Signed Tropicalizations of Convex Semialgebraic Sets

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Abstract: We study the signed valuations of convex semialgebraic sets defined over non-Archimedean fields. This is motivated by the efforts to understand the structure of semialgebraic sets that arise in convex optimization, such as the spectrahedra and the hyperbolicity cones. We give a full characterization of regular sets that are obtained as signed tropicalizations of convex semialgebraic sets, and we prove that the signed tropicalizations of hyperbolicity cones have a more restrictive structure. To obtain our results, we combine two recent advances in the area of tropical geometry: the study of signed valuations of general semialgebraic sets and the separation theorems for signed tropical convexities.

Keywords: Convex algebraic geometry, Tropical geometry, Hyperbolicity cones, Semialgebraic sets, Non-Archimedean fields

AMS Subject Classification: 90C25, 14T05, 14P10

1. INTRODUCTION

Convex semialgebraic sets arise naturally in convex optimization problems such as the semidefinite programming or the hyperbolic programming. As a result, questions motivated by optimization problems, such as the study of the expressivity of semidefinite programming, lead to questions in real algebraic geometry (e.g., study of classes of sets that are representable by linear matrix inequalities). We refer to Blekherman et al. (2013) for more information about the interactions between these disciplines. In this work, we apply techniques from tropical geometry to study convex semialgebraic sets.

One of the research directions in tropical geometry is to analyze the (semi)algebraic sets defined over non-Archimedean fields with the help of the valuation map. This idea was first applied to sets arising in convex optimization by Develin and Yu (2007), who studied the tropicalizations of polyhedra. This inspired numerous other works on the tropicalizations of polyhedra, see, e.g., Allamigeon et al. (2015, 2018); Allamigeon and Katz (2017); Joswig and Smith (2018). The study of tropical polyhedra was also extended to more general semialgebraic sets, such as spectrahedra (Yu, 2015; Allamigeon et al., 2020), hyperbolicity cones in dimension 3 (Le Texier, 2021), convex semialgebraic sets (Allamigeon et al., 2019), and arbitrary semialgebraic sets (Alessandrini, 2013; Allamigeon et al., 2020; Jell et al., 2022). A recent idea proposed by Jell et al. (2022) is to study the tropicalizations of semialgebraic sets using a *signed* valuation map. In other recent development, Loho and Végh (2020) and Loho and Skomra (2022b) started a systematic study of tropical convexities in the signed setting and proved new tropical analogues of the hyperplane separation theorem. In this work, we combine both ideas by studying the *signed* valuations of *convex* semialgebraic sets. Our main goal is to generalize the known results about tropical polyhedra and spectrahedra to the tropicalizations of hyperbolicity cones in arbitrary dimension. We present the first results in this direction, by characterizing the regular sets arising as tropicalizations of convex semialgebraic sets, and by showing that the tropicalizations of hyperbolicity cones have a more restricted, "tropically quadratic" structure.

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2. PRELIMINARIES

2.1 Generalized Puiseux series

In this work, we denote by $\mathbb{K}$ the field of *absolutely convergent generalized real Puiseux series*, i.e., series of the form

$$\boldsymbol{x} = \boldsymbol{x}(t) = c_1 t^{\alpha_1} + c_2 t^{\alpha_2} + \dots , \tag{1}$$

where both the coefficients c_i and the exponents α_i are real numbers. We further suppose that the sequence $(\alpha_i)_{i\geqslant 1}$ is strictly decreasing and either finite or unbounded and that the series $\boldsymbol{x}(t)$ is absolutely convergent for all sufficiently large $t > 0$. It is known that $\mathbb{K}$ is a real closed field (van den Dries and Speissegger, 1998). In particular, $\mathbb{K}$ is ordered by putting $\boldsymbol{x} \geqslant 0 \iff c_1 \geqslant 0$. We state our results for $\mathbb{K}$, but we note that a quantifier elimination argument discussed in Allamigeon et al. (2020) allows to transfer our main theorems from $\mathbb{K}$ to any real closed field equipped with a nontrivial and convex valuation whose value group is $\mathbb{R}$.

