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# Stochastic Gain-Scheduled Control of Discrete-Time Systems Characterized by Random Polytopes \*

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## Abstract

This paper discusses stability analysis and gain-scheduled controller synthesis for discrete-time non-stationary stochastic systems characterized by random polytopes. An associated deterministic time-varying parameter is assumed to be available online, and how to use its information in adjustment of controller gains is discussed in a framework of stochastic control using linear matrix inequalities. Conservativeness is reduced using our recent results and the  $S$ -variable approach.

## 1 Introduction

This paper is concerned with stability analysis and gain-scheduled controller synthesis for discrete-time linear systems with non-stationary random coefficient matrices. The coefficient matrices are assumed to depend on both an independent and identically distributed (i.i.d.) process (a sequence of i.i.d. random variables [Kni09]) and a deterministic time-varying parameter. The i.i.d. process is introduced for representing a sort of stochastic uncertainties of the systems, while the time-varying parameter is for describing the temporal variations of the distributions of such uncertainties. To couple such non-stationary stochastic systems with a gain-scheduling control approach [RS00, dSCNP21], we use the concept called random polytopes [Hug13] (details will be clearer later). We suppose that the time-varying parameter is available online, and discuss gain-scheduled control using its information.

Gain-schedule control is a well-known approach to controlling deterministic linear parameter-varying (LPV) systems, which naturally appear when dealing with nonlinear systems in control problems. For example, the behavior of a vehicle can be modeled by an LPV system in which the vehicle speed is regarded as the time-varying parameter [Raj11]. By adjusting a gain in accordance with a time-varying parameter such as the vehicle speed, the corresponding gain-scheduled controller can achieve better performance than usual robust controllers without such adjustment functions. The main idea of the present paper is to import this gain scheduling technique into control of stochastic systems.

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Control theory for stochastic systems depending only on an i.i.d. process has been discussed, e.g., in [DK92, OM13, HH19]. To deal with ambiguity of the distributions related to the i.i.d. process, the group of the authors also discussed in [HHP18, HPH20] robust control of such stochastic systems by using random polytopes. In robust control of deterministic systems, the introduction of auxiliary variables [dOBG99] (called  $S$ -variables in [EPA15]) is known to be effective for reducing conservativeness of associated analysis and synthesis. This idea was already exploited in the earlier study [HPH20] for robust control of stochastic systems. The present paper also exploits the  $S$ -variable approach in the derivations of inequality conditions for gain-scheduled control to reduce associated conservativeness. Then, one of the essential differences of the present study from these earlier studies is that *the coefficient matrices of the systems are no longer i.i.d. because of the dependence on the introduced time-varying parameter*. Hence, not the framework of these earlier studies about systems with an i.i.d. process but that in [HH22] about systems with general stochastic dynamics is required for ensuring stability of the closed-loop system, with which the main arguments will begin later.

The i.i.d. process is known to be compatible with networked control systems affected by random communication delays, and the results in [HH19] were confirmed, through experiments using a real automobile and the Internet, to be useful for achieving remote automated driving under the condition of constant speed running [KH22]. Although the details are omitted, if one uses the same remote control system in [KH22], then the discrete-time plant for which controllers are designed becomes stochastic and has an  $A$  matrix that can be approximated in the form

$$A(h_k) = A_0(h_k) + v^{-1}A_1(h_k), \quad (1)$$

where  $k$  denotes the discrete time instant,  $h_k$  denotes the randomly time-varying communication delay (which may be regarded as  $\xi_k$  introduced later), and  $v$  denotes the vehicle speed. In [KH22],  $v$  was fixed, and thus the distribution of  $A(h_k)$  was time-invariant. To make the remote automated driving more practical, however, time-varying  $v$  is required to be allowed, which leads to a non-stationary random  $A$  matrix. The results in the present paper about stochastic gain-scheduled control enable us, e.g., to upgrade the control law so that the time-varying  $v$  is not only allowed but also used for online adjustment of the controller gain to improve the control performance. Note that this kind of control cannot be achieved only with deterministic control approaches (at least theoretically), since the range of  $h_k$  is not bounded in advance. If one tries to ensure the worst case performance for such an unbounded range by deterministic robust control theory, then the corresponding problem would have no solutions; even when the range is assumed to be bounded as the problem setting, no use of the information on the distribution of  $h_k$  obviously leads to conservativeness.

It should be also noted that a similar problem of stochastic gain-scheduled control was once studied in the conference paper [NHH15] by the group of the authors; the present paper corresponds to the journal version of [NHH15]. At the time of this conference paper, the developed approach not only was overly conservative but also required sample-based evaluation of control performance, since nontrivial advances discussed in [HH19, HPH20, HH22] for stochastic control had not been obtained (for more details about the sample-based evaluation, see [NHH15]). The present journal version resolves all these inconvenience through fundamentally redeveloping the associated theory with the latest scientific findings.

We use the following notation in this paper. The set of real numbers, that of positive real numbers, and that of nonnegative integers are denoted by  $\mathbf{R}$ ,  $\mathbf{R}_+$  and  $\mathbf{N}_0$ , respectively. The set of  $n$ -dimensional real column vectors and that of  $m \times n$  real matrices are denoted by  $\mathbf{R}^n$  and  $\mathbf{R}^{m \times n}$ , respectively. The set of  $n \times n$  symmetric matrices and that of  $n \times n$  positive definite matrices are denoted by  $\mathbf{S}^{n \times n}$  and  $\mathbf{S}_+^{n \times n}$ , respectively. The identity matrix of size  $n$  is denoted by  $I_n$ ; the subscript will be dropped when the size is obvious. The Euclidean norm is denoted by  $\|\cdot\|$ . The vectorization of the matrix  $(\cdot)$  in the row direction is denoted by  $\text{row}(\cdot)$ , i.e.,  $\text{row}(\cdot) := [\text{row}_1(\cdot), \dots, \text{row}_m(\cdot)]$ , where  $m$  is the number of rows of the matrix and  $\text{row}_i(\cdot)$  denotes the  $i$ th row. The expectation is denoted by  $E[\cdot]$ . The Kronecker product is denoted by  $\otimes$ . For the real square matrix  $M$ ,  $\text{He}(M) := M + M^T$ , where  $M^T$  denotes the transpose of  $M$ . The (block-)diagonal matrix is denoted by  $\text{diag}(\cdot)$ .

