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# Approximation of Nonlinear Filters for Continuous-Time Markov Chains under Randomly-Sampled Observations

Olga Yufereva

Aneel Tanwani

**Abstract**—For a continuous-time Markov chain with finite state space and an observation process with additive Gaussian noise, we consider the problem of designing optimal filters when the measurements of the observation process are available at randomly sampled time instants. We first define the optimal filter in this setting, and derive a recursive expression for it in the form of a continuous-discrete filter. Our main result is oriented at comparing the performance of the proposed filter with the continuous-time counterpart, that is, the classical Wonham filter obtained from continuous observation process. In particular, we show that by taking the sampling process to be a Poisson counter, and increasing the mean sampling rate, the expected value of the posterior conditional distribution of continuous-discrete filter converges to the posterior distribution of a purely continuous Wonham filter.

**Index Terms**—Wonham filters; hidden Markov model; randomly sampled observations; Bayes' rule.

## I. INTRODUCTION

Filtering is a problem of fundamental interest in the study of dynamical systems. The basic underlying problem is to compute *a posteriori* distribution of the state process conditioned upon the measured observations. Early developments in this area provide solutions by expressing conditional density in a recursive manner, either in the form of a differential or difference equation. These solutions are particularly difficult to analyze and implement in the nonlinear setting. For this reason, nonlinear filtering continues to attract broad interest with the objective of developing more insights about analyzing the filters, or improve the design for computational feasibility. One may refer to [1], [2], [3], [4] for comprehensive overviews on this topic. In this article, our aim is to study an analysis related problem for a class of nonlinear filters which is motivated by implementation of filtering algorithms in the presence of a passive communication channel.

For the purpose of analytical tractability, we will consider the filtering problem for systems with finitely many states. In particular, we consider the evolution of states described by a continuous-time Markov chain with an observation process which is a nonlinear function of the state process with additive Gaussian noise. We refer the reader to [5] for a standard exposition on Markov chains and related stability

notions. The classical solution to the filter design problem for such systems is given by Wonham in [6], which is in the form of continuous-time stochastic differential equations evolving over a simplex. Over the past twenty-five years, we see a certain interest in better understanding some system-theoretic properties of these filters. Earlier papers in this direction studied stability with respect to initial conditions [7], [8], [9] under different assumptions on the underlying Markov chain, and tools such as Lyapunov exponents were developed to characterize the convergence rates for the decay in error due to mismatched initial conditions. The PhD dissertation [3] revisits some of these results and in addition, provides results on robustness with respect to unknown parameters in the models [3, Chapter 3]. Such connections between system-theoretic tools and nonlinear filtering continue to develop in more recent works [10], [11], where the authors develop a dual of the filtering problem in the form of a backward stochastic differential equation and reformulate the stability of nonlinear filter in terms of stabilizability properties of the dual system. The stability analysis of nonlinear filters continues to attract attention of different communities [12], [13], and in more recent work [14], the stability of filters is treated in the presence of communication channels, with connections to observability and information rates.

On a conceptual level, Wonham filter provides the optimal solution to the filtering problem and the aforementioned works analyze how the solution deviates from the optimal solution if the initial condition, or some system parameters, are changed. In our work, we also consider the deviation of a finite-state filter from the optimal solution. However, the source of this deviation is considerably different, that is, we stipulate that the continuous-time observation process is not available for measurement but instead the realizations of this observation process are available at some randomly drawn time instants. In other words, the information used to compute the conditional distribution is different in our setup. We are primarily interested in the question of defining an optimal filter for the continuous-time Markov chain subject to this discrete information structure. Such problems are primarily motivated by the implementation of continuous-time systems over digital communication channels, which are conveniently modeled using the framework of stochastic hybrid systems [15]. We also see instances of such information constraints on measurements in optimal control problems for continuous-time stochastic systems [16]. In our previous related work [17], we considered nonlinear filtering with continuous state-space for stochastic differential equation

O. Yufereva and A. Tanwani are affiliated with CNRS – LAAS, University of Toulouse, CNRS, 31400 Toulouse, France (e-mail: oiufereva@laas.fr, aneel.tanwani@cnrs.fr). O. Yufereva is also affiliated with Krasovskii Institute of Mathematics and Mechanics, Ural Branch of RAS, Yekaterinberg, Russia.

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subject to Poisson sampling. However, due to the complexity of the underlying dynamics, we only studied suboptimal solutions and analyzed boundedness of the error covariance.

In this article, we restrict our attention to a system with finite state space which makes the optimal filter a finite-dimensional differential equation. When the measurements are time-sampled, our optimal estimator takes the form of a stochastic hybrid system, where the flow of the conditional density is described by the continuous transition kernel of the Markov chain, and at the times when an output measurement is available, we reset the conditional density according to a Bayes' rule. We are then interested in analyzing how this hybrid filter is different from the continuous-time filter. In particular, we assign Poisson distribution to the randomly drawn time instants and look at the expectation of the conditional probability with respect to the distribution of the sampling process. In particular, we show that by letting sampling rate of the underlying process tend to infinity, the expected value of the conditional probability converges to the continuous-time conditional probability (obtained from the continuous observation process).