As a non-Archimedean field, $\mathbb{K}$ has a valuation function $\mathsf{val}\colon \mathbb{K} \to \mathbb{R} \cup \{-\infty\}$ that maps a series of the form (1) to its leading exponent, $\mathsf{val}(\boldsymbol{x}) = \alpha_1$, with $\mathsf{val}(0) = -\infty$. We note that the usual convention in the theory of valued fields would be to define the valuation as $-\alpha_1$ rather than α_1. We use the opposite convention for the sake of coherence

with the max-plus tropical semiring introduced below. One can easily adapt our results to the other setting. We also use an extended valuation map that keeps track not only of the leading exponent of the series but also of its sign.

Definition 1. We define the *signed valuation* $\mathsf{sval}\colon \mathbb{K} \to (\{-1,1\}\times\mathbb{R})\cup\{-\infty\}$ by setting $\mathsf{sval}(\boldsymbol{x}) = \big(\operatorname{sign}(\boldsymbol{x}), \mathsf{val}(\boldsymbol{x})\big)$, with the convention that $\mathsf{sval}(0) = -\infty$.

We extend the definitions of val and sval to vectors by applying them coordinatewise. A *signed tropicalization* of a semialgebraic set $\boldsymbol{X} \subset \mathbb{K}^n$ is then given by $\mathsf{sval}(\boldsymbol{X})$. An alternative viewpoint follows from the work of Alessandrini (2013), who showed that if $\boldsymbol{X} \subset \mathbb{K}^n_{>0}$, then $\mathsf{val}(\boldsymbol{X})$ coincides with the "log-limit" of sets $\boldsymbol{X}(t) \subset \mathbb{R}^n_{>0}$ obtained by fixing the parameter t, $\mathsf{val}(\boldsymbol{X}) = \lim_{t\to\infty} \log_t\big(\boldsymbol{X}(t)\big)$. In this way, if $\boldsymbol{X} \subset \mathbb{K}^n$, then $\mathsf{sval}(\boldsymbol{X})$ is obtained by "gluing" the log-limits given by all sign patterns.

2.2 Signed tropical numbers

Tropicalizations of semialgebraic sets are studied with the help of an algebraic structure known as the tropical semiring, see Maclagan and Sturmfels (2015); Joswig (2021) for more information. The *tropical (max-plus) semiring* is defined as $\mathbb{T} = (\mathbb{R}\cup\{-\infty\}, \oplus, \odot)$, where $a\oplus b = \max\{a,b\}$ and $a\odot b = a+b$. In order to replace valuation by the signed valuation, we extend $\mathbb{T}$ to signed tropical numbers. The set of signed tropical numbers is $\mathbb{T}_\pm = (\{-1,1\}\times\mathbb{R})\cup\{-\infty\}$. By convention, we denote the numbers of the form $(1,a)$ by a and call them *positive*. We also denote the numbers of the form $(-1,a)$ by $\ominus a$ and call them *negative*. Then, the set $\mathbb{T}_\pm$ is ordered by mimicking the order of the real line, with $-\infty$ having the role of zero, so that

$$\ominus 2 < \ominus 1 < \ominus(-1) < -\infty < (-1) < 1 < 2\,.$$

We denote by $[a,b]$ the interval from a to b in $\mathbb{T}_\pm$ and we embed the set $\mathbb{T}$ in $\mathbb{T}_\pm$ by identifying it with the set of signed tropical numbers that are not smaller than $-\infty$. We equip $\mathbb{T}_\pm$ with the topology induced by the order and use the product topology on $\mathbb{T}^n_\pm$. We also use a sign function $\operatorname{tsign}\colon \mathbb{T}_\pm \to \{-1,0,1\}$ that gives the sign of a signed tropical number, $\operatorname{tsign}(-\infty) = 0$, and an absolute value function $|\cdot|\colon \mathbb{T}_\pm \to \mathbb{T}$ that acts by forgetting the sign of a tropical number. In this way, the function $\phi\colon \mathbb{T}_\pm \to \mathbb{R}$ defined as $\phi(x) = \operatorname{tsign}(x)\exp(|x|)$ is an order-preserving homeomorphism. Thus, $\mathbb{T}^n_\pm$ and $\mathbb{R}^n$ are homeomorphic.

