Robust data-driven control design for linear systems subject to input saturation

Alexandre Seuret, Sophie Tarbouriech

Robust data-driven control design for linear systems subject to input saturation

A. Seuret, S. Tarbouriech

***Abstract*—This paper deals with the problem of providing a data-driven solution to the local stabilization of linear systems subject to input saturation. After presenting a model-based solution to this well-studied problem, a systematic method to transform model-driven into data-driven LMI conditions is presented. This technical solution is demonstrated to be equivalent to the recent advanced results on LMI formulations based on S-procedure or Peterson Lemmas. However, the advantage of the proposed method relies on its simplicity and its potential to be applicable to a wide class of problems of stabilization of (non)linear discrete-time systems. The method is then illustrated on both an academic example and the spacecraft Rendezvous problem.**

***Index Terms*—Linear systems, Saturation, LMI, Data-driven control design**

I. Introduction

The robustness properties for dynamical control systems have been studied in several works from an analysis or design context [11], [17], [18]. Then, robustness conditions to face uncertainty or presence of additive disturbances have been proposed allowing to ensure the stability and a certain level of performance for the systems under consideration. Most of these conditions are formulated in the form of linear matrix inequalities (LMIs), due to the powerful numerical and optimization procedures, as semi-definite programming, which can be handled. The general common feature of this kind of methods is that they are model-based, possibly modeling also the presence of uncertainties as norm-bounded or polytopic uncertainties. In front of complex systems to model, we need to consider very imprecise or even unknown mathematical models of the dynamic evolution, including for examples non-linearities. The consequence is then that the model-based methods for analysis and controller design may reveal to be difficult or even impossible to apply. Adapting the tools issued from the control theory, some works have emerged based on some data information: see, for example, [1], [2], [14], [24] and the references therein.

Hence, some studies addressing the stability or stabilization criteria from a data-driven point of view, have been published. In particular several problems related to the design of state/output feedback controllers for linear systems have been revisited [5], [8]–[10], [23], [24]. In these works considering the case of exact data experiments for linear time-invariant systems, i.e. without noise or uncertainties, equivalent formulations between model-based and data-based criteria both in stability analysis and control design have been exhibited.

Another important feature when dealing with dynamical control systems pertains the control input saturation due to limitations on the actuators [21]. At our knowledge, few works in the literature addressed the problem to deal with constrained control (see, for example, [6], [16] and the references therein) and more especially with saturating control input. The objective of the current paper is to bring some preliminary bricks in considering this aspect. To this end, we consider the design of a state-feedback saturated control law allowing to ensure the regional (local) asymptotic stability of the origin when the system is noisy-free and to ensure the convergence to an attractor when the system is affected by noise, which has not been fully considered in the literature, to the best of our knowledge. Indeed, the characterization of an inner-approximation of the basin of attraction of the origin together with an outer-approximation of the attractor are proposed first by model-based techniques and then by data-driven techniques. Taking inspiration from the model-based conditions (see Theorem 1 in Section III), which are written in a friendly-data form, the data-driven local stabilization is formulated through matrix inequalities conditions (see Theorem 2 in Section IV-C). The technique allowing to exhibit the sufficient conditions is based on the use of Lyapunov arguments, generalized sector-bounded conditions to deal with the saturation and S-procedure to handle the presence of noise. The implicit objectives are to maximize the inner-approximation of the basin of attraction of the origin and to minimize the outer-approximation of the attractor. The main rational behind the matrix inequalities formulation is due to the friendly form of the model-based conditions which allows to directly derive the data-driven conditions thanks to a matrix-constrained relaxation (see Lemma 3 in Section IV-B). Note that the matrix inequalities conditions are quasi-LMIs in the sense that there is the product between matrices and scalars.

The paper is organized as follows. Section II presents the system under consideration and formally states the problem. In Section III, sufficient conditions to solve the control problem are formulated as quasi-LMIs, in the sense that there is a product between a matrix and a scalar. Section IV deals with the local stabilization from a data-driven point of view. Section V illustrates the theoretical results and proposes some insights regarding the influence of the tuning parameters. Comparison between model-based and data-driven approaches is proposed for different collections of data to depict the trade-off between the approximation of the basin of attraction and that one of the attractor. Section VI draws some concluding remarks.

This work has been partially funded by the European Union - NextGenerationEU. Research by Sophie Tarbouriech is partially funded by ANR via project OLYMPIA, number ANR-23-CE48-0006

A: Seuret is with the Univ. de Sevilla, Camino de los Descubrimientos, s/n 41092 Sevilla, Spain. (e-mail: aseuret@us.es).

S. Tarbouriech is with LAAS-CNRS, Université de Toulouse, CNRS, Toulouse, France. (e-mail: tarbour@laas.fr)..

Notation. Throughout the paper, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{R}$ the real numbers, $\mathbb{R}^n$ the n-dimensional Euclidean space, $\mathbb{R}^{n\times m}$ the set of all real $n\times m$ matrices, $\mathbb{S}^n$ ($\mathbb{S}^n_+$) the set of symmetric (positive definite) matrices in $\mathbb{R}^{n\times n}$ and $\mathbb{D}^n_+$ the set of diagonal positive definite matrices in $\mathbb{R}^{n\times n}$. For any n and m in $\mathbb{N}$, matrices I_n and $\mathbf{0}_{n,m}$ ($\mathbf{0}_n = \mathbf{0}_{n,n}$) denote the identity matrix of $\mathbb{R}^{n\times n}$ and the null matrix of $\mathbb{R}^{n\times m}$, respectively. When no confusion is possible, the subscripts of these matrices that precise the dimension, will be omitted. For any matrices $A = A^\top, B, C = C^\top$ of appropriate dimensions, matrix $\left[\begin{smallmatrix} A & B \\ * & C \end{smallmatrix}\right]$ denotes the symmetric matrix $\left[\begin{smallmatrix} A & B \\ B^\top & C \end{smallmatrix}\right]$. For any matrix $N \in \mathbb{R}^{n\times m}$, notation $N_{(i)}$, for any $i = 1,\ldots,n$, stands for the i^{th} row of N and notation $\|N\|^2$ to denote the symmetric matrix $NN^\top$. If N is a square matrix, $\mathrm{Tr}(N)$ denote the trace of N. For any matrix M of $\mathbb{R}^{n\times n}$, the notation $M \succ 0$ means that M is in $\mathbb{S}^n_+$. For a matrix $M \in \mathbb{S}^n_+$ and any positive scalar $\alpha \in \mathbb{R}_{>0}$, we denote the ellipsoid $\mathcal{E}(M,\alpha) = \left\{x \in \mathbb{R}^n,\ x^\top M x \le \alpha^{-1}\right\}$.

II. Problem formulation

A. System data

Consider the discrete-time linear system subject to an input saturation and affected by an external perturbation. Such a system is described by the following equations

$$\left\{\begin{array}{lcl} x^+ & = & Ax + B\,\mathrm{sat}(u) + w, \\ x_0 & \in & \mathbb{R}^{n_x}, \end{array}\right. \tag{1}$$

where, for any given positive integers n_x and n_u, $x \in \mathbb{R}^{n_x}$ is the state vector, for which adopts the following notation $x^+ = x_{k+1}$ and $x = x_k$, $u = u_k \in \mathbb{R}^{n_u}$ is the control input. The initial condition is $x_0 \in \mathbb{R}^{n_x}$. The dynamics of the system is defined by matrices $A \in \mathbb{R}^{n_x\times n_x}$ and $B \in \mathbb{R}^{n_x\times n_u}$. In the sequel, notation $\mathcal{G}$ stands for the matrices of the system as $\mathcal{G} := \begin{bmatrix} A & B \end{bmatrix}$. The system is perturbed by the unknown noise signal $w = w_k \in \mathbb{R}^{n_x}$, assumed to be bounded leading to the following assumption.

