



HAL
open science

An Equivalence Class of Gradual Semantics

Leila Amgoud, Vivien Beuselinck

► **To cite this version:**

Leila Amgoud, Vivien Beuselinck. An Equivalence Class of Gradual Semantics. 7th European Conference on Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU 2023), Sep 2023, Arras, France. pp.95-108, <10.1007/978-3-031-45608-4_8>. <hal-04210432v2>

HAL Id: hal-04210432

<https://laas.hal.science/hal-04210432v2>

Submitted on 22 Oct 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

An Equivalence Class of Gradual Semantics

Leila Amgoud¹[0000–0002–1838–4271] and Vivien Beuselinck²[0000–0002–5722–7025]

¹ CNRS – IRIT, France

amgoud@irit.fr

² Toulouse University, France

vivien@beuselinck.fr

Abstract. Assessing argument strength is a key step in an argumentation process, therefore a plethora of methods, called *semantics*, have been developed for the purpose. Their comparison can be carried out using two complementary formal tools: the formal properties they satisfy and the ranking they produce.

It has been shown that the two gradual semantics Mbs and EMbs are *strongly equivalent*, i.e., they return the same ranking of arguments when applied to flat argumentation graphs. This paper goes even further by characterizing the whole equivalence class to which they belong. It shows that every instance of the class is based on a numerical series and *refines* the ranking provided by the grounded semantics. We discuss an instance, hence novel semantics, of the class.

Keywords: Argumentation · Semantics · Equivalence.

1 Introduction

Argumentation is a reasoning approach based on the justification of claims by arguments (see [8, 21, 20] for more on its multiple applications). An argumentation system is made of an *argumentation graph* and a *semantics*. The former is a graph whose nodes are arguments and edges represents attacks, i.e., conflicts. The latter is a formal method, called *semantics*, that is used to assess *argument strength*. Several methods have been proposed in the literature; they can be classified into three families: *extension* semantics that have been initiated in [15], *gradual* semantics introduced in [14] and *ranking* semantics defined in [2] (see [1] for a discussion on their differences and applications).

Several efforts have been made in the last ten years to understand the theoretical foundations of semantics and to compare the plethora of existing methods. For this purpose, two complementary approaches have been investigated. The first consists in defining formal properties and comparing methods based on these properties (see [3, 4, 7, 9, 10, 12, 13] for more on properties). We have shown in [5] that satisfying the same properties does not mean that semantics evaluate in a similar way arguments of a graph. Hence, we proposed the second approach which compares the rankings of semantics. We have shown that the two gradual semantics Mbs ([4]) and EMbs ([6]) are *strongly equivalent*, i.e., they produce the same ranking when applied to flat graphs.

This paper contributes to the understanding of the mathematical foundations of gradual semantics. It provides a *characterization* of the whole *equivalence class* to which belong Mbs and EMbs. We show that every instance of the class is based on a

certain type of numerical series, which are composed of two sub-series: an increasing and a decreasing one. Every instance of the class *refines* the ranking provided by the grounded semantics proposed in [15]. Indeed, it keeps its strict comparisons and breaks some ties. We discuss an instance, hence novel semantics, of the class.

The paper is structured as follows: Section 2 is devoted to background, Section 3 introduces the equivalence class whose instance is discussed in Section 4. The last section is devoted to some concluding remarks and perspectives.

2 Background

Throughout the paper, we focus on *flat argumentation graphs*, i.e., graphs whose nodes are arguments and edges represent *attacks* (i.e. conflicts) between arguments. We denote by Args the universe of all arguments.

Definition 1 (AG). An argumentation graph (AG) is a tuple $\mathbf{G} = \langle \mathcal{A}, \mathcal{R} \rangle$ where \mathcal{A} is a finite subset of Args and $\mathcal{R} \subseteq \mathcal{A} \times \mathcal{A}$. We denote by AG the set of all argumentation graphs built from Args .

Argument strength is assessed using formal methods, called *semantics*. There are three families of semantics in the literature: i) *extension* semantics that identify acceptable sets of arguments [15], ii) *gradual* semantics that assign a value from a given ordered scale to each argument [14], and iii) *ranking* semantics that rank-order arguments [2]. Below we focus on the second family and consider the unit interval as scale.

Definition 2 (Gradual Semantics). A gradual semantics is a function σ assigning to any $\mathbf{G} = \langle \mathcal{A}, \mathcal{R} \rangle \in \text{AG}$ a weighting $\text{Deg}_{\mathbf{G}}^{\sigma} : \mathcal{A} \rightarrow [0, 1]$. For any $a \in \mathcal{A}$, $\text{Deg}_{\mathbf{G}}^{\sigma}(a)$ represents the strength of a . We denote by Sem the set of all possible gradual semantics.

Examples of gradual semantics are Mbs and EMbs which were defined in [4] and [6] respectively for *weighted graphs*, i.e., graphs where arguments have initial weights. Their definitions in case of a flat graph $\mathbf{G} = \langle \mathcal{A}, \mathcal{R} \rangle$ are as follows: For any $a \in \mathcal{A}$,

$$\text{Deg}_{\mathbf{G}}^{\text{Mbs}}(a) = \frac{1}{1 + \max_{b \mathcal{R} a} \text{Deg}_{\mathbf{G}}(b)}$$

$$\text{Deg}_{\mathbf{G}}^{\text{EMbs}}(a) = e^{-\max_{b \mathcal{R} a} \text{Deg}_{\mathbf{G}}^{\text{EMbs}}(b)}$$

Let us now recall some basic notions on an arbitrary binary relation \succeq on a set X of objects, which may be arguments, criteria, . . .

- For any $x \in X$, $x \succeq x$ (Reflexivity)
- For all $x, y, z \in X$, if $x \succeq y$ and $y \succeq z$, then $x \succeq z$ (Transitivity)
- For all $x, y \in X$, $x \succeq y$ or $y \succeq x$ (Total)

The notation $x \succ y$ is a shortcut for $x \succeq y$ and $y \not\succeq x$, and $x \approx y$ is a shortcut for $x \succeq y$ and $y \succeq x$. If a binary relation \succeq is reflexive and transitive, then it is called a *preordering*; the latter is *total* if \succeq is total.

The values assigned by a gradual semantics to arguments of an argumentation graph can be used to rank the arguments from strongest to weakest as follows.

Definition 3 (Ranking). Let $\sigma \in \text{Sem}$ and $\mathbf{G} = \langle \mathcal{A}, \mathcal{R} \rangle \in \text{AG}$. A ranking induced from σ is a binary relation $\succeq_{\mathbf{G}}^{\sigma}$ on \mathcal{A} such that for all $a, b \in \mathcal{A}$,

$$a \succeq_{\mathbf{G}}^{\sigma} b \text{ iff } \text{Deg}_{\mathbf{G}}^{\sigma}(a) \geq \text{Deg}_{\mathbf{G}}^{\sigma}(b).$$

Property 1. For all $\sigma \in \text{Sem}$ and $\mathbf{G} = \langle \mathcal{A}, \mathcal{R} \rangle \in \text{AG}$, $\succeq_{\mathbf{G}}^{\sigma}$ is a total preordering.

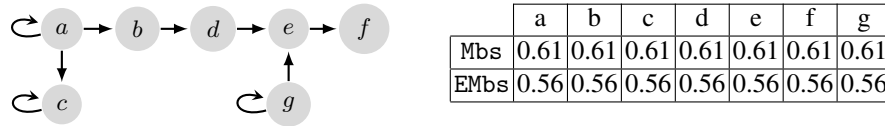
Example 1. Consider the argumentation graph \mathbf{G}_1 depicted in the figure below:



For any $x \in \{\text{Mbs}, \text{EMbs}\}$, we have the following ranking of the seven arguments:

$$a \succ^x d \succ^x f \succ^x g \succ^x e \succ^x b \approx^x c.$$

Example 2. Consider now the following argumentation graph \mathbf{G}_2 .