## II. PROBLEM STATEMENT

### A. State process

The dynamical systems considered in this paper are described by a finite-state continuous-time Markov chain  $(x_t)_{t \geq 0}$ , on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . The notation  $\mathcal{S} := \{a_1, \dots, a_d\}$  is used to denote the state space and the matrix  $\Lambda := \{\lambda_{ij}\} \in \mathbb{R}^{d \times d}$ ,  $i, j \in \{1, \dots, d\}$ , denotes the transition intensities. The transition rates are such that,  $\lambda_{ij} \geq 0$ ,  $j \neq i$  and  $\sum_{j \in \{1, \dots, d\}} \lambda_{ij} = 0$ , for each  $i \in \{1, \dots, d\}$ . The later condition is true only if  $\lambda_{ii} \leq 0$ , for each  $i \in \{1, \dots, d\}$ . For  $t \geq 0$ , let us denote the probability distribution of  $x_t$  by  $p(t)$ , so that,  $p_j(t) = \mathbb{P}(x(t) = a_j)$ . Similarly, the distribution conditioned upon initial condition is denoted by  $p_{ij}(t) = \mathbb{P}(x(t) = a_j \mid x_0 = a_i)$ , and we let  $P(t) := [p_{ij}(t)] \in \mathbb{R}^{d \times d}$  denote the matrix of transition probabilities. For each  $t \geq 0$ , the matrix  $P(t)$  is obtained from the transition matrix  $\Lambda$  by solving the following Chapman-Kolmogorov equation:

$$\dot{P}(t) = P(t)\Lambda, \quad P(0) = I_{d \times d}.$$

For a given initial distribution of  $x_0$ , given by  $p_j(0)$ , the distribution  $p_j(t)$ , for each  $t \geq 0$ , is

$$p(t) = P^\top(t)p(0) = p(0) + \int_0^t (\Lambda^\top p(\tau)) d\tau, \quad (1)$$

or, in the differential form, it can be written as  $\dot{p}(t) = \Lambda^\top p(t)$ , subject to the initial condition  $p(0)$ .

### B. Continuous-time optimal Wonham filter

Using the model from [6], an output process  $z_t$  is assumed to be generated by state process  $x_t$  and the observation noise

$\eta_t$ , which is assumed to be a Wiener process independent of  $x_t$ . It is described by the equation

$$z_t = \int_0^t h(x_s) ds + \eta_t \quad (2)$$

where  $h : \mathcal{S} \rightarrow \mathbb{R}$  is a measurable function on  $\mathcal{S}$ . For simplicity, we take our observation process to be one-dimensional. The noise covariance is assumed to be constant and denoted by  $R$ , that is,  $\mathbb{E}[d\eta^\top d\eta] = R dt$ . We denote by  $\sigma\{z_{[0,t]}\}$  the filtration generated by  $(z_s)_{s \leq t}$ . We recall that if  $\phi$  is some square integrable function of the signal process  $x_t$ , then the 'best estimate' (in mean square sense) of  $\phi(x_t)$  given the observations up to time  $t$  is

$$\mathbb{E}[\phi(x_t) \mid \sigma\{z_{[0,t]}\}].$$

To compute this conditional expectation, we are therefore interested in the corresponding conditional probability

$$\rho_t^j := \mathbb{P}(x_t = a_j \mid \sigma\{z_{[0,t]}\}), \quad j \in \{1, \dots, d\}. \quad (3)$$

The classical Wonham filter provides a recursive expression for  $\rho_t := [\rho_t^1, \dots, \rho_t^d]^\top$  in the form of following stochastic differential equation [3, Corollary 1.2.1]:

$$d\rho_t = \Lambda^\top \rho_t dt + (H - \bar{h})\rho_t(dz_t - \bar{h}dt), \quad \rho_0 = p(0) \quad (4)$$

where  $\bar{h} = \sum_{i=1}^d h(a_i)\rho_t^i$ , and  $H$  denotes a diagonal matrix,  $H := \text{diag}\{h(a_1), \dots, h(a_d)\}$ . Due to the presence of  $dz_t$  on the right-hand side of (4),  $\rho_t$  is a vector-valued random variable, for each  $t \geq 0$ . It is noted that the computation of  $\rho_t$  requires continuous measurements of the process  $dz_t$ .

### C. Randomly-sampled observation process

In this article, we study the filtering problem for continuous-time Markov chains under the premise that the measurements of the observation process are not available continuously. The motivation to work with randomly time-sampled measurements comes from several applications, such as, communication over networks which allow information packets to be sent at some discrete randomly distributed time instants. Thus, we consider a monotone nondecreasing sequence  $(\tau_n)_{n \in \mathbb{N}}$  taking values in  $\mathbb{R}_{\geq 0}$  which denote the time instants at which the measurements are available for estimation. We introduce the process  $N_t$  defined as

$$N_t := \sup\{n \in \mathbb{N} \mid \tau_n \leq t\} \quad \text{for } t \in \mathbb{R}, \quad (5)$$

and are particularly interested in the case where  $(N_t)_{t \geq 0}$  is a Poisson stochastic process independent of the noise and the state processes. The discretized and noisy observation process is thus defined as

$$\hat{z}_t = \hat{z}_{\tau_{N_t}} = z_{\tau_{N_t}}, \quad t \geq 0. \quad (6)$$

Associated to this process, we consider the filtration generated by  $(\hat{z}_s)_{s \leq t}$ , which is  $\sigma\{\hat{z}_{[0,t]}\} = \sigma\{\hat{z}_{\tau_1}, \dots, \hat{z}_{\tau_{N_t}}\}$ .

#### D. Problem statement

Our primary objective is to study the filtering problem subject to the random sampling of the observation process. Toward this end, we first consider the conditional probabilities

$$\hat{\rho}_t^j := P(x_t = a_j | \sigma\{\hat{z}_{[0,t]}\}), \quad j \in \{1, \dots, d\}, \quad (7)$$

and develop an expression for the vector  $\hat{\rho}_t := [\hat{\rho}_t^1, \dots, \hat{\rho}_t^d]^\top$ , for each  $t \geq 0$ , using  $\{\hat{z}_{\tau_k} | k \leq N_t\}$ . These developments are carried out in Section III.