Assumption 1: There exists $\lambda > 0$ such that the norm of the disturbance verifies $w_k^\top w_k < \lambda$, for all $k \ge 0$. In other words, the disturbance belongs to the following set Ω_λ

$$\Omega_\lambda := \{v \in \mathbb{R}^{n_x}, \quad v^\top v \le \lambda\}. \tag{2}$$

The saturation function $\mathrm{sat}(u)$ is the classical decentralized vector-valued saturation map from R^{n_u} to R^{n_u}, whose the components are defined by:

$$\mathrm{sat}(u_i) = \mathrm{sign}(u_i)\min(|u_i|, \bar{u}_i), \quad \forall i = 1,\ldots,n_u, \tag{3}$$

where u_i refers to the i^{th} control input and $\bar{u}_i$ is the i^{th} entry of the vector $\bar{u} \in \mathbb{R}^{n_u}$ that is the level vector of the saturation.

In this paper, we focus on the design of a static state-feedback control law of the form

$$u = Kx, \tag{4}$$

with $K \in \mathbb{R}^{n_u\times n_x}$, that regionally (locally) stabilizes the trajectories of the closed-loop system (1) with (4) in the absence of perturbation.

B. Preliminaries on generalized sector conditions

Due to the presence of the saturation map in system (1), stability of the closed-loop system (1)-(4) should be addressed with attention [21]. To this hand, we follow the approach proposed in [12] and [21], which consists in rewriting (1) as

$$x^+ = (A + BK)x + B\phi(u) + w, \tag{5}$$

where $\phi(u)$ is the dead-zone function defined by

$$\phi(u) = \mathrm{sat}(u) - u. \tag{6}$$

Following Remark 7.3 in [21], we use the following *Generalized sector condition* lemma to handle the dead-zone (6).

Lemma 1: ([21]) Consider a matrix $G \in \mathbb{R}^{n_u\times n_x}$. The following relation holds

$$\phi(u)^\top T[\mathrm{sat}(u) + Gx] \le 0, \tag{7}$$

with any diagonal positive definite matrix $T \in \mathbb{R}^{n_u\times n_u}$ provided that $x \in \mathcal{S}(G)$ where the polyhedral set $\mathcal{S}(G)$ is defined as follows

$$\mathcal{S}(G) = \{x \in \mathbb{R}^{n_x} : |G_{(i)}x| \le \bar{u}_i, \quad \forall i = 1,\ldots,n_u\}. \tag{8}$$

C. Control objectives

It is well-known (see [15], [21] and the references therein) that ensuring global asymptotic stability of the origin for saturated systems (1) (without perturbation) is in general impossible, unless the open-loop dynamics is not exponentially unstable. That means that the basin of attraction of the origin is not the whole state space and needs to be determined, that turns out to be a complex task. In this case, the objective is to characterize an inner approximation of the basin of attraction of the origin by considering level sets built from a Lyapunov function. Furthermore, in presence of perturbation satisfying Assumption 1 (bounded but not vanishing), the closed-loop trajectories cannot converge to the origin but only to a neighborhood of it. Therefore, it is of interest to characterize the attractor toward which the closed-loop trajectories converge to, by using again level sets of the same Lyapunov function. Hence, this paper deals with the following problems:

(P1) In the case $w = 0$, characterize an approximation of the basin of attraction, expressed as a level of the Lyapunov function given by $\mathcal{E}(P,1)$, where P is a symmetric positive definite matrix to be designed. Thus, for any initial condition belonging to $\mathcal{E}(P,1)$, the closed-loop trajectories will asymptotically converge to the origin.

(P2) In the case $w \ne 0$, characterize an approximation of the attractor capturing the closed-loop trajectories expressed as a level set of of the same Lyapunov function given by $\mathcal{E}(P,\varepsilon)$, where P is a symmetric positive definite matrix and $\varepsilon > 0$ are to be designed.

(P3) In addition to the previous problems, maximize the approximation of the basin of attraction $\mathcal{E}(P,1)$, as well as minimize the approximation of the attractor $\mathcal{E}(P,\varepsilon)$.

Moreover, as we consider that the model is not assumed to be perfectly known but only approached via finite set of data experiments, the final objective is to solve Problems (P1), (P2) and (P3) but from a data-driven point of view. Hence, we want to revisit the solutions obtained from a model-based approach to provide a new data-driven practical stabilization criterion, which arises from matrix manipulations of a suitable model-based preliminary result. This transformation is made possible thanks to a lemma, which can be seen as a particular case of those from the literature, but which is particularly well-suited for data-driven design for saturated systems.

III. Model-based local practical stabilization

The following theorem follows the methods presented in Chapter 3 in [21] and deals with the model-based design of stabilizing controllers (4) for system (1) under Assumption 1. Therefore, the following theorem states a solution to solve Problems (P1), (P2) and (P3) through a model-based approach.

Theorem 1: Under Assumption 1, and for a given $\mu \in (0,1)$, $\alpha_1 > 0$ and $\alpha_2 > 0$, assume that there exists

$$\begin{aligned} \mathcal{D}^1_{\mathcal{V}} &:= \{\varepsilon, \mu, W, S, Y, Z\} \\ &\in \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \mathbb{S}^{n_x}_{+} \times \mathbb{D}^{n_x}_{+} \times \mathbb{R}^{n_x \times n_u} \times \mathbb{R}^{n_x \times n_u} \end{aligned}$$

solution to the following optimization problem

$$\begin{array}{ll} \max_{\mathcal{D}^1_{\mathcal{V}}} & \alpha_1 \varepsilon + \alpha_2 \mathrm{Tr}(W) \\ s.t. & \varepsilon > 1,\ \Phi(\mathcal{G}) \succ 0,\ \begin{bmatrix} W & Z^\top_{(i)} \\ * & \bar{u}^2_i \end{bmatrix} \succ 0, \end{array} \tag{9}$$

for all $i = 1, \dots, n_u$, where

$$\Phi(\mathcal{G}) = \begin{bmatrix} (1-\mu)W & Y^\top + Z^\top & WA^\top + Y^\top B^\top \\ * & 2S & SB^\top \\ * & * & W - \frac{\lambda\varepsilon}{\mu} I_{n_x} \end{bmatrix}.$$

Then, the control law (4) with $K = YW^{-1}$ ensures that Problems (P1), (P2) and (P3) are solved, i.e.

- When $w = 0$, the ellipsoid $\mathcal{E}(W^{-1}, 1)$ is an approximation of the basin of attraction of the origin for the closed-loop system (1)-(4).
- When $w \in \Omega_\lambda$ and $w \neq 0$, the solutions to the closed-loop system (1)-(4) initialized in $\mathcal{E}(W^{-1}, 1) \setminus \mathcal{E}(W^{-1}, \varepsilon)$ converge to the attractor $\mathcal{E}(W^{-1}, \varepsilon)$.
- Sets $\mathcal{E}(W^{-1}, 1)$ and $\mathcal{E}(W^{-1}, \varepsilon)$ are optimal with respect to the cost function $\alpha_1 \varepsilon + \alpha_2 Tr(W)$, which aims deriving a compromise between enlarging $\mathcal{E}(W^{-1}, 1)$ and minimizing $\mathcal{E}(W^{-1}, \varepsilon)$.