## 2 Problem of Synthesis for Stochastic Gain-Scheduled Control

### 2.1 Random Polytopes

In the usual robust control theory, polytopes are used to describe a class of uncertainties in deterministic systems. Since the coefficients of deterministic systems are naturally deterministic, such polytopes are usually defined with deterministic vertices. In the case of stochastic systems, however, coefficients of systems may be random. Hence, uncertainties or ambiguities of coefficients of stochastic systems cannot be described only with the usual deterministic polytopes, in general. As one of the ways to resolve this issue partially, the use of the concept called random polytopes [Hug13] is proposed in [HHP18, HPH20].

Let us consider a parameter vector  $\vartheta = [\vartheta^{(1)}, \dots, \vartheta^{(L)}]^T$  belonging to the set

$$\mathbf{E}^L := \left\{ [\vartheta^{(1)}, \dots, \vartheta^{(L)}]^T \in \mathbf{R}^L : \vartheta^{(l)} \geq 0 \quad (l = 1, \dots, L), \sum_{l=1}^L \vartheta^{(l)} = 1 \right\}. \quad (2)$$

Then, for given  $Z$ -dimensional random vector  $\zeta$  with support  $\Xi \subset \mathbf{R}^Z$  and Borel measurable matrix-valued functions  $A^{(l)} : \Xi \rightarrow \mathbf{R}^{n \times n}$  ( $l = 1, \dots, L$ ), a convex combination of random matrices

$$A(\zeta; \vartheta) := \sum_{l=1}^L \vartheta^{(l)} A^{(l)}(\zeta) \quad (3)$$

can be defined. For each sample of  $\zeta$ , the set  $\mathbf{A}(\zeta) := \{A(\zeta; \vartheta)\}_{\vartheta \in \mathbf{E}^L}$  becomes a standard convex polytope. Since  $\zeta$  is actually random, the set  $\mathbf{A}(\zeta)$  also becomes random. Such  $\mathbf{A}(\zeta)$  is called a random polytope. By choosing  $\zeta$  and  $A^{(l)}$  ( $l = 1, \dots, L$ ) appropriately, a class of uncertainties in the distributions of coefficients in stochastic systems can be described; e.g., uncertainties (or ambiguities) in the hyper-parameters of Gaussian distributions, exponential distributions and continuous uniform distributions.

Random polytopes are already used for tackling robust control of stochastic systems in our previous research [HHP18, HPH20] through viewing the aforementioned  $\vartheta$  as an unknown deterministic time-invariant parameter vector. This paper, on the other hand, introduces a time-varying parameter vector  $\theta_k$  available online, and discusses gain-scheduled control using such  $\theta_k$  to achieve some adaptation in control of non-stationary stochastic systems. Since our  $\theta_k$  can describe a class of temporal variations of distributions of random coefficients in stochastic systems, the controller designed by our theory can adapt to such variations online; e.g., the time-varying vehicle speed in the networked control system for remote automated driving referred to in the Introduction can be represented with  $\theta_k$ .

**Remark 1** *In this paper, the  $k$ -independent symbols  $\vartheta$  and  $\zeta$  are used in the arguments irrelevant to the discrete time, while  $\theta$ ,  $\xi$ ,  $\theta_k$  and  $\xi_k$  are used in the arguments related to the discrete time.*

### 2.2 Systems Characterized by Random Polytopes and Problem of Synthesis

Let us first introduce a  $Z$ -dimensional discrete-time stochastic process  $\xi = (\xi_k)_{k \in \mathbf{N}_0}$  and a deterministic time-varying parameter vector  $\theta_k$  characterizing our system described later. The process  $\xi$  is supposed to satisfy the following assumption.

**Assumption 1** *The process  $\xi$  is independent and identically distributed (i.i.d.), i.e.,  $\xi_0, \xi_1, \dots$  are i.i.d.*

The process  $\xi$  satisfying this assumption is stationary and ergodic. The time-invariant support of such  $\xi_k$  is denoted by  $\Xi$ . The time-varying vector  $\theta_k$  is supposed to belong to  $\mathbf{E}^L$  for each  $k \in \mathbf{N}_0$ , i.e., the sequence  $\theta = (\theta_k)_{k \in \mathbf{N}_0}$  belongs to the set

$$\mathcal{E}^L := \{(\theta_k)_{k \in \mathbf{N}_0} : \theta_k \in \mathbf{E}^L \quad (\forall k \in \mathbf{N}_0)\}. \quad (4)$$

The  $l$ th entry of  $\theta_k$  is denoted by  $\theta_k^{(l)}$ .

By using these  $\xi$  and  $\theta$ , we introduce the system

$$x_{k+1} = A(\xi_k; \theta_k)x_k + B(\xi_k)u_k, \quad (5)$$

where  $x_k$  is the state,  $u_k$  is the input, and  $A : \mathcal{E} \times \mathbf{E}^L \rightarrow \mathbf{R}^{n \times n}$  and  $B : \mathcal{E} \rightarrow \mathbf{R}^{n \times m}$  are Borel measurable matrix-valued functions. The function  $A$  is supposed to be described by (3) with given vertex functions  $A^{(l)} : \mathcal{E} \rightarrow \mathbf{R}^{n \times n}$  ( $l = 1, \dots, L$ ). In addition, those vertices and  $B$  are also supposed to satisfy the following assumption.

**Assumption 2** For each  $l = 1, \dots, L$ , the squares of entries of  $A^{(l)}(\xi_0)$  are all Lebesgue integrable, i.e.,

$$E[A_{ij}^{(l)}(\xi_0)^2] < \infty \quad (\forall i, j = 1, \dots, n), \quad (6)$$

where  $A_{ij}^{(l)}(\xi_0)$  is the  $(i, j)$ -entry of  $A^{(l)}(\xi_0)$ . Similarly, the squares of entries of  $B(\xi_0)$  are also all Lebesgue integrable.