For any $x \in \{\text{Mbs}, \text{EMbs}\}$, we have the following ranking of the seven arguments:

$$a \approx^x b \approx^x c \approx^x d \approx^x e \approx^x f \approx^x g.$$

We proposed in [5] a novel approach to understand *links* between semantics. It consists of comparing the rankings induced by pairs of semantics. Two notions were particularly investigated: *refinement* and *strong equivalence*. A semantics refines another if it agrees with its strict comparisons of arguments and may break some of its ties. Two semantics are strongly equivalent if they produce the same ranking.

Definition 4. Let $\sigma_1, \sigma_2 \in \text{Sem}$.

- σ_1 refines σ_2 iff $\forall \mathbf{G} \in \text{AG}, \succ_{\mathbf{G}}^{\sigma_2} \subseteq \succ_{\mathbf{G}}^{\sigma_1}$.
- σ_1 and σ_2 are strongly equivalent iff $\forall \mathbf{G} \in \text{AG}, \succeq_{\mathbf{G}}^{\sigma_1} = \succeq_{\mathbf{G}}^{\sigma_2}$.

Property 2 ([5]). The following properties hold:

- Let $\sigma_1, \sigma_2, \sigma_3 \in \text{Sem}$. If σ_1, σ_2 are strongly equivalent and σ_1 refines σ_3 , then σ_2 refines σ_3 .
- Mbs and EMbs are strongly equivalent.

Below is a list of notations that are used in the rest of the paper.

Notations: Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R} \rangle \in \text{AG}$ and $\mathcal{E} \subseteq \mathcal{A}$. We denote by \mathcal{E}^+ the set $\{a \in \mathcal{A} \mid \exists b \in \mathcal{E} \text{ s.t. } (b, a) \in \mathcal{R}\}$ and $\mathcal{E}^o = \mathcal{A} \setminus (\mathcal{E} \cup \mathcal{E}^+)$. Let \succeq be a preordering on a set X and $X_1, X_2 \subseteq X$. We abuse notation and write $X_1 \succ X_2$ (resp. $X_1 \approx X_2$) iff $\forall x \in X_1, \forall y \in X_2, x \succ y$ (resp. $x \approx y$).

3 Equivalence Class of Semantics

The notion of strong equivalence structures the universe Sem of gradual semantics into equivalence classes. In this section, we characterize the whole equivalence class which contains Mbs and EMbs. Let us first introduce a family of *numerical series* that are mainly made of two (increasing and decreasing) sub-sequences.

Definition 5 (\mathbf{S}^*). We define \mathbf{S}^* to be the set containing any numerical series $S = (S^n)_{n \geq 1}$ which satisfies the following conditions:

- for any $n \geq 1$, $S^n \in [0, 1]$
- S contains two sub-series $S_1 = (S_1^n)_{n \geq 1}$ and $S_2 = (S_2^n)_{n \geq 1}$ s.t. for any $n \geq 1$:
 - S_1^n is strictly decreasing and S_2^n is strictly increasing
 - $\lim_{n \rightarrow \infty} S_1^n \geq \lim_{n \rightarrow \infty} S_2^n$

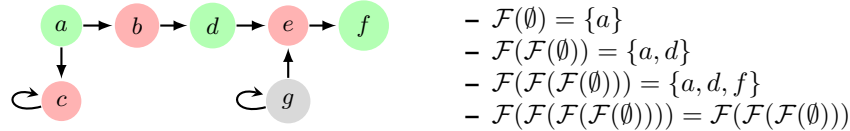
We present next a sequence that is based on the Fibonacci numbers and show that it is a member of the set \mathbf{S}^* .

Proposition 1. Let $\{F^n\}_{n \geq 0}$ be the Fibonacci series where $F^0 = 0, F^1 = 1$, and $F^n = F^{n-1} + F^{n-2}$ for $n > 1$. The series $\{U^n\}_{n \geq 1}$ where $U^n = \frac{F^n}{F^{n+1}}$ is in \mathbf{S}^* . Consequently, $\mathbf{S}^* \neq \emptyset$.

Let us now recall the notion of *defence* between arguments as introduced in [15].

Definition 6 (Defence). Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R} \rangle \in \text{AG}$, $\mathcal{E} \subseteq \mathcal{A}$ and $a \in \mathcal{A}$. We say that \mathcal{E} defends a if $\forall b \in \mathcal{A}$ such that $(b, a) \in \mathcal{R}$, $\exists c \in \mathcal{E}$ such that $(c, b) \in \mathcal{R}$, and define $\mathcal{F}(\mathcal{E}) = \{a \in \mathcal{A} \mid \mathcal{E} \text{ defends } a\}$.³

Example 1 (Cont.) Consider again the flat graph \mathbf{G}_1 .



In what follows, we define a family of gradual semantics that are based on series of the set \mathbf{S}^* . Every instance (i.e., semantics) of the family partitions the set of arguments into three groups. It assigns to arguments of the first group values taken from the decreasing sub-sequence of the series on which it is based. The exact value of an argument depends on the iteration at which it appears for the first time in the set $\text{Gr} = \bigcup_{n \geq 1} \mathcal{F}^n(\emptyset)$. The arguments of the second group are assigned values from the increasing sub-sequence and the value of an argument depends on the first appearance of its strongest attacker in Gr . Finally, the semantics ascribes the same value, which is between the limits of the two sub-sequences, to all arguments of the third group.

³ \mathcal{F} is the so-called *characteristic function* in [15].

Definition 7 (Sem^*). Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R} \rangle \in \text{AG}$. A gradual semantics $\sigma \in \text{Sem}$ based on a series $\mathcal{S} \in \mathbf{S}^*$ is a mapping from \mathcal{A} to $\mathcal{S} \cup \{\delta\}$, with $\lim_{n \rightarrow \infty} \mathcal{S}_1^n \geq \delta \geq \lim_{n \rightarrow \infty} \mathcal{S}_2^n$, such that for any $a \in \mathcal{A}$,

- $\text{Deg}_{\mathbf{G}}^\sigma(a) = \mathcal{S}_1^i$ if $a \in \mathcal{F}^i(\emptyset)$ and $a \notin \bigcup_{j=1}^{i-1} \mathcal{F}^j(\emptyset)$.
- $\text{Deg}_{\mathbf{G}}^\sigma(a) = \mathcal{S}_2^i$ if $\mathcal{F}^i(\emptyset)$ attacks a and $\bigcup_{j=1}^{i-1} \mathcal{F}^j(\emptyset)$ does not attack a .
- $\text{Deg}_{\mathbf{G}}^\sigma(a) = \delta$ otherwise.

We denote by Sem^* the set of all semantics that are based on a series from \mathbf{S}^* .

Example 1 (Cont.) Assume $\sigma \in \text{Sem}^*$ and σ is based on $\mathcal{S} \in \mathbf{S}^*$. According to Definition 7, $\text{Deg}_{\mathbf{G}_1}^\sigma(g) = \delta$ with $\lim_{n \rightarrow \infty} \mathcal{S}_1^n \geq \delta \geq \lim_{n \rightarrow \infty} \mathcal{S}_2^n$.