Proof. Consider the Lyapunov function given by $V(x) = x^\top P x$, for a given $P \in \mathbb{S}^{n_x}_{+}$. In order to address Problems (P1) and (P2), the objective is to ensure that the forward increment of V, i.e. $\Delta V(x) = x^{+\top} P x^{+} - x^\top P x$ verifies:

$$\Delta V(x) \preceq 0,\ \forall (x, w) \text{ s.t.} \begin{cases} x \in \mathcal{E}(P, 1), & (\text{i.e. } x^\top P x \leq 1) \\ x \notin \mathcal{E}(P, \varepsilon), & (\text{i.e. } x^\top P x \geq \varepsilon^{-1}) \\ w \in \Omega_\lambda, & (\text{i.e. } w^\top w \leq \lambda) \end{cases} \tag{10}$$

which is implied, thanks to the use of S-procedure [4], by the following condition

$$\begin{aligned} \mathcal{L}(x, w) = {} & \Delta V(x) + \bar{\mu}_1 (x^\top P x - \varepsilon^{-1}) + \bar{\mu}_2 (\lambda - w^\top w) \\ & + \bar{\mu}_3 (1 - x^\top P x) \preceq 0 \end{aligned} \tag{11}$$

with $\bar{\mu}_i \geq 0$, $i = 1, 2, 3$. Let us introduce the following augmented vector $\xi = (x, \phi(Kx), w)$, then $\Delta V(x)$ reads:

$$\Delta V(x) = \xi^\top \left(\begin{bmatrix} (A+BK)^\top \\ B^\top \\ I_{n_x} \end{bmatrix} P \begin{bmatrix} (A+BK)^\top \\ B^\top \\ I_{n_x} \end{bmatrix}^\top - \begin{bmatrix} P & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \xi.$$

The last inequalities in (9) ensure that the ellipsoid $\mathcal{E}(W^{-1}, 1)$ is included in the set $\mathcal{S}(G)$. From Lemma 1, for $x \in \mathcal{E}(W^{-1}, 1)$ inequality (7) holds and reads using ξ as follows:

$$\xi^\top \begin{bmatrix} 0 & (K+G)^\top T & 0 \\ T(K+G) & 2T & 0 \\ 0 & 0 & 0 \end{bmatrix} \xi \leq 0. \tag{12}$$

Then, merging (12) and (11) yields

$$\mathcal{L}_1(x, w) := \mathcal{L}(x, w) - \xi^\top \begin{bmatrix} 0 & (K+G)^\top T & 0 \\ T(K+G) & 2T & 0 \\ 0 & 0 & 0 \end{bmatrix} \xi \geq \mathcal{L}(x, w).$$

Therefore, having $\mathcal{L}_1(x, w)$ negative implies that $\mathcal{L}(x, w)$ is also negative for all (x, w). Let us rewrite the expression of $\mathcal{L}_1(x, w)$ in a more compact form.

$$\mathcal{L}_1(x, w) := -\xi^\top \Phi_1(\mathcal{G}) \xi - \Theta_\lambda,$$

where

$$\begin{aligned} \Phi_1(\mathcal{G}) &= \begin{bmatrix} (1-\bar{\mu}_1+\bar{\mu}_3)P & (K+G)^\top T & 0 \\ T(K+G) & 2T & 0 \\ 0 & 0 & \bar{\mu}_2 I_{n_x} \end{bmatrix} - \begin{bmatrix} (A+BK)^\top \\ B^\top \\ I_{n_x} \end{bmatrix} P \begin{bmatrix} (A+BK)^\top \\ B^\top \\ I_{n_x} \end{bmatrix}^\top, \\ \Theta_\lambda &= \bar{\mu}_1 \varepsilon^{-1} - \bar{\mu}_2 \lambda - \bar{\mu}_3. \end{aligned}$$

In the next developments, we will show that the existence of a solution to condition $\Phi(\mathcal{G}) \succ 0$ ensures the existence of $\bar{\mu}_1 > 0$, $\bar{\mu}_2 > 0$ and $\bar{\mu}_3 > 0$ such that $\mathcal{L}_1(x, w) \leq 0$.

Applying the Schur complement to $\Phi_1(\mathcal{G})$, pre- and post-multiplying it by $\mathrm{diag}(W, S, I_{2n_x})$, with $W = P^{-1}$ and $S = T^{-1}$, and selecting $Y = KW$ and $Z = GW$ yield

$$\begin{bmatrix} (1-\bar{\mu}_1+\bar{\mu}_3)W & Y^\top + Z^\top & 0 & WA^\top + Y^\top B^\top \\ * & 2S & 0 & SB^\top \\ * & * & \bar{\mu}_2 I_{n_x} & I_{n_x} \\ * & * & * & W \end{bmatrix} \succ 0,$$

which is equivalent to

$$\Phi_2(\mathcal{G}) := \begin{bmatrix} (1-\bar{\mu}_1+\bar{\mu}_3)W & Y^\top + Z^\top & WA^\top + Y^\top B^\top \\ * & 2S & SB^\top \\ * & * & W - \bar{\mu}_2^{-1} I_{n_x} \end{bmatrix} \succ 0. \tag{13}$$

We have shown so far that having $\mathcal{L}_1 \leq 0$ is equivalent to $\Phi_2(\mathcal{G}) \succ 0$ and $\Theta_\lambda > 0$. Enforcing the introduction of $\Phi(\mathcal{G})$, we get the following expression

$$\Phi_2(\mathcal{G}) = \Phi(\mathcal{G}) + \begin{bmatrix} (\mu-\bar{\mu}_1+\bar{\mu}_3)W & 0 & 0 \\ * & 0 & 0 \\ * & * & \left(\frac{\lambda\varepsilon}{\mu} - \frac{1}{\bar{\mu}_2}\right) I_{n_x} \end{bmatrix}. \tag{14}$$

Then, consider the following selection

$$\bar{\mu}_1 := \mu \in (0,1), \quad \bar{\mu}_2 := \frac{\mu}{\lambda\varepsilon}\left(1 - \frac{\varepsilon\gamma}{\mu}\right), \quad \bar{\mu}_3 < \gamma,$$

for sufficiently small $\gamma > 0$ such that $\bar{\mu}_2 > 0$. This selection ensures that $\Theta_\lambda = \gamma - \bar{\mu}_3 > 0$, which is required. Using this selection for $\bar{\mu}_2$, we note that

$$\frac{\lambda\varepsilon}{\mu} - \frac{1}{\bar{\mu}_2} = \frac{\lambda\varepsilon}{\mu}\left(1 - \frac{1}{1-\gamma\varepsilon/\mu}\right) = \frac{-\gamma\lambda\varepsilon^2}{\mu^2 - \gamma\varepsilon\mu} > \frac{-\bar{\mu}_3\lambda\varepsilon^2}{\mu^2 - \gamma\varepsilon\mu},$$

so that the following inequality holds

$$\Phi_2(\mathcal{G}) \succeq \Phi(\mathcal{G}) + \bar{\mu}_3 \begin{bmatrix} W & 0 & 0 \\ * & 0 & 0 \\ * & * & -\frac{\lambda\varepsilon^2}{\mu^2 - \gamma\varepsilon\mu} I_{n_x} \end{bmatrix}. \tag{15}$$

Hence, if condition $\Phi(\mathcal{G}) \succ 0$ holds, then there exists a sufficiently small $\bar{\mu}_3$ such that $\Phi_2(\mathcal{G}) \succ 0$, which ensures that there exist $\bar{\mu}_1 > 0, \bar{\mu}_2 > 0, \bar{\mu}_3 > 0$ such that $\mathcal{L}_1(x, w)$ is negative, which ensures (10).

Therefore, the satisfaction of relation $\Phi(\mathcal{G}) \succ 0$ ensures the local stability of the closed-loop system with the control gain $K = YW^{-1}$, for any initial condition in $x \in \mathcal{E}(W^{-1}, 1)$.