In order to equip $\mathbb{T}_\pm$ with an algebraic structure, we first define the multiplication on $\mathbb{T}_\pm$ (still denoted by $\odot$) as $a\odot b = -\infty$ if $-\infty \in \{a,b\}$ and

$$a \odot b = \big(\operatorname{tsign}(a)\operatorname{tsign}(b), |a|+|b|\big)$$

otherwise. As an example, $(\ominus 2)\odot 3 = \ominus 5$, $(\ominus 3)\odot(\ominus 4) = 7$. Defining the addition is more problematic as there is no way to define an addition that extends $\oplus$ and turns $\mathbb{T}_\pm$ into a semiring. As a way to overcome this difficulty, we equip $\mathbb{T}_\pm$ with a multivalued addition $\boxplus\colon \mathbb{T}_\pm \to 2^{\mathbb{T}_\pm}$ defined as

$$a \boxplus b = \begin{cases} a & \text{if } |a| > |b| \text{ or } a = b,\\ b & \text{if } |b| > |a|,\\ [\ominus|a|, |a|] & \text{otherwise.}\end{cases}$$

In this way $(\ominus 2)\boxplus 3 = 3$, but $(\ominus 2)\boxplus 2 = [\ominus 2, 2]$. We extend the addition to vectors by applying it coordinatewise and to sets $A, B \subset \mathbb{T}_\pm$ by putting $A\boxplus B = \cup\{a\boxplus b\colon a\in A, b\in B\}$. These operations turn $(\mathbb{T}_\pm, \boxplus, \odot)$ into a *hyperfield*, see Baker and Bowler (2019) for more information. The following lemma gives a link between the signed valuation and the hyperfield operations.

Lemma 2. For any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{K}$ we have $\mathsf{sval}(\boldsymbol{xy}) = \mathsf{sval}(\boldsymbol{x}) \odot \mathsf{sval}(\boldsymbol{y})$ and $\mathsf{sval}(\boldsymbol{x}+\boldsymbol{y}) \in \mathsf{sval}(\boldsymbol{x}) \boxplus \mathsf{sval}(\boldsymbol{y})$.

2.3 Tropical polynomials

A *(signed) tropical polynomial* $P(x) \in \mathbb{T}_\pm[x]$ is an expression of the form

$$\bigboxplus_{\alpha\in\Lambda} c_\alpha \odot x_1^{\odot\alpha_1} \odot \ldots \odot x_n^{\odot\alpha_n}\,,$$

where $c_\alpha \in \mathbb{T}_\pm$. A tropical polynomial defines a multivalued function $P\colon \mathbb{T}^n_\pm \to 2^{\mathbb{T}_\pm}$. One can check that if we evaluate P on a point $x \in \mathbb{T}^n_\pm$, then the result is either a singleton in $\mathbb{T}_\pm$ or an interval of the form $[\ominus a, a]$ for some $a > -\infty$. We define the *(signed) tropical hypersurface* of P as the set $\{x \in \mathbb{T}^n_\pm\colon -\infty \in P(x)\}$. Furthermore, given a polynomial $\boldsymbol{P} \in \mathbb{K}[x]$ of the form $\boldsymbol{P}(\boldsymbol{x}) = \sum_{\alpha\in\Lambda} \boldsymbol{c}_\alpha \boldsymbol{x}_1^{\alpha_1}\ldots\boldsymbol{x}_n^{\alpha_n}$, we define its *formal tropicalization* as $\operatorname{trop}(\boldsymbol{P}) = \bigboxplus_{\alpha\in\Lambda} \mathsf{sval}(\boldsymbol{c}_\alpha) \odot x_1^{\odot\alpha_1} \odot \ldots \odot x_n^{\odot\alpha_n}$. The next lemma follows from Lemma 2 and links the classical and tropical hypersurfaces.