- | | | |
|---|---|---|
| $- a \in \mathcal{F}^1(\emptyset)$ | $- \text{Deg}_{\mathbf{G}_1}^\sigma(a) = \mathcal{S}_1^1$ | $- \text{Deg}_{\mathbf{G}_1}^\sigma(b) = \mathcal{S}_2^1$ |
| $- d \in \mathcal{F}^2(\emptyset) \setminus \mathcal{F}^1(\emptyset)$ | $- \text{Deg}_{\mathbf{G}_1}^\sigma(d) = \mathcal{S}_1^2$ | $- \text{Deg}_{\mathbf{G}_1}^\sigma(c) = \mathcal{S}_2^1$ |
| $- f \in \mathcal{F}^3(\emptyset) \setminus \bigcup_{i=1}^2 \mathcal{F}^i(\emptyset)$ | $- \text{Deg}_{\mathbf{G}_1}^\sigma(f) = \mathcal{S}_1^3$ | $- \text{Deg}_{\mathbf{G}_1}^\sigma(e) = \mathcal{S}_2^2$ |

Example 2 (Cont.) In the argumentation graph \mathbf{G}_2 , for any $i \in \mathbb{N}$, $\mathcal{F}^i(\emptyset) = \emptyset$. Hence, for any $\sigma \in \text{Sem}^*$ such that σ is based on $\mathcal{S} \in \mathbf{S}^*$, with $\lim_{n \rightarrow \infty} \mathcal{S}_1^n \geq \delta \geq \lim_{n \rightarrow \infty} \mathcal{S}_2^n$, the following holds: $\forall x \in \mathcal{A}_2, \text{Deg}_{\mathbf{G}_2}^\sigma(x) = \delta$ (i.e., all arguments in \mathbf{G}_2 get the value δ).

We show next that the set Sem^* is non-empty as it contains Mbs and EMbs.

Proposition 2. *It holds that $\{\text{Mbs}, \text{EMbs}\} \subseteq \text{Sem}^*$.*

We show below that all semantics in the set Sem^* are pairwise strongly equivalent.

Theorem 1. *For all $\sigma, \sigma' \in \text{Sem}^*$, σ and σ' are strongly equivalent.*

We go one step further by showing that Sem^* is the whole equivalence class as it contains all semantics that are strongly equivalent to Mbs.

Theorem 2. *For any $\sigma \in \text{Sem}$, if σ and Mbs are strongly equivalent, then $\sigma \in \text{Sem}^*$.*

We show next that every semantics of the class Sem^* refines the *grounded semantics* from [15]. Indeed, it keeps all its strict rankings and breaks some ties. Before presenting the formal result, let us first recall the grounded extension and the ranking it induces. The latter considers arguments of the grounded extension as equally strong and strictly stronger than all remaining arguments. Arguments that are attacked by the extension are equally strong and strictly weaker than all remaining arguments.

Definition 8 ($\succeq^{\mathbf{g}}$). Let $\mathcal{A} \in \text{AG}$. The grounded extension of \mathbf{G} is the set $\text{Gr} = \bigcup_{n=1}^{\infty} \mathcal{F}^n(\emptyset)$. It induces a total preordering $\succeq^{\mathbf{g}}$ such that $\text{Gr} \succ^{\mathbf{g}} \text{Gr}^o \succ^{\mathbf{g}} \text{Gr}^+$ and for any $x \in \{\text{Gr}, \text{Gr}^+, \text{Gr}^o\}$, for all $a, b \in x$, $a \approx^{\mathbf{g}} b$.

Theorem 3. For any $\sigma \in \text{Sem}^*$, σ refines the grounded semantics.

From the definition of the ranking $\succeq^{\mathbf{g}}$ and the previous result, it follows that any semantics $\sigma \in \text{Sem}^*$ preserves the strict ordering of $\succeq^{\mathbf{g}}$.

Theorem 4. For any $\sigma \in \text{Sem}^*$, the strict relations $\text{Gr} \succ^{\sigma} \text{Gr}^o \succ^{\sigma} \text{Gr}^+$ hold.

Example 1 (Cont.) The grounded extension of the flat graph is $\text{Gr} = \{a, d, f\}$. Hence, $a \approx^{\mathbf{g}} d \approx^{\mathbf{g}} f \succ^{\mathbf{g}} g \succ^{\mathbf{g}} b \approx^{\mathbf{g}} c \approx^{\mathbf{g}} e$. However, for any $\sigma \in \text{Sem}^*$, we have the following ranking: $a \succ^{\sigma} d \succ^{\sigma} f \succ^{\sigma} g \succ^{\sigma} e \succ^{\sigma} b \approx^{\sigma} c$.

Example 2 (Cont.) The grounded extension of the graph is empty (i.e., $\text{Gr} = \emptyset$). Hence, $a \approx^{\mathbf{g}} b \approx^{\mathbf{g}} c \approx^{\mathbf{g}} d \approx^{\mathbf{g}} e \approx^{\mathbf{g}} f \approx^{\mathbf{g}} g$. Furthermore, for any $\sigma \in \text{Sem}^*$, we have the same ranking, i.e., $\succeq^{\sigma} = \succeq^{\mathbf{g}}$.

4 Another Instance of the Equivalence Class

We have seen in the previous section that the equivalence class Sem^* contains at least two semantics: Mbs and EMbs. In what follows, we discuss another instance of the class. It is based on the well-known Jacobsthal series [16] $\mathcal{J} = (\mathcal{J}^n)_{n \geq 1}$ where:

$$\mathcal{J}_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ \mathcal{J}_{n-1} + 2\mathcal{J}_{n-2} & \text{if } n \geq 2 \end{cases} \quad (1)$$

From \mathcal{J} , we define a novel series which belongs to the set \mathbf{S}^* .

Proposition 3. It holds that $\mathbf{J} = (\frac{\mathcal{J}^{n+1}}{2^n})_{n \geq 0} \in \mathbf{S}^*$.

We now characterize the gradual semantics which is based on the above series \mathbf{J} . Like Mbs and EMbs, it only considers the strongest attacker of an argument when computing its strength. Furthermore, it takes half of that attacker's strength.

Theorem 5. Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R} \rangle \in \text{AG}$, $a \in \mathcal{A}$ and Jac a mapping from \mathcal{A} to $[0, 1]$ such that for any $a \in \mathcal{A}$, $\text{Deg}_{\mathbf{G}}^{\text{Jac}}(a) = 1 - \frac{\max_{b/\mathcal{R}a} \text{Deg}_{\mathbf{G}}^{\text{Jac}}(b)}{2}$. The following hold:

- Jac is based on the series \mathbf{J} ,
- $\text{Jac} \in \text{Sem}^*$,
- for any $a \in \mathcal{A}$, $\text{Deg}_{\mathbf{G}}^{\text{Jac}}(a) \in [\frac{1}{2}, 1]$.

Example 1 (Cont.) The values of the arguments are summarized in the table below.

	a	b	c	d	e	f	g
Mbs	1	0.50	0.50	0.66	0.60	0.62	0.61
EMbs	1	0.36	0.36	0.69	0.50	0.60	0.56
Jac	1	0.5	0.5	0.75	0.625	0.69	0.66

In the example, Jac assigns higher values than Mbs, EMbs. The following result confirms that this property is valid in general. It also shows that Mbs assigns higher values than EMbs but only to arguments that are attacked by the grounded extension.

Theorem 6. Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R} \rangle \in \text{AG}$ and $a \in \mathcal{A}$. It holds that: $\text{Deg}_{\mathbf{G}}^{\text{Jac}}(a) \geq \text{Deg}_{\mathbf{G}}^{\text{Mbs}}(a)$, $\text{Deg}_{\mathbf{G}}^{\text{Jac}}(a) \geq \text{Deg}_{\mathbf{G}}^{\text{EMbs}}(a)$, If $a \in \text{Gr}^+$, then $\text{Deg}_{\mathbf{G}}^{\text{Mbs}}(a) > \text{Deg}_{\mathbf{G}}^{\text{EMbs}}(a)$

5 Conclusion

The paper contributed to setting up the mathematical foundations of computational argumentation and more precisely of gradual semantics. It defined a mathematical counterpart of a large class of semantics. In particular, it showed that a number of semantics including Mbs, EMbs and Jac can be defined with numerical series having specific characteristics. Furthermore, those semantics provide all the same ranking of arguments of a flat graph while they may assign different values to the same argument. This shows that a ranking is more expressive than the numbers assigned to arguments. Finally, the semantics of the class refine the grounded extension and the value of an argument depends on the iteration at which it appears for the first time in the grounded extension, or is attacked by the latter. This characterization allowed us to define a very efficient algorithm which computes strengths of arguments.