Assumptions 1 and 2 imply the coefficient matrices to be Lebesgue integrable regardless of  $k$ . This is a sort of minimal requirement for defining second-moment stability of the system [HH22]. Since  $\theta_k$  is time-varying, the distribution of  $A(\xi_k; \theta_k)$  becomes time-varying. We mainly consider the situation where this deterministic  $\theta_k$  is available online in control, which is used for adjusting the controller gain as follows.

$$u_k = F(\theta_k)x_k, \quad (7)$$

$$F(\theta_k) := \sum_{l=1}^L \theta_k^{(l)} F^{(l)}, \quad (8)$$

$$F^{(l)} \in \mathbf{R}^{m \times n} \quad (l = 1, \dots, L) \quad (9)$$

This is nothing but gain-scheduled control for stochastic systems. The closed-loop system, denoted by  $G_\theta$ , consisting of system (5) and this state feedback controller is given by

$$x_{k+1} = A_{\text{cl}}(\xi_k; \theta_k)x_k, \quad (10)$$

$$A_{\text{cl}}(\xi_k; \theta_k) = \sum_{l=1}^L \theta_k^{(l)} A_{\text{cl}}^{(l)}(\xi_k), \quad (11)$$

$$A_{\text{cl}}^{(l)}(\xi_k) = A^{(l)}(\xi_k) + B(\xi_k)F^{(l)}. \quad (12)$$

For given state feedback (7), second-moment exponential stability of the closed-loop system  $G_\theta$  can be defined as follows [Koz69].

**Definition 1** Suppose that Assumptions 1 and 2 are satisfied and state feedback (7) is given. For a given  $\theta \in \mathcal{E}^L$ , the system  $G_\theta$  is said to be exponentially stable in the second moment, if there exist  $a = a(\theta) \in \mathbf{R}_+$  and  $\lambda = \lambda(\theta) \in (0, 1)$  such that

$$\sqrt{E[\|x_k\|^2]} \leq a \|x_0\| \lambda^k \quad (\forall k \in \mathbf{N}_0, \forall x_0 \in \mathbf{R}^n). \quad (13)$$

The problem to be tackled in this paper is to design state feedback (7) such that the closed-loop system  $G_\theta$  is exponentially stable in the second moment robustly with respect to  $\mathcal{E}^L$ , i.e., regardless of  $\theta \in \mathcal{E}^L$ .

## 3 Stability Conditions

### 3.1 Lyapunov Inequality

Since the present stochastic system is not stationary because of the presence of the time-varying parameter vector  $\theta_k$ , we cannot apply the usual i.i.d. results in [HH19] about Lyapunov inequalities. Hence, by temporarily regarding  $\theta_k$  as a part of non-stationary stochastic process, we exploit the results in [HH22] about Lyapunov inequalities for general stochastic systems. Then, although the details are omitted in this paper, we have the following theorem.

**Theorem 1** *Suppose that Assumptions 1 and 2 are satisfied and state feedback (7) is given. For each  $\theta \in \mathcal{E}^L$ , the following two conditions are equivalent.*

1. *The system  $G_\theta$  is exponentially stable in the second moment.*
2. *There exist  $\epsilon_1, \epsilon_2 \in \mathbf{R}_+$ ,  $\lambda \in (0, 1)$  and  $P_k \in \mathbf{S}^{n \times n}$  ( $k \in \mathbf{N}_0$ ) such that*

$$P_k \geq \epsilon_1 I \quad (\forall k \in \mathbf{N}_0), \quad (14)$$

$$P_k \leq \epsilon_2 I \quad (\forall k \in \mathbf{N}_0), \quad (15)$$

$$E[\lambda^2 P_k - A_{\text{cl}}(\xi_k; \theta_k)^T P_{k+1} A_{\text{cl}}(\xi_k; \theta_k)] \geq 0 \quad (\forall k \in \mathbf{N}_0). \quad (16)$$

The inequality (16) is nothing but a Lyapunov inequality for the present non-stationary stochastic system. This inequality is not only infinite dimensional with respect to  $k$  but also expectation-based with some nonlinearity. Hence, the direct use of the inequality in numerical analysis and synthesis is difficult. In the following two subsections, we resolve this difficulty in two steps by using a modified version of the linearizing method discussed in [HPH20], and finally derive a standard LMI condition for analysis, which will be further extended toward synthesis in Section 4.

### 3.2 First Step of Linearization

The first step is to obtain a  $\theta_k$ -independent inequality condition from (16) so that the size of the inequality condition becomes finite, as is the case with time-invariant  $\theta$  in [HPH20]. The following is the key lemma about this first step, which is uniquely used in the case of stochastic systems.

**Lemma 1** *For a given random vector  $\zeta$  with support  $\mathcal{E}$ , given constant vectors  $\vartheta, \vartheta_+ \in \mathbf{E}^L$  and a given mapping  $R : \mathcal{E} \times \mathbf{E}^L \times \mathbf{E}^L \rightarrow \mathbf{S}^{n_0 \times n_0}$ , the following two conditions are equivalent.*

1. *The following inequality holds.*

$$E[R(\zeta; \vartheta, \vartheta_+)] \geq 0 \quad (17)$$

2. *There exists  $T : \mathcal{E} \times \mathbf{E}^L \times \mathbf{E}^L \rightarrow \mathbf{S}^{n_0 \times n_0}$  such that*

$$E[T(\zeta; \vartheta, \vartheta_+)] \leq 0, \quad (18)$$

$$T(\zeta_\star; \vartheta, \vartheta_+) + R(\zeta_\star; \vartheta, \vartheta_+) \geq 0 \quad (\forall \zeta_\star \in \mathcal{E}). \quad (19)$$

This lemma is a modified version of Lemma 1 in [HPH20] so that time-varying  $\theta_k$  can be dealt with later. This lemma leads us to the following corollary, which is also a modified version of Corollary 1 in [HPH20].