Lemma 3. Let $\boldsymbol{P} \in \mathbb{K}[x]$ be a polynomial. Then, the set

$$\mathsf{sval}\big(\{\boldsymbol{x} \in \mathbb{K}^n\colon \boldsymbol{P}(\boldsymbol{x}) = 0\}\big)$$

is included in the tropical hypersurface of $\operatorname{trop}(\boldsymbol{P})$. In general, this inclusion may be strict.

Example 4. Consider the bivariate polynomial $\boldsymbol{P} \in \mathbb{K}[x]$ defined as $\boldsymbol{P}(\boldsymbol{x}) = (\boldsymbol{x}_1 - 1 - t^{-1})^4 + \boldsymbol{x}_2^4 - 1$. By opening the parentheses, one can see that the formal tropicalization of $\boldsymbol{P}$ is given by $x_2^{\odot 4} \boxplus x_1^{\odot 4} \boxplus (\ominus x_1^{\odot 3}) \boxplus x_1^{\odot 2} \boxplus (\ominus x_1) \boxplus (-1)$. The tropical hypersurface of $\operatorname{trop}(\boldsymbol{P})$ is the boundary of the set depicted in Figure 1. In this case, the inclusion from Lemma 3 is satisfied as an equality.

2.4 Tropical convexity

In the unsigned case, we say that a subset $X \subset \mathbb{T}^n$ is *tropically convex* if for every $x, y \in X$ and for all $\lambda, \mu \in \mathbb{T}$ such that $\lambda\oplus\mu = 0$ we have $(\lambda\odot x)\oplus(\mu\odot y) \in X$. We note that this mimics the definition of convexity over $\mathbb{R}^n$, since 0 is the neutral element of $\odot$ and the weights λ, μ are always "nonnegative", as they satisfy $\lambda, \mu \geqslant -\infty$. The following lemma relates the classical and tropical convexity, see, e.g., Develin and Sturmfels (2004), Develin and Yu (2007), Allamigeon et al. (2019) for more information.

Lemma 5. If $\boldsymbol{X} \subset \mathbb{K}^n$ is convex, then $\mathsf{val}(\boldsymbol{X}) \subset \mathbb{T}^n$ is tropically convex.

The following extension of tropical convexity to $\mathbb{T}_\pm$ was introduced by Loho and Végh (2020).

Definition 6. We say that a set $X \subset \mathbb{T}^n_\pm$ is *TO-convex* if for every $x, y \in X$ and for all $\lambda, \mu \in \mathbb{T}_\pm$ such that $\lambda, \mu \geqslant -\infty$ and $\lambda \boxplus \mu = 0$ we have $(\lambda\odot x)\boxplus(\mu\odot y) \subset X$.

The drawback of multivalued addition is that the TO-convexity is a rather strong property. In particular, Lemma 5 does not generalize to the signed setting (this requires to use a weaker notion of TC-convexity). Nevertheless, we still have a weaker property.

Lemma 7. (Loho and Skomra (2022a)). If $\boldsymbol{X} \subset \mathbb{K}^n$ is convex, then the interior of $\mathsf{sval}(\boldsymbol{X}) \subset \mathbb{T}^n_\pm$ is TO-convex.

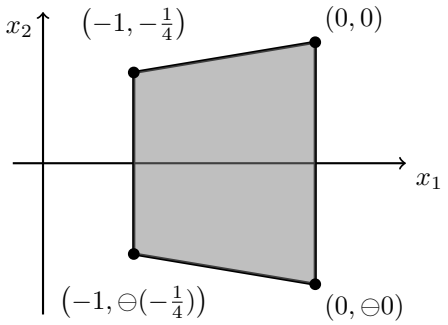

Fig. 1. Signed valuation of a convex semialgebraic set.