This work lends itself to several developments, in order to have a full understanding of existing semantics. First, we plan to analyse the same set of semantic when applied to weighted graphs (nodes and/or edges have basic weights). Second, we will investigate the equivalence class to which belongs the well-known h-Categorizer [11].

Appendix: Proofs

Lemma 1. *Let $\mathcal{S} \in \mathbf{S}^*$. For all $x \in \mathcal{S}_1$, $y \in \mathcal{S}_2$, the following hold: $x > y$, $x > \lim_{n \rightarrow \infty} \mathcal{S}_1^n$, $y < \lim_{n \rightarrow \infty} \mathcal{S}_2^n$.*

Proof Let $\mathcal{S} \in \mathbf{S}^*$. Let $\lim_{n \rightarrow \infty} \mathcal{S}_1^n = x$ and $\lim_{n \rightarrow \infty} \mathcal{S}_2^n = y$. Since \mathcal{S}_1 is strictly decreasing, then $x < \mathcal{S}_1^n, \forall n$. Since \mathcal{S}_2 is strictly increasing, then $y > \mathcal{S}_2^n, \forall n$. Since $\lim_{n \rightarrow \infty} \mathcal{S}_1^n \geq \lim_{n \rightarrow \infty} \mathcal{S}_2^n$, then we get for any $n > 1$, $\mathcal{S}_1^n > x \geq y > \mathcal{S}_2^n$, so $\mathcal{S}_1^n > \mathcal{S}_2^n$. ■

Lemma 2. *Let $\sigma \in \text{Sem}^*$ be based on a series \mathcal{S} and takes its values from $\mathcal{S} \cup \{\delta\}$. For any $\mathbf{G} = \langle \mathcal{A}, \mathcal{R} \rangle \in \text{AG}$, for any $a \in \mathcal{A}$, the following hold: i) If $a \in \bigcup_{n \geq 1} \mathcal{F}^n(\emptyset)$, then $\text{Deg}_{\mathbf{G}}^\sigma(a) > \delta$. ii) If a is attacked by $\bigcup_{n \geq 1} \mathcal{F}^n(\emptyset)$, then $\text{Deg}_{\mathbf{G}}^\sigma(a) < \delta$.*

Proof Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R} \rangle \in \text{AG}$. Since \mathcal{A} is finite, the authors in [18] proposed an algorithm which runs in $\mathcal{O}(|\mathcal{A}| + |\mathcal{R}|)$ for computing the grounded extension, i.e., the set $\bigcup_{j \geq 1} \mathcal{F}^j(\emptyset)$. Hence, $\exists i \leq |\mathcal{A}| + |\mathcal{R}|$ such that $\bigcup_{j=1}^i \mathcal{F}^j(\emptyset) = \bigcup_{j=1}^{i-1} \mathcal{F}^j(\emptyset)$. From Definition 7 and Lemma 1, if $a \in \bigcup_{n \geq 1} \mathcal{F}^n(\emptyset)$, then $\text{Deg}_{\mathbf{G}}^\sigma(a) > \delta$, and if a is attacked by $\bigcup_{n \geq 1} \mathcal{F}^n(\emptyset)$, then $\text{Deg}_{\mathbf{G}}^\sigma(a) < \delta$. ■

Lemma 3. *For any $i \in \mathbb{N}$, $J^i = 1 - \frac{J^{i-1}}{2}$.*

Proof. From [17], $\forall n \geq 0$, $\mathcal{J}^n = \frac{2^n - (-1)^n}{3}$. Then, $\mathcal{J}^{n+1} = 2^n - \mathcal{J}^n$. Since $J^n = \frac{\mathcal{J}^{n+1}}{2^n}$, we get $J^n = \frac{2^n - \mathcal{J}^n}{2^n} = 1 - \frac{\mathcal{J}^n}{2^n} = 1 - \frac{2^n - (-1)^n}{2^{n+1}} = 1 - \frac{J^{n-1}}{2}$. ■

Lemma 4. Let $\langle \mathcal{A}, \mathcal{R} \rangle \in \text{AG}$, $a, b \in \mathcal{A}$ and $i \in \mathbb{N}$.

- If $a \in \bigcup_{j=1}^i \mathcal{F}^j(\emptyset)$ and $b \notin \bigcup_{j=1}^i \mathcal{F}^j(\emptyset)$, then $\text{Deg}_{\mathbf{G}}^{\text{Jac}}(a) > \text{Deg}_{\mathbf{G}}^{\text{Jac}}(b)$.
- If the set $\bigcup \mathcal{F}^{i \geq 1}(\emptyset)$ attacks a and does not attack b , then $\text{Deg}_{\mathbf{G}}^{\text{Jac}}(a) < \text{Deg}_{\mathbf{G}}^{\text{Jac}}(b)$.

Proof Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R} \rangle \in \text{AG}$ and $a, b \in \mathcal{A}$.

► Let us show the first property.

Let (P) be the following property: If $a \in \bigcup_{j=1}^i \mathcal{F}^j(\emptyset)$ and $b \notin \bigcup_{j=1}^i \mathcal{F}^j(\emptyset)$, then $\text{Deg}_{\mathbf{G}}^{\text{Jac}}(a) > \text{Deg}_{\mathbf{G}}^{\text{Jac}}(b)$. We show by induction that (P) holds for every $i \in \mathbb{N}$.

Case $i = 1$ $a \in \mathcal{F}^1(\emptyset)$, thus $\text{Deg}_{\mathbf{G}}^{\text{Jac}}(a) = 1$. Since $b \notin \mathcal{F}^1(\emptyset)$, then $\text{Attackers}(b) \neq \emptyset$ and so $\text{Deg}_{\mathbf{G}}^{\text{Jac}}(b) < 1$.

Case $i > 1$ Assume that (P) holds at step i and let us show that it holds also for $i + 1$.

Assume $a \in \bigcup_{j=1}^{i+1} \mathcal{F}^j(\emptyset)$ and $b \notin \bigcup_{j=1}^{i+1} \mathcal{F}^j(\emptyset)$. If $a \in \bigcup_{j=1}^i \mathcal{F}^j(\emptyset)$, then by assumption $\text{Deg}_{\mathbf{G}}^{\text{Jac}}(a) > \text{Deg}_{\mathbf{G}}^{\text{Jac}}(b)$.

Assume now that $a \in \mathcal{F}^{i+1}(\emptyset) \setminus (\bigcup_{j=1}^i \mathcal{F}^j(\emptyset))$. Since $a, b \notin \mathcal{F}^1(\emptyset)$, then $\text{Att}(a) \neq \emptyset$ and $\text{Att}(b) \neq \emptyset$. There are two cases:

- $\bigcup_{j=1}^{i+1} \mathcal{F}^j(\emptyset)$ **does not attack** b . Thus, $\forall y \in \text{Attackers}(b)$, $\text{Attackers}(y) \neq \emptyset$. Let \mathcal{D} be the set of arguments defending b . Note that $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$ where $\mathcal{D}_1 \subseteq \bigcup_{j=1}^{i+1} \mathcal{F}^j(\emptyset)$ and $\mathcal{D}_2 \cap (\bigcup_{j=1}^{i+1} \mathcal{F}^j(\emptyset)) = \emptyset$. Note also that $\mathcal{D}_2 \neq \emptyset$ since $b \notin \bigcup_{j=1}^{i+1} \mathcal{F}^j(\emptyset)$.