**Corollary 1** *Suppose that  $\vartheta, \vartheta_+ \in \mathbf{E}^L$ ,  $\lambda \in (0, 1)$ ,  $\mathcal{P}, \mathcal{P}_+ \in \mathbf{S}_+^{n \times n}$ ,  $S \in \mathbf{R}^{n_1 \times n_2}$ ,  $T : \mathcal{E} \times \mathbf{E}^L \times \mathbf{E}^L \rightarrow \mathbf{S}^{n_0 \times n_0}$ ,  $J : \mathbf{S}^{n_0 \times n_0} \rightarrow \mathbf{S}^{n_1 \times n_1}$ ,  $U : \mathbf{R} \times \mathbf{S}^{n \times n} \times \mathbf{S}^{n \times n} \rightarrow \mathbf{S}^{n_1 \times n_1}$ ,  $N_1 : \mathcal{E} \times \mathbf{E}^L \rightarrow \mathbf{R}^{n_1 \times n_0}$ ,  $N_2 : \mathcal{E} \times \mathbf{E}^L \rightarrow \mathbf{R}^{n_2 \times n_1}$  and the distribution of the  $Z$ -dimensional random vector  $\zeta$  with support  $\mathcal{E}$  are given. If they satisfy (18) and*

$$J(T(\zeta_\star; \vartheta, \vartheta_+)) + U(\lambda, \mathcal{P}, \mathcal{P}_+) + \text{He}(SN_2(\zeta_\star; \vartheta)) \geq 0 \quad (\forall \zeta_\star \in \mathcal{E}), \quad (20)$$

$$N_1(\zeta_\star; \vartheta)^T J(\Pi) N_1(\zeta_\star; \vartheta) = \Pi \quad (\forall \Pi \in \mathbf{S}^{n_0 \times n_0}, \forall \zeta_\star \in \mathcal{E}), \quad (21)$$

$$N_2(\zeta_\star; \vartheta) N_1(\zeta_\star; \vartheta) = 0 \quad (\forall \zeta_\star \in \mathcal{E}), \quad (22)$$

then

$$E[N_1(\zeta; \vartheta)^T U(\lambda, \mathcal{P}, \mathcal{P}_+) N_1(\zeta; \vartheta)] \geq 0. \quad (23)$$

**Proof 1** *Post-multiplying  $N_1(\zeta_\star; \vartheta)$  and pre-multiplying its transpose on (20), together with the use of (21) and (22), lead to*

$$\begin{aligned} T(\zeta_\star; \vartheta, \vartheta_+) + N_1(\zeta_\star; \vartheta)^T U(\lambda, \mathcal{P}, \mathcal{P}_+) N_1(\zeta_\star; \vartheta) &\geq 0 \\ (\forall \zeta_\star \in \mathcal{E}). \end{aligned} \quad (24)$$

Hence, (18), (24) and Lemma 1 with  $R(\zeta_\star; \vartheta, \vartheta_+) = N_1(\zeta_\star; \vartheta)^T U(\lambda, \mathcal{P}, \mathcal{P}_+) N_1(\zeta_\star; \vartheta)$  lead to (23).

By using the above lemma and corollary, as well as the restriction

$$P_k = P(\theta_k) := \sum_{l=1}^L \theta_k^{(l)} P^{(l)}, \quad (25)$$

$$P^{(l)} \in \mathbf{S}^{n \times n} \quad (l = 1, \dots, L) \quad (26)$$

on the Lyapunov matrix, we obtain the following theorem, which can be proved in a similar manner to the case of time-invariant  $\theta$  in [HPH20], through appropriately dealing with  $\vartheta$  and  $\vartheta_+$  associated with the time-varying  $\theta_k$ .

**Theorem 2** *Suppose that Assumptions 1 and 2 are satisfied and state feedback (7) is given. The closed-loop system  $G_\theta$  is exponentially stable in the second moment robustly with respect to  $\mathcal{E}^L$ , if there exist  $S_1, S_2 \in \mathbf{R}^{n \times n}$ ,  $P^{(l)} \in \mathbf{S}_+^{n \times n}$  ( $l = 1, \dots, L$ ) and  $\lambda \in (0, 1)$  satisfying*

$$S_2 + S_2^T - P^{(l)} > 0 \quad (l = 1, \dots, L), \quad (27)$$

$$E[\lambda^2 P^{(l)} + \text{He}(S_1 A_{\text{cl}}^{(l)}(\xi_0)) - (S_1^T + S_2 A_{\text{cl}}^{(l)}(\xi_0))^T (S_2 + S_2^T - P^{(l)})^{-1} (S_1^T + S_2 A_{\text{cl}}^{(l)}(\xi_0))] \geq 0 \quad (l, l_+ = 1, \dots, L). \quad (28)$$

**Proof 2** *Lemma 1 (or that in [HPH20]) and the Schur complement technique immediately imply that for given  $S_1, S_2 \in \mathbf{R}^{n \times n}$ ,  $P^{(l)} \in \mathbf{S}_+^{n \times n}$  ( $l = 1, \dots, L$ ) and  $\lambda \in (0, 1)$ , all the inequalities (28) hold if and only if there exist  $T^{(l, l_+)} : \mathcal{E} \rightarrow \mathbf{S}^{n \times n}$  ( $l, l_+ = 1, \dots, L$ ) satisfying*

$$E[T^{(l, l_+)}(\zeta)] \leq 0 \quad (l, l_+ = 1, \dots, L), \quad (29)$$

$$\begin{bmatrix} T^{(l, l_+)}(\zeta_\star) + \lambda^2 P^{(l)} & 0 \\ 0 & -P^{(l_+)} \end{bmatrix} + \text{He}(S[A_{\text{cl}}^{(l)}(\zeta_\star) \quad I]) \geq 0 \quad (l, l_+ = 1, \dots, L; \forall \zeta_\star \in \mathcal{E}), \quad (30)$$