Next, we are interested in comparing  $\hat{\rho}_t$  with  $\rho_t$ . It is noted that  $\hat{\rho}_t$ , as defined in (7) is random not only due to the observation noise  $\hat{z}_t$ , but it also depends on the random time instants  $(\tau_k)_{k \in \mathbb{N}}$ . If we average out the randomness due to sampling times and take the mean sampling rate of the Poisson counter  $N_t$  to be large enough, then it is natural to compare the resulting random variable with  $\rho_t$ . To formally state this intuition, we look at the expectation of  $\hat{\rho}_t$  with respect to the sampling times, and show in Section IV that, for each  $t \geq 0$ , the resulting random variable converges to  $\rho_t$  in the mean as the sampling rate gets large.

### III. OPTIMAL FILTER FOR A SAMPLE PATH OF $N_t$

In this section, we develop expressions that allow us to compute the conditional probability  $\hat{\rho}_t^j$ , for each  $j \in \{1, \dots, d\}$  and each  $t \geq 0$ , for a single realization of the random time instants  $\{\tau_k\}_{k \in \mathbb{N}}$ . By developing a recursive expression, it is possible to compute  $\hat{\rho}_t^j$  simply by updating the last stored value as a function of the newly received information.

To present the recursive filter, we consider the innovation sequence  $\Delta z_{N_t}$  obtained from taking the difference of two consecutive measurements:

$$\Delta z_{N_t} = \hat{z}_{\tau_{N_t}} - \hat{z}_{\tau_{N_t-1}} = \xi_{N_t} + \eta_{\tau_{N_t}} - \eta_{\tau_{N_t-1}},$$

where we used the notation

$$\xi_k := \int_{\tau_{k-1}}^{\tau_k} h(x_s) ds, \quad k \in \mathbb{N}.$$

With probability one, a Poisson process  $N_t$  yields finite number of random variables  $\Delta z_i$  and  $\xi_i$  over a finite interval  $[0, t]$ . Notice that

$$\begin{aligned} & p(\Delta z_1, \dots, \Delta z_{N_t} | x_0, x_t) \\ &= E \left[ p(\Delta z_1, \dots, \Delta z_{N_t}, \xi_1, \dots, \xi_{N_t}) | x_0, x_t \right] \\ &= E \left[ \prod_{k=1}^{N_t} \frac{1}{\sqrt{2\pi R \Delta \tau_k}} \exp \left[ -\frac{(\Delta z_k - \xi_k)^2}{2R \Delta \tau_k} \right] | x_0, x_t \right] \\ &= \prod_{k=1}^{N_t} \frac{1}{\sqrt{2\pi R \Delta \tau_k}} E \left[ \exp \left[ -\frac{(\Delta z_k - \xi_k)^2}{2R \Delta \tau_k} \right] | x_0, x_t \right] \end{aligned}$$

since  $\Delta z_i - \xi_i$  are independent for each  $i$  and are Gaussian with mean 0 and variance  $R \Delta \tau_i$ . Let

$$\Psi_n = \Psi_n(\Delta z, \xi, \Delta \tau) := \exp \left( -\sum_{k=1}^n \frac{(\Delta z_k - \xi_k)^2}{2R \Delta \tau_k} \right). \quad (8)$$

To get the expression for  $\hat{\rho}_t^j$  at  $t = \tau_{N_t}$  for  $N_t \in \mathbb{N}$ , we observe that

$$\begin{aligned} \hat{\rho}_t^j &= P(x(t) = a_j | \hat{z}_{\tau_k}, k = 0, \dots, N_t) \\ &= P(x(t) = a_j | \Delta z_k, k = 1, \dots, N_t) \\ &= \frac{\sum_{i=1}^d p_i(0) p_{ij}(t) E[\Psi_{N_t} | x_0 = a_i, x_t = a_j]}{\sum_{i=1}^d \sum_{l=1}^d p_i(0) p_{il}(t) E[\Psi_{N_t} | x_0 = a_i, x_t = a_l]} \end{aligned}$$

as the term  $\prod_{k=1}^{N_t} \frac{1}{\sqrt{2\pi R \Delta \tau_k}}$  cancels out from the numerator and the denominator. Then the numerator is the unnormalized conditional density. Let us denote it as follows.

**Definition 1.** Let  $\bar{z}_{[0,t]}$  be a pure jump process realization with jumps only at  $\{\tau_k\}_{k=1}^{N_t}$ . For each  $t \geq 0$ , and for each  $j \in \{1, \dots, d\}$ , define the function  $U^j(t, \bar{z}_{[0,t]})$  as follows

$$U^j(t, \bar{z}_{[0,t]}) := \sum_{i=1}^d p_i(0) p_{ij}(t) E \left[ \Psi_{N_t} \middle| x_0 = a_i, x_t = a_j \right],$$

where  $\Psi_{N_t}$  is from (8) and  $x_t$  is the state process.

*Remark III.1.* One can consider  $U_t^j$  as a function of a pure jump process  $\bar{z}_t$  since  $\{\tau_k\}$  is also defined by  $\bar{z}_t$ . The Markov chain  $x$  affects the value of  $U_t^j$  only by its transitional probabilities, not by realizations.