Example 8. The *TV screen set* is a convex set in $\mathbb{R}^2$ defined as $\{x \in \mathbb{R}^2 \colon x_1^4 + x_2^4 \leqslant 1\}$. The convexity of the TV screen set implies that the set $\boldsymbol{S}_1 = \{\boldsymbol{x} \in \mathbb{K}^2 \colon (\boldsymbol{x}_1 - 1 - t^{-1})^4 + \boldsymbol{x}_2^4 \leqslant 1\}$ is also convex. Figure 1 depicts the set $\mathsf{sval}(\boldsymbol{S}_1)$. We note that this set is TO-convex.

Example 9. The *bean curve* is the curve defined by the polynomial $p(x) = x_1(x_1^2 + x_2^2) - x_1^4 - x_1^2x_2^2 - x_2^4$. The region $\{x \in \mathbb{R}^2 \colon p(x) \geqslant 0\}$ is convex. This implies that the set $\boldsymbol{S}_2 = \{\boldsymbol{x} \in \mathbb{K}^2 \colon \boldsymbol{x}_1 \geqslant t^{-1}, p(1 - \boldsymbol{x}_1, \boldsymbol{x}_2) \geqslant 0\}$ is also convex. Its signed valuation is the same as in the previous example, $\mathsf{sval}(\boldsymbol{S}_1) = \mathsf{sval}(\boldsymbol{S}_2)$.

3. MAIN RESULTS

We now state our main results concerning the signed valuations of convex semialgebraic sets. Some of these results were obtained in collaboration with Georg Loho.

For the purpose of this work, we say that a set $X \subset \mathbb{T}_\pm^n$ is *regular* if it is equal to the closure of its interior. Regular sets arise naturally in the study of valuations of convex sets—it is known that generic tropical polyhedra are regular (Allamigeon et al., 2015) and the same is true for tropical Metlzer spectrahedra (Allamigeon et al., 2020).

3.1 Polyhedra

The signed valuations of polyhedra are studied in Loho and Végh (2020); Loho and Skomra (2022a,b). In particular, we have the following result.

Theorem 10. (Loho and Skomra (2022a)). Let $X \subset \mathbb{T}_\pm^n$ be a regular set. Then, the following are equivalent:

(1) X is an intersection of finitely many signed tropical halfspaces.
(2) X is a signed valuation of a polyhedron.

Here, a *signed tropical halfspace* is a set of the form

$$\{x \in \mathbb{T}_\pm^n \colon P(x) \cap [-\infty, +\infty) \neq \emptyset\},$$

where $P(x)$ is an affine tropical polynomial,

$$P(x) = c_0 \boxplus (c_1 \odot x_1) \boxplus \ldots \boxplus (c_n \odot x_n).$$

For the unsigned valuation, it is known that Theorem 10 holds even without the regularity assumption, see Develin and Yu (2007); Gaubert and Katz (2011). On the other hand, some assumption is necessary in the signed case. Indeed, an intersection of finitely many signed tropical halfspaces does not need to be connected, whereas a signed valuation of a convex set is always connected. It is an open question if Theorem 10 is true if we replace the assumption "X is regular" by "X is connected".

3.2 General convex semialgebraic sets

In order to characterize the regular sets that arise as images of convex semialgebraic sets, we need the following definition.

Definition 11. Let $X \subset \mathbb{T}_\pm^n$ be any set and let $\sigma \in \{-1, 1\}^n$ be a vector of signs. Then, the *maximal stratum of X given by σ* is the subset of $\mathbb{R}^n$ defined as

$$\mathrm{str}(\sigma, X) = \{(|x_1|, \ldots, |x_n|) \colon x \in X \wedge \forall i, \mathrm{tsign}(x_i) = \sigma_i\}.$$

We also say that a closed subset of $\mathbb{R}^n$ is *semilinear* if it is a union of polyhedra of the form $\{x \colon Ax \geqslant b\}$, where $b \in \mathbb{R}^m$ and $A \in \mathbb{Q}^{m \times n}$ is a rational matrix. The following theorem is our main result about tropicalizations of convex semialgebraic sets.