Since $\varepsilon > 1$, one gets $\mathcal{E}(W^{-1}, \varepsilon) \subseteq \mathcal{E}(W^{-1}, 1)$. In order to prove that $\mathcal{E}(W^{-1}, \varepsilon)$ is an attractor for the closed-loop system, it remains to demonstrate that $\mathcal{E}(W^{-1}, \varepsilon)$ is invariant, i.e.

$$x \in \mathcal{E}(W^{-1}, \varepsilon) \quad \Rightarrow \quad x^+ \in \mathcal{E}(W^{-1}, \varepsilon).$$

To do so, the satisfaction of $\Phi(\mathcal{G}) \succ 0$ ensures that

$$\begin{aligned} V(x^+) = &\ V(x) + \underbrace{\mathcal{L}_1(x, w)}_{\leq 0} + \underbrace{\xi^\top \begin{bmatrix} 0 & (K+G)^\top T & 0 \\ * & 2T & 0 \\ * & * & 0 \end{bmatrix} \xi}_{\leq 0} \\ & - \bar{\mu}_1 (V(x) - \varepsilon^{-1}) \underbrace{- \bar{\mu}_2 (\lambda - w^\top w)}_{\leq 0} - \bar{\mu}_3 (1 - V(x)) \\ \leq &\ (1 - \bar{\mu}_1 + \bar{\mu}_3) V(x) + \bar{\mu}_1 \varepsilon^{-1} - \bar{\mu}_3 \end{aligned}$$

Since $\bar{\mu}_1 = \mu$ is in $(0, 1)$, there exists a sufficiently small $\bar{\mu}_3$ such that $1 - \bar{\mu}_1 + \bar{\mu}_3 > 0$ and, consequently, as $V(x) \leq \varepsilon^{-1}$, we have

$$V(x^+) \leq (1 - \bar{\mu}_1 + \bar{\mu}_3) \varepsilon^{-1} + \bar{\mu}_1 \varepsilon^{-1} - \bar{\mu}_3 = \varepsilon^{-1} - \bar{\mu}_3 (1 - \varepsilon^{-1}),$$

which implies that $V(x^+) \leq \varepsilon^{-1}$, since $\varepsilon > 1$.

The last step of the proof addresses the particular case when $w = 0$ (or $\lambda = 0$). In this situation, ε disappears from the condition and can be selected as large as possible, so that $\mathcal{E}(W^{-1}, \varepsilon)$ shrinks to $\{0\}$. Therefore, the solution to the system, initialized in $\mathcal{E}(W^{-1}, 1)$ converges to the origin. $\square$

Remark 1: Note that $\Phi(\mathcal{G}) \succ 0$ is not an LMI due to the terms μW and $\lambda \varepsilon \mu^{-1}$. Nevertheless, once μ is fixed a priori, the resulting LMI can easily be solved. A gridding on the parameter μ should be included to the LMI solver.

IV. Local data-driven control design

In the previous section, the model $\mathcal{G}$ was assumed to be constant and known. However, as $\Psi(\mathcal{G})$ is convex with respect to $\mathcal{G}$, i.e. the system's matrices A and B, Theorem 1 enables the extension of the condition to the case of uncertain (polytopic or norm-bounded) systems, following the classical manipulations. In this section, we will now assume that $\mathcal{G}$ is unknown and possibly time-varying. Therefore, the objective aims to find a method to guarantee that the conditions of Theorem 1 holds for all $\mathcal{G}$ that belongs to an unknown and bounded set $\mathcal{Z} \subset \mathbb{R}^{n_x \times (n_x + n_u)}$. Unfortunately, such an extension requires additional information on as $\mathcal{Z}$, to be stated and solved properly. The motivation of this section is first to use several data experiments to estimate the set of uncertainties. In a second step, the objective is to use a generic transformation to adapt the previous model-based theorem to a data-based one. After presenting the main features and assumptions on the data experiments, we introduce a lemma to achieve this transformation in a systematic manner and to exhibit the main result of this paper.

A. Data collections and assumption

Unlike the usual situation where the system model is available, our objective is here to formulate a data-driven design result. To do this, let us first specify the notion of data, we are considering here. We define the data collection $\mathcal{D}$ as follows

$$\mathcal{D} := (\mathcal{X}^+, \mathcal{X}, \mathcal{U}) \in \mathbb{R}^{n_x \times p} \times \mathbb{R}^{n_x \times p} \times \mathbb{R}^{n_u \times p}. \qquad (16)$$

where the matrices that collect the available measurements for the control design are given by

$$\begin{array}{rcl} \mathcal{X}^+ & := & [\ x_1^+ \quad x_2^+ \quad \dots \quad x_p^+\ ], \\ \mathcal{X} & := & [\ x_1 \quad x_2 \quad \dots \quad x_p\ ], \\ \mathcal{U} & := & [\ \mathrm{sat}(u_1) \quad \mathrm{sat}(u_2) \quad \dots \quad \mathrm{sat}(u_p)\ ], \end{array} \qquad (17)$$

Let us formulate the first assumption on the data, which refers to the persistence of excitation or the informativity of the data

Assumption 2 (Data Informativity): The data collection $\mathcal{D}$ in (17) is such that matrix $[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}][\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}]^\top$ is non singular.

This assumption is weak since, if it is not satisfied, it suffices to consider additional data to complete it. For any pair of matrices A and B, let us define the matrix $\mathcal{W}$, given by

$$\mathcal{W} := \mathcal{X}^+ - A\mathcal{X} - B\mathcal{U} \quad \in \mathbb{R}^{n_x \times p}. \qquad (18)$$

Matrix $\mathcal{W}$ collects the error between the measurements $\mathcal{X}^+$ and the expected values of the data, i.e. $A\mathcal{X} + B\mathcal{U}$, built with some matrices A and B. Of course, the matrix depends on the available data and on the selection of A and B.

Following the principles of the least squares approximation and if Assumption 2 is verified, the optimal pair of matrices A and B that minimizes the norm of $\mathcal{W}$ is the solution of the least squares optimization given by $\mathbb{G} := \mathcal{X}^+ [\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}]^\top \left([\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}][\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}]^\top\right)^{-1}$. Then, the computation of the optimal value of $\|\mathcal{W}^*\|^2$ obtained by selecting $[A\ B] = \mathbb{G}$, we get

$$\|\mathcal{W}^*\|^2 = \|\mathcal{X}^+ - \mathbb{G}[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}]\|^2 = \|\mathcal{X}^+ \mathbb{P}\|^2 = \mathcal{X}^+ \mathbb{P} \mathcal{X}^{+\top}.$$

where $\mathbb{P}$ is the symmetric projection matrix given by

$$\mathbb{P} = I_p - [\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}]^\top \left([\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}][\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}]^\top\right)^{-1} [\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}], \qquad (19)$$

which verifies $\mathbb{P}^2 = \mathbb{P}\mathbb{P}^\top = \mathbb{P}$. The following lemma holds

Lemma 2: For any $\sigma \geq 1$, the set $\mathcal{C}_\sigma$, given by

$$\mathcal{C}_\sigma := \{[A\ B], \|\mathcal{X}^+ - [A\ B][\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}]\|^2 \preceq \sigma \mathcal{X}^+ \mathbb{P} \mathcal{X}^{+\top}\} \subset \mathbb{R}^{n_x \times (n_x + n_u)},$$

with $\mathbb{P}$ given in (19), is nonempty.

Proof. Following the previous calculations related to the least squares optimization, it holds

$$\mathcal{X}^+ \mathbb{P} \mathcal{X}^{+\top} = \left\|\mathcal{X}^+ - \mathbb{G}[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}]\right\|^2 \preceq \left\|\mathcal{X}^+ - [A\ B][\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}]\right\|^2 \preceq \sigma \mathcal{X}^+ \mathbb{P} \mathcal{X}^{+\top}.$$

Therefore, set $\mathcal{C}_\sigma$ is nonempty since $\sigma \geq 1$ and contains, at least, the least squares estimate $\mathbb{G}$. $\square$

Remark 2: Note that in the case of a perfect LTI system, the data experiments are such that $\mathcal{X}^+ \mathbb{P} \mathcal{X}^{+\top} = 0$. This means that $\mathcal{C}_\sigma$ captures inherently the uncertainties in the data, independently of the values of σ.