We define the evolution of  $U_t^j$  with differential equations over the intervals  $]\tau_{N_t}, \tau_{N_t+1}[$ , and via jumps at  $\tau_{N_t}$ , for all  $t \geq 0$ . Firstly, if  $t \in ]\tau_{N_t}, \tau_{N_t+1}[$ , we obtain

$$\begin{aligned} U_t^j &= \sum_{i=1}^d p_i(0) p_{ij}(t) E \left[ \Psi_{N_t} \middle| x_0 = a_i, x_t = a_j \right] \\ &= \sum_{\ell=1}^d \sum_{i=1}^d p_i(0) p_{i\ell}(\tau_{N_t}) p_{\ell j}(t - \tau_{N_t}) \\ &\quad E \left[ \Psi_{N_t} \middle| x_0 = a_i, x_{\tau_{N_t}} = a_\ell, x_t = a_j \right] \\ &= \sum_{\ell=1}^d p_{\ell j}(t - \tau_{N_t}) \sum_{i=1}^d p_i(0) p_{i\ell}(\tau_{N_t}) \\ &\quad E \left[ \Psi_{N_t} \middle| x_0 = a_i, x_{\tau_{N_t}} = a_\ell \right] \\ &= \sum_{\ell=1}^d p_{\ell j}(t - \tau_{N_t}) U_{\tau_{N_t}}^\ell. \end{aligned}$$

It yields  $\dot{U} = U\Lambda$  for  $t \in ]\tau_{N_t}, \tau_{N_t+1}[$ . Secondly, for  $t = \tau_{N_t}$  with  $n = N_t > 1$ , the strong Markov property and independence of the different  $\eta_k$  give

$$\begin{aligned} & E[\Psi_{N_t} | x_0 = a_i, x_{\tau_n} = a_j] \\ &= \sum_{l=1}^d E[\Psi_n | x_0 = a_i, x_{\tau_n} = a_j, x_{\tau_{n-1}} = a_l] \\ &= \sum_{l=1}^d E \left[ \exp \left( -\frac{(\Delta z_n - \xi_n)^2}{2R \Delta \tau_n} \right) | x_{\tau_n} = a_j, x_{\tau_{n-1}} = a_l \right] \\ &= E[\Psi_{n-1} | x_0 = a_i, x_{\tau_{n-1}} = a_l] \\ &= P(x_{\tau_{n-1}} = a_l | x_0 = a_i, x_{\tau_n} = a_j) \end{aligned}$$

By Bayes rule and Markov property

$$p_{ij}(t) P(x_{\tau_{n-1}} = a_l | x_0 = a_i, x_{\tau_n} = a_j)$$

$$\begin{aligned}
&= \mathbb{P}(x_{\tau_{n-1}} = a_l \mid x_0 = a_i) \mathbb{P}\left(x_{\tau_n} = a_l \mid \begin{array}{l} x_0 = a_i, \\ x_{\tau_n} = a_j \end{array}\right) \\
&= p_{li}(\tau_{n-1}) p_{jl}(\Delta\tau_n)
\end{aligned}$$

and then  $U_{\tau_n}^j$  equals

$$\begin{aligned}
&\sum_{l=1}^d \sum_{i=1}^d p_i(0) p_{ij}(t) \mathbb{E}\left[\exp\left(-\frac{(\Delta z_n - \xi_n)^2}{2R\Delta\tau_n}\right) \mid \begin{array}{l} x_{\tau_n} = a_j, \\ x_{\tau_{n-1}} = a_l \end{array}\right] \\
&\mathbb{E}\left[\Psi_{n-1} \mid \begin{array}{l} x_0 = a_i, \\ x_{\tau_{n-1}} = a_l \end{array}\right] \mathbb{P}\left(x_{\tau_{n-1}} = a_l \mid \begin{array}{l} x_0 = a_i, \\ x_{\tau_n} = a_j \end{array}\right) \\
&= \sum_{l=1}^d \sum_{i=1}^d p_i(0) p_{li}(\tau_{n-1}) p_{jl}(\Delta\tau_n) \\
&\mathbb{E}\left[\exp\left(-\frac{(\Delta z_n - \xi_n)^2}{2R\Delta\tau_n}\right) \mid x_{\tau_n} = a_j, x_{\tau_{n-1}} = a_l\right] \\
&\mathbb{E}\left[\Psi_{n-1} \mid x_0 = a_i, x_{\tau_{n-1}} = a_l\right] \\
&= \sum_{l=1}^d p_{jl}(\Delta\tau_n) U_{n-1}^l \\
&\mathbb{E}\left[\exp\left(-\frac{(\Delta z_n - \xi_n)^2}{2R\Delta\tau_n}\right) \mid x_{\tau_n} = a_j, x_{\tau_{n-1}} = a_l\right].
\end{aligned}$$

Thus we state the recursive rule for  $U$  as follows.

**Proposition III.2.** For a Markov chain  $(x_t)_{t \geq 0}$ , a fixed sample path of  $N_t$  associated with jump times  $\tau_{N_t}$ , the observations  $\hat{z}_t$  defined in (6), the conditional density  $\hat{\rho}_t = (\hat{\rho}_t^1, \dots, \hat{\rho}_t^d)^\top$  is

$$\hat{\rho}_t^j = \frac{U_t^j}{\sum_{i=1}^d U_t^i},$$

where the vector  $U_t = (U_t^1, \dots, U_t^d)$  satisfies

$$\begin{aligned}
\dot{U}_t &= U_t \Lambda & t \in ]\tau_{N_t}, \tau_{N_t+1}[ \\
U_{\tau_n} &= K_n \lim_{s \nearrow \tau_n} U_s & \forall \tau_n = \tau_{N_t}
\end{aligned} \tag{9}$$

where  $lj$ -th component of the matrix  $K_n$  is the following

$$\mathbb{E}\left[\exp\left(-\frac{(\Delta z_n - \int_{\tau_{n-1}}^{\tau_n} h(x_s) ds)^2}{2R\Delta\tau_n}\right) \mid \begin{array}{l} x_{\tau_n} = a_j, \\ x_{\tau_{n-1}} = a_l \end{array}\right].$$