where  $S = [S_1^T, S_2^T]^T$  and  $\zeta = \xi_0$ . Multiplying  $\vartheta^{(l)} \vartheta_+^{(l_+)}$  ( $\vartheta, \vartheta_+ \in \mathbf{E}^L$ ) on (29) and (30) and taking the sums for  $l, l_+ = 1, \dots, L$  lead us to (18) and (20) with

$$\begin{aligned} U(\lambda, \mathcal{P}, \mathcal{P}_+) &= \begin{bmatrix} \lambda^2 \mathcal{P} & 0 \\ 0 & -\mathcal{P}_+ \end{bmatrix} \quad (\mathcal{P}, \mathcal{P}_+ \in \mathbf{S}_+^{n \times n}), \\ J(\cdot) &= \begin{bmatrix} (\cdot) & 0 \\ 0 & 0 \end{bmatrix}, \quad N_1(\cdot; \cdot) = \begin{bmatrix} I \\ -A_{\text{cl}}(\cdot; \cdot) \end{bmatrix}, \\ N_2(\cdot; \cdot) &= [A_{\text{cl}}(\cdot; \cdot) \quad I], \end{aligned} \quad (31)$$

where

$$T(\zeta_\star; \vartheta, \vartheta_+) = \sum_{l=1}^L \sum_{l_+=1}^L \vartheta^{(l)} \vartheta_+^{(l_+)} T^{(l, l_+)}(\zeta_\star). \quad (32)$$

The above  $U$ ,  $J$ ,  $N_1$  and  $N_2$  are given to satisfy (21) and (22). Hence, by Corollary 1, we have (23) under  $\mathcal{P} = P(\vartheta)$  and  $\mathcal{P}_+ = P(\vartheta_+)$ . Since  $\xi_0, \xi_1, \dots$  are i.i.d., for each  $k \in \mathbf{N}_0$ , we substitute  $\zeta = \xi_k$ ,  $\vartheta = \theta_k$  and  $\vartheta_+ = \theta_{k+1}$  into the inequality. Then, we obtain (16) with  $P_k$  given by (25). Since  $P^{(l)} > 0$  ( $l = 1, \dots, L$ ), the existence of  $\epsilon_1, \epsilon_2 \in \mathbf{R}_+$  satisfying (14) and (15) is also ensured under such  $P_k$ . Hence, the condition 2 in Theorem 1 is satisfied. This, together with Theorem 1, completes the proof.

As we can see, the inequality condition (28) is independent of  $\theta_k$ , and thus is finite dimensional. This condition gives a robust stability condition. The auxiliary variables  $S_1$  and  $S_2$  are introduced so that the conservativeness of the corresponding analysis is reduced as much as possible as in the deterministic control [EPA15].

### 3.3 Second Step of Linearization

We next linearize (28) as the aforementioned second step. This involves making the decision variables uncontained in the expectation operation<sup>1</sup>. With Lemma 2 in [HPH20] and the standard LMI techniques, we obtain the following theorem.

**Lemma 2** *For matrices*

$$M_1 = \begin{bmatrix} M_{11} \\ \vdots \\ M_{1p} \end{bmatrix} \quad (M_{1i} \in \mathbf{R}^{r \times m_1}), \quad (33)$$

$$M_2 = \begin{bmatrix} M_{21} \\ \vdots \\ M_{2q} \end{bmatrix} \quad (M_{2i} \in \mathbf{R}^{r \times m_2}) \quad (34)$$

and  $H \in \mathbf{R}^{p \times q}$ ,

$$M_1^T (H \otimes I_r) M_2 = \sum_{i=1}^p \sum_{j=1}^q h_{ij} M_{1i}^T M_{2j} \quad (35)$$

holds, where  $h_{ij}$  is the  $(i, j)$ -entry of  $H$ .

This lemma, together with the standard LMI techniques, leads us to the following theorem, which gives an LMI for robust stability analysis of  $G_\theta$  with respect to  $\theta \in \mathcal{E}^L$ .

**Theorem 3** *Suppose that Assumptions 1 and 2 are satisfied and state feedback (7) is given. For given  $S_1, S_2 \in \mathbf{R}^{n \times n}$ ,  $P^{(l)} \in \mathbf{S}_+^{n \times n}$  ( $l = 1, \dots, L$ ) and  $\lambda \in (0, 1)$ , the inequalities (27) and (28) hold if and only if (27) and*

$$\begin{bmatrix} \lambda^2 P^{(l)} & 0 \\ 0 & -P^{(l)} \otimes I_{\bar{n}_l} \end{bmatrix} + \text{He} \left( \begin{bmatrix} S_1 \otimes \bar{X}_1^{(l)T} \\ S_2 \otimes I_{\bar{n}_l} \end{bmatrix} \begin{bmatrix} \tilde{X}_A^{(l)} & I_{n\bar{n}_l} \end{bmatrix} \right) \geq 0 \quad (l, l_+ = 1, \dots, L) \quad (36)$$

hold, where for each  $l = 1, \dots, L$ ,

$$\tilde{X}_A^{(l)} := \begin{bmatrix} \bar{X}_{A1}^{(l)} \\ \vdots \\ \bar{X}_{An}^{(l)} \end{bmatrix} \in \mathbf{R}^{n\bar{n}_l \times n}, \quad (37)$$

$$\begin{aligned} [\bar{X}_1^{(l)}, \bar{X}_{A1}^{(l)}, \dots, \bar{X}_{An}^{(l)}] &:= \bar{X}_{1A}^{(l)} \\ (\bar{X}_1^{(l)} \in \mathbf{R}^{\bar{n}_l \times 1}, \bar{X}_{Ai}^{(l)} \in \mathbf{R}^{\bar{n}_l \times n} \quad (i = 1, \dots, n)) & \end{aligned} \quad (38)$$

with  $\bar{X}_{1A}^{(l)} \in \mathbf{R}^{\bar{n}_l \times (n^2+1)}$  satisfying

$$\bar{X}_{1A}^{(l)T} \bar{X}_{1A}^{(l)} = E[X_{1A}^{2(l)}(\xi_0)], \quad (39)$$

$$X_{1A}^{2(l)}(\xi_0) := [1, \text{row}(A_{cl}^{(l)}(\xi_0))]^T [1, \text{row}(A_{cl}^{(l)}(\xi_0))]. \quad (40)$$