Recall that matrix $\mathbb{G}$ corresponds to the best values of $[A\ B]$ that minimizes $\|\mathcal{W}^*\|^2$. It does not mean that the dynamics of the system follows the linear dynamics (5) with $[A\ B] =$

$\mathbb{G}$. To include robustness to the analysis, we will introduce the following assumption, which allows considering pairs of matrices $[A\ B]$ that are close to $\mathbb{G}$ in the sense of the norm of the approximation errors. This assumption is stated below.

Assumption 3: Assume that the data verifies Assumption 2 and that there exists a known scalar $\sigma \geq 1$ such that $\mathcal{Z} \subset \mathcal{C}_\sigma$.

B. Matrix-constrained Relaxation

The following lemma, which is the main brick of the paper, presents a generic method to transform a problem of a particular matrix inequality which depends on parameters verifying a quadratic constraint into a formulation that is independent of these parameters. It is stated below.

Lemma 3: For given positive integers n_1, n_2, n_3, and given matrices $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3)$ in $\mathbb{S}_+^{n_1} \times \mathbb{R}^{n_1 \times n_2} \times \mathbb{S}_+^{n_3}$ and $(\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3)$ in $\mathbb{S}^{n_3} \times \mathbb{R}^{n_3 \times n_2} \times \mathbb{S}_+^{n_3}$, the following statements are equivalent

(i) Inequality

$$\mathcal{M}(\mathcal{A}) = \begin{bmatrix} \mathcal{M}_1 & \mathcal{M}_2 \mathcal{A}^\top \\ * & \mathcal{M}_3 \end{bmatrix} \succ 0, \quad \forall \mathcal{A} \in \Sigma_\mathcal{N}, \qquad (20)$$

holds true, where $\Sigma_\mathcal{N} \subset \mathbb{R}^{n_3 \times n_2}$ represents the set of allowable uncertain matrices $\mathcal{A}$ characterized by

$$\Sigma_\mathcal{N} := \left\{ \mathcal{A} \in \mathbb{R}^{n_3 \times n_2}, \begin{bmatrix} I_{n_3} \\ \mathcal{A}^\top \end{bmatrix}^\top \begin{bmatrix} \mathcal{N}_1 & \mathcal{N}_2 \\ * & \mathcal{N}_3 \end{bmatrix} \begin{bmatrix} I_{n_3} \\ \mathcal{A}^\top \end{bmatrix} \preceq 0 \right\}. \qquad (21)$$

(ii) There exists $\eta > 0$ such that

$$\begin{bmatrix} \mathcal{M}_1 & \mathbf{0}_{n_1, n_3} & \mathcal{M}_2 \\ * & \mathcal{M}_3 + \eta \mathcal{N}_1 & \eta \mathcal{N}_2 \\ * & * & \eta \mathcal{N}_3 \end{bmatrix} \succ 0. \qquad (22)$$

The proof can be found in [19, Lemma 1] and is therefore omitted. Lemma 3 provides an alternative formulation in robust analysis for uncertain matrices subject to quadratic constraints of the form (21) compared to the one presented in [23], [24]. For the sake of consistency, the S-Lemma provided in [23] is recalled in the following lemma.

Lemma 4: [23, Th.9] Let $\mathcal{M}_s, \mathcal{N}_s \in \mathbb{R}^{(n_s+m_s)\times(n_s+m_s)}$ be symmetric matrices and assume that inequality $\begin{bmatrix} I_{n_s} \\ \mathcal{A}_s^\top \end{bmatrix}^\top \mathcal{N}_s \begin{bmatrix} I_{n_s} \\ \mathcal{A}_s^\top \end{bmatrix} \succeq 0$ for at least one matrix $\mathcal{A}_s \in \mathbb{R}^{n_s \times m_s}$. Then the next statements are equivalent:

(i) $\begin{bmatrix} I_{n_s} \\ \mathcal{A}_s^\top \end{bmatrix}^\top \mathcal{M}_s \begin{bmatrix} I_{n_s} \\ \mathcal{A}_s^\top \end{bmatrix} \succ 0, \ \forall \mathcal{A}_s \in \Sigma_{\mathcal{N}_s}$, where set $\Sigma_{\mathcal{N}_s}$ has the same definition as in (21) but replacing $\mathcal{A}$ and $\mathcal{N}$ by $\mathcal{A}_s$ and $\mathcal{N}_s$, respectively.

(ii) There exists $\eta > 0$ such that $\mathcal{M}_s - \eta \mathcal{N}_s \succ 0$.

Note that both lemmas address the problem of the satisfaction of an inequality subject to uncertain matrices characterized by a quadratic constraint. The main interest of both lemmas is to derive equivalent inequalities that are independent of the uncertain matrix $\mathcal{A}$ (or $\mathcal{A}_s$), resulting from the applications of the Schur Complement, the Finsler's lemma and the S-procedure. Despite their similarities, both lemmas have substantial differences. First, Lemma 4 requires that matrix $\mathcal{M}_s$ has the same size as $\mathcal{N}_s$, the matrix that characterizes the quadratic constraint on $\mathcal{A}$. Lemma 3 is more flexible in this sense, as there is no relationship between matrices $\mathcal{M}_1$ and $\mathcal{N}$, which are independent. This flexibility has the benefit of reducing the initial manipulations to derive, from usual stability or control problems, the appropriate expressions of $\mathcal{M}_i$ and $\mathcal{N}_i$, for $i = 1, 2, 3$, to fit the framework of Lemma 4. In fact, the relationship between both lemmas can be seen by selecting $n_s = n_1 + n_3$, $m_s = n_2$, $\mathcal{A}_s^\top = \begin{bmatrix} 0_{n_2 \times n_3} & \mathcal{A}^\top \end{bmatrix}$ and

$$\mathcal{M}_s = \begin{bmatrix} \mathcal{M}_1 & 0 & \mathcal{M}_2 \\ * & \mathcal{M}_3 & 0 \\ * & * & 0 \end{bmatrix}, \quad \mathcal{N}_s = -\begin{bmatrix} 0 & 0 & 0 \\ * & \mathcal{N}_1 & \mathcal{N}_2 \\ * & * & \mathcal{N}_3 \end{bmatrix}.$$

From this selection, it is clear that item (ii) of Lemma 4 is equivalent to item (ii) of Lemma 3, showing that Lemma 3 is a particular case of Lemma 4. That being said, the main advantage of Lemma 3 is that the structure of $\mathcal{M}(\mathcal{A})$ arises in many LMI problems in control as it will be shown hereafter. It avoids enforcing an initial LMI problem to fit with the structure of Lemma 4, which is not an easy task in general.

Remark 3: A similar lemma was already presented in [19], where block $\mathcal{M}_2 \mathcal{A}^\top$ is extended to $\bar{\mathcal{M}}_2 + \mathcal{M}_2 \mathcal{A}^\top$, where $\bar{\mathcal{M}}_2$ is a term that is independent of the uncertain matrix. In addition, a deeper discussion on the similarities with the existing lemmas from the literature has been proposed therein.

C. Data-driven local stabilization of saturated systems

This section provides a new contribution on the data-based design of stabilizing control law for linear systems subject to input saturation. The method is highly inspired from [23] but has been reformulated to get a simpler and user-friendly formulation. In this section, we will demonstrate how Lemma 3 can be easily applied to the stabilization problem of saturated systems.