#### IV. CONVERGENCE IN MEAN

We now turn to the question of comparing the filter obtained from randomly sampled observations with the continuous Wonham filter. For this purpose, we assume that the state process is an ergodic Markov chain. In what follows, we use the notation  $\mathbb{E}^z$  to denote expectation with respect to noise in  $z$ -process, and  $\mathbb{E}^\lambda$  to denote expectation with respect to the distribution of the Poisson sampling process with intensity  $\lambda > 0$ . We also recall that  $\sigma(\cdot)$  is used to denote the filtration generated by its argument. Our main result is formulated as follows:

**Theorem IV.1.** Let  $(x_t)_{t \geq 0}$  be an ergodic Markov chain with non-zero initial distribution. For each  $t > 0$ , it holds that

$$\lim_{\lambda \rightarrow \infty} \mathbb{E}^\lambda \mathbb{E}^z \left| \mathbb{E}[x_t \mid \sigma\{\hat{z}_{[0,t]}\}] - \mathbb{E}[x_t \mid \sigma\{z_{[0,t]}\}] \right| = 0.$$

We will carry out the proof of this result in the remainder of this section. An outline of the main steps of the proof is drawn in Figure 1.

#### A. Intermediate results

We start with the definition of operators  $\theta$  and  $H$  that are used excessively in the sequel.

**Definition 2.** For  $n \in \mathbb{N}, t > 0$ , let  $\theta_n^t$  and  $H_n^t$  be the following mappings:

$$\begin{aligned}
\theta_n^t(s) &:= \max\left\{\frac{mt}{n} \in [0, s] \mid m = 0, \dots, n\right\}, \\
(H_n^t v)_s &:= H_n^t(v)(s) = v(\theta_n^t(s)).
\end{aligned}$$

Using these definitions, we can now state the intermediate results which are necessary for the proof of Theorem IV.1.

**Lemma IV.2.** Let  $(x_t)_{t \geq 0}$  be an ergodic Markov chain with non-zero initial distribution. For each  $t > 0$ , there is convergence in  $\mathbb{E}^z$ -mean:

$$\lim_{n \rightarrow \infty} \mathbb{E}^z \left| \mathbb{E}[x_t \mid \sigma\{(H_n^t z)_{[0,t]}\}] - \mathbb{E}[x_t \mid \sigma\{z_{[0,t]}\}] \right| = 0.$$

The proof of Lemma IV.2 is a direct consequence of the arguments appearing in [6, Appendix 2] and is not carried out here. The vector  $U(t, (H_n^t \hat{z})_t)$  below consists of the corresponding entries  $U^j(t, (H_n^t \hat{z})_t)$  as considered in Definition 1.

**Proposition IV.3.** Let  $x_t$  be an ergodic Markov chain with non-zero initial distribution. For each  $t > 0, n \in \mathbb{N}$ , there is convergence in  $\mathbb{E}^\lambda \mathbb{E}^z$ -mean:

$$\lim_{\lambda \rightarrow \infty} \mathbb{E}^\lambda \mathbb{E}^z \left| \frac{U(t, (H_n^t \hat{z})_t)}{\|U(t, (H_n^t \hat{z})_t)\|_1} - \mathbb{E}[x_t \mid \sigma\{H_n^t z_{[0,t]}\}] \right| = 0.$$

The proof of Proposition IV.3 appears in Section IV-C. The final statement, that we need for the proof of Theorem IV.1 is the following:

**Lemma IV.4.** Let  $(x_t)_{t \geq 0}$  be an ergodic Markov chain with non-zero initial distribution. For each  $t > 0$  and each  $\lambda > 0$  there is convergence in  $\mathbb{E}^\lambda \mathbb{E}^z$ -mean:

$$\lim_{n \rightarrow \infty} \mathbb{E}^\lambda \mathbb{E}^z \left| \frac{U(t, (H_n^t \hat{z})_t)}{\|U(t, (H_n^t \hat{z})_t)\|_1} - \mathbb{E}[x_t \mid \sigma\{\hat{z}_{[0,t]}\}] \right| = 0.$$

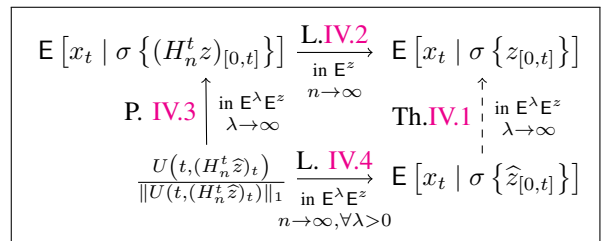


Fig. 1: Schematic of convergences

### B. Proof of Theorem IV.1

The proof is essentially based on the schematic shown in Figure 1.