If $\mathcal{D}_1 \subseteq \bigcup_{j=1}^i \mathcal{F}^j(\emptyset)$, then by assumption $\mathcal{D}_1 >^m \mathcal{D}_2$, hence

$$\max_{y \mathcal{R} b} \text{Deg}_{\mathbf{G}}^{\text{Jac}}(y) = \max_{(y \mathcal{R} b) \text{ and } \text{Attackers}(y) \subseteq \mathcal{D}_2} \text{Deg}_{\mathbf{G}}^{\text{Jac}}(y).$$

By assumption $\mathcal{F}^i >^m \mathcal{D}_2$, hence

$$\max_{x \mathcal{R} a} \text{Deg}_{\mathbf{G}}^{\text{Jac}}(x) < \max_{(y \mathcal{R} b) \text{ and } \text{Attackers}(y) \subseteq \mathcal{D}_2} \text{Deg}_{\mathbf{G}}^{\text{Jac}}(y).$$

Thus, $\text{Deg}_{\mathbf{G}}^{\text{Jac}}(a) > \text{Deg}_{\mathbf{G}}^{\text{Jac}}(b)$.

Assume now that $\mathcal{D}_1 \subseteq \bigcup_{j=1}^{i+1} \mathcal{F}^j(\emptyset)$. Hence, $\mathcal{D}_1 = \mathcal{D}'_1 \cup \mathcal{D}''_1$ such that $\mathcal{D}'_1 \subseteq$

$\mathcal{F}^{i+1}(\emptyset) \setminus (\bigcup_{j=1}^i \mathcal{F}^j(\emptyset))$ and $\mathcal{D}''_1 \subseteq \bigcup_{j=1}^i \mathcal{F}^j(\emptyset)$. By assumption $\mathcal{D}''_1 >^m \mathcal{D}'_1$ and

$\mathcal{D}''_1 >^m \mathcal{D}_2$. So, $\max_{y \mathcal{R} b} \text{Deg}_{\mathbf{G}}^{\text{Jac}}(y) = \max_{(y \mathcal{R} b) \text{ and } \text{Attackers}(y) \subseteq \mathcal{D}'_1 \cup \mathcal{D}_2} \text{Deg}_{\mathbf{G}}^{\text{Jac}}(y)$. Fur-

thermore, $\mathcal{F}^i >^m \mathcal{D}'_1, \mathcal{D}_2$.

Then, $\max_{x \mathcal{R} a} \text{Deg}_{\mathbf{G}}^{\text{Jac}}(x) < \max_{(y \mathcal{R} b) \text{ and } \text{Attackers}(y) \subseteq \mathcal{D}'_1 \cup \mathcal{D}_2} \text{Deg}_{\mathbf{G}}^{\text{Jac}}(y)$.

- $\bigcup_{j=1}^{i+1} \mathcal{F}^j(\emptyset)$ **attacks** b . Let j be the smallest integer such that $\mathcal{F}^j(\emptyset)$ attacks b . There are two cases:

- Case $j \leq i$. $\mathcal{F}^j(\emptyset)$ contains the strongest attackers of b since by assumption $\mathcal{F}^j(\emptyset) >^m \mathcal{F}^{j+k}(\emptyset)$, $\forall k \in \mathbb{N}$, and $\mathcal{F}^j(\emptyset) >^m \text{Attackers}(b) \setminus (\bigcup_{j=1}^{i+1} \mathcal{F}^j(\emptyset))$.

Since $a \in \mathcal{F}^{i+1}(\emptyset)$ and $\bigcup_{j=1}^{i+1} \mathcal{F}^j(\emptyset)$ is conflict-free, then $\text{Attackers}(a) \cap (\bigcup_{j=1}^{i+1} \mathcal{F}^j(\emptyset)) = \emptyset$ and thus $\mathcal{F}^j(\emptyset) >^m \text{Attackers}(a)$. Consequently, $\text{Deg}^{\text{Jac}}(a) > \text{Deg}^{\text{Jac}}(b)$.

- Case $j = i + 1$. Let $\text{Attackers}(b) = X \cup Y$ such that $X \subseteq \mathcal{F}^{i+1}(\emptyset) \setminus (\bigcup_{j=1}^i \mathcal{F}^j(\emptyset))$ and $Y \subseteq \mathcal{A}$ such that $Y \cap (\bigcup_{j=1}^{i+1} \mathcal{F}^j(\emptyset)) = \emptyset$. Since $b \notin \bigcup_{j=1}^{i+1} \mathcal{F}^j(\emptyset)$,

then b is defended by arguments that do not belong to $\bigcup_{j=1}^{i+1} \mathcal{F}^j(\emptyset)$. Let \mathcal{D} be the set of those defenders. By assumption, $\mathcal{F}^i(\emptyset) >^m \mathcal{D}$, hence

$$\max_{x \mathcal{R} a} \text{Deg}^{\text{Jac}}(x) < \max_{(y \mathcal{R} b) \text{ and } \text{Attackers}(y) \subseteq \mathcal{D}} \text{Deg}^{\text{Jac}}(y).$$

Furthermore, $\max_{(y \mathcal{R} b) \text{ and } \text{Attackers}(y) \subseteq \mathcal{D}} \text{Deg}^{\text{Jac}}(y) \leq \max_{(y \mathcal{R} b)} \text{Deg}^{\text{Jac}}(y)$. Thus, $\text{Deg}^{\text{Jac}}(a) > \text{Deg}^{\text{Jac}}(b)$.

► Let us show the second property. Assume that $\bigcup \mathcal{F}^{i \geq 1}(\emptyset)$ attacks a and does not attack b . Since a is attacked by $\bigcup \mathcal{F}^{i \geq 1}(\emptyset)$, then $\exists a' \in \text{Attackers}(a) \cap (\bigcup \mathcal{F}^{i \geq 1}(\emptyset))$. Furthermore, $\text{Attackers}(b) \cap (\bigcup \mathcal{F}^{i \geq 1}(\emptyset)) = \emptyset$. From the first property of this Lemma 4, $\text{Deg}^{\text{Jac}}(a') > \text{Deg}^{\text{Jac}}(b')$, $\forall b' \in \text{Attackers}(b)$. Hence, $\max_{x \mathcal{R} a} \text{Deg}^{\text{Jac}}(x) > \max_{y \mathcal{R} b} \text{Deg}^{\text{Jac}}(y)$.

Thus, $\text{Deg}^{\text{Jac}}(b) > \text{Deg}^{\text{Jac}}(a)$. ■

Proof of Proposition 1. Recall that the *Fibonacci numbers*, i.e. elements of the Fibonacci sequence $\{F^n\}_{n \geq 0}$, are defined as follows: $F^0 = 0$ $F^1 = 1$ $F^n = F^{n-1} + F^{n-2}$ for $n > 1$. We get the sequence of so-called Fibonacci numbers: $\langle 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \dots \rangle$. Consider now the sequence $\{\mathcal{U}^n\}_{n \geq 1}$ defined as follows: $\mathcal{U}^n = \frac{F^n}{F^{n+1}}$. It contains two sub-sequences: The *decreasing* sub-sequence

$$\mathcal{U}_1 = \langle 1, \frac{2}{3}, \frac{5}{8}, \frac{13}{21}, \frac{34}{55}, \frac{89}{144}, \frac{233}{377}, \dots \rangle \quad (2)$$

made of the numbers that are at odd positions in \mathcal{S} , and the *increasing* sub-sequence

$$\mathcal{U}_2 = \langle \frac{1}{2}, \frac{3}{5}, \frac{8}{13}, \frac{21}{34}, \frac{55}{89}, \frac{144}{233}, \dots \rangle \quad (3)$$

which contains the numbers that are at even positions in \mathcal{U} .