The matrices  $\bar{X}_1^{(l)}$  and  $\tilde{X}_A^{(l)}$  are constructed so as to satisfy

$$\tilde{X}_A^{(l)T} (H \otimes I_{\bar{n}_l}) \tilde{X}_A^{(l)} = E[A_{cl}^{(l)}(\xi_0)^T H A_{cl}^{(l)}(\xi_0)], \quad (41)$$

$$\tilde{X}_I^{(l)T} (H \otimes I_{\bar{n}_l}) \tilde{X}_A^{(l)} = E[H A_{cl}^{(l)}(\xi_0)], \quad (42)$$

$$\tilde{X}_I^{(l)T} (H \otimes I_{\bar{n}_l}) \tilde{X}_I^{(l)} = E[H] = H \quad (43)$$

<sup>1</sup>If one desires only to make the decision variables uncontained in the expectation operation, it suffices to extract all the entries of the  $\xi_0$ -independent decision variables, e.g., as in Section IV of [OWHP16] about stochastic switched systems. The point of the present arguments is how we achieve such an extraction so that LMI techniques such as the Schur complement and the change of variables can be applied toward the final goal of linearization.

simultaneously for any  $H \in \mathbf{R}^{n \times n}$ , where

$$\tilde{X}_I^{(l)} := I_n \otimes \bar{X}_1^{(l)} \in \mathbf{R}^{n\bar{m}_l \times n}. \quad (44)$$

These matrices are expectation-based but deterministic, which can be pre-calculated without using any decision variables. Through solving the LMI condition consisting of (27) and (36), we can analyze robust stability of the closed-loop system  $G_\theta$  with a given gain-scheduled controller.

## 4 Gain-Scheduled Controller Synthesis

In the preceding section, we obtained a  $\theta$ -independent LMI (36) for robust stability analysis. In this section, we directly extend the obtained LMI toward gain-scheduling controller synthesis. This kind of direct extension has not been discussed in any of the articles about stochastic control by the authors. Because of the complexity of the dependency of the expectation-based coefficients such as  $\bar{X}_1^{(l)}$  and  $\tilde{X}_A^{(l)}$  on the involved controller parameters, we always returned to the expectation-based matrix inequalities such as (16) and (28), and then discuss their linearization again, separately from the LMI for analysis. We resolve this redundant situation without affecting the final result, whose idea can be used also for other synthesis problems including previous ones.

Let us first consider deterministic matrices  $\tilde{Y}_1^{(l)}$ ,  $\tilde{Y}_A^{(l)}$  and  $\tilde{Y}_B^{(l)}$  given by

$$\begin{aligned} \tilde{Y}_A^{(l)} &:= \begin{bmatrix} \bar{Y}_{A1}^{(l)} \\ \vdots \\ \bar{Y}_{An}^{(l)} \end{bmatrix} \in \mathbf{R}^{n\bar{m}_l \times n}, \quad \tilde{Y}_B^{(l)} := \begin{bmatrix} \bar{Y}_{B1}^{(l)} \\ \vdots \\ \bar{Y}_{Bn}^{(l)} \end{bmatrix} \in \mathbf{R}^{n\bar{m}_l \times n} \\ [\bar{Y}_1^{(l)}, \bar{Y}_{A1}^{(l)}, \dots, \bar{Y}_{An}^{(l)}, \bar{Y}_{B1}^{(l)}, \dots, \bar{Y}_{Bn}^{(l)}] &:= \bar{Y}_{1AB}^{(l)} \\ (\bar{Y}_1^{(l)} \in \mathbf{R}^{\bar{m}_l \times 1}, \bar{Y}_{Ai}^{(l)} \in \mathbf{R}^{\bar{m}_l \times n}, \bar{Y}_{Bi}^{(l)} \in \mathbf{R}^{\bar{m}_l \times m} \\ (i = 1, \dots, n)) & \end{aligned} \quad (45)$$

with  $\bar{Y}_{1AB}^{(l)} \in \mathbf{R}^{\bar{m}_l \times (n^2 + nm + 1)}$  satisfying

$$\bar{Y}_{1AB}^{(l)T} \bar{Y}_{1AB}^{(l)} = E[\bar{Y}_{1AB}^{2(l)}(\xi_0)] \in \mathbf{R}^{(n^2 + nm + 1) \times (n^2 + nm + 1)}, \quad (47)$$

$$\bar{Y}_{1AB}^{2(l)}(\xi_0) := [1, \text{row}(A^{(l)}(\xi_0)), \text{row}(B(\xi_0))]^T [1, \text{row}(A^{(l)}(\xi_0)), \text{row}(B(\xi_0))]. \quad (48)$$

These matrices are constructed so as to satisfy

$$\tilde{Y}_A^{(l)T} (H \otimes I_{\bar{m}_l}) \tilde{Y}_A^{(l)} = E[A^{(l)}(\xi_0)^T H A^{(l)}(\xi_0)], \quad (49)$$

$$\tilde{Y}_A^{(l)T} (H \otimes I_{\bar{m}_l}) \tilde{Y}_B^{(l)} = E[A^{(l)}(\xi_0)^T H B(\xi_0)], \quad (50)$$

$$\tilde{Y}_A^{(l)T} (H \otimes I_{\bar{m}_l}) \tilde{Y}_I^{(l)} = E[A^{(l)}(\xi_0)^T H], \quad (51)$$

$$\tilde{Y}_B^{(l)T} (H \otimes I_{\bar{m}_l}) \tilde{Y}_B^{(l)} = E[B(\xi_0)^T H B(\xi_0)], \quad (52)$$

$$\tilde{Y}_B^{(l)T} (H \otimes I_{\bar{m}_l}) \tilde{Y}_I^{(l)} = E[B(\xi_0)^T H], \quad (53)$$

$$\tilde{Y}_I^{(l)T} (H \otimes I_{\bar{m}_l}) \tilde{Y}_I^{(l)} = E[H] = H \quad (54)$$

simultaneously for any  $H \in \mathbf{R}^{n \times n}$ , where

$$\tilde{Y}_I^{(l)} := I_n \otimes \bar{Y}_1^{(l)} \in \mathbf{R}^{n\bar{m}_l \times n}. \quad (55)$$