*Proof.* By Proposition IV.3, for any  $\varepsilon_1$  there is large enough  $\lambda_1 = \lambda(\varepsilon_1)$  such that for any  $\lambda \geq \lambda_1$ ,

$$\mathbb{E}^\lambda \mathbb{E}^z \left| \frac{U(t, (H_n^t \hat{z})_t)}{\|U(t, (H_n^t \hat{z})_t)\|_1} - \mathbb{E}[x_t \mid \sigma\{(H_n^t z)_{[0,t]}\}] \right| \leq \varepsilon_1.$$

By Lemmas IV.2 and IV.4, for any  $\varepsilon_2$ , we can find  $n_2 = n(\varepsilon_2)$ , such that for every  $n \geq n_2$ , the following two inequalities hold:

$$\begin{aligned} \mathbb{E}^\lambda \mathbb{E}^z \left| \mathbb{E}[x_t \mid \sigma\{(H_n^t z)_{[0,t]}\}] - \mathbb{E}[x_t \mid \sigma\{z_{[0,t]}\}] \right| &\leq \varepsilon_2, \\ \mathbb{E}^\lambda \mathbb{E}^z \left| \frac{U(t, (H_n^t \hat{z})_t)}{\|U(t, (H_n^t \hat{z})_t)\|_1} - \mathbb{E}[x_t \mid \sigma\{\hat{z}_{[0,t]}\}] \right| &\leq \varepsilon_2. \end{aligned}$$

In the first of these two inequalities, the expectation  $\mathbb{E}^\lambda$  does not change anything as inner parts are independent of  $N_t$ .

So for a given  $\varepsilon > 0$ , taking  $\varepsilon_1 = \varepsilon_2 = \varepsilon/3$  and the corresponding  $\lambda_1$  and  $n_2$ , we immediately obtain that there exists  $\lambda(\varepsilon)$  such that for any  $\lambda \geq \lambda(\varepsilon)$  the required convergence property holds:

$$\mathbb{E}^\lambda \mathbb{E}^z \left| \mathbb{E}[x_t \mid \sigma\{\hat{z}_{[0,t]}\}] - \mathbb{E}[x_t \mid \sigma\{z_{[0,t]}\}] \right| \leq \varepsilon. \quad \square$$

### C. Proof of Proposition IV.3

*Proof.* Recall that  $\frac{U^j(t, (H_n^t z)_t)}{\|U(t, (H_n^t z)_t)\|_1}$  is the  $j$ -th component of the conditional expectation  $\mathbb{E}[x_t \mid \sigma\{(H_n^t z)_{[0,t]}\}]$  as showed in Section III. Then it is sufficient to prove that for any  $\varepsilon > 0$  there is sufficiently large  $\lambda > 0$  such that

$$\mathbb{E}^\lambda \mathbb{E}^z \left| \frac{U^j(t, (H_n^t \hat{z})_t)}{\|U(t, (H_n^t \hat{z})_t)\|_1} - \frac{U^j(t, (H_n^t z)_t)}{\|U(t, (H_n^t z)_t)\|_1} \right| \leq \varepsilon. \quad (10)$$

In what follows, we use the notation  $\tau_k$  for jump times of a Poisson process and  $\frac{mt}{n}$  for jump times of the ‘uniform’ process  $\theta_n^t$ . Moreover, we recall that a Poisson process  $N_t$  is associated with  $\{\tau_k\}$  as defined in (5).

In order to make  $H_n^t \hat{z}$  somehow similar to  $H_n^t z$ , we need the probability that at least one Poisson arrival time  $\tau_k \in \{\tau_k\}_{k=0}^{N_t}$  occurs between  $\frac{mt}{n} - \delta$  and  $\frac{mt}{n}$  for all  $m = 1, \dots, n$  and an arbitrary  $\delta \in (0, \frac{t}{n})$ . Since  $\tau_k \leq \frac{mt}{n}$  if  $k \leq N_{\frac{mt}{n}}$ , this condition can be formulated as

$$\max_{m=1, \dots, n} \min_{k=0, \dots, N_{\frac{mt}{n}}} \left( \frac{mt}{n} - \tau_k \right) \leq \delta. \quad (11)$$

The opposite event is

$$\max_{m=1, \dots, n} \min_{k=0, \dots, N_{\frac{mt}{n}}} \left( \frac{mt}{n} - \tau_k \right) > \delta \quad (12)$$

and its probability is the sum  $\sum_{m=1}^n (1 - e^{-\lambda \delta})^{m-1} e^{-\lambda \delta}$ , that is  $(1 - e^{-\lambda \delta})^n$ . Using this, we can decompose the expectation

$\mathbb{E}^\lambda \left[ \frac{U^j(t, (H_n^t \hat{z})_t)}{\|U(t, (H_n^t \hat{z})_t)\|_1} \right]$  into the sum

$$\begin{aligned} &(1 - e^{-\lambda \delta})^n \mathbb{E}^\lambda \left[ \frac{U^j(t, (H_n^t \hat{z})_t)}{\|U(t, (H_n^t \hat{z})_t)\|_1} \mid \text{eq. (12)} \right] + \\ &+ (1 - (1 - e^{-\lambda \delta})^n) \mathbb{E}^\lambda \left[ \frac{U^j(t, (H_n^t \hat{z})_t)}{\|U(t, (H_n^t \hat{z})_t)\|_1} \mid \text{eq. (11)} \right]. \end{aligned}$$

Let us take  $\lambda = 1/\sqrt{\delta}$  and define  $\delta$  later. Notice that  $\frac{U^j(t, (H_n^t \hat{z})_t)}{\|U(t, (H_n^t \hat{z})_t)\|_1} \leq 1$  and  $(1 - e^{-\lambda \delta})^n \rightarrow 0$  as  $\lambda \rightarrow \infty$ . Then it remains to prove that  $\mathbb{E}^\lambda \mathbb{E}^z \left( \frac{U^j(t, (H_n^t \hat{z})_t)}{\|U(t, (H_n^t \hat{z})_t)\|_1} \mid \text{eq. (11)} \right) \rightarrow \mathbb{E}^z \left( \frac{U^j(t, (H_n^t \hat{z})_t)}{\|U(t, (H_n^t \hat{z})_t)\|_1} \right)$ .