Theorem 12. Suppose that the set $X \subset \mathbb{T}_\pm^n$ is regular. Then, the following are equivalent:

(1) X has a TO-convex interior and semilinear maximal strata.
(2) X is a signed valuation of a convex semialgebraic set.

For the unsigned valuation, a full characterization that does not require regularity was given by Allamigeon et al. (2019). As in the case of polyhedra, it is an open question to obtain a generalization of Theorem 12 that does not require regularity.

The next corollary gives more insight into the structure of signed valuations of convex cones.

Corollary 13. Suppose that $\boldsymbol{X} \subset \mathbb{K}^n$ is a convex semialgebraic cone such that $\mathsf{sval}(\boldsymbol{X})$ is regular. Let $\mathcal{W}$ be any nonempty maximal stratum of $\mathsf{sval}(\boldsymbol{X})$. Then, $\mathcal{W}$ is a support of a pure polyhedral complex of dimension n. Furthermore, if F is an $(n-1)$-dimensional face of this complex, then the affine space spanned by F is of the form

$$\{x \in \mathbb{R}^n \colon \lambda + x_k = p^T x\}$$

for some $k \in [n]$, $\lambda \in \mathbb{R}$, and $p \in \mathbb{Q}^n_{\geqslant 0}$ that satisfies $p_k = 0$ and $\sum_{i=1}^n p_i = 1$.

Example 14. Consider the cone $\boldsymbol{S}_3 = \{\boldsymbol{x} \in \mathbb{K}^3 \colon \big(\boldsymbol{x}_1 - (1 + t^{-1})\boldsymbol{x}_3\big)^4 + \boldsymbol{x}_2^4 - \boldsymbol{x}_3^4 \leqslant 0, \boldsymbol{x}_3 \geqslant 0\}$, which is a homogenized version of the set $\boldsymbol{S}_1$ from Example 8. Its signed valuation $S = \mathsf{sval}(\boldsymbol{S}_3)$ is therefore a (tropically) homogenized version of the set depicted in Figure 1. If we fix $\sigma = (1, 1, 1)$, then $\mathrm{str}(\sigma, S)$ is a polyhedral complex with three faces of dimension 2. These faces are included in affine spaces given by $x_3 = x_1$, $x_3 - 1 = x_1$, and

$$x_2 = \frac{1}{4}x_1 + \frac{3}{4}x_3. \tag{2}$$

3.3 Hyperbolicity cones

Let us recall that a homogeneous polynomial $\boldsymbol{P} \in \mathbb{K}[x]$ is *hyperbolic with respect to* $\boldsymbol{e} \in \mathbb{K}^n$ if $\boldsymbol{P}(\boldsymbol{e}) > 0$ and, for all $\boldsymbol{x} \in \mathbb{K}^n$, all the roots of the univariate polynomial $\boldsymbol{\lambda} \mapsto \boldsymbol{P}(\boldsymbol{e} - \boldsymbol{\lambda}\boldsymbol{x})$ belong to $\mathbb{K}$. If $\boldsymbol{P}$ is hyperbolic with respect to $\boldsymbol{e}$, then the set

$$\{\boldsymbol{x} \in \mathbb{K}^n \colon \boldsymbol{\lambda} \mapsto \boldsymbol{P}(\boldsymbol{e} - \boldsymbol{\lambda}\boldsymbol{x}) \text{ has only nonnegative roots}\}$$

is called its *hyperbolicity cone*. The following lemmas summarize basic properties of hyperbolicity cones. They are stated over $\mathbb{R}$ in Renegar (2006), but they hold in all real closed fields.

Lemma 15. Hyperbolicity cones are convex.