Theorem 2: Under Assumptions 1 and 3, i.e. for a given matrix $\Delta_w > 0$, and for given $\mu \in (0, 1)$, $\alpha_1 > 0$ and $\alpha_2 > 0$, assume that there exist

$$\begin{array}{rcl} \mathcal{D}_\mathcal{V}^2 & := & \{\varepsilon, \eta, W, S, Y, Z\} \\ & \in & \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \mathbb{S}_+^{n_x} \times \mathbb{D}^{n_x} \times \mathbb{R}^{n_x \times n_u} \times \mathbb{R}^{n_x \times n_u} \end{array}$$

that are solution to the following optimization problem

$$\begin{array}{ll} \max_{\mathcal{D}_\mathcal{V}^2} & \alpha_1 \varepsilon + \alpha_2 \mathrm{Tr}(W) \\ s.t. & \varepsilon > 1, \ \eta > 0, \ \Psi(\mathcal{D}) \succ 0, \ \begin{bmatrix} W & Z_{(i)}^\top \\ * & \bar{u}_i^2 \end{bmatrix} \succ 0, \end{array} \qquad (23)$$

for all $i = 1, \dots, n_u$, where

$$\Psi(\mathcal{D}) = \begin{bmatrix} (1-\mu)W & Y^\top + Z^\top & 0 & W & Y^\top \\ * & 2S & 0 & 0 & S \\ * & * & \Psi_3 & -\eta \mathcal{X}^+ \mathcal{X}^\top & -\eta \mathcal{X}^+ \mathcal{U}^\top \\ * & * & * & \eta \mathcal{X} \mathcal{X}^\top & \eta \mathcal{X} \mathcal{U}^\top \\ * & * & * & * & \eta \mathcal{U} \mathcal{U}^\top \end{bmatrix},$$
$$\Psi_3 = W - \tfrac{\lambda \varepsilon}{\mu} I_{n_x} + \eta \mathcal{X}^+ (I - \sigma \mathbb{P}) \mathcal{X}^{+\top}. \qquad (24)$$

The control law (4) with $K = Y W^{-1}$ ensures that Problems (P1) and (P2) are solved, i.e. the following statements hold:

- When $w = 0$, the ellipsoid $\mathcal{E}(W^{-1}, 1)$ is an approximation of the basin of attraction of the origin for the closed-loop system (1)-(4);

- When $w \in \Omega_\lambda$ and $w \neq 0$, the solutions to the closed-loop system (1)-(4) initialized in $\mathcal{E}(W^{-1},1) \setminus \mathcal{E}(W^{-1},\varepsilon)$ converge to the attractor $\mathcal{E}(W^{-1},\varepsilon)$.
- The estimations of the basin of attraction and of the attractor are optimal with respect to the cost function $\alpha_1\varepsilon + \alpha_2 \mathrm{Tr}(W)$, leading to a compromise between enlarging $\mathcal{E}(W^{-1},1)$ and minimizing $\mathcal{E}(W^{-1},\varepsilon)$.

Proof. Let us first note that, in Theorem 1, conditions $\varepsilon > 1$ and $\begin{bmatrix} W & Z_{(i)}^\top \\ * & \bar{u}_i \end{bmatrix} \succ 0$ do not involve the matrices of the system and thus remains unchanged. On the other side, condition $\Phi(\mathcal{G}) \succ 0$ has to be adapted to remove the system's matrices. To do so, let us note that $\Phi(\mathcal{G})$ is rewritten as follows

$$\Phi(\mathcal{G}) = \begin{bmatrix} \begin{bmatrix} (1-\mu)W & Y^\top + Z^\top \\ * & 2S \end{bmatrix} & \begin{bmatrix} W & Y^\top \\ 0 & S \end{bmatrix} \begin{bmatrix} A^\top \\ B^\top \end{bmatrix} \\ * & W - \frac{\lambda\varepsilon}{\mu} I_{n_x} \end{bmatrix},$$

which has the same structure as $\mathcal{M}(\mathcal{A})$ in Lemma 3. Indeed, selecting $n_1 = n_2 = n_x + n_u$, $n_3 = n_x$ and

$$\begin{aligned} &\mathcal{M}_1 = \begin{bmatrix} (1-\mu)W & Y^\top + Z^\top \\ * & 2S \end{bmatrix}, \ \mathcal{M}_2 = \begin{bmatrix} W & Y^\top \\ 0 & S \end{bmatrix}, \\ &\mathcal{M}_3 = W - \tfrac{\lambda\varepsilon}{\mu} I_{n_x}, \ \mathcal{N}_1 = \mathcal{X}^+(I_p - \sigma\mathbb{P})\mathcal{X}^{+\top}, \\ &\mathcal{N}_2 = -\mathcal{X}^+ \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}^\top, \ \mathcal{N}_3 = \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}^\top, \end{aligned}$$

and with the uncertain matrix $\mathcal{A} = \mathcal{G} = \begin{bmatrix} A & B \end{bmatrix} \in \mathcal{C}_\sigma \subset \mathbb{R}^{n_x \times (n_x + n_u)}$. Altogether, the problem can be expressed as in (20) with this set of matrices. Then, the problem resumes to the satisfaction of condition $\Phi(\mathcal{G}) \succ 0$ for all matrices $\mathcal{G} \in \mathcal{Z} \subset \mathcal{C}_\sigma$. Note that $\mathcal{N}_3 \succ 0$ is required in Lemma 3, which refers to the informativity of the data and is also required for the satisfaction of condition $\Psi(\mathcal{D}) \succ 0$. We are thus in position to apply Lemma 3, ensuring that $\Phi(\mathcal{G}) \succ 0$ for all $\mathcal{G} \in \mathcal{C}_\sigma$ is equivalent to the existence of $\eta > 0$ such that $\Psi(\mathcal{D}) \succ 0$. □

Remark 4: The informativity of the data, which refers to the persistence of excitation [26] is a necessary condition for the solvability of the LMI optimization problem. In other words, if matrix $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]^\top$ is singular, then the LMI cannot be solved.

Remark 5: Note that, contrary to [1] or [23], our method does not require to study the dual system involving $(A + BK)^\top$. First, this trick is not possible for nonlinear systems. Second, our method directly treats the initial problem without using this artifice of calculus to avoid a technical problem.

Remark 6: Differently from [3], the additional variable η cannot be "absorbed" by the other decision variables. This is due to the nature of the local stabilization problem.

Remark 7: It is worth noting that the resulting LMI condition is very similar to the ones presented in [3] based on the application of the Petersen's lemma for the case of linear systems. Indeed, selecting any $S \succ 0$, and $Z = 0$ and $\mu = \varepsilon = 0$, the same LMI condition as the one of Th.2 in [3] is retrieved. The advantage of Lemma 3 is that the structure of many LMI problems in control fits with the structure in Lemma 3, so that no additional manipulation is required.

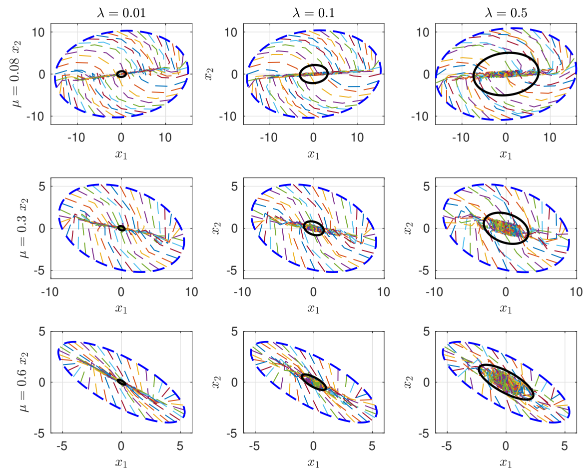


Fig. 1: Estimations of the basin of attraction and of the attractor obtained by solving the model-based optimization problem (9) for various λ and μ.