Clearly, for any $n \geq 1$, $F^n \in [0, 1]$. From [19], the two sub-sequences converge, furthermore they have the same limit. Indeed, $\lim_{n \rightarrow \infty} \mathcal{U}_1^n = \lim_{n \rightarrow \infty} \mathcal{U}_2^n = \frac{1}{\varphi}$, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the so-called *golden ratio*. Finally, it is also well-known that $\mathcal{U}_2^n < \frac{1}{\varphi} < \mathcal{U}_1^n$, $\forall n \geq 1$. Hence, $\mathcal{U} \in \mathbf{S}^*$. Consequently, $\mathbf{S}^* \neq \emptyset$. ■

Proof of Proposition 2. In [5], Theorem 6 shows that Mbs is based on the series $(\mathcal{U}^n)^{n \geq 1}$ recalled in Lemma 1. Lemma 1 states that $\mathcal{U} \in \mathbf{S}^*$, then $\text{Mbs} \in \text{Sem}^*$. Theorem 7 in [5] shows that EMbs is based on the well-known exponential series which is also in \mathbf{S}^* , then $\text{EMbs} \in \text{Sem}^*$. ■

Proof of Proposition 3. From [17], $\forall n \geq 0$, $\mathcal{J}^n = \frac{2^n - (-1)^n}{3}$ (called Binet formulae). Since $J = (\frac{\mathcal{J}^{n+1}}{2^n})^{n \geq 0}$, it follows that $J^n = \frac{1}{3}(2 + (-\frac{1}{2})^n)$. Let $J_1 = (J_1^n)^{n \geq 0}$ and $J_2 = (J_2^n)^{n \geq 0}$ be two sub-series of J such that for any $n \geq 0$: $J_1^n = J^{2n}$ and $J_2^n = \mathcal{S}^{2n+1}$. We show that J_1 is strictly decreasing while J_2 is strictly increasing. Obviously, $J_1^{n+1} - J_1^n = \frac{1}{3}(2 + (\frac{1}{2})^{n+1}) - \frac{1}{3}(2 + (\frac{1}{2})^n) = -\frac{1}{3} \times \frac{1}{2^{n+1}}$. Since $\frac{1}{2^{n+1}} > 0$, then $J_1^{n+1} - J_1^n < 0$ and so $J_1^n > J_1^{n+1}$.

In a similar way we have $J_2^{n+1} - J_2^n = \frac{1}{3}(2 - (\frac{1}{2})^{n+1}) - \frac{1}{3}(2 - (\frac{1}{2})^n) = \frac{1}{3} \times \frac{1}{2^{n+1}}$. Since $\frac{1}{2^{n+1}} > 0$, then $J_2^{n+1} - J_2^n > 0$ and so $J_2^n < J_2^{n+1}$.

Obviously, $J_1^n > J_2^n$ because $\frac{2^{n+1}+1}{2^n} > \frac{2^{n+1}-1}{2^n}$. Since J_1^n is strictly decreasing with $J_1^1 = 1$ then $\forall n$, $J_1^n \leq 1$. From [16], $\forall n \geq 0$, $\mathcal{J}^n \geq 0$ and thus $J^n \geq 0$.

Finally, it is easy to check that $\lim_{n \rightarrow \infty} J_1^n = \lim_{n \rightarrow \infty} J_2^n = \frac{2}{3}$. Then, $J \in \mathbf{S}^*$. ■

Proof of Theorem 1. Let $\sigma, \sigma' \in \text{Sem}^*$ which are based on the series \mathcal{S} and \mathcal{S}' respectively. Let $\langle \mathcal{A}, \mathcal{R} \rangle \in \text{AG}$, $a, b \in \mathcal{A}$ and $\text{Gr} = \bigcup_{j \geq 1} \mathcal{F}^j(\emptyset)$.

► Let us show the implication: $(a \succ_{\sigma'} b) \Rightarrow (a \succ_{\sigma} b)$. Assume that $a \succ_{\sigma} b$, thus $\text{Deg}^{\sigma}(a) > \text{Deg}^{\sigma}(b)$. From Lemma 2, there are four possibilities:

Case 1 $a \in \text{Gr}$ and $b \notin \text{Gr}$. From Definition 7, $\text{Deg}^{\sigma'}(a) \in \mathcal{S}'_1$ and $\text{Deg}^{\sigma'}(b) \in \mathcal{S}'_2 \cup \{\delta'\}$. From Lemma 1, $\text{Deg}^{\sigma'}(a) > \text{Deg}^{\sigma'}(b)$.

Case 2 $a \in \text{Gr}^o$ and $b \in \text{Gr}^+$. From Definition 7, $\text{Deg}^{\sigma'}(a) = \delta'$ and $\text{Deg}^{\sigma'}(b) \in \mathcal{S}'_2$. From Lemma 1, $\text{Deg}^{\sigma'}(a) > \text{Deg}^{\sigma'}(b)$.

Case 3 $a, b \in \text{Gr}$. Since \mathcal{S}_1 is decreasing, then $\exists i, j \in \mathbb{N}$ such that $i < j$ and $a \in \mathcal{F}^i(\emptyset) \setminus (\bigcup_{k=1}^{i-1} \mathcal{F}^k(\emptyset))$ and $b \in \mathcal{F}^j(\emptyset) \setminus (\bigcup_{k=1}^{j-1} \mathcal{F}^k(\emptyset))$. From Definition 7, $\text{Deg}^{\sigma'}(a) = \mathcal{S}'_1^i$ and $\text{Deg}^{\sigma'}(b) = \mathcal{S}'_1^j$. Since \mathcal{S}'_1 is also decreasing and $i < j$, then $\text{Deg}^{\sigma'}(a) > \text{Deg}^{\sigma'}(b)$.

Case 4 $a, b \in \text{Gr}^+$. From Definition 7, $\exists i, j \in \mathbb{N}$ such that $i > j$ and a is attacked by $\mathcal{F}^i(\emptyset)$ and not attacked by $\bigcup_{k=1}^{i-1} \mathcal{F}^k(\emptyset)$, and b is attacked by $\mathcal{F}^j(\emptyset)$ and not attacked by $\bigcup_{k=1}^{j-1} \mathcal{F}^k(\emptyset)$. It follows that $\text{Deg}^{\sigma'}(a) = \mathcal{S}'_2^i$ and $\text{Deg}^{\sigma'}(b) = \mathcal{S}'_2^j$. Since \mathcal{S}'_2 is increasing and $i > j$, then $\text{Deg}^{\sigma'}(a) > \text{Deg}^{\sigma'}(b)$.

► To show the implication: $(a \succ_{\sigma'} b) \Rightarrow (a \succ_{\sigma} b)$, it is sufficient to follow exactly the same reasoning as for the above case. ■

Proof of Theorem 2. Let $\sigma \in \text{Sem}$ and assume σ is strongly equivalent to Mbs, i.e., for any $\mathbf{G} = \langle \mathcal{A}, \mathcal{R} \rangle \in \text{AG}$, $\succeq_{\mathbf{G}}^{\text{Mbs}} = \succeq_{\mathbf{G}}^{\sigma}$. Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R} \rangle \in \text{AG}$. We denote by $\text{Gr}, \text{Gr}_+, \text{Gr}_o$

the following sets: $\text{Gr} = \text{Gr}^1 \cup \dots \cup \text{Gr}^n$ where $\text{Gr}^i = \mathcal{F}^i(\emptyset) \setminus (\bigcup_{j=1}^{i-1} \mathcal{F}^j(\emptyset))$

$\text{Gr}_+ = \text{Gr}_+^1 \cup \dots \cup \text{Gr}_+^n$ where $\text{Gr}_+^i = \{a \in \mathcal{A} \mid \text{Gr}^i \text{ attacks } a\}$,

$\text{Gr}_o = \mathcal{A} \setminus (\text{Gr} \cup \text{Gr}_+)$. From strong equivalence of σ and Mbs, it follows that:

- $\forall a, b \in \text{Gr}^i$ ($i \in \{1, \dots, n\}$), $\text{Deg}_{\mathbf{G}}^\sigma(a) = \text{Deg}_{\mathbf{G}}^\sigma(b) = v_i$ and $\forall c \in \text{Gr}^j$, with $j > i$, $\text{Deg}_{\mathbf{G}}^\sigma(a) > \text{Deg}_{\mathbf{G}}^\sigma(c)$. Hence, the sequence $V = (v_1, \dots, v_n)$ is strictly decreasing.
- $\forall a, b \in \text{Gr}_+^i$, $\text{Deg}_{\mathbf{G}}^\sigma(a) = \text{Deg}_{\mathbf{G}}^\sigma(b) = u_i$ and $\forall c \in \text{Gr}_+^j$, with $j > i$, $\text{Deg}_{\mathbf{G}}^\sigma(a) < \text{Deg}_{\mathbf{G}}^\sigma(c)$. Hence, the sequence $U = (u_1, \dots, u_n)$ is strictly increasing.
- $\exists \delta \in [0, 1]$ such that for any $a \in \text{Gr}_o$, $\text{Deg}_{\mathbf{G}}^\sigma(a) = \delta$.

