathbf{x}) \leq \overline{a}'\|\mathbf{x}\|,$$
$$D^+V_i(\mathbf{x}, A_i\mathbf{x}) \leq -\frac{\alpha}{\overline{a}}V_i(\mathbf{x})$$

for all $\mathbf{x}$ in the interior of $\mathcal{D}_{\mathcal{T}_K^{\mathbf{F}}}$. Note that we proved (17) directly from the constraints and did not go through constraints (8) with $\overline{a} = \overline{a}'$, which would lead to a worse estimate on τ_a.

By extending V_i to $\mathbb{R}^n$ in the obvious way, i.e. for every $\mathbf{x} \in \mathbb{R}^n$ there exists a $\mathfrak{S}_\nu$ and unique numbers $\lambda_j \geq 0$ such that $\mathbf{x} = \sum_{j=1}^n \lambda_j \mathbf{x}_j^\nu$ (a cone defined by the vertices of $\mathfrak{S}_\nu$) and we set $V_i(\mathbf{x}) = \sum_{j=1}^n \lambda_j V_{\mathbf{x}_j^\nu, i}$, we see that the V_i fulfill the constraints (8) on $\mathbb{R}^n$, for each $i \in \mathcal{P}$.

4. Simulations

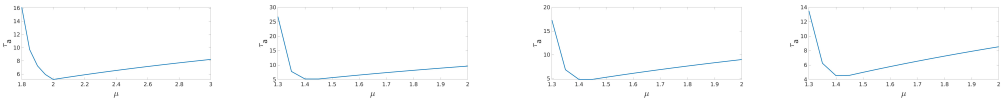
We will now test our LP algorithm for several examples from the literature. The class of systems for these simulations is (13). The set $\mathcal{P}$ and the matrices $\{A_i\}_{i\in\mathcal{P}}$ will be specified differently for the examples considered here. In the examples, we always fix $\underline{a} = 10^{-5}$ and $\overline{a} = 10$. We used YALMIP / sdpt3 and Gurobi to solve the LMI and LP problems, respectively.

4.1. Example 1: Dwell-time stable but not under arbitrary switching

Consider the switched system (13) with

$$A_1 = \begin{pmatrix} -0.1 & -1 \\ 2 & -0.1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -0.1 & -2 \\ 1 & -0.1 \end{pmatrix}$$

This example is taken from [16, p. 26]. It is stable for a certain minimum value for the average dwell-time, but it is not stable under arbitrary switching. Solving (10) using the LMI approach, the best τ_a obtained is 5.1929 with $\mu = 2$. Using the LP approach, the best τ_a obtained is 4.5283 with $\mu = 1.4$ and using $K = 500$ in the triangulation. Using triangulations with fewer triangles delivers a higher lower bound τ_a for the average dwell time; $K = 50$ gives $\tau_a = 5.16493$ with $\mu = 1.45$, $K = 100$ gives $\tau_a = 4.79315$ with $\mu = 1.4$, and $K = 200$ gives $\tau_a = 4.62407$ with $\mu = 1.4$. In all cases they are better than the bounds from the LMI approach.



(a) LMIs with quadratics (b) LP with $K = 50$ (c) LP with $K = 100$ (d) LP with $K = 500$

Figure 2: Plots for μ versus τ_a in Example 1.

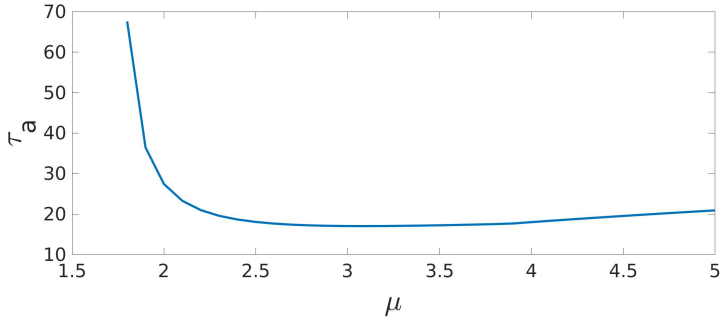


Figure 3: LMI based plot for μ vs τ_a in Example 2.