V. Numerical applications

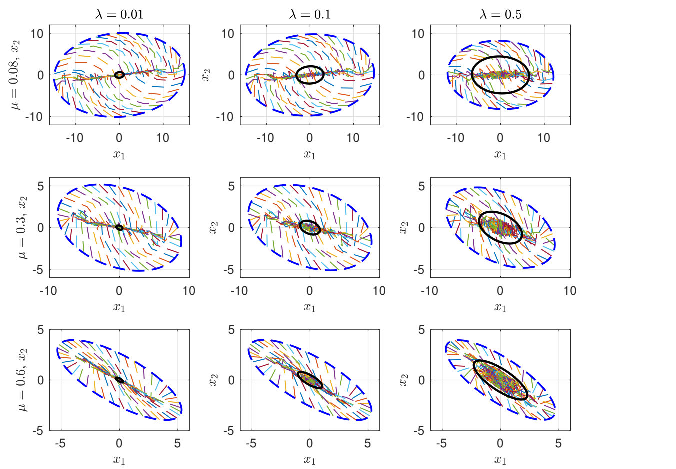
A. Example 1

Consider the discrete-time systems (5) borrowed from [22], [25], with the following matrices

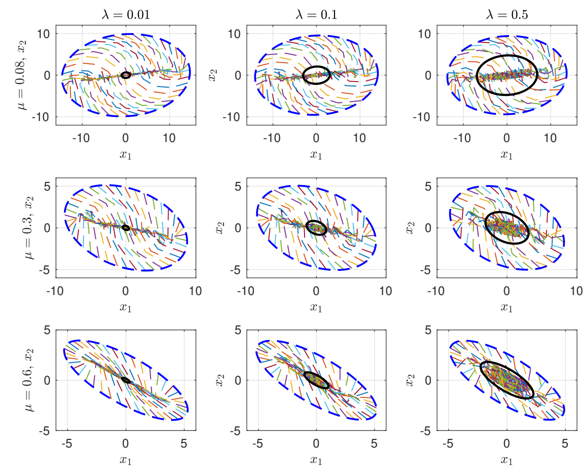
$$A = \begin{bmatrix} 0.8 & 0.5 \\ -0.4 & 1.2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \bar{u}_1 = 5. \tag{25}$$

1) Model-based solution of Theorem 1: Solving the optimization problem (9) with $\alpha_1 = 1$ and $\alpha_2 = 10^{-3}$ for $\lambda = 0.01, 0.1$ and 0.5 and $\mu = 0.08, 0.3$ and 0.6. Figure 1 shows the estimations of the basin of attraction (dashed blue line) and of the attractor (black line) for all combinations of (λ, μ). In addition, each sub-figure also depicts 40 simulations that are initiated at the boundary of $\mathcal{E}(W^{-1},1)$, which all converge the attractor $\mathcal{E}(W^{-1},\varepsilon)$. For the particular case $(\lambda,\mu) = (0.5, 0.6)$, we have obtained $W = \left[\begin{smallmatrix} 28.963 & -15.869 \\ -15.869 & 15.791 \end{smallmatrix}\right]$ and $\varepsilon = 5.389$.

One can observe on Figure 1 that increasing the magnitude of the noise (i.e., λ), increases the size of the attractor, while the size of approximation of the basin of attraction remains almost the same. Interestingly, Figure 1 shows that the tuning parameter μ has an important effect on the solution. Indeed, for small values of μ, the approximation of the basin of attraction is larger, but the approximation of the attractor is rather poor, since the black ellipsoids are very large compared to the chattering around the origin. Reversely, when μ is large, the optimization process provides a more accurate approximation of the attractor but at the price of drastically reducing the size of the approximation of the basin of attraction. This shows that the tuning parameter μ plays the role of the balance between the optimization of estimations of the basin of attraction and of the attractor, which cannot be performed simultaneously. This compromise can be expected from the LMI condition, since having $(1-\mu)$ small, implies that W has to be large (block $(1,1)$ of $\Phi(\mathcal{G})$) and having μ small implies that $W\varepsilon^{-1}$ has to be large (block $(3,3)$ of $\Phi(\mathcal{G})$).



(a) Simulations obtained with a collection of $p = 10$ data.



(b) Simulations obtained with a collection of $p = 20$ data.

Fig. 2: Estimations of the basin of attraction and of the attractor obtained by solving the data-based optimization problem (9) for various λ and μ.

2) Data-driven local stabilization: This section illustrates the impact of the data-driven approach compared to the model-driven one. Let us first present the construction of the data. Following the dynamics of the system, we have selected $p = 10$ and $p = 20$ random values of x_i and u_i in (17) where each component of x_i and u_i lies in $[-10, 10]$. Then, we have created the random noise signal w_i, each vector satisfying Assumption 1, and we have built matrix $\mathcal{X}^+$ using equation (1) with matrices A and B given in (25). The resulting values of $\mathcal{X}^+\mathbb{P}\mathcal{X}^{+\top}$ obtained for the particular case $\lambda = 0.5$ are

$$\mathcal{X}^+\mathbb{P}\mathcal{X}^{+\top} = \begin{bmatrix} 0.9051 & -0.3857 \\ -0.3857 & 1.9002 \end{bmatrix} \text{ and } \begin{bmatrix} 2.9838 & -0.8361 \\ -0.8361 & 2.1481 \end{bmatrix}$$

for $p = 10$ and 20, respectively.

More precisely, Figures 2a and 2b illustrate the results obtained by solving the data-driven stabilization conditions of Theorem 2 with $p = 10$ and $p = 20$ experiments, respectively with $\sigma = 1.1$. As for the model-based condition of Theorem 1, increasing the magnitude of the noise leads to a reduction of the size of the estimated basin of attraction (dashed blue ellipsoids), and also an increase of the size of the estimated attractor (black ellipsoids).

Compared to the model-driven results in Figure 1, it occurs a notable reduction of both ellipsoids for each case. This is due to the fact that the model-based solution has no uncertainties in the matrices A and B, while the data-driven condition includes the inherent uncertainties due to the noise affecting the data.

Comparing now both data-driven result, one can see that each estimation of the basin of attraction obtained for $p = 10$ are smaller than the one derived with $p = 20$. This can be interpreted by the fact that increasing the number of data experiments give more information on the system so that the uncertainties due to the noise are reduced. In other words the set of allowable matrices in $\mathcal{C}_\sigma$ becomes "smaller" as the number of data increases.

B. Example 2: Spacecraft Rendezvous problem

To illustrate our theoretical contributions to a more complex and more practical example, consider the problem of spacecrafts rendezvous and more particularly the case of a target evolving in a *circular* Keplerian orbit and the approaching vehicle (chaser) is close to the target. This problem can be efficiently modelled by the linear Hill-Clohessy-Wiltshire (HCW) equations, introduced in [13] and [7], which describes the relative position of the spacecraft. Following [20], the planar dynamics in the relative plan (x, y) is driven by the discrete-time system (1), where $x = [r_x \ v_x \ r_y \ v_y]^\top$ is the state vector, which gathers the relative positions (r_x, r_y) and velocities (v_x, v_y), and the impulsive control inputs $u = [u_x \ u_y]$ and with matrices $A = e^{A_0 T_0}$ and $B = e^{A_0 T_0} B_0$, where $T_0 = 1s$ represents the period of the impulses. The matrices defining the impulsive motion are given by

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3n^2 & 0 & 0 & 2n \\ 0 & 0 & 0 & 1 \\ 0 & -2n & 0 & 0 \end{bmatrix}, \ B_0 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The HCW model assumes that the target vehicle is passive and moving along a circular orbit of radius R. For a typical orbit at an altitude of $R = 500km$, we would get $n = 0.0011$. In this context, reducing the consumption of the control action is a crucial issue for safety mission. In addition, the impulses are of limited amplitude. Moreover, due to the possible impression on the sampling period, or assuming that the target is not following a perfect circular orbit at 500km, the model may suffer from disturbance and uncertainties. Altogether, this motivates the application of a data-driven control design. To compensate for these uncertainties, we collect $N = 50$ data presented as in (17) with random matrices of $\mathcal{X}$ and $\mathcal{U}$ such that some columns of $\mathcal{U}$ includes saturated values and arbitrary values of w verifying Assumption 1 for $\lambda = 0.005$, which is of the same range as n. Here, we consider that Assumption 3 is fulfilled with $\sigma = 1.1$. The optimization problem (23) with