Let us consider an arbitrary realization  $N_t$  satisfying (11). For an arbitrary fixed  $z_t$ , we obtain the process  $\hat{z}_t$ . Then both processes  $H_n^t \hat{z}$  and  $H_n^t z$  are pure jump and all jumps occur at times  $\frac{mt}{n}$  for  $m = 1, \dots, n$  only. Hence both processes  $U^j(\cdot, H_n^t \hat{z})$  and  $U^j(\cdot, H_n^t z)$  differ in their exponent terms only. For  $U^j(t, (H_n^t z)_t)$  we have jump times only at  $\{\frac{mt}{n}\}$  and the exponent term straightforwardly contains the following sum

$$- \sum_{m=1}^n \left( z_{\frac{mt}{n}} - z_{\frac{(m-1)t}{n}} - \int_{\frac{(m-1)t}{n}}^{\frac{mt}{n}} h(\bar{x}_s) ds \right)^2 \quad (13)$$

For  $U^j(\cdot, H_n^t \hat{z})$ , employing the condition (11), we obtain that for each  $m = 1, \dots, n$ , there is a certain  $\delta_m \in (0, \delta)$  such that  $H(\hat{z}_{\frac{mt}{n}}) = \hat{z}_{\frac{mt}{n}} = z_{\frac{mt}{n} - \delta_m}$ . So the exponent term of  $U^j(t, (H_n^t \hat{z})_t)$  contains a similar sum

$$- \sum_{m=1}^n \left( z_{\frac{mt}{n} - \delta_m} - z_{\frac{(m-1)t}{n} - \delta_{m-1}} - \int_{\frac{(m-1)t}{n}}^{\frac{mt}{n}} h(\bar{x}_s) ds \right)^2 \quad (14)$$

Thus a realization of Poisson process  $N_t$  defines only the sequence of  $\delta_m$ , which are uniformly bounded by  $\delta$ . It immediately yields the convergence of each component:  $\mathbb{E}^\lambda \mathbb{E}^z \{ |U^j(t, H_n^t(z)) - U^j(t, H_n^t(\hat{z}))| \} \rightarrow 0$  as  $\delta \rightarrow 0$ , i.e.  $\lambda \rightarrow \infty$ . To get the convergence of normalized values, notice that both fractions are less than one and the denominators are bounded away from zero in the following sense. Denote  $\omega = \min_i p_i(0)$ ; by ergodicity for all  $t$  there is a pair  $(i, j)$  such that  $p_i(0)p_{i,j}(t) \geq \omega/d$ , and  $\|U(t, H_n^t(\hat{z}))\|_1 \geq U^j(t, H_n^t(\hat{z})) \geq \omega/d$   $\mathbb{E} \left[ \exp \frac{\text{eq. (14)}}{2Rt/n} \mid x_0 = a_i, x_t = a_j \right]$ . Definition (2) of the process  $z$  implies the following chain of estimations:

$$\begin{aligned} &\left( \int_{-M}^M \frac{\exp\{-\frac{1}{2}(\frac{x}{R(t/n+\delta)})^2\}}{\sqrt{2\pi R(t/n+\delta)}} dx \right)^n \\ &\leq \mathbb{P}[\max_m |\eta_{\frac{mt}{n} - \delta_m} - \eta_{\frac{(m-1)t}{n} - \delta_{m-1}}| \leq M] \\ &\leq \mathbb{P}[\text{eq. (14)} \leq n(M + 2 \max_l a_l)^2 (t/n + \delta)^2] \\ &\leq \mathbb{P}[\|U(t, (H_n^t \hat{z})_t)\|_1 \geq \kappa_M] \rightarrow 1 \text{ as } M \rightarrow \infty \end{aligned}$$

where  $\kappa_M = \omega/d \exp \left\{ \frac{n(M + 2 \max_l a_l)^2 (t/n + \delta)^2}{2Rt/n} \right\}$  and tends to 0 as  $M \rightarrow \infty$ . For  $\|U(t, (H_n^t \hat{z})_t)\|_1$  we have the same estimation, but with zeros instead of  $\delta_m$  and  $\delta$ . These inequalities defines a suitable  $\delta = \delta(\varepsilon)$  so that (10) holds.  $\square$

#### D. Proof of Lemma IV.4

*Proof.* In Section III, we showed that  $\frac{U^j(t, \hat{z}_t)}{\|U(t, \hat{z}_t)\|_1}$  is the  $j$ -th component of  $\mathbb{E}[x_t | \sigma\{\hat{z}_{[0,t]}\}]$ . Then it is sufficient to prove that for any  $\varepsilon$ , there exists  $D = D(\varepsilon) > 0$  such that for any  $n > t/D$

$$\mathbb{E}^\lambda \mathbb{E}^z \left| \frac{U^j(t, (H_n^t \hat{z})_t)}{\|U(t, (H_n^t \hat{z})_t)\|_1} - \frac{U^j(t, \hat{z}_t)}{\|U^j(t, \hat{z}_t)\|_1} \right| \leq \varepsilon. \quad (15)$$

If  $t/D =: M_D \in \mathbb{N}$ , we can consider  $M_D$  disjoint intervals  $\left(\frac{tM}{M_D}, \frac{t(M+1)}{M_D}\right]$  of length  $D$  (for  $M = 1, \dots, M_D$ ). Now we are interested in the case where at most one jump of  $N_t$  occurs at each interval  $\left(\frac{tM}{M_D}, \frac{t(M+1)}{M_D}\right]$ , i.e.