Lemma 16. The hyperbolicity cone of $\boldsymbol{P}$ with respect to $\boldsymbol{e}$ is equal to the closure of the connected component of the set $\{\boldsymbol{x} \in \mathbb{K}^n \colon \boldsymbol{P}(\boldsymbol{x}) > 0\}$ that contains $\boldsymbol{e}$.

Our first result gives a tropical analogue of Lemma 16.

Proposition 17. Suppose that $\boldsymbol{X} \subset \mathbb{K}^n$ is a hyperbolicity cone of $\boldsymbol{P}$ with respect to some point. Furthermore, suppose that $\mathsf{sval}(\boldsymbol{X})$ is regular and let e be any point in the interior of $\mathsf{sval}(\boldsymbol{X})$. Then, $\mathsf{sval}(\boldsymbol{X})$ is equal to the closure of the connected component of the set

$$\{x \in \mathbb{T}_{\pm}^n \colon \operatorname{trop}(\boldsymbol{P})(x) \text{ is a positive singleton}\}$$

that contains e.

We note that the boundary of the connected component mentioned in Proposition 17 belongs to the tropical hypersurface of $\operatorname{trop}(\boldsymbol{P})$. Proposition 17 implies that this boundary is a subset of $\mathsf{sval}\big(\{\boldsymbol{x} \in \mathbb{K}^n \colon \boldsymbol{P}(\boldsymbol{x}) = 0\}\big)$.

Our final result implies that signed valuations of hyperbolicity cones have more restricted structure than signed valuations of general convex semialgebraic sets.

Theorem 18. Suppose that $\boldsymbol{X} \subset \mathbb{K}^n$ is a hyperbolicity cone such that $\mathsf{sval}(\boldsymbol{X})$ is regular. Let $\mathcal{W}$ be a nonempty maximal stratum of $\mathsf{sval}(\boldsymbol{X})$ and let F be an $(n-1)$-dimensional face of a polyhedral complex with support $\mathcal{W}$. Then, the affine space spanned by F is of the form

$$\{x \in \mathbb{R}^n \colon \lambda + x_k = p^T x\}, \tag{3}$$

for some $k \in [n]$, $\lambda \in \mathbb{R}$, and $p \in \{0, \frac{1}{2}, 1\}^n$ that satisfies $p_k = 0$ and $\sum_{i=1}^n p_i = 1$. In particular, p has either one or two nonzero coefficients.

Example 19. Equation (2) implies that the set $\boldsymbol{S}_3$ from Example 14 does not satisfy the condition of Theorem 18. In particular, $\boldsymbol{S}_3$ is not a hyperbolicity cone. By the discussion in Examples 8 and 9, we recover the known fact that neither the TV screen set nor the set bounded by the bean curve are spectrahedral, see Helton and Vinnikov (2007); Henrion (2010) for more discussion.

Informally, Theorem 18 can be interpreted by saying that signed valuations of hyperbolicity cones are "tropically quadratic" since the hyperplanes of the form (3) are given by tropical polynomials of degree 2. In this sense, Theorem 18 generalizes the result of Allamigeon et al. (2020) who showed that generic tropical Metzler spectrahedral cones are described by systems of tropical quadratic inequalities. We note however that Theorem 18 is only a local result: it states that every face separately can be described by a tropically quadratic polynomial, but does not give a global characterization of $\mathsf{sval}(\boldsymbol{X})$. I particular, the following is the main open question about the tropicalizations of hyperbolicity cones. This question is the tropical analogue of the generalized Lax conjecture, see Amini and Brändén (2018) for more information.

Problem 20. Suppose that $X \subset \mathbb{T}_{\pm}^n$ is a signed valuation of a hyperbolicity cone. Does this imply that X is a signed valuation of a spectrahedron?

We note that if the answer to Problem 20 is negative and a counterexample is obtained from a linear transformation of a real polynomial (like in Example 14), then we would obtain a negative answer to the generalized Lax conjecture over the real numbers.