4.2. *Example 2: Stable under arbitrary switching but no common quadratic Lyapunov function*

Take $\mathcal{P} = \{1, 2\}$, with the matrices

$$A_1 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -1 & -10 \\ 0.1 & -1 \end{pmatrix}$$

This example is taken from [6]. It is stable under arbitrary switching but the matrices A_1 and A_2 do not share a common quadratic Lyapunov function. This example helps us see the limitation of using LMIs because searching for quadratic certificates in this case is not the best choice.

Solving (10) using LMIs, the minimum value for τ_a is $\tau_a = 17.0394$ with $\mu = 3.1$. Whereas, with our LP approach, $K = 20$ gives a solution with $\mu = 1$. Hence, the origin is stable under arbitrary switching ($\tau_a = 0$), and we get a common piecewise linear Lyapunov function although no quadratic Lyapunov function exists.

4.3. *Example 3: Exponentially stable system under arbitrary switching with 5 modes*

We consider the switched system (13) with $\mathcal{P} = \{1, 2, 3, 4, 5\}$, where

$$A_1 = \begin{pmatrix} -5 & 1 & 2 \\ 0 & -5 & 1 \\ 0 & 1 & -2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -1 & 3 & 1 \\ 0 & -2 & 0 \\ 0 & 1 & -1 \end{pmatrix},$$
$$A_3 = \begin{pmatrix} 0 & 0 & 3 \\ -2 & -1 & -3 \\ -1 & 0 & -2 \end{pmatrix}, \quad A_4 = \begin{pmatrix} -4 & 0 & -3 \\ 2 & -2 & 4 \\ 1 & 0 & -1 \end{pmatrix},$$
$$A_5 = \begin{pmatrix} -1 & 0 & 0 \\ -1 & -1 & -1 \\ -3 & 0 & -4 \end{pmatrix}$$

This example was also considered in [17] with a graph that determines the switching sequence, and in this particular, we have the star topology. With $K = 6$, we get a solution with $\mu = 1$, i.e. the origin is exponentially stable under arbitrary switching (which is then arbitrary without the graph too).

With the LMI approach in (10), the minimum value for the average dwell-time is $\tau_a = 4.6870$ with $\mu = 2.7$.

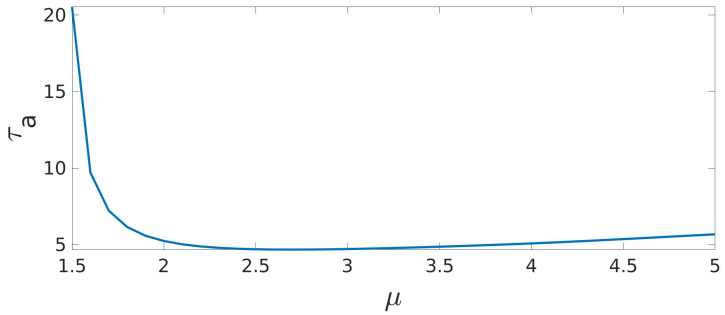


Figure 4: LMI based plot for μ vs τ_a in Example 3.

5. Conclusions

We considered a linear programming (LP) based computational algorithm for computing lower bounds on average dwell-times that ensure asymptotic stability of switched systems. The algorithm is essentially based on gridding the state space into simplices and computing values for the corresponding Lyapunov functions at the vertices of these simplices. By choosing appropriate values of the parameters in the inequalities defining the linear program, the solution provides

us lower bounds on the average dwell-time necessary to assure stability. From the simulations, we see in several case studies, that LP based bounds are better than the ones based on linear matrix inequalities (LMIs). This is not really surprising since LMIs restrict the Lyapunov functions to be quadratic, whereas the proposed LPs can potentially approximate a broader class of Lyapunov function templates. Computing dwell-time via inequalities in (8) introduces some conservatism because we optimize over a single parameter α while keeping μ fixed. The same conservatism is observed in going from (6) to (10). As a topic of ongoing investigation, we are working out algorithm to optimize α and μ simultaneously directly using an LP version of (6). Other than understanding the complexity of the proposed algorithm, we also aim to study the extensions of the proposed algorithm in different directions, which includes the study of generalized lower bounds on average dwell-time [15, 18].

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