From Theorem 8 in [5], $\text{Gr} \succ^{\text{Mbs}} \text{Gr}^o \succ^{\text{Mbs}} \text{Gr}^+$. From strong equivalence of σ and Mbs, we have $\text{Gr} \succ^\sigma \text{Gr}^o \succ^\sigma \text{Gr}^+$. Hence, $\forall a, b \in \text{Gr}_o$, $\text{Deg}_{\mathbf{G}}^\sigma(a) > \delta$. Hence, the sequence $(V^n)^{n \geq 1}$ is bounded (and strictly decreasing) and so it converges, i.e., $\lim_{n \rightarrow \infty} V^n = l \geq \delta$. Similarly, $\forall a, b \in \text{Gr}_+$, $\text{Deg}_{\mathbf{G}}^\sigma(a) < \delta$. Thus, the sequence $(U^n)^{n \geq 1}$ is thus bounded (and strictly increasing). So, it converges, i.e., $\lim_{n \rightarrow \infty} U^n = l' \leq \delta$. It follows that $\lim_{n \rightarrow \infty} V^n \geq \lim_{n \rightarrow \infty} U^n$. Finally, the sequence $(S^n)^{n \geq 1}$ composed of (V^n) and (U^n) is an element of \mathbf{S}^* . Thus, $\sigma \in \text{Sem}^*$. ■

Proof of Theorem 3. Let $\sigma \in \text{Sem}^*$. From Proposition 2, $\text{Mbs} \in \text{Sem}^*$, thus it is strongly equivalent to σ . From Theorem 10 in [5], Mbs refines the grounded semantics. From Property 2, σ refines the grounded semantics. ■

Proof of Theorem 5. Let $\langle \mathcal{A}, \mathcal{R} \rangle \in \text{AG}$, $a \in \mathcal{A}$, and $i \in \mathbb{N}$.

$$\blacktriangleright \text{Let: (P) } \text{Deg}_{\mathbf{G}}^{\text{Jac}}(a) = J_1^i \text{ if } a \in \mathcal{F}^i(\emptyset) \setminus \bigcup_{j=1}^{i-1} \mathcal{F}^j(\emptyset),$$

$$\text{(Q) } \text{Deg}_{\mathbf{G}}^{\text{Jac}}(a) = J_2^i \text{ if } \mathcal{F}^i(\emptyset) \text{ attacks } a \text{ and } \bigcup_{j=1}^{i-1} \mathcal{F}^j(\emptyset) \text{ does not attack } a.$$

We prove by induction that the property $P \wedge Q$ is true for any $i \in \mathbb{N}$.

Case $i = 1$. $\mathcal{F}^1(\emptyset) = \{x \in \mathcal{A} \mid \text{Attackers}(a) = \emptyset\}$. Furthermore, $\text{Deg}^{\text{Jac}}(a) = 1 = J_1^1$ if $\text{Attackers}(a) = \emptyset$, and so (P) holds for $i = 1$.

By definition of Jac, for any $a \in \mathcal{A}$, $\text{Deg}^{\text{Jac}}(a) = \frac{1}{2}$ iff $\exists b \in \text{Attackers}(a)$ such that $\text{Deg}^{\text{Jac}}(b) = 1$. Hence, $\text{Deg}^{\text{Jac}}(a) = \frac{1}{2} = J_2^1$. Thus, (Q) holds for $i = 1$.

Case $i > 1$. Assume that the property P&Q is true at step i and let us show that it holds at step $i + 1$.

Assume $a \in \mathcal{F}^{i+1}(\emptyset) \setminus \left(\bigcup_{j=1}^i \mathcal{F}^j(\emptyset) \right)$. Hence, $\text{Attackers}(a) \neq \emptyset$ and a is defended by $\mathcal{F}^i(\emptyset)$, i.e., $\forall b \in \text{Attackers}(a)$, $\exists c \in \mathcal{F}^i(\emptyset)$ such that $c \mathcal{R} b$. There are two sub-cases:

- $\bigcup_{j=1}^{i-1} \mathcal{F}^j(\emptyset)$ does not attack any $b \in \text{Attackers}(a)$. Thus, $\forall b \in \text{Attackers}(a)$, $\text{Deg}^{\text{Jac}}(b) = J_2^i$ (by assumption). Then, $\text{Deg}^{\text{Jac}}(a) = 1 - \frac{J_2^i}{2} = J_1^{i+1}$. (from Lemma 3).
- $\bigcup_{j=1}^{i-1} \mathcal{F}^j(\emptyset)$ attacks some $b \in \text{Attackers}(a)$. This means that $\exists j < i$ such that $\mathcal{F}^j(\emptyset)$ attacks b and $\bigcup_{k=1}^{j-1} \mathcal{F}^k(\emptyset)$ does not attack b . By assumption, $\text{Deg}^{\text{Jac}}(b) = J_2^j$.

But since $a \notin \bigcup_{k=1}^i \mathcal{F}^k(\emptyset)$, then $\exists b' \in \text{Attackers}(a)$ such that $\mathcal{F}^i(\emptyset)$ attacks b' and $\bigcup_{k=1}^{i-1} \mathcal{F}^k(\emptyset)$ does not attack b' . Thus, by assumption, $\text{Deg}^{\text{Jac}}(b') = J_2^i$. Since the subsequence $\{J_2\}^n$ is strictly increasing, then $J_2^i > J_2^j$ and so $\max_{b \in \mathcal{R}_a} \text{Deg}^{\text{Jac}}(b) = J_2^i$ and $\text{Deg}^{\text{Jac}}(a) = 1 - \frac{J_2^i}{2} = J_1^{i+1}$ (from Lemma 3).

Assume now that $\mathcal{F}^{i+1}(\emptyset)$ attacks a and $\bigcup_{j=1}^i \mathcal{F}^j(\emptyset)$ does not attack a . Thus, $\text{Attackers}(a) = X_1 \cup X_2$ such that: $X_1 \subseteq \mathcal{F}^{i+1}(\emptyset) \setminus (\bigcup_{j=1}^i \mathcal{F}^j(\emptyset))$ and $X_2 \cap (\bigcup_{j=1}^{i+1} \mathcal{F}^j(\emptyset)) = \emptyset$. From Property (P) above, $\forall b \in X_1, \text{Deg}^{\text{Jac}}(b) = J_1^{i+1}$. There are two possibilities:

- i) $X_2 = \emptyset$. Hence, $\text{Deg}^{\text{Jac}}(a) = 1 - \frac{J_1^{i+1}}{2} = J_2^{i+1}$.
- ii) $X_2 \neq \emptyset$. From Lemma 4, $X_1 >^m X_2$. So, $\max_{b \in \mathcal{R}_a} \text{Deg}^{\text{Jac}}(b) = \max_{(b \in \mathcal{R}_a) \text{ and } b \in X_1} \text{Deg}^{\text{Jac}}(b)$.

So, $\text{Deg}^{\text{Jac}}(a) = 1 - \frac{J_1^{i+1}}{2} = J_2^{i+1}$.