REFERENCES

Alessandrini, D. (2013). Logarithmic limit sets of real semi-algebraic sets. *Adv. Geom.*, 13(1), 155–190.

Allamigeon, X., Benchimol, P., Gaubert, S., and Joswig, M. (2015). Tropicalizing the simplex algorithm. *SIAM J. Discrete Math.*, 29(2), 751–795.

Allamigeon, X., Benchimol, P., Gaubert, S., and Joswig, M. (2018). Log-barrier interior point methods are not strongly polynomial. *SIAM J. Appl. Algebra Geom.*, 2(1), 140–178.

Allamigeon, X., Gaubert, S., and Skomra, M. (2019). The tropical analogue of the Helton–Nie conjecture is true. *J. Symbolic Comput.*, 91, 129–148.

Allamigeon, X., Gaubert, S., and Skomra, M. (2020). Tropical spectrahedra. *Discrete Comput. Geom.*, 63, 507–548.

Allamigeon, X. and Katz, R.D. (2017). Tropicalization of facets of polytopes. *Linear Algebra Appl.*, 523, 79–101.

Amini, N. and Brändén, P. (2018). Non-representable hyperbolic matroids. *Adv. Math.*, 334, 417–449.

Baker, M. and Bowler, N. (2019). Matroids over partial hyperstructures. *Adv. Math.*, 343, 821–863.

Blekherman, G., Parrilo, P.A., and Thomas, R.R. (2013). *Semidefinite Optimization and Convex Algebraic Geometry*, volume 13 of *MOS-SIAM Ser. Optim.* SIAM, Philadelphia, PA.

Develin, M. and Sturmfels, B. (2004). Tropical convexity. *Doc. Math.*, 9, 1–27 and 205–206 (erratum).

Develin, M. and Yu, J. (2007). Tropical polytopes and cellular resolutions. *Exp. Math.*, 16(3), 277–291.

Gaubert, S. and Katz, R.D. (2011). Minimal half-spaces and external representation of tropical polyhedra. *J. Algebraic Combin.*, 33(3), 325–348.

Helton, J.W. and Vinnikov, V. (2007). Linear matrix inequality representation of sets. *Comm. Pure Appl. Math.*, 60(5), 654–674.

Henrion, D. (2010). Detecting rigid convexity of bivariate polynomials. *Linear Algebra Appl.*, 432(5), 1218–1233.

Jell, P., Scheiderer, C., and Yu, J. (2022). Real tropicalization and analytification of semialgebraic sets. *Int. Math. Res. Not.*, 2022(2), 928–958.

Joswig, M. (2021). *Essentials of Tropical Combinatorics*, volume 219 of *Grad. Stud. Math.* AMS, Providence, RI.

Joswig, M. and Smith, B. (2018). Convergent Hahn series and tropical geometry of higher rank. arXiv:1809.01457.

Le Texier, C. (2021). Hyperbolic plane curves near the non-singular tropical limit. arxiv:2109.14961.

Loho, G. and Skomra, M. (2022a). Applications of signed tropical convexity. Unpublished.

Loho, G. and Skomra, M. (2022b). Signed tropical halfspaces and convexity. arxiv:2206.13919.

Loho, G. and Végh, L.A. (2020). Signed Tropical Convexity. In *Proceedings of ITCS 2020*, 24:1–24:35.

Maclagan, D. and Sturmfels, B. (2015). *Introduction to Tropical Geometry*, volume 161 of *Grad. Stud. Math.* AMS, Providence, RI.

Renegar, J. (2006). Hyperbolic programs, and their derivative relaxations. *Found. Comput. Math.*, 6(1), 59–79.

van den Dries, L. and Speissegger, P. (1998). The real field with convergent generalized power series. *Trans. Amer. Math. Soc.*, 350(11), 4377–4421.

Yu, J. (2015). Tropicalizing the positive semidefinite cone. *Proc. Amer. Math. Soc.*, 143(5), 1891–1895.