Here, if we take

$$\bar{X}_1^{(l)} = \bar{Y}_1^{(l)}, \quad \tilde{X}_A^{(l)} = \tilde{Y}_A^{(l)} + \tilde{Y}_B^{(l)} F^{(l)}, \quad (56)$$

then it follows from the direct calculation using (49)–(54) that the relations (41)–(43) with  $\bar{n}_l$  replaced by  $\bar{m}_l$  hold for any  $H \in \mathbf{R}^{n \times n}$ . This implies that the LMI condition (36) for analysis

of the closed-loop system can be rewritten in the form using  $\bar{Y}_1^{(l)}$ ,  $\tilde{Y}_A^{(l)}$  and  $\tilde{Y}_B^{(l)}$ . That is, we immediately obtain

$$\begin{bmatrix} \lambda^2 P^{(l)} & 0 \\ 0 & -P^{(l_+)} \otimes I_{\bar{m}_l} \end{bmatrix} + \text{He} \left( \begin{bmatrix} S_1 \otimes \bar{Y}_1^{(l)T} \\ S_2 \otimes I_{\bar{m}_l} \end{bmatrix} \begin{bmatrix} \tilde{Y}_A^{(l)} + \tilde{Y}_B^{(l)} F^{(l)} & I_{n\bar{m}_l} \end{bmatrix} \right) \geq 0 \quad (l, l_+ = 1, \dots, L) \quad (57)$$

as a closed-loop stability condition from (36); this simple technique has not been discussed in our previous research. By linearizing this deterministic matrix inequality, we can obtain a synthesis-oriented LMI condition. The following theorem is our main result about stochastic gain-scheduled controller synthesis.

**Theorem 4** *Suppose that Assumptions 1 and 2 are satisfied. If there exist  $A_s \in \mathbf{R}^{n \times n}$ ,  $V \in \mathbf{R}^{n \times n}$ ,  $K^{(l)} \in \mathbf{R}^{m \times n}$ ,  $Q^{(l)} \in \mathbf{S}_+^{n \times n}$  ( $l = 1, \dots, L$ ) and  $\lambda \in (0, 1)$  satisfying*

$$V + V^T - Q^{(l)} > 0 \quad (l = 1, \dots, L), \quad (58)$$

$$\begin{bmatrix} \lambda^2 Q^{(l)} & 0 \\ 0 & -Q^{(l_+)} \otimes I_{\bar{m}_l} \end{bmatrix} + \text{He} \left( \begin{bmatrix} A_s^T \otimes \bar{Y}_1^{(l)T} \\ I_{n\bar{m}_l} \end{bmatrix} \begin{bmatrix} \tilde{Y}_A^{(l)} V + \tilde{Y}_B^{(l)} K^{(l)} & V \otimes I_{\bar{m}_l} \end{bmatrix} \right) \geq 0 \quad (l, l_+ = 1, \dots, L), \quad (59)$$

then there exists gain-scheduled state feedback (7) such that the closed-loop system (10) is exponentially stable in the second moment robustly with respect to  $\mathcal{E}^L$ . In particular,  $V$  becomes nonsingular, and one such state feedback is given by (7) with  $F^{(l)} = K^{(l)} V^{-1}$  ( $l = 1, \dots, L$ ).

**Proof 3** *The matrix  $V$  is nonsingular by (58). Consider the change of variables  $G = V^{-1}$ ,  $P^{(l)} = G^T Q^{(l)} G$ ,  $F^{(l)} = K^{(l)} G$  and*

$$S = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = \begin{bmatrix} G^T A_s^T \\ G^T \end{bmatrix}. \quad (60)$$

Then, (58) equivalently reduces to (27) through appropriate congruence transformation using  $G$ . Similarly, (59) also equivalently reduces to (57) through appropriate congruence transformation using  $\text{diag}(G, G \otimes I_{\bar{m}_l})$ . This, together with the relationship (56) and Theorems 2 and 3, completes the proof.

To use (59) as an LMI, one needs to fix  $A_s$  a priori; otherwise (59) is a bilinear matrix inequality (BMI). Taking  $A_s = 0$  is empirically known to lead to a relatively good solution for this type of BMI. The reader is also invited to consult [EPA15] for details about other heuristic choices of the  $A_s$  matrix in the deterministic case, which would apply as well to the present stochastic case. For example, the following lemma illustrates that  $A_s$  needs to be chosen Schur stable, as in the deterministic case.

**Lemma 3** *If there exist  $V \in \mathbf{R}^{n \times n}$ ,  $K \in \mathbf{R}^{m \times n}$ ,  $Q^{(l)} \in \mathbf{S}_+^{n \times n}$  ( $l = 1, \dots, L$ ) and  $\lambda \in (0, 1)$  satisfying (59) and (58) for a given  $A_s \in \mathbf{R}^{n \times n}$ , then  $A_s$  is a Schur stable matrix.*

**Proof 4** *Post-multiplying  $[-I_n \quad A_s \otimes \bar{Y}_1^{(l)}]$  and pre-multiplying its transpose on (59) lead to*

$$\lambda^2 Q^{(l)} - A_s^T Q^{(l_+)} A_s \geq 0 \quad (l, l_+ = 1, \dots, L). \quad (61)$$

For  $l = l_+ = 1$ , this inequality is a deterministic Lyapunov inequality ensuring Schur stability of  $A_s$ . Hence, Schur stability of  $A_s$  follows.