$$\max_{M=1, \dots, M_D} N_{\frac{tM}{M_D}} - N_{\frac{t(M-1)}{M_D}} \leq 1. \quad (16)$$

The probability of the event (16) is  $e^{-\lambda t}(1 + \lambda D)^{\frac{t}{D}}$ , and so it tends to 1 as  $D \rightarrow 0$ . Therefore, for any  $D > 0$ , we decompose and estimate the expectations of (15) as

$$2 \left(1 - e^{-\lambda t}(1 + \lambda D)^{\frac{t}{D}}\right) + e^{-\lambda t}(1 + \lambda D)^{\frac{t}{D}} \sum_i p_i(0) p_{ij}(t) \times \mathbb{E}^\lambda \mathbb{E}^z \left[ \left| \frac{U^j(t, (H_n^t \hat{z})_t)}{\|U(t, (H_n^t \hat{z})_t)\|_1} - \frac{U^j(t, \hat{z}_t)}{\|U^j(t, \hat{z}_t)\|_1} \right| \text{eq. (16)} \right],$$

and the first addend is less than  $\varepsilon/4$  for every  $D \in ]0, D^*[$  for a certain  $D^*$ . In the same way, we can separate two cases: if total number of jumps of  $\hat{z}$  on  $[0, t]$  is more than  $K = K(\lambda, \varepsilon)$  or if it is less or equal. The expectations of (15) conditioned on the first case are less than  $\varepsilon/4$  if  $K$  is such that  $e^{-\lambda t} \sum_{i=K}^{\infty} \frac{(\lambda t)^i}{i!} < \varepsilon/8$ . Indeed, it means that  $\varepsilon/4$  is greater than the value  $\left| \frac{U^j(t, (H_n^t \hat{z})_t)}{\|U(t, (H_n^t \hat{z})_t)\|_1} - \frac{U^j(t, \hat{z}_t)}{\|U^j(t, \hat{z}_t)\|_1} \right|$  times the probability that  $\hat{z}$  has more than  $K$  jumps. Finally, let us get rid of the case of Poisson processes that contain relatively small inter-arrival times, i.e. let us find  $L > 0$  such that  $\varepsilon/4$  bounds from below the expectations of (15) conditioned on Poisson process with inter-arrival time less than  $L$ . For a Poisson process, the probability of containing all inter-arrival times greater than  $L$  is not less than  $e^{-\lambda L K}$  (since  $K$  is the maximum number of jumps), i.e. it suffices to take  $L = L(\lambda, \varepsilon)$  such that  $1 - e^{-\lambda L K} \leq \varepsilon/8$ .

Let us denote by  $\mathcal{A}$  the event that correspond to the inequality (16), the condition that the number of jumps of  $\hat{z}$  is not more than  $K$  and the condition that the inter-arrival times are greater than  $L$ . Consider a realization of  $N_t$  from  $\mathcal{A}$ . Denote by  $\tau_k$  the corresponding jump times of  $H_n^t \hat{z}$ . Then jumps of  $\hat{z}$  occur at  $\tau_k - \delta_k$ . So that  $H_n^t \hat{z}(\frac{tM}{M_D}) = \hat{z}(\frac{tM}{M_D})$  for  $M = 1, \dots, M_D$  and so the exponent term in  $U^j(t, H_n^t \hat{z})$  becomes  $\exp \sum_k \frac{-(\hat{z}(\tau_k) - \hat{z}(\tau_{k-1}) - \int_{\tau_{k-1}}^{\tau_k} h(\bar{x}_s) ds)^2}{2R(\tau_k - \tau_{k-1})}$  while the exponent term in  $U^j(t, H_n^t \hat{z})$  contains same values at times  $\tau_k - \delta_k$ . Thus it is easy to see that we have convergence  $\mathbb{E}^\lambda \mathbb{E}^z \left[ \left| U^j(t, (H_n^t \hat{z})_t) - U^j(t, \hat{z}_t) \right| \middle| \mathcal{A} \right] \rightarrow 0$  for all components  $j$ . To get (15) it remains to estimate the

denominators. Denote  $\omega = \min_i p_i(0)$ ; by ergodicity we get that there is a pair  $(i, j)$  such that  $p_i(0) p_{i,j}(t) \geq \omega/d$ , and so  $\|U(t, H_n^t \hat{z})\|_1 \geq U^j(t, H_n^t \hat{z}) \geq \omega/d$ . Here the sum contains not more than  $K$  elements,  $\tau_k - \tau_{k-1} > L - D$ ,  $x_s$  are from a finite set  $S$ , and the only remaining randomness in  $\eta$  is bounded as  $\left( \int_{-M}^M \frac{\exp\{-\frac{1}{2}(\frac{x}{Rt})^2\}}{\sqrt{2\pi R t}} dx \right)^K \leq (\mathbb{P}[|\eta_t - \eta_0| \leq M])^K \leq \mathbb{P}[\max_k |\eta_{\tau_k} - \eta_{\tau_{k-1}}| \leq M]$ . Thus, we have

$$\mathbb{P}[\|U(t, H_n^t \hat{z})\|_1 \geq \kappa] \rightarrow 0, \quad \text{as } \kappa \rightarrow 0.$$

The same holds for  $\|U(t, \hat{z})\|_1$ . This allows us to choose a suitable  $D \in ]0, D^*[$  and conclude the proof.  $\square$

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