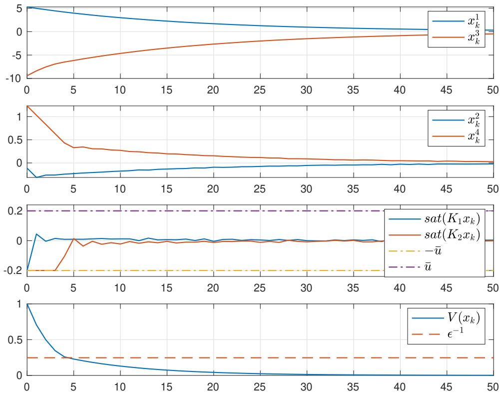


Fig. 3: Graphs showing from top to bottom, the relative positions r_x, r_y, the relative velocities v_x, v_y, the two saturated control inputs u_x, u_y, and the Lyapunov function.

$\alpha_1 = 10$ and $\alpha_2 = 1$, $\lambda = 0.005$ and $\mu = 0.1$ yields

$$K = \begin{bmatrix} -0.0771 & -1.4047 & 0.0012 & -0.0018 \\ -0.0007 & 0.0028 & -0.0779 & -1.4057 \end{bmatrix},$$
$$W = \begin{bmatrix} 250.8938 & -13.7299 & 9.2323 & -0.6880 \\ -13.7299 & 1.8397 & -0.2889 & 0.0254 \\ 9.2323 & -0.2889 & 239.4707 & -13.2245 \\ -0.6880 & 0.0254 & -13.2245 & 1.8176 \end{bmatrix}.$$

The simulation results are depicted in Figure 3, which shows the trajectories of the system with disturbances w verifying Assumption 1, and with the initial condition $x_0 = \begin{bmatrix} 5.2813 & -0.1126 & -9.4150 & 1.2316 \end{bmatrix}^\top$, at the boundary of the ellipsoid $\mathcal{E}(W^{-1}, 1)$. Figure 3 shows that the resulting trajectory converges asymptotically to $\mathcal{E}(W^{-1}, 3.05^{-1})$, despite the saturation of both control inputs, demonstrating the efficiency of our method even for systems with multiple input.

VI. Conclusion

This paper addressed the problem of providing a data-driven solution to the local stabilization of linear systems subject to input saturation. Using model-based solution to this well-studied problem, a systematic method to transform model-driven into data-driven LMI conditions is presented thanks to some adequate rewriting of the conditions. Although this technical solution is shown to be equivalent to some recent advanced results, its main advantage relies on its simplicity and its potential to be applicable to a broad class of problems of stabilization of (non)linear discrete-time systems.

The proposed results pave the way for future work, as in particular the possibility to consider more complex dynamics or isolated non-linearities.

References

[1] J. Berberich, A. Koch, C.W. Scherer, and F. Allgöwer. Robust data-driven state-feedback design. In *IEEE American Control Conference*, pages 1532–1538. IEEE, 2020.

[2] J. Berberich, J. Köhler, M. A Müller, and F. Allgöwer. Data-driven model predictive control with stability and robustness guarantees. *IEEE Trans. on Automatic Control*, 66(4):1702–1717, 2020.

[3] A. Bisoffi, C. De Persis, and P. Tesi. Data-driven control via Petersen's lemma. *Automatica*, 145:110537, 2022.

[4] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*. Society for Industrial and Applied Mathematics, 1997.

[5] V. Breschi, C. De Persis, S. Formentin, and P. Tesi. Direct data-driven model-reference control with Lyapunov stability guarantees. In *IEEE Conference on Decision and Control*, pages 1456–1461. IEEE, 2021.

[6] V. Breschi, L. Zaccarian, and S. Formentin. Data-driven stabilization of input-saturated systems. *IEEE Control Systems Letters*, 2023.

[7] W. H. Clohessy and R. S. Wiltshire. Terminal guidance systems for satellite rendezvous. *Journal of the Aerospace Sciences*, 27(9):653–658, 1960.

[8] T. Dai and M. Sznaier. A convex optimization approach to synthesizing state feedback data-driven controllers for switched linear systems. *Automatica*, 139:110190, 2022.

[9] C. De Persis and P. Tesi. Formulas for data-driven control: Stabilization, optimality, and robustness. *IEEE Trans. on Automatic Control*, 65(3):909–924, 2019.

[10] W. D'Amico and M. Farina. Data-based control design for linear discrete-time systems with robust stability guarantees. In *IEEE Conference on Decision and Control*, pages 1429–1434, 2022.

[11] Y. Ebihara, D. Peaucelle, and D. Arzelier. *S-variable approach to LMI-based robust control*, volume 6. Springer, 2015.

[12] J.M. Gomes Da Silva and S. Tarbouriech. Antiwindup design with guaranteed regions of stability: an LMI-based approach. *IEEE Trans. on Automatic Control*, 50(1):106–111, 2005.

[13] G.W. Hill. Researches in lunar theory. *American Journal of Mathematics*, 1(3):5–26, 129–147, 245–260, 1878.

[14] Z.S. Hou and Z. Wang. From model-based control to data-driven control: Survey, classification and perspective. *Information Sciences*, 235:3–35, 2013.

[15] T. Hu and Z. Lin. *Control systems with actuator saturation: analysis and design*. Birkhauser, Boston, 2001.

[16] D. Piga, S. Formentin, and A. Bemporad. Direct data-driven control of constrained systems. *IEEE Trans. on Control Systems Technology*, 26:1422–1429, 2018.

[17] I. Postlethwaite, M.C. Turner, and G. Herrmann. Robust control applications. *Annual Reviews in Control*, 31(1):27–39, 2007.

[18] C.W. Scherer. *Theory of robust control*. Citeseer, 2001.

[19] A. Seuret, C. Albea, and F. Gordillo. LMI relaxations and its application to data-driven control design for switched affine systems. *Int. Journal of Robust and Nonlinear Control*, 33(12):6597–6618, 2023.

[20] A. Seuret and S. Tarbouriech. Data-driven event-triggering mechanism for linear systems subject to input saturations. In *European Control Conference*. EUCA, 2024.

[21] S. Tarbouriech, G. Garcia, J.M. Gomes da Silva Jr., and I. Queinnec. *Stability and Stabilization of Linear Systems with Saturating Actuators*. Springer, 2011.

[22] S. Tarbouriech and J.M. Gomes Da Silva Jr. Admissible polyhedra for discrete-time linear systems with saturating controls. In *IEEE American Control Conference*, volume 6, pages 3915–3919. IEEE, 1997.

[23] H.J. Van Waarde, M.K. Camlibel, and M. Mesbahi. From noisy data to feedback controllers: non-conservative design via a matrix S-lemma. *IEEE Trans. on Automatic Control*, 67(1):162 – 175, 2022.

[24] H.J. Van Waarde, J. Eising, H.L. Trentelman, and M.K. Camlibel. Data informativity: a new perspective on data-driven analysis and control. *IEEE Trans. on Automatic Control*, 65(11):4753–4768, 2020.

[25] M. Vassilaki, J.C. Hennet, and G. Bitsoris. Feedback control of linear discrete-time systems under state and control constraints. *International Journal of control*, 47(6):1727–1735, 1988.

[26] J.C. Willems, P. Rapisarda, I. Markovsky, and B.L.M. De Moor. A note on persistency of excitation. *Systems & Control Letters*, 54(4):325–329, 2005.