If one has computational power and desires to improve the result as much as possible, a brute-force approach or metaheuristic algorithms such as the genetic algorithm may be used for the search of  $A_s$  leading to a good solution. Then, the above lemma implies that the search range of such  $A_s$  can be confined within the set of Schur stable matrices. Once a solution of (59) is obtained, it is also possible to find a better (at least not worse)  $A_s$  through searching for  $A_s$  and  $Q^{(l)}$  satisfying (59) under the obtained fixed  $V$  and  $K^{(l)}$ . Similar iterations can be further continued if needed.

By Theorem 4 and Lemma 3, a gain-scheduled controller can be systematically designed for the present non-stationary stochastic system through searching for a solution of the LMI condition consisting of (58) and (59) under an appropriately given  $A_s$ . Although we have mainly focused on the situation where  $\theta_k$  is available online in this paper, we can also design a  $\theta_k$ -independent robust controller by solving (58) and (59) under the constraint  $K^{(l)} = K$  ( $l = 1, \dots, L$ ) when  $\theta_k$  is unavailable.

## 5 Numerical Example

As stated in Section 1, a similar gain-scheduled control problem was tackled by the group of the authors in the conference paper [NHH15]. This earlier result did not use the progress in stochastic control obtained, e.g., in [HH19, HPH20] and thus was overly conservative, in addition to the need to use a sample-based approach. To compare the present result with this earlier result from the viewpoint of conservativeness, this section deals with the same numerical example used in [NHH15].

Let us consider  $\xi$  that satisfies Assumption 1 and is given by  $\xi_k = [\xi_k^1, \xi_k^2, \xi_k^3]^T$  (i.e.,  $Z = 3$ ) with the mutually independent random variables  $\xi_k^1$ ,  $\xi_k^2$  and  $\xi_k^3$  respectively obeying  $N(0, 0.1^2)$ ,  $N(0, 0.3^2)$  and  $N(0, 0.4^2)$ , where  $N(\mu, \sigma^2)$  stands for the Gaussian distribution with mean  $\mu$  and standard deviation  $\sigma$ . Let us further consider the open-loop system (5) with the coefficients  $A^{(l)}(\xi_k)$  ( $l = 1, 2$ ) and  $B(\xi_k)$  given in Section V of [NHH15].

$$\begin{aligned} A^{(1)}(\xi_k) &= \begin{bmatrix} \xi_k^1 & 0.2 & 0.6 \\ 0.3 & 0.7 + \xi_k^1 & 1.0 \\ -1.0 - \xi_k^3 & 0.6 & 1.7 + \xi_k^1 \end{bmatrix}, \\ A^{(2)}(\xi_k) &= \begin{bmatrix} -0.6 + \xi_k^1 & 0.2 & 0.6 + (\xi_k^2)^2 \\ 0.3 - 2\xi_k^2 & 0.1 + \xi_k^1 & 1.0 \\ -1.0 - \xi_k^3 & 0.6 & 1.1 + \xi_k^1 \end{bmatrix}, \\ B(\xi_k) &= [0, 1, 2]^T \end{aligned} \quad (62)$$

(i.e.,  $L = 2$ ). This system is not robustly stable without feedback. To stabilize this system, the following gain was designed in [NHH15] using a conservative inequality condition.

$$\begin{aligned} F^{(1)} &= [0.3934 \quad -0.3383 \quad -0.8276], \\ F^{(2)} &= [0.2804 \quad -0.2477 \quad -0.5267] \end{aligned} \quad (63)$$

For the closed-loop system  $G_\theta$  using this feedback gain, we numerically minimized  $\lambda$  with respect to the LMI consisting of (27) and (36) (i.e., our new result for analysis in Section 3), which gives an upper bound of the minimal  $\lambda$  robustly satisfying (13). Then, we obtained 0.6449 ( $< 1$ ) as the upper bound, which implies that the designed gain indeed stabilizes  $G_\theta$ . We used MATLAB and Symbolic Math Toolbox for computing the expectations in (39), and additionally used YALMIP [LÖ4] and SDPT3 [TTT03] for solving LMIs. The same tools were used also for the synthesis discussed soon later.

For the same open-loop system, we also designed a gain-scheduled feedback controller by using the new result in Section 4. Specifically, we minimized  $\lambda$  with respect to (58) and (59) with  $A_s = 0$ , and obtained  $K$ ,  $V^{(l)}$ ,  $Q^{(l)}$  ( $l = 1, 2$ ) leading to

$$\begin{aligned} F^{(1)} &= [0.4037 \quad -0.3135 \quad -0.7523], \\ F^{(2)} &= [0.2557 \quad -0.2302 \quad -0.4592]. \end{aligned} \quad (64)$$

The achieved minimal  $\lambda$  at this synthesis stage was 0.6269. For the closed-loop system  $G_\theta$  using this new feedback gain, we also perform post-synthesis analysis using (27) and (36) to make the comparison fair. Then, we obtained 0.6251 as an upper bound of the minimal  $\lambda$  robustly satisfying (13). The small gap between 0.6269 and 0.6251 is considered to come from the restriction  $A_s = 0$ , since no other restriction is additionally used in the derivation of the LMI for synthesis in Section 4.

Since 0.6251 achieved by our new result is smaller than 0.6449 achieved by the earlier result, the reduction of conservativeness in stochastic gain-scheduled controller synthesis is successfully demonstrated. It would be also worth recalling that the new result does not necessarily require the use of a sample-based approach, in contrast to the earlier result in [NHH15]. The value of 0.6449 was the analysis result obtained using our new LMI condition, and even this value was impossible to be ensured only with the earlier result. The present new result successfully resolved all these inconvenience.

## 6 Conclusions

This paper discussed LMI conditions for stochastic gain-scheduled control of discrete-time systems characterized by random polytopes. Further extensions of the proposed approach from the

viewpoint of flexibility in system class might be possible through using the results on descriptor systems [EPA15] and/or ROLMIP [AFOP19] as in the case of deterministic systems. Control performance such as stochastic  $H_2$  performance could also be discussed in a fashion similar to the present stabilization.

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