► Let now $a \in \mathcal{A}$ such that $a \notin \bigcup \mathcal{F}^{i \geq 1}(\emptyset)$ and a is not attacked by $\bigcup \mathcal{F}^{i \geq 1}(\emptyset)$. From Lemma 4, for all $b \in \text{Gr}$, $\text{Deg}^{\text{Jac}}(a) < \text{Deg}^{\text{Jac}}(b)$ and from the first property of Theorem 5, $\text{Deg}^{\text{Jac}}(a) < J_1^i$ for all $i \in \mathbb{N}$. From Lemma 4, for all $c \in \text{Gr}^+$, $\text{Deg}^{\text{Jac}}(a) > \text{Deg}^{\text{Jac}}(c)$ and from the second property of Theorem 5, $\text{Deg}^{\text{Jac}}(a) > J_2^i$ for all $i \in \mathbb{N}$. Hence, for all $i \in \mathbb{N}$, $J_1^i > \text{Deg}^{\text{Jac}}(a) > J_2^i$. $\text{Deg}^{\text{Jac}}(a) = \lim_{n \rightarrow \infty} J_1^n = \lim_{n \rightarrow \infty} J_2^n = \frac{2}{3}$. ■

Proof of Theorem 6. Let $\langle \mathcal{A}, \mathcal{R} \rangle \in \text{AG}$ and $a \in \mathcal{A}$. Let \mathcal{U} and \mathcal{E} be the series on which Mbs and EMbs are based.

Case 1 $a \in \text{Gr}$. If $\text{Att}(a) = \emptyset$, then $\text{Deg}_G^x(a) = 1$ for any $x \in \{\text{Jac}, \text{Mbs}, \text{EMbs}\}$. Assume now $\text{Att}(a) \neq \emptyset$. Then, for any $n \geq 1$, $J^n > \frac{2}{3}$ since J_1^n is decreasing and $\lim_{n \rightarrow \infty} J_1^n = \frac{2}{3}$. It is also easy to check that $J_1^2 > \mathcal{E}_1^2 > \mathcal{U}_1^2$ and $J_1^3 > \mathcal{E}_1^3 > \mathcal{F}_1^3$. Note also that $\mathcal{E}_1^3 < \lim_{n \rightarrow \infty} J_1^n$ and since it is strictly decreasing then, for any $i \geq 3$, $\mathcal{E}_1^i < J_1^i$. Similarly, $\mathcal{U}_1^3 < \lim_{n \rightarrow \infty} J_1^n$ and since it is strictly decreasing then, for any $i \geq 3$, $\mathcal{U}_1^i < J_1^i$.

Case 2 $a \in \text{Gr}^+$. From Proposition 2, both \mathcal{U}_2 and \mathcal{E}_2 are decreasing and $\lim_{n \rightarrow \infty} \mathcal{E}_2^n = \Omega$ and $\lim_{n \rightarrow \infty} \mathcal{U}_2^n = \frac{1}{\varphi}$. Note that $J_2^2 = 0.625 > \Omega$ and $J_2^2 > \frac{1}{\varphi}$. Since \mathcal{E}_2 and \mathcal{U}_2 are increasing, then $\forall i \geq 2$, $\mathcal{E}_2^i < \Omega$ and $\mathcal{U}_2^i < \frac{1}{\varphi}$. Then, $\forall i \geq 2$, $\text{Deg}_G^{\text{Jac}}(a) > \text{Deg}_G^{\text{EMbs}}(a)$ and $\text{Deg}_G^{\text{Jac}}(a) > \text{Deg}_G^{\text{Mbs}}(a)$. For specific case of $i = 1$, $\text{Deg}_G^{\text{Jac}}(a) = 0.5$ and $\text{Deg}_G^{\text{Mbs}}(a) = 0.5$ and $\text{Deg}_G^{\text{EMbs}}(a) \approx 0.36$.

Case 3 $a \in \text{Gr}^o$. Then, $\text{Deg}_G^{\text{Jac}}(a) = \frac{2}{3}$, $\text{Deg}_G^{\text{Mbs}}(a) = \frac{1}{\varphi} \approx 0.61$ and $\text{Deg}_G^{\text{EMbs}}(a) = \Omega \approx 0.571$. Hence, $\frac{1}{\varphi} < \frac{2}{3}$ and $\Omega < \frac{2}{3}$. ■

References

1. Amgoud, L.: A replication study of semantics in argumentation. In: Kraus, S. (ed.) Proceedings of the Twenty-Eighth International Joint Conference on Artificial Intelligence, IJCAI. pp. 6260–6266 (2019)
2. Amgoud, L., Ben-Naim, J.: Ranking-based semantics for argumentation frameworks. In: Proceedings of the 7th International Conference Scalable on Uncertainty Management SUM. pp. 134–147 (2013)
3. Amgoud, L., Ben-Naim, J.: Evaluation of arguments from support relations: Axioms and semantics. In: Proceedings of the International Joint Conference on Artificial Intelligence IJCAI. pp. 900–906 (2016)
4. Amgoud, L., Ben-Naim, J., Doder, D., Vesic, S.: Acceptability semantics for weighted argumentation frameworks. In: Proceedings of the International Joint Conference on Artificial Intelligence IJCAI. pp. 56–62 (2017)
5. Amgoud, L., Beuselinck, V.: Equivalence of semantics in argumentation. In: Proceedings of the 18th International Conference on Principles of Knowledge Representation and Reasoning, KR. pp. 32–41 (2021)
6. Amgoud, L., Doder, D.: Gradual semantics accounting for varied-strength attacks. In: Proceedings of the 18th International Conference on Autonomous Agents and MultiAgent Systems, AAMAS. pp. 1270–1278 (2019)
7. Amgoud, L., Doder, D., Vesic, S.: Evaluation of argument strength in attack graphs: Foundations and semantics. *Artificial Intelligence* **302**, 103607 (2022)
8. Baroni, P., Gabbay, D., Giacomin, M., van der Torre, L. (eds.): Handbook of Formal Argumentation, Volume 1. College Publications (2018)
9. Baroni, P., Giacomin, M.: On principle-based evaluation of extension-based argumentation semantics. *Artificial Intelligence* **171**(10-15), 675–700 (2007)
10. Baroni, P., Rago, A., Toni, F.: From fine-grained properties to broad principles for gradual argumentation: A principled spectrum. *International Journal of Approximate Reasoning* **105**, 252–286 (2019)
11. Besnard, P., Hunter, A.: A logic-based theory of deductive arguments. *Artificial Intelligence* **128**(1-2), 203–235 (2001)
12. Beuselinck, V., Delobelle, J., Vesic, S.: A principle-based account of self-attacking arguments in gradual semantics. *Journal of Logic and Computation* **33**(2), 230–256 (2023)
13. Bonzon, E., Delobelle, J., Konieczny, S., Maudet, N.: A comparative study of ranking-based semantics for abstract argumentation. In: Proceedings of the AAAI Conference on Artificial Intelligence (AAAI). pp. 914–920 (2016)
14. Cayrol, C., Lagasquie-Schiex, M.: Graduality in argumentation. *Journal of Artificial Intelligence Research* **23**, 245–297 (2005)
15. Dung, P.M.: On the Acceptability of Arguments and its Fundamental Role in Non-Monotonic Reasoning, Logic Programming and n-Person Games. *Artificial Intelligence* **77**, 321–357 (1995)
16. Horadam, A.F.: Jacobsthal and pell curves. *Fibonacci Quart* **26**, 77–83 (1988)
17. Horadam, A.F.: Jacobsthal representation numbers. *Fibonacci Quart* **34**, 40–54 (1996)
18. Nofal, S., Atkinson, K., Dunne, P.E.: Computing grounded extensions of abstract argumentation frameworks. *Computer Journal* **64**(1), 54–63 (2021)
19. Philippou, A.: Fibonacci numbers, probability and gambling. In: Proceedings of the International Conference on Mathematics Education and Mathematics in Engineering and Technology, (ICMET). pp. 13–21 (2015)
20. Rahwan, I., Simari, G. (eds.): Argumentation in Artificial Intelligence. Springer (2009)
21. Simari, G., Giacomin, M., Gabbay, D., Thimm, M. (eds.): Handbook of Formal Argumentation, Volume 2. College Publications (2021)